

A NOTE ON THE BLOCH-TAMAGAWA SPACE AND SELMER GROUPS

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ABSTRACT. For any abelian variety A over a number field, we construct an extension of the Tate-Shafarevich group by the Bloch-Tamagawa space using the recent work of Lichtenbaum and Flach. This gives a new example of a Zagier sequence for the Selmer group of A .

Introduction. Let A be an abelian variety over a number field F and A^\vee its dual. B. Birch and P. Swinnerton-Dyer, interested in defining the Tamagawa number $\tau(A)$ of A , were led to their celebrated conjecture [2, Conjecture 0.2] for the L-function $L(A, s)$ (of A over F) which predicts both its order r of vanishing and its leading term c_A at $s = 1$. The difficulty in defining $\tau(A)$ directly is that the adelic quotient $\frac{A(\mathbb{A}_F)}{A(F)}$ is Hausdorff only when $r = 0$, i.e., $A(F)$ is finite. S. Bloch [2] has introduced a semiabelian variety G over F with quotient A such that $G(F)$ is discrete and cocompact in $G(\mathbb{A}_F)$ [2, Theorem 1.10] and famously proved [2, Theorem 1.17] that the Tamagawa number conjecture - recalled briefly below, see (5) - for G is equivalent to the Birch-Swinnerton-Dyer conjecture for A over F . Observe that G is not a linear algebraic group. The Bloch-Tamagawa space $X_A = \frac{G(\mathbb{A}_F)}{G(F)}$ of A/F is compact and Hausdorff.

The aim of this short note is to indicate a functorial construction of a locally compact group Y_A

$$(1) \quad 0 \rightarrow X_A \rightarrow Y_A \rightarrow \text{III}(A/F) \rightarrow 0,$$

an extension of the Tate-Shafarevich group $\text{III}(A/F)$ by X_A . The compactness of Y_A is clearly equivalent to the finiteness of $\text{III}(A/F)$. This construction would be straightforward if $G(L)$ were discrete in $G(\mathbb{A}_L)$ for all finite extensions L of F . But this is not true (Lemma 4): the quotient

$$\frac{G(\mathbb{A}_L)}{G(L)}$$

is not Hausdorff, in general.

The very simple idea for the construction of Y_A is: *Yoneda's lemma*. Namely, we consider the category of topological \mathcal{G} -modules as a subcategory of the classifying topos $B\mathcal{G}$ of \mathcal{G} (natural from the context of the continuous cohomology of a topological group \mathcal{G} , as in S. Lichtenbaum [10], M. Flach [5]) and construct Y_A via the classifying topos of the Galois group of F .

D. Zagier [18] has pointed out that the Selmer groups $\text{Sel}_m(A/F)$ (6) can be obtained from certain two-extensions (7) of $\text{III}(A/F)$ by $A(F)$; these we call Zagier sequences. We show how Y_A provides a new natural Zagier sequence. In particular, this shows that Y_A is not a split sequence. The Zagier sequence

$$0 \rightarrow A(F) \rightarrow A(\mathbb{A}_F) \rightarrow \left(\frac{A(\mathbb{A}_{\bar{F}})}{A(\bar{F})}\right)^\Gamma \rightarrow \text{III}(A/F) \rightarrow 0$$

seems to be new; this description of $\text{III}(A/F)$ is simpler than the usual definition [2, Lemma 1.16] as it does not involve any higher cohomology groups.

Bloch's construction has been generalized to one-motives; it led to the Bloch-Kato conjecture on Tamagawa numbers of motives [3]; it is close in spirit to Scholl's method of relating non-critical values of L-functions of pure motives to critical values of L-functions of mixed motives [9, p. 252] [13, 14].

Notations. We write $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ for the ring of adèles over \mathbb{Q} ; here $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the ring of finite adèles. For any number field K , we let \mathcal{O}_K be the ring of integers, \mathbb{A}_K denote the ring of adèles $\mathbb{A} \otimes_{\mathbb{Q}} K$ over K ; write \mathbb{I}_K for the ideles. Let \bar{F} be a fixed algebraic closure of F and write $\Gamma = \text{Gal}(\bar{F}/F)$ for the Galois group of F . For any abelian group P and any integer $m > 0$, we write P_m for the m -torsion subgroup of P . A topological abelian group is Hausdorff.

Construction of Y_A . This will use the continuous cohomology of Γ via classifying spaces as in [10, 5] to which we refer for a detailed exposition.

