# From Jacobians to one-motives: exposition of a conjecture of Deligne

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ABSTRACT. Deligne has conjectured that certain mixed Hodge theoretic invariants of complex algebraic invariants are motivic. This conjecture specializes to an algebraic construction of the Jacobian for smooth projective curves, which was done by A. Weil. The conjecture (and one-motives) are motivated by means of Jacobians, generalized Jacobians of Rosenlicht, and Serre's generalized Albanese varieties. We discuss the connections with the Hodge and the generalized Hodge conjecture. We end with some applications to number theory by providing partial answers to questions of Serre, Katz and Jannsen.

*Parmi toutes les choses mathématiques que j'avais eu le privilège de découvrir et d'amener au jour, cette realité des motifs m'apparaît encore comme la plus fascinante, la plus chargée de mystère - au cœur même de l'identité profonde entre la "geométrie" et l' "arithmétique". Et le "yoga des motifs" auquel m'a conduit cette réalité longtemps ignorée est peut-être le plus puissant instrument de découverte que j'ai dégagé dans cette première période de ma vie de mathématicien.*

— **Alexandre Grothendieck**
"Récoltes et Semailles"

One of the most profound contributions of Grothendieck to mathematics is the concept of *motives*. Even though Grothendieck himself wrote very little on this subject, the philosophy of motives played a very important role in his research. While the influence and importance of motives is clear, it is hard to fathom the true extent of their potential impact. Motives provide fascinating bridges among the mathematical trinity: algebra, geometry, and analysis ([**50**] is a beautiful introduction). The question *What is a motive?* eludes an answer even today (cf. Problems of present day mathematics in [**10**] pp. 39-42).

The prototype of motives are abelian varieties. One aspect of the vision of Grothendieck, namely that motives (= pure motives) are attached to smooth projective varieties, is itself inspired by the theory of the Picard and Albanese varieties of smooth projective varieties. The latter theory can be viewed as a *purely algebraic* definition of the first cohomology and homology groups of smooth projective

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varieties. One thinks of the Picard and Albanese varieties as the motivic $h^1$ and $h_1$ of smooth projective varieties. The amazingly potent nature of a motive is already evident from the rich theory of smooth projective curves and their Jacobians. We begin by reviewing briefly this classical episode, surely one of the most beautiful edifices of nineteenth century mathematics. We consider generalizations of Albanese varieties to smooth varieties which naturally leads us to semiabelian varieties. From there, it is a short step to one-motives. Deligne's conjecture is motivated as a counterpart for arbitrary complex varieties of the generalized Hodge conjecture for smooth projective complex varieties. We end with a few arithmetical applications of the proof of Deligne's conjecture.

Even though one-motives were defined by Deligne in 1972, they do not seem to be that well known. Abelian varieties provide a testing ground for many conjectures in the theory of pure motives (just as toric varieties do for higher dimensional algebraic geometry). In a similar vein, one could argue (and successfully so!) that one-motives provide a fertile testing ground for the theory of mixed motives (a theory that remains mysterious even today!). Deligne's view (cf. §6) is that one-motives are precisely the *mixed motives of level one*[1]. In particular, these should enable one to describe the $H^1$ of arbitrary varieties thereby generalizing the description by abelian varieties of the $H^1$ of smooth projective varieties. Many of the hypothesized properties of mixed motives can be verified for one-motives [29]. The role of one-motives in the development of mixed Hodge structures underscores their importance. Deligne admits that it was exactly the close relation between one-motives over $\mathbb{C}$ and mixed Hodge structures that convinced him of the validity of the philosophy of mixed Hodge structures ([19], 2.1):

Pour aller plus loin, il fallait se convaincre que tout motif a une filtration par le poids $W$, croissante, avec $Gr_i^W(M)$ pur de poids $i$ (= facteur direct de $H_{mot}^i(X)$ pour $X$ projectif non singulier). C'est sur le $H^1$ des courbes, i.e., sur les 1-motifs que je m'en suis convaincu, et le premier test qu'a dû passer la définition des structures de Hodge mixtes est qu'elles redonnent comme cas particulier les 1-motifs sur $\mathbb{C}$.

The appearance of one-motives in the cohomology of algebraic varieties is not limited to $H^1$. In fact, Deligne's conjecture (the focus of this paper) predicts their appearance in the higher cohomology of higher dimensional complex varieties.

There has been a slow but surely growing interest in the theory of one-motives. We list a few instances of their appearance: Mumford-Tate groups and special values of G-functions (Y. André [1] [2]), transcendence issues (D. Bertrand [7], D. Masser [36], K. Ribet [43]), automorphic forms and Shimura varieties (J.-L. Brylinski [11], C. Brinkmann [9]), crystalline cohomology (J.M. Fontaine- K. Joshi [22]), geometric monodromy (M. Raynaud [42]), Tate curves and rigid analytic uniformization (Raynaud [19] 2.1), moduli spaces of abelian varieties and cubical structures (C. L. Chai and G. Faltings [14], L. Breen [8]), Fourier transform and geometric Langlands correspondence (G. Laumon [32], A. Beilinson and V. Drinfeld), function field analogue of Stark's conjecture (J. Tate [52]).

There are no proofs in this paper; its intention is to be purely expository and motivational. For details and proofs, we refer to the bibliography at the end of the paper.

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[1] This is the reason for the terminology of one-motives.

## 1. Jacobians

Consider a smooth projective complex curve $C$, i.e., a compact Riemann surface $C^{an}$. Associated with it is an abelian variety (its Jacobian variety $J$), usually constructed via the period mapping (a transcendental construction):

$$J = \frac{H^0(C^{an}; \Omega^1_{an})^*}{Im H_1(C^{an}; \mathbb{Z})}.$$

where $H^0(C^{an}; \Omega^1_{an})$ denotes the complex vector space of global differential forms of degree one and $*$ denotes the dual complex vector space.

There is a morphism from the curve to its Jacobian; this is unique up to translations in the Jacobian. It induces an isomorphism on $H_1(C; \mathbb{Z}) \xrightarrow{\sim} H_1(J; \mathbb{Z})$. This is actually an isomorphism of polarized Hodge structures. One can even show that the polarized Hodge structure on $H_1(C; \mathbb{Z})$ uniquely determines the curve up to isomorphism (Torelli's theorem).

The first amazing fact (proved by Weil) is that the Jacobian variety admits a *purely algebraic construction*. What does one mean by the phrase "purely algebraic construction" ? We mean that this invariant can be constructed without ever leaving the universe of algebraic geometry. One is allowed any (and every) tool – geometric, cohomological, sheaf-theoretic, cycle theoretic methods – strictly in the realm of algebraic geometry. One is not allowed to use, for example, the classical topology of the complex numbers. Such a construction for *a priori* transcendental invariants lends their definition a wider scope, that is, a setting far more general than originally envisioned or intended. For instance, the "purely algebraic construction" of the Jacobian allows it to be defined in positive characteristic, the starting point of Weil's proof of the Riemann hypothesis for curves over finite fields; as is well known, it was the Riemann hypothesis that later developed into the celebrated Weil conjectures (aka Deligne's theorem).

Other consequences of the algebraic construction of the Jacobian include

(1) a definition of the Jacobian variety (an abelian variety over $k$) valid for a smooth projective curve over any field $k$.

(2) generalization of Torelli's theorem to any perfect field: The canonically polarized Jacobian of a smooth projective curve determines the curve up to isomorphism.

(3) Galois representations on the $\ell$-adic Tate module of the Jacobian, a powerful tool in the study of rational points of curves over number fields, e.g., Mordell's conjecture, modular curves.

