Math Club Problems

Rich Schwartz

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Problems with answers:
Suppose P_1 and P_2 are convex polygons in the plane, with P_1 contained in the interior of the region bounded by P_2. Prove that the length of P_1 is less than the length of P_2. This result is due to Archimedes.

Say that a rectangle is good if at least one of its sides has integer length. Suppose that a rectangle R is tiled by smaller rectangles, each of which is good. Prove that R is good.

Consider the sequence $\{a_n\}$ defined recursively by the rule $a_1 = 1$ and $a_{n+1} = 1/a_n + a_n/2$. Prove that $\lim a_n = \sqrt{2}$.

Suppose that a quadrilateral Q has its vertices contained in a circle. Suppose that the sides of Q have length A, B, C, D. Prove that the area of Q is given by $\frac{1}{4}\sqrt{(A+B+C-D)(A+B+D-C)(A+C+D-B)(B+C+D-A)}$. This is called Brahmagupta's formula.

Suppose T is a triangle, with sides s_1, s_2, s_3. For $j = 1, 2, 3$ let Δ_j be the equilateral triangle which shares the edge e_j with T and otherwise is disjoint from T. Let m_j be the center point of Δ_j. Prove that m_1, m_2, m_3 are the vertices of an equilateral triangle. This is known as Napoleon's theorem.

Suppose that every point in the plane is colored one of 3 colors. Prove that there are two points in the plane, one unit apart, with the same color.

Prove that it is possible to color the plane with 7 colors such that no two points, which are one unit apart, have the same color.

Say that two polygons P and P' are equivalent if they have the same number of sides, and if their sides can be matched up in a way such that each side of P has the same length as the corresponding side of P'. Say that $P' \prec P$ if the region bounded by P has at least as much area as the region bounded by P'. Say that P is maximal if $P' \prec P$ for all P' equivalent to P. Prove that P is maximal if and only if all the vertices of P lie on a circle. This result is essentially the isoperimetric inequality.

Say that two triangles T_1 and T_2 are neighbors if you can cut T_1 into two triangles and rearrange them to make T_2. Say that triangles T and T' are equivalent if there is a finite sequence $T = T_0, ..., T_n = T'$ such that T_j and T_{j+1} are neighbors for all j. Prove that every triangle is equivalent to an acute triangle.

Let P be a polygon in the plane. Prove that there are 4 points on P which form the vertices of a square.

Say that two polyhedra P and P' are equivalent if there is a bijection between the faces of P and P' such that each face of P is isometric to the corresponding face of P'. Prove that P and P' are isometric to each other if they are equivalent and both convex. This is known as the Cauchy rigidity theorem.

Given a polygon P, let P' be the polygon obtained by joining together the midpoints of the sides of P. Define $P^{(0)} = P$ and $P^{(n)} = (P^{(n-1)})'$. Prove that $P^{(n)}$ is convex, for sufficiently large n.

Give an example of a connected subset $S \subset \boldsymbol{R}^2$ which does not admit any nonconstant continuous maps $f : \boldsymbol{R} \to S$. So, no two points in S can be joined by a path in S.

Let $\boldsymbol{Z}^2$ be the integer lattice in the plane. Let T be a triangular region in the plane such that $T \cap \boldsymbol{Z}^2$ is precisely the three vertices of T. Prove that the area of T is $1/2$. This result is part of Pick's theorem.

Prove that it is possible to tile $\boldsymbol{R}^3$ using only regular tetrahedra and regular octahedra.

Let $D_1, ..., D_n$ be disks in the plane. Also let $D'_1, ..., D'_n$ be disks in the plane. Suppose that D'_j is the same size as D_j, for all j. Suppose also that the distance from the center of D_i to the center of D_j is greater than the distance from the center of D'_i to the center of D'_j for all pairs (i, j) with $i \neq j$. Suppose that $\bigcap D_j \neq \emptyset$. Prove that $\bigcap D'_j \neq \emptyset$.

Let $\boldsymbol{Z}^2$ be the set of integer points in the plane. Suppose there is a constant K and a map $f : \boldsymbol{Z}^2 \to \boldsymbol{Z}^2$ such that $\|f(x) - f(y)\|/\|x - y\| \in [1/K, K]$ for all $x, y \in \boldsymbol{Z}^2$. Prove that there is some other constant K' such that every point of $\boldsymbol{Z}^2$ is within K' of some point of $f(\boldsymbol{Z}^2)$.

Suppose that $\{D_j\}_{j=1}^{\infty}$ is a collection of disks in the plane whose total area is less than 1. Prove that it is possible to rearrange the disks so that they are disjoint from each other and all contained inside the disk of radius 100.