For each field L with $F \subset L \subset \bar{F}$, the group $G(\mathbb{A}_L)$ is a locally compact group. If L/F is Galois, then

$$G(\mathbb{A}_L)^{\text{Gal}(L/F)} = G(\mathbb{A}_F).$$

So

$$\mathbb{E} = \lim_{\rightarrow} G(\mathbb{A}_L),$$

the direct limit of locally compact abelian groups, is equipped with a continuous action of Γ . The natural map

$$(2) \quad E := G(\bar{F}) \hookrightarrow \mathbb{E}$$

is Γ -equivariant. Though the subgroup $G(F) \subset G(\mathbb{A}_F)$ is discrete, the subgroup

$$E \subset \mathbb{E}$$

fails to be discrete; this failure happens at finite level (see Lemma 4 below). The non-Hausdorff nature of the quotient

$$\mathbb{E}/E$$

directs us to consider the classifying space/topos.

Let Top be the site defined by the category of (locally compact) Hausdorff topological spaces with the open covering Grothendieck topology (as in the "gros topos" of [5, §2]). Any locally compact abelian group M defines a sheaf yM of abelian groups on Top ; this (Yoneda) provides a fully faithful embedding of the (additive, but not abelian) category Tab of locally compact abelian groups into the (abelian) category $\mathcal{T}ab$ of sheaves of abelian groups on Top . Write $\mathcal{T}op$ for the category of sheaves of sets on Top and let $y : Top \rightarrow \mathcal{T}op$ be the Yoneda embedding. A generalized topology on a given set S is an object F of $\mathcal{T}op$ with $F(*) = S$.

For any (locally compact) topological group \mathcal{G} , its classifying topos $B\mathcal{G}$ is the category of objects F of $\mathcal{T}op$ together with an action $y\mathcal{G} \times F \rightarrow F$. An abelian group object F of $B\mathcal{G}$ is a sheaf on $\mathcal{T}op$, together with actions $y\mathcal{G}(U) \times F(U) \rightarrow F(U)$, functorial in U ; we write $H^i(\mathcal{G}, F)$ (objects of $\mathcal{T}ab$) for the continuous/topological group cohomology of \mathcal{G} with coefficients in F . These arise from the global section functor

$$B\mathcal{G} \rightarrow \mathcal{T}ab, \quad F \mapsto F^{y\mathcal{G}}.$$

Details for the following facts can be found in [5, §3] and [10].

- (a) (Yoneda) Any topological \mathcal{G} -module M provides an (abelian group) object yM of $B\mathcal{G}$; see [10, Proposition 1.1].

- (b) If $0 \rightarrow M \rightarrow N$ is a map of topological \mathcal{G} -modules with M homeomorphic to its image in N , then the induced map $yM \rightarrow yN$ is injective [5, Lemma 4].
- (c) Applying Propositions 5.1 and 9.4 of [5] to the profinite group Γ and any continuous Γ -module M provide an isomorphism

$$H^i(\Gamma, yM) \simeq yH_{cts}^i(\Gamma, M)$$

between this topological group cohomology and the continuous cohomology (computed via continuous cochains). This is also proved in [10, Corollary 2.4].

For any continuous homomorphism $f : M \rightarrow N$ of topological abelian groups, the cokernel of $yf : yM \rightarrow yN$ is well-defined in $\mathcal{T}ab$ even if the cokernel of f does not exist in Tab . If f is a map of topological \mathcal{G} -modules, then the cokernel of the induced map $yf : yM \rightarrow yN$, a well-defined abelian group object of $B\mathcal{G}$, need not be of the form yP .

By (a) and (b) above, the pair of topological Γ -modules $E \hookrightarrow \mathbb{E}$ (2) gives rise to a pair $yE \hookrightarrow y\mathbb{E}$ of objects of $B\Gamma$. Write \mathcal{Y} for the quotient object $\frac{y\mathbb{E}}{yE}$. As \mathbb{E}/E is not Hausdorff (Lemma 4), \mathcal{Y} is not yN for any topological Γ -module N .

Definition 1. We set $\mathcal{Y}_A = H^0(\Gamma, \mathcal{Y}) \in \mathcal{T}ab$.

Our main result is the

Theorem 2. (i) \mathcal{Y}_A is the Yoneda image yY_A of a Hausdorff locally compact topological abelian group Y_A .