(3) *geometric class field theory* for curves (M. Rosenlicht, S. Lang [**31**], and J.-P. Serre [**48**]).

The classical theory of differentials of the first, second and third kind on a complex curve $C$ are subsumed in the theory of the Jacobian [**53**, **37**] (cf. 3.1).

From the Jacobian $J$ of a smooth projective curve $C$ over a field $k$, one can obtain all the cohomological invariants associated with the curve by applying "realization" functors. Let us be more explicit as to what this means.

• **(Hodge)** For every embedding $\iota : k \hookrightarrow \mathbb{C}$, we get a Hodge structure $H_\iota :=$ $H^1(C_\iota; \mathbb{Z}(1))$ [2] by considering the compact Riemann surface $C_\iota^{an}$ associated to the complex curve $C_\iota$. This Hodge structure is of type $\{(-1,0),(0,-1)\}$; it is polarizable, by Poincaré duality. A classical theorem of Riemann asserts that every such structure $H$ corresponds to an essentially unique complex abelian variety $A(H)$. More precisely, one has :

THEOREM 1. *The functor $A \mapsto H_1(A; \mathbb{Z})$ provides an equivalence of categories between the category of complex abelian varieties and torsion-free polarizable Hodge structures of type $\{(-1,0),(0,-1)\}$.*

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The complex abelian variety $J_\iota$ obtained by base change via $\iota$, is canonically isomorphic to the $A(H_\iota)$ corresponding to $H_\iota$; this isomorphism is induced by the exponential sequence on $C_\iota$. These are valid for every $\iota : k \hookrightarrow \mathbb{C}$.

Even though the process of obtaining the Hodge structure $H_\iota$ is a transcendental one (it uses the classical topology on the complex curve $C_\iota$), one can separate the algebro-geometric step - that of associating the Jacobian $J$ with $C$ which makes no reference to $\iota$ - in this process from the transcendental step (which depends on $\iota$). One has a factorization

$$C \to J \xrightarrow{\iota} J_\iota \leftrightarrows H^1(C_\iota; \mathbb{Z}(1))$$

where the first association is completely algebraic (and independent of $\iota$) and the last association is given by theorem 1.

• **(Étale)** Let $G = Gal(\bar{k}/k)$ be the Galois group of an algebraic closure $\bar{k}$ of $k$. Let $\ell$ be a prime different from the characteristic $p$ of $k$. One has the étale cohomology group $H^1_{et}(C \times_k \bar{k}; \mathbb{Z}_\ell(1))$ for each $\ell$; this is a representation of $G$.

From every abelian variety $A$ over $k$, one can fabricate an $\ell$-adic representation of $G$ by considering the $\ell$-adic Tate module $T_\ell(A)$, the projective limit of the $\ell$-power torsion points of $A(\bar{k})$. This construction, applied to the Jacobian $J$, yields the Galois representation $T_\ell J$, canonically isomorphic to the Galois representation $H^1_{et}(C \times_k \bar{k}; \mathbb{Z}_\ell(1))$, via the Kummer sequence.

Here too, the main point is that the association $C \mapsto J$ is the common step in obtaining any of the étale cohomology groups $H^1_{et}(C \times_k \bar{k}; \mathbb{Z}_\ell(1))$.

• **(De Rham)** Assume that the characteristic of $k$ is zero. Consider the first De Rham cohomology group $H^1_{dR}(C)$ of $C$; it is a vector space over $k$. One can obtain this as well from the Jacobian $J$: We may assume that $C$ has a rational point and we use it to get a morphism $\alpha$ from $C$ to $J$. The induced map $\alpha^* : H^1_{dR}(J) \to H^1_{dR}(C)$ on the DeRham cohomology groups, independent of the rational point, is an isomorphism. This isomorphism is usually stated as follows: *every global differential one-form on $C$ is the pullback, via $\alpha^*$, of a unique translation-invariant differential one-form on $J$.*

There are various relations between the above cohomology theories; these are formalized as "compatibility isomorphisms". Consider a smooth projective curve $C$ over $\mathbb{Q}$. The De Rham cohomology group $H^1_{dR}(C)$ is a vector space over $\mathbb{Q}$. The

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[2]Here $\mathbb{Z}(1) = 2\pi i \mathbb{Z}$ is the Tate Hodge structure; it has rank one and of type $(-1,-1)$. We have shifted from homology to cohomology, for ease of exposition.

complex curve $C_\iota$ obtained from $C$ for the unique embedding $\iota : \mathbb{Q} \hookrightarrow \mathbb{C}$ ("the infinite prime") provides $H^1(C_\iota; \mathbb{Q})$, a vector space over $\mathbb{Q}$, using the usual singular homology theory. One has an isomorphism $H^1_{dR}(C) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^1(C_\iota; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ of $\mathbb{C}$-vector spaces. This isomorphism usually does *not* preserve the underlying $\mathbb{Q}$-vector spaces. The determinant of the isomorphism (= period) is usually a transcendental number and arises in deep arithmetic questions about $C$ [**18**].

## 2. Intermediate Jacobians and the Hodge conjecture.

One may ask whether the Jacobian of a smooth projective curve discussed in the previous section admits a generalization to higher dimensions. It turns out that it admits two such generalizations.

Associated with any smooth projective complex variety $X$ are two abelian varieties: the Albanese variety $Alb(X)$ and the Picard variety $Pic(X)$. Let $X^{an}$ be the compact complex analytic manifold associated with $X$. Let $\Omega^i_{an}$ be the locally free sheaf of $i$-forms on $X^{an}$ and $\Omega^0_{an} = \mathcal{O}_{X^{an}}$, the structure sheaf of regular functions. The cohomology groups of $X^{an}$ admit a canonical decomposition as complex vector spaces:

$$H^n(X^{an}; \mathbb{C}) = \bigoplus_{i=0}^{n} H^{n-i}(X^{an}; \Omega^i_{an})$$

This decomposition is known as the Hodge decomposition of the cohomology of $X^{an}$. One says that $H^n(X^{an}; \mathbb{Z})$ is a Hodge structure of weight $n$ and type $\{(0,n), ..., (n,0)\}$.

The pairing of (homological) cycles and differential forms on $X^{an}$ gives us, by Stokes' theorem, a natural map from $H_1(X^{an}; \mathbb{Z})$ to $H^0(X^{an}; \Omega^1_{an})^*$; it is an injection (mod torsion). We denote its image by $Im H_1(X^{an}; \mathbb{Z})$. Let $\mu : \mathbb{Z}(1) \to \mathcal{O}_{X^{an}}$ be the natural map of sheaves on $X^{an}$. We denote the image of the induced map $\mu_* : H^1(X^{an}; \mathbb{Z}(1)) \to H^1(X^{an}; \mathcal{O}_{X^{an}})$ by $Im H^1(X^{an}; \mathbb{Z}(1))$.

One possible description of the Albanese variety and the Picard variety of $X$ is:

$$Alb(X) = \frac{H^0(X^{an}; \Omega^1_{an})^*}{Im H_1(X^{an}; \mathbb{Z})}$$

$$Pic^0(X) = \frac{H^1(X^{an}; \mathcal{O}_{X^{an}})}{Im H^1(X^{an}; \mathbb{Z}(1))}$$

It follows that one has isomorphisms

$$H_1(X; \mathbb{Z})/torsion \xrightarrow{\sim} H_1(Alb(X); \mathbb{Z}) \qquad H_1(Pic(X); \mathbb{Z}) \xrightarrow{\sim} H^1(X; \mathbb{Z}(1))$$

of polarizable Hodge structures. Therefore, it is possible to recover the polarizable Hodge structure on $H_1(X; \mathbb{Z})/torsion$ and $H^1(X; \mathbb{Z})$ from the Albanese and the Picard variety. The key point here is theorem 1.