Given a geometric proof that the map $z \to 1/z$ maps circles in the plane to either circles or straight lines.

Suppose that $f : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is a continuous bijection which maps straight lines to straight lines. Suppose also that $f(0) = 0$. Prove that f is a linear transformation. This result is essentially the fundamental theorem of projective geometry.

Let $a(1) = 10$ and $a(n+1) = 10^{a(n)}$. Let $b(1) = 10$ and $b(n+1) = a(b(n))$. Which is larger, $a(b(100))$ or $b(a(100))$?

Construct a continuous onto map $f : S^1 \to S^2$ such that $f^{-1}(x)$ is either 1 or 3 points for each $x \in S^2$. I've never actually seen this construction, but I think its probably known.

Say that a finite union S of line segments in the plane blocks the unit circle if every line which intersects the unit circle also intersects S. Prove that S has total length at least $1/2$ if it blocks the unit circle.

Suppose that A is a closed and bounded subset of $\boldsymbol{R}^2$ which has the following property: If $x, y \in A$ then $(x + y)/2 \notin A$. Prove that $\text{area}(A) = 0$.

On an island off the coast of California there live a group of people who share a peculiar religion and genetic mutation. Each person has been born with a spot on their forehead, either red or blue. There are no mirrors on the island, and the greatest crime is that one person tells another the color of their spot. This crime is never committed, so people cannot figure out the color of their spot. The basic tenet of the religion is that a person must kill themselves, at midnight, if they discover the color of their spot. These people spend every day seated in an enormous circle, looking at everyone else's spot. So, in short, everyone knows everyone else's spot color except their own. Currently there are M people with red spots and N people with blue spots. Life proceeds happily and without incident until one day a parachutist plummets to the earth−his parachute having failed to open−right in the center of the circle. The parachutist gets to his feet, looks around the circle, says "What a lovely red spot" and then drops dead. Prove that everyone on the island eventually kills themself, independent of the values of M and N.

Some Open Problems:
Most of these problems are well-known unsolved problems. They're easy to state and fun to think about, but the chances of solving them are probably infinitesimal. Lots of people have tried and failed. On the other hand, thinking unsuccessfully about them might still lead to interesting new reults.

Given a positive real number x let (x) denote the decimal part of x. For instance $(3.1415) = .1415$. Does the sequence $\{(3^n/2^n)\}_{n=1}^{\infty}$ have $1/2$ as an accumulation point? More generally, is the sequence above dense in $[0,1]$? One of the best partial results on questions like this is due to our very own Dan Rudolph.

If d is an integer then the coefficients of the continued fraction expansion of d are bounded. Are the coefficients in the continued fraction expansion for the cube-root of 2 bounded or unbounded?

Is there a smooth non-zero vector field on $\boldsymbol{R}^3$ whose flow lines are all closed loops? This is an open problem in the smooth case, though 10 years ago someone named Vogt constructed an example of a differentiable vector field with this property.

Prove that there is some $\epsilon > 0$ with the following property: If P is a polygon in the plane which bounds a region containing the unit disk then there are 4 points on P which form the vertices of a square whose side length is at least ϵ.

Let $n \in \{4,5,6\}$. Is it possible to color the plane with n colors so that no two points in the plane, which are one unit apart, get the same color?

Suppose that $X = \{x_1, ..., x_n\}$ is a set of $n \geq 3$ points in the plane, no three of which are on the same line. Let X_3 denote the set of ordered triples of distinct points in X. There is a map $\phi : X_3 \to \{-1, 1\}$ defined as follows: $\phi(x_i, x_j, x_k) = 1$ if the points x_i, x_j, x_k go clockwise around the triangle they determine and $\phi(x_i, x_j, x_k) = -1$ if the points x_i, x_j, x_k go counterclockwise around the triangle they determine. Say that X_0 and X_1 are equivalent if the maps ϕ_0 and ϕ_1 are the same. If X_0 and X_1 are equivalent, are they part of a continuous family $\{X_t |\ t \in [0,1]\}$ of equivalent configurations.

Define a function f, on the natural numbers, as follows: $f(n) = 3n + 1$ if n is odd and $f(n) = n/2$ if n is even. Given any n consider the sequence n,f(n), $f(f(n))$, etc. Must this sequence contain the number 1?

Say that a finite union S of line segments in the plane blocks the unit circle if every line which intersects the unit circle also intersects S. Prove that there is some $\epsilon > 0$ so that S has total length at least $\epsilon + 1/2$ provided that S blocks the unit circle. This ϵ is meant to be independent of S.