(ii) X_A is an open subgroup of Y_A .

(iii) The group Y_A is compact if and only if $\text{III}(A/F)$ is finite. If Y_A is compact, then the index of X_A in Y_A is equal to $\#\text{III}(A/F)$.

As $\text{III}(A/F)$ is a torsion discrete group, the topology of Y_A is determined by that of X_A .

Proof. (of Theorem 2) The basic point is the proof of (iii). From the exact sequence

$$0 \rightarrow yE \rightarrow y\mathbb{E} \rightarrow \mathcal{Y} \rightarrow 0$$

of abelian objects in $B\Gamma$, we get a long exact sequence (in $\mathcal{T}ab$)

$$\begin{aligned} 0 \rightarrow H^0(\Gamma, yE) \rightarrow H^0(\Gamma, y\mathbb{E}) \rightarrow \\ \rightarrow H^0(\Gamma, \mathcal{Y}) \rightarrow H^1(\Gamma, yE) \xrightarrow{j} H^1(\Gamma, y\mathbb{E}) \rightarrow \cdots \end{aligned}$$

We have the following identities of topological groups: $H^0(\Gamma, yE) = yG(F)$ and $H^0(\Gamma, y\mathbb{E}) = yG(\mathbb{A}_F)$, and by [5, Lemma 4], $\frac{yG(\mathbb{A}_F)}{yG(F)} \simeq yX_A$. This exhibits yX_A as a sub-object of \mathcal{Y}_A and provides the exact sequence

$$0 \rightarrow yX_A \rightarrow \mathcal{Y}_A \rightarrow \text{Ker}(j) \rightarrow 0.$$

If $\text{Ker}(j) = y\text{III}(A/F)$, then $\mathcal{Y}_A = yY_A$ for a unique topological abelian group Y_A because $\text{III}(A/F)$ is a torsion discrete group. Thus, it suffices to identify $\text{Ker}(j)$ as $y\text{III}(A/F)$. Let \mathbb{E}^δ denote \mathbb{E} endowed with the discrete topology; the identity map on the underlying set provides a continuous Γ -equivariant map $\mathbb{E}^\delta \rightarrow \mathbb{E}$. Since E is a discrete Γ -module, the inclusion $E \rightarrow \mathbb{E}$ factorizes via \mathbb{E}^δ . By item (c) above, $\text{Ker}(j)$ is isomorphic to the Yoneda image of the kernel of the composite map

$$H_{cts}^1(\Gamma, E) \xrightarrow{j'} H_{cts}^1(\Gamma, \mathbb{E}^\delta) \xrightarrow{k} H_{cts}^1(\Gamma, \mathbb{E}).$$

Since E and \mathbb{E}^δ are discrete Γ -modules, the map j' identifies with the map of ordinary Galois cohomology groups

$$H^1(\Gamma, E) \xrightarrow{j''} H^1(\Gamma, \mathbb{E}^\delta).$$

The traditional definition [2, Lemma 1.16] of $\text{III}(G/F)$ is as $\text{Ker}(j'')$. As

$$\text{III}(A/F) \simeq \text{III}(G/F)$$

[2, Lemma 1.16], to prove Theorem 2, all that remains is the injectivity of k . This is straightforward from the standard description of H^1 in terms of crossed homomorphisms: if $f : \Gamma \rightarrow \mathbb{E}^\delta$ is a crossed homomorphism with kf principal, then there exists $\alpha \in \mathbb{E}$ with $f : \Gamma \rightarrow \mathbb{E}$ satisfies

$$f(\gamma) = \gamma(\alpha) - \alpha \quad \gamma \in \Gamma.$$

This identity clearly holds in both \mathbb{E} and \mathbb{E}^δ . Since the Γ -orbit of any element of \mathbb{E} is finite, the left hand side is a continuous map from Γ to \mathbb{E}^δ . Thus, f is already a principal crossed (continuous) homomorphism. So k is injective, finishing the proof of Theorem 2. \square

Remark 3. The proof above shows: If the stabilizer of every element of a topological Γ -module N is open in Γ , then the natural map $H^1(\Gamma, N^\delta) \rightarrow H^1(\Gamma, N)$ is injective.