The second amazing fact is that the Albanese variety and the Picard variety also admit a *purely algebraic construction*. Therefore it is possible to develop a theory of the Albanese and the Picard variety for a smooth projective variety over any field. This has been used by S. Lang to extend *geometric class field theory* to higher dimensions. The theory of Albanese and Picard varieties should be viewed as an algebraic version of the classical theory of the first homology and cohomology groups for smooth complex projective varieties. For example, a classical observation (due to J. Igusa and Weil) is that the Albanese and Picard varieties of any smooth

projective variety over a field are *dual* abelian varieties. This can be viewed as a special case of a universal coefficient formula. The Albanese and the Picard varieties of a smooth projective curve are both isomorphic to the Jacobian. One obtains the

**Theorem.** (Abel-Jacobi) *The Jacobian of a smooth projective curve is a self-dual abelian variety.*

The properties of the Jacobian about the realizations in the various cohomologies of the curve carry over *mutatis mutandis* to the Picard variety of a smooth projective variety.

These previous remarks indicate a strong tie between the theory of abelian varieties and the cohomology of smooth projective varieties over a field. The question that poses itself is whether the entire cohomology $H^*(X)$ (with its Hodge structure) of a smooth complex projective variety $X$ admits a *purely algebraic* construction. It turns out that, in this generality, the answer is no. However, one can still hope that parts of the cohomology admit an algebraic construction. The Hodge conjecture and its generalization formulate the expectations.

**2.1. Hodge conjecture.** Let $X$ be a smooth projective complex variety. Let $d$ be the dimension of $X$. For the even-dimensional cohomology groups, one has the

**Conjecture.** (W. V. D. Hodge [**27**]) *The $\mathbb{Q}$-vector space*

$$H_{\mathbb{Q}}^{i,i}(X) := (H^{2i}(X;\mathbb{Q}) \cap H^{i,i}) \subset H^{2i}(X;\mathbb{C})$$

*is generated by the image of codimension $i$ algebraic cycles on $X$ under the cycle class map.*

**Corollary** (of the conjecture). *Let $V$ be a smooth projective variety over a field $k$. Fix a natural number $i$. The dimension of the $\mathbb{Q}$-vector space $H_{\mathbb{Q}}^{i,i}(V \times_\iota \mathbb{C})$ is independent of the complex imbedding $\iota : k \hookrightarrow \mathbb{C}$.*

The only cases in which the above conjecture is known for any complex smooth projective variety are $i = 1, d-1$; this is due to S. Lefschetz, K. Kodaira and D. C. Spencer. For $i = 1$, one obtains that the $\mathbb{Q}$-vector space $NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ obtained from the Néron-Severi group of $X$ is isomorphic to $H_{\mathbb{Q}}^{1,1}(X)$, the famous $(1,1)$-theorem of Lefschetz. This yields the

THEOREM 2. *For any smooth projective variety $V$ over any field $k$, the dimension of the $\mathbb{Q}$-vector space $H_{\mathbb{Q}}^{1,1}(V \times_\iota \mathbb{C})$ is independent of the complex imbedding $\iota : k \hookrightarrow \mathbb{C}$.*

For the odd-dimensional cohomology groups, one has the theory of intermediate Jacobians due to P. Griffiths [**25**]; these compact complex tori however are *not* always abelian varieties. A famous problem asks for a direct and purely algebraic construction of the maximal abelian varieties contained in them. This may be reformulated as the purely algebraic construction of the abelian variety $A^i(X)$ corresponding to the maximal $\mathbb{Z}$-Hodge structure of level 1 (i.e. of type $\{(i-1,i),(i,i-1)\}$) contained in $H^{2i-1}(X;\mathbb{C})$. This problem is completely answered in general only for $i = 1, d$: one has $A^1(X) = Pic(X)$ and $A^d(X) = Alb(X)$. For other $i$, it would follow from the *generalized Hodge conjecture* (as corrected by Grothendieck [**26**]). Conjecturally, $A^i(X)$ is the image under the Abel-Jacobi map of codimension $i$ cycles on $X$ which are algebraically equivalent to zero. Results in this direction can be found in [**35**], [**45**], [**38**].

A counterpart of these famous problems for *arbitrary* complex varieties is furnished by the conjecture of Deligne. This asks for a purely algebraic construction of algebraic objects (one-motives) that correspond to certain *mixed Hodge structures of level one* contained in the cohomology of complex algebraic varieties.

## 3. One-motives

In envisaging motives, Grothendieck naturally did not restrict himself to smooth and projective varieties. His vision was that mixed motives are attached to arbitrary varieties. Since Poincaré duality does not hold in this generality, one must differentiate between cohomology mixed motives and homology mixed motives. Grothendieck did provide a precise definition of pure motives but not of mixed motives. His definition allows one to interpret *abelian varieties* as examples of pure motives. As regards mixed motives, one still does not have a satisfactory definition (let alone a satisfactory theory!) for them.

Despite the sad state of affairs in mixed motives, we do have a precise definition (by Deligne in 1972) of one-motives. It is hoped that one-motives are prototypes of mixed motives. One believes that the properties of one-motives are true for mixed motives in general. Namely, it is supposed that

- there exists an increasing weight filtration $W$ on mixed motives

- $W$ is strict for morphisms of mixed motives, i.e.,

if $M \xrightarrow{f} N$ is a morphism of mixed motives, then $f(W_i(M)) = f(M) \cap W_i(N)$

- the graded pieces of the weight filtration are (pure) motives.

As we shall see, one-motives over $\mathbb{C}$ give rise to mixed Hodge structures.

Before I turn to the actual definition of one-motives, let me dwell a little bit longer on the theme of Albanese and Picard varieties. If one wants a purely algebraic description of the mixed Hodge structure $H_1(X; \mathbb{Z})/torsion$ of a smooth complex variety $X$, then one sees that complex abelian varieties do not suffice: the possible weights on mixed Hodge structure on $H_1(X; \mathbb{Z})/torsion$ are $-1$ and $-2$. It turns out that the weight $-2$ part is a direct sum of the Tate Hodge structures $\mathbb{Z}(1)$. The mixed Hodge structure $H := H_1(X; \mathbb{Z})/torsion$ defines an extension

$$(**) \qquad 0 \to H' \to H \to H'' \to 0$$

where $H'$ is of type $(-1, -1)$ ($=$ a direct sum of Tate Hodge structures) and $H''$ is a polarizable Hodge structure (pure of weight $-1$) of type $\{(-1, 0), (0, -1)\}$. Polarizability of the graded pieces of mixed Hodge structures on the cohomology of complex algebraic varieties is a consequence of Chow's lemma and Hironaka's resolution of singularities. The identity $H_1(\mathbb{C}^*; \mathbb{Z}) = \mathbb{Z}(1)$ suggests bringing algebraic tori into the picture; one is lead by the extension $(**)$ to the consideration of semi-abelian varieties ($=$ extensions of abelian varieties by algebraic tori). In general, these extensions do not split (they need not even be torsion). The correspondence of Riemann (cf. theorem 1) extends to

THEOREM 3. *The functor $G \mapsto H_1(G; \mathbb{Z})$ from the category of complex semia-belian varieties to the category of torsion-free mixed Hodge structures $\mathcal{H}$ of type*

$$\{(-1, -1), (-1, 0), (0, -1)\}$$

*(with $Gr_{-1}^W \mathcal{H}$ polarizable) is an equivalence of categories.*

Any mixed Hodge structure $\mathcal{H}$ as in this theorem comes from an essentially unique complex semiabelian variety $A(\mathcal{H})$. In addition, the pure Hodge structure $Gr_{-1}^W \mathcal{H}$ corresponds to the maximal abelian variety quotient $A$ of the semiabelian variety $A(\mathcal{H})$. The complex dimension of the maximal torus of $A(\mathcal{H})$ is the rank of the group $Gr_{-2}^W \mathcal{H}$.