Bloch's semi-abelian variety G . [2, 11]

Write $A^\vee(F) = B \times \text{finite}$. By the Weil-Barsotti formula,

$$\text{Ext}_F^1(A, \mathbb{G}_m) \simeq A^\vee(F).$$

Every point $P \in A^\vee(F)$ determines a semi-abelian variety G_P which is an extension of A by \mathbb{G}_m . Let G be the semiabelian variety determined by B :

$$(3) \quad 0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0,$$

an extension of A by the torus $T = \text{Hom}(B, \mathbb{G}_m)$. The semiabelian variety G is the Cartier dual [4, §10] of the one-motive

$$[B \rightarrow A^\vee].$$

The sequence (3) provides (via Hilbert Theorem 90) [2, (1.4)] the following exact sequence

$$(4) \quad 0 \rightarrow \frac{T(\mathbb{A}_F)}{T(F)} \rightarrow \frac{G(\mathbb{A}_F)}{G(F)} \rightarrow \frac{A(\mathbb{A}_F)}{A(F)} \rightarrow 0.$$

It is worthwhile to contemplate this mysterious sequence: the first term is a Hausdorff, non-compact group and the last is a compact non-Hausdorff group, but the middle term is a compact Hausdorff group!

Lemma 4. Consider a field L with $F \subset L \subset \bar{F}$. The group $G(L)$ is a discrete subgroup of $G(\mathbb{A}_L)$ if and only if $A(K) \subset A(L)$ is of finite index.

Proof. Pick a subgroup $C \simeq \mathbb{Z}^s$ of $A^\vee(L)$ such that $B \times C$ has finite index in $A^\vee(L)$. The Bloch semiabelian variety G' over L determined by $B \times C$ is an extension of A by $T' = \text{Hom}(B \times C, \mathbb{G}_m)$. One has an exact sequence $0 \rightarrow T'' \rightarrow G' \rightarrow G \rightarrow 0$ defined over L where $T'' = \text{Hom}(C, \mathbb{G}_m)$ is a split torus of dimension s . Consider the commutative diagram with exact rows and columns

$$\begin{array}{ccccc}
& 0 & & 0 & \\
& \downarrow & & \downarrow & \\
& \frac{T''(\mathbb{A}_L)}{T''(L)} & \xlongequal{\quad} & \frac{T''(\mathbb{A}_L)}{T''(L)} & \\
& \downarrow & & \downarrow & \\
0 & \longrightarrow & \frac{T'(\mathbb{A}_L)}{T'(L)} & \longrightarrow & \frac{G'(\mathbb{A}_L)}{G'(L)} & \longrightarrow & \frac{A(\mathbb{A}_L)}{A(L)} \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & \frac{T(\mathbb{A}_L)}{T(L)} & \longrightarrow & \frac{G(\mathbb{A}_L)}{G(L)} & \longrightarrow & \frac{A(\mathbb{A}_L)}{A(L)} \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0.
\end{array}$$

The proof of surjectivity in the columns follows Hilbert Theorem 90 applied to T'' [2, (1.4)]. The Bloch-Tamagawa space $X'_A = \frac{G'(\mathbb{A}_L)}{G'(L)}$ for A over L is compact and Hausdorff; its quotient by

$$\frac{T''(\mathbb{A}_L)}{T''(L)} = \left(\frac{\mathbb{I}_L}{L^*}\right)^s$$

is $\frac{G(\mathbb{A}_L)}{G(L)}$. The quotient is Hausdorff if and only if $s = 0$. \square

A more general form of Lemma 4 is implicit in [2]: For any one-motive $[N \xrightarrow{\phi} A^\vee]$ over F , write V for its Cartier dual (a semiabelian variety), and put

$$X = \frac{V(\mathbb{A}_F)}{V(F)}.$$

We assume that the Γ -action on N is trivial. Then X is compact if and only if $\text{Ker}(\phi)$ is finite; X is Hausdorff if and only if the image of ϕ has finite index in $A^\vee(F)$.

Tamagawa numbers. Let H be a semisimple algebraic group over F . Since $H(F)$ embeds discretely in $H(\mathbb{A}_F)$, the adelic space $X_H = \frac{H(\mathbb{A}_F)}{H(F)}$ is Hausdorff. The Tamagawa number $\tau(H)$ is the volume of X_H relative to a canonical (Tamagawa) measure [15]. The Tamagawa number theorem [8, 1] (which was formerly a conjecture) states

$$(5) \quad \tau(H) = \frac{\#\text{Pic}(H)_{\text{torsion}}}{\#\text{III}(H)}$$

where $\text{Pic}(H)$ is the Picard group and $\text{III}(H)$ the Tate-Shafarevich set of H/F (which measures the failure of the Hasse principle). Taking $H = SL_2$ over \mathbb{Q} in (5) recovers Euler's result

$$\zeta(2) = \frac{\pi^2}{6}.$$

The above formulation (5) of the Tamagawa number theorem is due to T. Ono [12, 17] whose study of the behavior of τ under an isogeny explains the presence of $\text{Pic}(H)$, and reduces the semisimple case to the simply connected case. The original form of the theorem (due to A. Weil) is that $\tau(H) = 1$ for split simply connected H . The Tamagawa number theorem (5) is valid, more

generally, for any connected linear algebraic group H over F . The case $H = \mathbb{G}_m$ becomes the Tate-Iwasawa [16, 7] version of the analytic class number formula: the residue at $s = 1$ of the zeta function $\zeta(F, s)$ is the volume of the (compact) unit idele class group $\mathbb{J}_F^1 = \text{Ker}(|-| : \frac{\mathbb{I}_F}{F^*} \rightarrow \mathbb{R}_{>0})$ of F . Here $|-|$ is the absolute value or norm map on \mathbb{I}_F .

Zagier extensions. [18] The m -Selmer group $\text{Sel}_m(A/F)$ (for $m > 0$) fits into an exact sequence

$$(6) \quad 0 \rightarrow \frac{A(F)}{mA(F)} \rightarrow \text{Sel}_m(A/F) \rightarrow \text{III}(A/F)_m \rightarrow 0.$$

D. Zagier [18, §4] has pointed out that while the m -Selmer sequences (6) (for all $m > 1$) cannot be induced by a sequence (an extension of $\text{III}(A/F)$ by $A(F)$)

$$0 \rightarrow A(F) \rightarrow ? \rightarrow \text{III}(A/F) \rightarrow 0,$$

they can be induced by an exact sequence of the form

$$(7) \quad 0 \rightarrow A(F) \rightarrow \mathcal{A} \rightarrow \mathcal{S} \rightarrow \text{III}(A/F) \rightarrow 0$$

and gave examples of such (Zagier) sequences. Combining (1) and (4) above provides the following natural Zagier sequence

$$0 \rightarrow A(F) \rightarrow A(\mathbb{A}_F) \rightarrow \frac{Y_A}{T(\mathbb{A}_F)} \rightarrow \text{III}(A/F) \rightarrow 0.$$

Write $A(\mathbb{A}_{\bar{F}})$ for the direct limit of the groups $A(\mathbb{A}_L)$ over all finite subextensions $F \subset L \subset \bar{F}$. The previous sequence discretized (neglect the topology) becomes

$$0 \rightarrow A(F) \rightarrow A(\mathbb{A}_F) \rightarrow \left(\frac{A(\mathbb{A}_{\bar{F}})}{A(\bar{F})}\right)^\Gamma \rightarrow \text{III}(A/F) \rightarrow 0.$$

Remark 5. (i) For an elliptic curve E over F , Flach has indicated how to extract a canonical Zagier sequence via $\tau_{\geq 1}\tau_{\leq 2}R\Gamma(S_{et}, \mathbb{G}_m)$ from any regular arithmetic surface $S \rightarrow \text{Spec } \mathcal{O}_F$ with $E = S \times_{\text{Spec } \mathcal{O}_F} \text{Spec } F$.

(ii) It is well known that the class group $\text{Pic}(\mathcal{O}_F)$ is analogous to $\text{III}(A/F)$ and the unit group \mathcal{O}_F^\times is analogous to $A(F)$. Iwasawa [6, p. 354] proved that the compactness of \mathbb{J}_F^1 is equivalent to the two basic finiteness results of algebraic number theory: (i) $\text{Pic}(\mathcal{O}_F)$ is finite; (ii) \mathcal{O}_F^\times is finitely generated. His result provided a beautiful new proof of these two finiteness theorems. Bloch's result [2, Theorem 1.10] on the compactness of X_A uses the Mordell-Weil theorem (the group $A(F)$ is finitely generated) and the non-degeneracy of the Néron-Tate pairing on $A(F) \times A^\vee(F)$ (modulo torsion).

Question 6. *Can one define directly a space attached to A/F whose compactness implies the Mordell-Weil theorem for A and the finiteness of $\text{III}(A/F)$?*

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