The main point of mixed Hodge structures is that there are *nontrivial extensions*. One can ask for the effect of the correspondence in theorem 3 on the extension groups. Let the pure Hodge structure $H'$ of type $(-1,-1)$ correspond to an algebraic torus $T$. Let $H''$ of type $\{(-1,0),(0,-1)\}$ correspond to a complex abelian variety $A$. We assume that $H''$ and $H'$ are torsion-free. Denote by $Ext^1_{MHS}(H'',H')$ the set of isomorphism classes of extensions $(**)$ in the category of mixed Hodge structures. Let us denote by $Ext^1(A,T)$ the set of isomorphism classes of extensions $G$ of $A$ by $T$ in the category of commutative complex algebraic groups ($G$ is a semiabelian variety). These two sets of isomorphism classes form an abelian group under Yoneda addition. The correspondence of theorem 3 actually induces an isomorphism

$$\varepsilon : Ext^1(A,T) \xrightarrow{\sim} Ext^1_{MHS}(H'',H') \qquad G \mapsto H_1(G;\mathbb{Z}).$$

**Theorem.** (Weil-Barsotti) *Let $k$ be an algebraically closed field. Let $\mathcal{A}$ be an abelian variety over $k$ and let $\mathcal{A}^*$ be the dual abelian variety. Let $\mathcal{T}$ be a torus (a variety) over $k$ and let $\mathcal{D}$ be the character group of $\mathcal{T}$, i.e., $\mathcal{D} = Hom(\mathcal{T}, \mathbb{G}_m)$.*

*The group $Ext^1(\mathcal{A}, \mathcal{T})$ of isomorphism classes of extensions in the category of commutative group varietes is naturally isomorphic to the group $Hom(\mathcal{D}, \mathcal{A}^*)$ of homomorphisms from $\mathcal{D}$ to $\mathcal{A}^*$.*

In particular, when $k = \mathbb{C}$, $A$ is a complex abelian variety, and $\mathcal{T} = \mathbb{G}_m$ , we obtain an isomorphism $Ext^1(A, \mathbb{G}_m) \xrightarrow{\sim} Hom(\mathbb{Z}, A^*) \xrightarrow{\sim} A^*(\mathbb{C})$ of abelian groups. So, for the mixed Hodge structures in theorem 3, we obtain not only that the objects and their morphisms are algebraic in nature but that the extension groups also are algebraic.

**Problem I.** *Provide a purely algebraic construction of the semiabelian variety $A(H)$ associated with the mixed Hodge structure $H := H_1(X;\mathbb{Z})/torsion$ of a complex smooth variety $X$.*

For $X$ assumed to be also projective (i.e. $H_1(X)$ is pure of weight $-1$), this semiabelian variety is the classical Albanese variety $Alb(X)$ of $X$. Recall that the Albanese variety $Alb(X)$ enjoys a universal property (which uniquely characterizes it): given a base point $x$ of $X$, there is a unique morphism $f_x : X \to Alb(X)$ sending $x$ to the identity point $e$ of $Alb(X)$; this morphism $f_x$ is universal for morphisms from $X$ to abelian varieties $A$ which send $x$ to the identity point $e_A$ of $A$.

**3.1. Generalized Jacobians.** We consider problem I for smooth curves. Let $U$ be a smooth complex curve (assumed to be connected) and let $C$ be the unique smooth compactification of $U$. (A good example to have in mind is that of $C = E$, an elliptic curve.) Let $\{P_i\}$ be the set $C(\mathbb{C}) - U(\mathbb{C})$ of points ("at infinity"). Put $H := H_1(U;\mathbb{Z})$. We wish to describe the semiabelian variety $A(H)$. The Hodge structure $Gr_{-1}^W H$, isomorphic to $H_1(C;\mathbb{Z})$, corresponds to the Jacobian $J$. Therefore the semiabelian variety $A(H)$ is an extension of $J$ by an algebraic torus $T$.

Since $J$ is characterized as being universal for morphisms from $C$ into abelian varieties (modulo the choice of a base-point), one is tempted to pose a variant for $U$ using semiabelian varieties. Rosenlicht [44] has defined (purely algebraically) a semiabelian variety for any given modulus (= a divisor) on a smooth projective curve. These are known as the generalized Jacobians of Rosenlicht. Consider the generalized Jacobian $J_m$ corresponding to the modulus $m = \Sigma_i P_i$ on $C$. Any base point $x$ in $U$ determines a canonical map $g_x : U \to J_m$. One can show [48] that $g_x$ is universal for morphisms from $U$ into semiabelian varieties $G$ which send $x$ to the identity $e_G$ of $G$. Furthermore, it can be shown that $A(H) = J_m$! In fact, one has the

**Corollary.** *The map* $g_x : U \to J_m$ *induces an isomorphism*

$$g_x : H_1(U; \mathbb{Z}) \xrightarrow{\sim} H_1(J_m; \mathbb{Z}).$$

Rosenlicht's work thus answers Problem I for curves.

Let $d$ denote the cardinality of the set $D := C(\mathbb{C}) - U(\mathbb{C})$. We have an exact sequence

$$0 \to Gr^W_{-2}H \to H = H_1(U; \mathbb{Z}) \to H_1(C; \mathbb{Z}) \to 0.$$

One sees easily that the rank of $Gr^W_{-2}H$ is $d-1$. Heuristically speaking, deleting one point from $C$ does not change the rank of the first homology and every subsequent point that is removed increases the rank by one.

Consider the complex vector space $H_D := H^0(C; \Omega^1(D))$ of differential forms regular on $U$ and allowed to have only logarithmic poles along $D$. These are the classical *differential forms of the third kind*[3]. The dimension of $H_D$ is $g+d-1$ where $g$ is the genus of the curve. Given any regular one-form $\omega$ on $J_m$, one can certainly pull it back via $g_x$ to obtain a regular one-form $g_x^*(\omega)$ on $U$. If $\omega$ is invariant under translations of $J_m$, it turns out that the poles of $g_x^*(\omega)$ are logarithmic which yields $g_x^*(\omega) \in H_D$. The De Rham analog of the corollary is encapsulated in [48]: *Every element of $H_D$ is the pullback via $g_x$ of a unique translation-invariant differential one-form on $J_m$.*

Other natural questions in this context are: How does one describe $T$? How does one determine the extension $J_m$? When is $J_m$ isomorphic to $J \times T$?

Rosenlicht's work provides the answer, as summarized in the next proposition. We have a natural map from $\mathbb{Z}(D)$ (the free abelian group on the elements of $D$) to $H^2(C; \mathbb{Z}(1))$ by sending elements of $D$ (viewed as divisors on $C$) to their degree. Let $B_D$ denote the kernel of this map; it is a free abelian group of rank $d-1$. There is a natural map $\phi : B_D \to J$ by $a \mapsto \mathcal{O}(a)$.

**Proposition.** *With notations as above, one has the following:*

(i) $T = Hom(B_D, \mathbb{G}_m)$.

(ii) *The map $\phi$ corresponds to $J_m$ under the isomorphism in the Weil-Barsotti theorem.*

(iii) *The semiabelian variety $J_m$ is isomorphic to the direct product $J \times T$ if and only if $\phi$ is the zero map.*

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[3]Weil [53] was the first to notice the connection between these and extensions of $J$ by $\mathbb{G}_m$.

From (iii), we see that $J_m$ is generally a nontrivial extension [4]. We also get an answer to when $J_m$ is isogenous to a direct product $J \times T$: this happens exactly when the image of $\phi$ is torsion.

If $U'$ is obtained by deleting a finite number of points from $U$, then one has a surjection $H' := H_1(U'; \mathbb{Z}) \twoheadrightarrow H_1(U; \mathbb{Z})$; hence the rank of $H'$ is larger than that of $H_1(U; \mathbb{Z})$. Translated to abelian varieties, we obtain a surjection $A(H') \twoheadrightarrow A(H)$.

**3.2. Generalized Albaneses.** The existence of universal morphisms into semiabelian varieties for smooth varieties (over algebraically closed fields) has been proved by Serre [**47**]; these are the "generalized Albanese varieties". For any smooth complex variety $X$, one can show

**Proposition.** [**40**] *The generalized Albanese $Alb(X)$ corresponds to the mixed Hodge structure $H_1(X; \mathbb{Z})/torsion$.*

Therefore, the semiabelian variety $A(H)$ associated with the mixed Hodge structure $H := H_1(X; \mathbb{Z})/torsion$ of any *smooth* complex algebraic variety $X$ admits a purely algebraic construction!

A few words about Serre's construction of the generalized Albanese are in order. Let $V$ be a smooth projective complex variety. Let $D_i$ be a finite number of effective integral divisors on $V$. Let $U$ be the open subvariety of $V$ corresponding to the complement of the support of the divisor $D := \Sigma_i D_i$. Consider the group $B_D$ of divisors which are

(a) algebraically equivalent to zero and
(b) supported on $D$.

Let $T_D$ be the torus $Hom(B_D, \mathbb{G}_m)$ obtained from $B_D$ (a free abelian group of finite rank). The natural homomorphism

$$\phi_D : B_D \to Pic(V) \qquad a \mapsto \mathcal{O}(a)$$

determines, by the Weil-Barsotti theorem, a complex semiabelian variety $G_D$. The group variety $G_D$, an extension of $Alb(C)$ by the torus $T_D$, is the generalized Albanese of $U$. Any base point $x$ of $U$ determines a canonical map $g_x : U \to G_D$. it induces an isomorphism

$$g_x : H_1(U; \mathbb{Z})/torsion \xrightarrow{\sim} H_1(G_D; \mathbb{Z}).$$

Consider $H_D := H^0(V; \Omega^1(D))$ the complex vector space of differential one-forms which are regular on $U$ and which are allowed to have logarithmic poles along $D$. Every element of $H_D$ is the pullback of a unique translation-invariant differential one-form on $G_D$ [**47**].

So far, we have concentrated on extending the Albanese variety to smooth varieties. Generalizing the Picard variety to smooth varieties cannot be handled without bringing in one-motives. In fact, they are hidden in the map $\phi_D$[5] To motivate this, let us look at the mixed Hodge structure $H := H^1(U; \mathbb{Z}(1))$, notations as above. The possible weights on it are 0 and $-1$. It sits in an exact sequence:

$$0 \to H^1(V; \mathbb{Z}(1)) \to H \to B_D \to 0.$$

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[4]A consequence of the fact that the universal morphism for $C$, i.e., $g_x : C \to J$ is an imbedding.

[5]Amusingly enough, $\phi_D$ is the starting point of Serre's construction of the generalized Albanese; in other words, the Picard 1-motive of $U$ precedes the generalized Albanese of $U$.

Therefore, $H$ is an extension of $B_D$ by $H^1(V; \mathbb{Z}(1))$. Since $H^1(V; \mathbb{Z}(1))$ corresponds to $Pic(V)$ under 1, we see that the algebraic object that we need to describe $H$ must be an extension of $B_D$ by $Pic(V)$. If we look closely at $\phi_D$ and think of

$$B_D \xrightarrow{\phi_D} Pic(V)$$

as a (two-term) complex $M$ of group schemes, then we see that $M$ provides such an extension. This turns out to be the desired one. The complex $M$ is an example of a one-motive. Indeed, it is the Picard one-motive of the smooth variety $U$ [**40**].

In general, the mixed Hodge structures $H_1(X; \mathbb{Z})/torsion$ and $H^1(X; \mathbb{Z}(1))$ of a complex algebraic variety $X$ are
   (a) of level one with weights $-2, -1, 0$.
   (b) of type $\{(0,0), (-1,0), (0,-1), (-1,-1)\}$ and
   (c) their graded weight $-1$ piece $Gr^W_{-1}$ is polarizable.

**Question II.** *Do the mixed Hodge structures $H_1(X; \mathbb{Z})/torsion$ and $H^1(X; \mathbb{Z}(1))$ admit a purely algebraic construction?*

As we shall see next, one-motives enable us to answer this question.

**3.3. One-motives.** Let $k$ be a perfect field and let $\mathbb{G}$ be the Galois group of an algebraic closure $\bar{k}$ of $k$. A 1-motive $M$ over $k$ consists of a semiabelian variety $G$ over $k$, a finitely generated torsion-free abelian group $B$ with a structure of a $\mathbb{G}$-module, and a homomorphism $u : B \to G(\bar{k})$ of $\mathbb{G}$-modules. In particular, if $k$ is algebraically closed, then $u$ is a homomorphism of abelian groups. We write the 1-motive $M$ as $[B \xrightarrow{u} G]$. A morphism $\phi$ between 1-motives $M$ and $M'$ consists of a pair of morphisms $\phi_1 : B \to B'$ (of $\mathbb{G}$-modules) and $\phi_2 : G \to G'$ (of group schemes) satisfying $\phi_2 u = u'\phi_1$. It is convenient to regard $B$ as corresponding to a group scheme, locally constant in the étale topology on Spec $k$. The category of 1-motives over $k$ is additive (but not abelian).

A morphism $\phi$ (defined over $k$) consisting of $\phi_1 : B \to B'$ and $\phi_2 : G \to G'$ is termed an *isogeny* if $\phi_1$ is injective with finite cokernel and $\phi_2$ is surjective with finite kernel. One can consider the $\mathbb{Q}$-linear category (an abelian category) of isogeny classes of 1-motives over $k$ obtained by formally inverting all isogenies defined over $k$. For any 1-motive $M$, let us denote by $M \otimes \mathbb{Q}$ its isogeny class; we call this an isogeny 1-motive. The objects of the category of isogeny 1-motives are the same as those of 1-motives but the morphisms have changed: $Hom(M \otimes \mathbb{Q}, N \otimes \mathbb{Q}) = Hom(M, N) \otimes_{\mathbb{Z}} \mathbb{Q}$. In effect, isogenies of 1-motives have been transformed into isomorphisms of isogeny 1-motives.

One-motives have weight filtrations: Let $M := [B \xrightarrow{u} G]$ be a 1-motive. The semiabelian variety $G$ is an extension of an abelian variety $A$ by a torus $T$. The weight filtration is defined as $W_{-3}M = 0$, $W_{-2}M = [0 \to T], W_{-1}M = [0 \to G], W_0 M = M$. The graded quotients are $T$, $A$ and $B$. Of course, the triple $(T, A, B)$ does not determine $M$ because of the main point in mixed motives: there are nontrivial extensions in the game!

Deligne has defined various realization functors from the category of 1-motives over $k$ including the Hodge realization $\mathfrak{T}_{\mathbb{Z}}$ (for each imbedding of $k$ into $\mathbb{C}$), étale realization $\mathfrak{T}_\ell$ (for each prime $\ell$ different from the characteristic of $k$), and the De Rham realization $\mathfrak{T}_{DR}$ (when the characteristic of $k$ is zero). One deduces realization functors for isogeny 1-motives: Hodge realization $\mathfrak{T}_{\mathbb{Z}}$ in the category of

$\mathbb{Q}$-mixed Hodge structures, étale realization $\mathfrak{T}_\ell$ in the category of $\mathbb{Q}_\ell$-vector spaces (together with a $\mathbb{G}$-action), De Rham realization $\mathfrak{T}_{DR}$ in $k$-vector spaces.

All these realization functors generalize the familiar constructions with abelian varieties such as forming the Tate module. Cartier duality of tori (sending tori to their character group and vice versa) and duality of abelian varieties can be simultaneously generalized to a theory of duality of one-motives. This is compatible with the realization functors.

## 4. The conjecture

The Hodge realization is the one most relevant to our discussion. In [16] §10.1, Deligne has shown that the Hodge realization embeds the category of 1-motives over $\mathbb{C}$ as a full subcategory of the category of mixed Hodge structures and he provided a description of the image. Namely, he has shown that every torsion-free mixed Hodge structure $H$ of the form

$$(*) \qquad \{(0,0), (-1,0), (0,-1), (-1,-1)\},$$

with $Gr_{-1}^W H$ polarizable, arises from an essentially unique 1-motive $I(H)$ over $\mathbb{C}$. (Note the usual convention that a mixed Hodge structure always refers to a $\mathbb{Z}$-mixed Hodge structure but here the weight filtration $W$ is on $H_\mathbb{Q} := H \otimes_\mathbb{Z} \mathbb{Q}$.) The association $I(H)$ with $H$ is the functor inverse to the Hodge realization $\mathfrak{T}_\mathbb{Z}$. The Hodge realization preserves the natural weight filtrations on both categories.

Question II (3.2) can be rephrased as

**Problem III.** *Do the one-motives associated with the mixed Hodge structures $H^1(X; \mathbb{Z}(1))$ and $H_1(X; \mathbb{Z})/torsion$ of an arbitrary complex algebraic variety $X$ admit a purely algebraic description?*

In fact, mixed Hodge structures of the type $(*)$ can appear in higher cohomology groups as well and one can pose the same problem for the associated one-motives. To do so, let us begin with the

**Theorem.** (Deligne[16]) *Let $V$ be a complex algebraic variety. Denote by $d$ the dimension of $V$.*

(i) *The cohomology groups $H^n(V; \mathbb{Z}(1))$ are endowed with a mixed Hodge structure (functorial for morphisms).*

(ii) *The possible weights on $H^n(V; \mathbb{Z}(1))$ are $-2, -1, 0, ..., 2n-2$ and the possible Hodge numbers are $(p, q) \in [-1, n-1] \times [-1, n-1]$.*

(iii) *If $n \geq d$, then the possible Hodge numbers on $H^n(V; \mathbb{Z}(1))$ are $(p, q) \in [n-d-1, d-1] \times [n-d-1, d-1]$.*

(iv) *If $V$ is proper, then $p + q \leq n - 2$.*

(v) *If $V$ is smooth, then $p + q \geq n - 2$.*

By virtue of the previous theorem, one can define the maximal mixed Hodge substructure $t^n(V; \mathbb{Z}(1))$ of type $(*)$ of $H^n(V; \mathbb{Z}(1))/torsion$. From the previous theorem we deduce that $t^n(V; \mathbb{Z}(1)) = 0$ in these cases:

- $V$ is smooth and $n > 2$.
- $n > 1 + d$.

Let us consider $t^n(X; \mathbb{Z}(1))$ in the case of a smooth projective complex variety $X$. It is enough to look at the cases $n = 0, 1, 2$. For $n = 0, 1$, we obtain $t^n(X; \mathbb{Z}(1)) = H^n(X; \mathbb{Z}(1))$. For the case $n = 0$, we get $H^0(X; \mathbb{Z}(1)) = \mathbb{Z}(1)$. This Hodge structure corresponds to the torus $C^* = H^0(X; \mathbb{G}_m)$, where the last

cohomology group can be taken to be the étale cohomology group (or the Zariski cohomology). For the case $n = 1$, the Hodge structure $H^1(X; \mathbb{Z}(1))$ is pure of weight $-1$ and is of type $\{(-1, 0), (0, -1)\}$. We saw earlier that this structure corresponds under theorem 1 to $Pic(X)$, the Picard variety of $X$. The structure $t^2(X; \mathbb{Z}(1))$ is the $(0,0)$-part of $H^2(X)$. In other words, we have $t^2(X; \mathbb{Z}(1)) = Hom_{MHS}(\mathbb{Z}, H^2(X; \mathbb{Z}(1)))$. The theorem of Lefschetz identifies it as the (maximal torsion-free quotient of the) Néron-Severi group $NS(X)/torsion$ of $X$.

Therefore, in the case of smooth projective varieties $X$, the mixed Hodge structures $t^n(X; \mathbb{Z}(1))$ correspond to algebraic objects and these algebraic objects admit a purely algebraic construction. One is led to the following

**Conjecture.** (Deligne, ibid. 10.4.1, 1973) *For any complex algebraic variety $V$, the 1-motives $I^n(V)$ associated with $t^n(V; \mathbb{Z}(1))$ admit a purely algebraic construction.*

**Remark.** For integers $n > 1+ \dim V$, the conjecture is *vacuously true* by weight considerations as discussed earlier (cf. [**17**]).

**Remark.** For smooth projective complex varieties $X$, we have the Hodge (resp. the generalized Hodge conjecture) which provides a purely algebraic description of the maximal pure Hodge substructure of level zero (resp. one) contained in $H^{2n}(X)$ (resp. $H^{2n-1}(X)$). We see that the weight of the Hodge substructure in question is $2n$ (resp. $2n - 1$). Deligne's conjecture asks for a purely algebraic description of the maximal mixed Hodge structure of level one contained in the low weight part, contained in $W_0 H^n(V; \mathbb{Z}(1))$ to be precise, of the mixed Hodge structure $H^n(V; \mathbb{Z}(1))$ for an *arbitrary* complex algebraic variety. This is why we view Deligne's conjecture as a counterpart to the Hodge conjecture. These two conjectures, considered for smooth projective complex varieties, overlap for small values of $n$.

**4.1. Examples.** For a smooth proper variety $X$, the conjecture is equivalent to an algebraic description of the Picard scheme $Pic_X$. The 1-motive $I^1(X)$ is $[0 \to Pic_X^0]$; the neutral component $Pic_X^0$ of $Pic_X$ is classically known as the Picard variety $Pic(X)$ of $X$. The 1-motive $I^2(X)$ is $[NS(X)/torsion \to 0]$.

For a variety $Y$ assumed to be proper, $I^1(Y)$ is of the form $[0 \to G]$. This is because the mixed Hodge structure $H^1(Y; \mathbb{Z}(1))$ has only negative weights. The semiabelian variety $G$ turns out to be the maximal semiabelian quotient of the neutral component $Pic_Y^0$ of the Picard scheme of $Y$.

We resume notations as in 3.2. The 1-motive $I^1(U)$ is $[B_D \xrightarrow{\phi_D} Pic(V)]$; its dual is the 1-motive $[0 \to G_D]$ where $G_D$ is the "generalized Albanese" of $U$ (3.2). If $\dim U = 1$ i.e. $U$ is a smooth curve, then one obtains the generalized Jacobian $J_D \; (= G_D)$ of Rosenlicht (cf. 3.1).

For these cases, Deligne's conjecture is clearly true. In the case of curves, one has the following

**Theorem.** (Deligne) [**16**] §10.3 *The 1-motive $I^1(X)$ of an arbitrary curve $X$ over $\mathbb{C}$ admits a purely algebraic construction.*

This theorem was, in fact, provided by Deligne as evidence for the possible veracity of the conjecture. Other special cases of the conjecture which were proved earlier:

- J. Carlson has proved the conjecture for $H^2$ of surfaces [12] (assumed to be either projective or smooth). He has also announced results for projective normal crossing schemes [13] which remain unpublished.
- The proof of the conjecture for $H^1$ of arbitrary schemes can be found in [41], [40] where we also formulated a homological version of Deligne's conjecture and proved it for $H_1$ ([6] contains the same results and more).

These new motivic invariants (i.e. the one-motives $I^n$) are generally nontrivial for singular schemes; normal crossing schemes are typical examples. For related results, I refer to the work of S. Lichtenbaum [33, 34] on the connections of one-motives of curves with Suslin homology, of H. Önsiper [39] on the generalized Albanese and zero-cycles, of L. Barbieri-Viale, C. Pedrini and C. Weibel [3] on the one-motive $I^3(S)$ of a projective surface, and of L. Barbieri-Viale and V. Srinivas [4, 5] on the Néron-Severi group of a projective surface.

**Construction.** [41] *For any algebraic scheme $U$ over a perfect field $k$, one defines (in a purely algebraic manner) isogeny one-motives $L^n(U) \otimes \mathbb{Q}$. These are contravariant functorial for arbitrary morphisms. If $d$ is the dimension of $U$, then $L^n(U) \otimes \mathbb{Q} = 0$ for $n > d + 1$.*

The main result of [41] is the following:

**Theorem.** (Deligne's conjecture up to isogeny). *For any complex algebraic variety $V$, the (Hodge-theoretic) isogeny one-motive $I^*(V) \otimes \mathbb{Q}$ is canonically isomorphic to the (purely algebraic) $L^*(V) \otimes \mathbb{Q}$. The isomorphism is furnished by the exponential sequence.*

**Remark.** It seems likely that the methods of H.Gillet-C.Soulé [24] should suffice to prove Deligne's conjecture integrally and thereby refine this theorem. However, details have not yet been worked out.

I would like to mention a few points regarding the proof of the previous theorem. The main idea [6] can be explained easily for a complex *projective* variety $V$. Consider the construction by Deligne of the mixed Hodge structure $H^n(V; \mathbb{Z}(1))$. One takes a smooth proper hypercovering of $V$, in other words, a map $\alpha : X_\bullet \to V$ from a smooth projective simplicial scheme $X_\bullet$ which induces an isomorphism

$$\alpha^* : H^*(V; \mathbb{Z}(1)) \xrightarrow{\sim} H^*(X_\bullet; \mathbb{Z}(1))$$

of mixed Hodge structures. The existence of smooth hypercoverings depend on the resolution of singularities. There is a spectral sequence which calculates the cohomology of $V$ in terms of the cohomology of the various constituents $X_i$. One finds that the Hodge structure $Gr_k^W H^n(V; \mathbb{Z}(1))$ is a subquotient of $H^{k+2}(X_{n-k-2}; \mathbb{Z}(1))$. Since the conjecture deals with only $k = -2, -1, 0$, we see immediately that we need subquotients of $H^0(X_n; \mathbb{Z}(1))$, $H^1(X_{n-1}; \mathbb{Z}(1))$ and $H^2(X_{n-2}; \mathbb{Z}(1))$. Furthermore, since we are interested in $t^n(V; \mathbb{Z}(1))$, instead of $H^2(X_{n-2}; \mathbb{Z}(1))$, we need to consider a subquotient of $t^2(X_{n-2}; \mathbb{Z}(1))$. Since each $X_j$ is smooth projective, our previous discussion tells us that the mixed Hodge structures $H^0(X_n; \mathbb{Z}(1))$, $H^1(X_{n-1}; \mathbb{Z}(1))$ and $t^2(X_{n-2}; \mathbb{Z}(1))$ admit a purely algebraic description. Namely, they are described by the torus $H^0(X_n; \mathbb{G}_m)$, the abelian variety $Pic(X_{n-1})$ and the group $NS(X_{n-2})$. It can be checked that the relevant differentials in the spectral sequence mentioned earlier are actually algebraic in origin; they are either induced by honest geometric maps or Gysin maps. Therefore, we obtain that the required

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[6]This is greatly inspired by the work of Carlson [12].

subquotients of $H^0(X_n; \mathbb{Z}(1))$, $H^1(X_{n-1}; \mathbb{Z}(1))$ and $t^2(X_{n-2}; \mathbb{Z}(1))$ can be obtained by taking subquotients of the torus $H^0(X_n; \mathbb{G}_m)$, the abelian variety $Pic(X_{n-1})$ and the group $NS(X_{n-2})$.

What we have done so far is to describe the graded pieces of the 1-motive $L^n(X_\bullet)$. One still needs to figure out the extensions to complete the description. This can be done elegantly using the Picard scheme of truncated simplicial schemes. A difficult problem is to show that the $L^n(X_\bullet)$ is isomorphic to $I^n(X_\bullet) \xrightarrow{\sim} I^n(V)$. The degeneration (with $\mathbb{Z}$-coefficients) of the spectral sequence mentioned earlier is enough to deduce this isomorphism. However, the degeneration is known at present only with rational coefficients [16]; this weaker degeneration is enough to assure that $L^n(X_\bullet)$ and $I^n(X_\bullet)$ are isogenous whereby the proof of the previous theorem for projective varieties. There is another problem, namely that of showing that the one-motives $L^n(X_\bullet)$ do not depend upon the choice of the hypercovering $X_\bullet$ of $V$. Here also one obtains that, given another hypercovering $Y_\bullet$ of $V$, the one-motive $L^n(X_\bullet)$ is isogenous to the one-motive $L^n(Y_\bullet)$. Therefore, the isogeny one-motives $L^n(X_\bullet) \otimes \mathbb{Q}$ depend only on $V$ and deserve to be denoted by $L^n(V) \otimes \mathbb{Q}$. This finishes the brief description of the construction for a projective variety $V$. The arguments are just a little bit harder for an arbitrary variety [7].

## 5. Applications

The construction in the previous section has several consequences. The last theorem of the previous section is about the Hodge realization of these isogeny one-motives. Considerations of their $\ell$-adic realizations lead to "independence of $\ell$" results for varieties over number fields or finite fields.

### I. A Lefschetz $(1,1)$ theorem for complex varieties.[8]

For any complex variety $X$, the previous theorem provides a *purely algebraic* characterization of the $(1,1)$ part of $H^n(X; \mathbb{C})$. For a complex projective variety $V$, the $(1,1)$-part of $H^2(V; \mathbb{C})$ is provided in terms of divisors on any smooth resolution $\tilde{V}$ of $V$. This is a higher-dimensional generalization of the results in [12], [4, 5] on the Néron-Severi group of complex projective surfaces.

As a byproduct of the purely algebraic description, one obtains the following [41] (compare with theorem 2)

**Theorem.** *For any variety $V$ over a field $k$ and any integer $n$, the rank of $H^{1,1}_{\mathbb{Q}}(V_\iota)^n := Hom_{MHS}(\mathbb{Q}(0), Gr^W_0 H^n(V_\iota; \mathbb{Q}(1)))$ is independent of the complex imbedding $\iota : k \hookrightarrow \mathbb{C}$.*

The usual Néron-Severi group is obtained for $n = 2$ and $V$ projective.

### II. Independence of $\ell$ for parts of étale cohomology.

Let $k$ be a field. Let $\ell$ be a prime different from $p$, the characteristic of $k$. Let $G = Gal(\bar{k}/k)$ be the Galois group. Let $V$ be an *arbitrary* variety over $k$ with

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[7]One can try the same over a perfect field $k$ of positive characteristic using de Jong's weak resolution of singularities. However, in order to use the $\ell$-adic cohomology, one has to isolate the Néron-Severi group in the $\ell$-adic $H^2$; since the $\ell$-adic analog of the Hodge conjecture for divisors, i.e., the Tate conjecture is not known, the generalization to positive characteristic is not at all straightforward.

[8]The terminology is inspired by a talk of V. Srinivas at the ICM'98 Satellite conference on Arithmetic Geometry in Essen. In it, he described his results on the $(1,1)$ part of $H^2$ of complex projective varieties. His methods, compared to mine, have the great advantage of being completely intrinsic. His results do not neglect torsion.

$\bar{V} := V \times_k \bar{k}$. The $\ell$-adic étale cohomology groups $H^i_{et}(\bar{V}; \mathbb{Q}_\ell)$, for varying $\ell$, furnish a system of $\ell$-adic representations of $G$. There are many similarities amongst these $\ell$-adic cohomology theories (which would be explained by the theory of motives). Examples of such similarities: (i) the "compatibility" of Galois representations [49] over number fields. (ii) the "independence of $\ell$" of the characteristic polynomial of Frobenius, over finite fields.

If $k$ is a number field or a finite field, one has a $G$-equivariant weight filtration $W_j$ on the étale cohomology $H^i_{et}(\bar{V}; \mathbb{Q}_\ell)$ for any prime $\ell \neq p$; in other words, the $\mathbb{Q}_\ell$-subspace $W_j H^i_{et}(\bar{V}; \mathbb{Q}_\ell)$ is a subrepresentation of $G$ [17]. They provide new systems of $\ell$-adic representations of $G$.

### 1. $k$ is a number field

**Question** (Serre) [49, 51] *For fixed $V$ and $i$, is the system of Galois representations $H^i_{et}(\bar{V}; \mathbb{Q}_\ell)$ "compatible"?*

The only case in which one has an affirmative answer is that of smooth proper $V$; this depends on Deligne's proof of the Weil conjectures [17]. We prove [41]

**Theorem.** *The system of Galois representations $W_j H^i_{et}(\bar{V}; \mathbb{Q}_\ell)$ is "compatible" for $j = 0, 1$.*

### 2. $k$ is a finite field of characteristic $p$

The Galois group $G$ has a canonical generator $\Phi$, the Frobenius element. Let $\Phi^*_\ell$ denote the endomorphism of $H^i_{et}(\bar{V}; \mathbb{Q}_\ell)$ induced by $\Phi$. For smooth and proper varieties $V$, it is known by Deligne's proof of the Weil conjectures [17] that the system $H^i_{et}(\bar{V}; \mathbb{Q}_\ell)$ of $\ell$-adic representations of $G$ satisfy:

(1) the dimension of the $\mathbb{Q}_\ell$-vector space $H^i_{et}(\bar{V}; \mathbb{Q}_\ell)$ is independent of $\ell$.

(2) the characteristic polynomial of $\Phi^*_\ell$ has $\mathbb{Q}$-coefficients independent of $\ell$.

**Question** (N. Katz) [30] *Are (1) and (2) true for the $\ell$-adic cohomology of arbitrary varieties? Are they true for the weight graded pieces of the cohomology of arbitrary varieties?*

The second question clearly refines the first. I remark that neither is answered by a formal application of Deligne's proof of the Weil conjectures and de Jong's "weak resolution" [21]. Neither is answered even under the assumption of "strong resolution of singularities". Genuinely new ingredients are needed for an answer.

However, utilizing the one-motive theoretic interpretation of parts of $\ell$-adic cohomology, one proves [41] the

**Theorem.** (a) *For any variety $V$ and any integer $i$, the following systems of $\ell$-adic representations of $G$ satisfy (1) and (2) above:*
   (i) $W_0 H^i_{et}(\bar{V}; \mathbb{Q}_\ell)$, (ii) $Gr^W_1 H^i_{et}(\bar{V}; \mathbb{Q}_\ell)$.
(b) *the dimension of the $\mathbb{Q}_\ell$-vector space $Gr^W_1 H^i_{et}(\bar{V}; \mathbb{Q}_\ell)$ is even.*   (cf. [17] 8.1)

The Hodge analog of (b) over $\mathbb{C}$ is trivial: Hodge structures of odd weight are of even dimension. In this context, the dimension in (b) turns out to be the rank of the Tate module of an abelian variety and this rank is always twice the dimension of the abelian variety.

Cases of folklore conjectures about mixed motives [29] can also be proved [41].

## 6. After words

We have seen that one-motives provide a description of several *mixed motives of level one* of varieties over perfect fields. For instance, they provide an algebraic description of the motivic $H^1$ of arbitrary varieties over perfect fields. Doubts have been expressed by Grothendieck about extending these to the case of imperfect fields. In a letter to L. Illusie (reproduced as an appendix to [28]), he writes:

En caractéristique $p > 0$, j'ai des doutès tres sérieux pour l'existence d'une théorie des motifs pas iso du tout à cause des phénomènes de $p$-torsion (surtout pour les schémas qui ne sont projectifs et lisses). Ainsi, si on admet la description de Deligne des "motifs mixtes" de niveau 1 comme le genre de chose permettant de définir un $H^1$ motivique d'un schema pas projectif ou pas lisse, on voit déjà pour une courbe algébrique sur un corps imparfait $k$, la construction ne peut fournir en général qu'un objet du type voulu sur la clôture parfaite de $k$.

The method of hypercoverings (which we use to construct the isogeny one-motives $L^n \otimes \mathbb{Q}$ may not appeal to everyone (certainly it is not my preferred method). However, to this day, one has not found a completely intrinsic description of the mixed Hodge structure on the cohomology of a complex algebraic variety $V$. Until this is achieved, one cannot even dream of providing a purely algebraic construction of the one-motives $I^n(V)$. Needless to say, an intrinsic construction of the mixed Hodge structure on the cohomology of $V$ is one of the outstanding problems in Hodge theory. The folklore analogy [15] between Galois modules and Hodge structures exhibits the weak points of the latter [23]:

"In the mid-sixties, analogies between $(p, q)$-decompositions and representations of the Galois group in étale cohomology was understood. The formal side of this similarity is quite simple: $(p, q)$-decomposition is the action of the torus $\mathbb{C}^*$ while class field theory identifies the multiplicative group of a non-archimedian local field with the maximal abelian quotient of its Galois group, so that the $(p, q)$-decomposition is the Archimedian analog of its Galois representation. On the other hand, the non-formal part of this symmetry is mysterious: Galois representations result from the Galois symmetries in the étale topology, while the roots of the Hodge structures are lost in darkness: hidden "Hodge symmetries" generating these structures are still unknown.

...a natural construction (of the mixed Hodge structure on the cohomology of complex algebraic varieties) is still unknown; in particular, it is not known what kind of analysis lies behind the notion of a mixed Hodge structure [9]."

### Acknowledgments.

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[9]Note the pioneering work of M. Saito [46] in this direction.

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