Motivic complexes over finite fields and the ring of correspondences at the generic point

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Abstract

Already in the 1960s Grothendieck understood that one could obtain an almost entirely satisfactory theory of motives over a finite field when one assumes the Tate conjecture. In this note we prove a similar result for motivic complexes. In particular Beilinson's $\mathbb{Q}$-algebra of "correspondences at the generic point" is then defined for all connected varieties. We compute this for all smooth projective varieties (hence also for varieties birational to such a variety).

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Introduction

Forty years after Grothendieck predicted that the standard cohomology functors factor through a tannakian category of pure motives, we still do not know how to construct such a category. However, when the field is finite and one assumes the Tate conjecture, there is an almost entirely satisfactory theory of pure motives. Deligne tells us that this was known to Grothendieck, but it was re-discovered by Langlands and Rapoport (1987), who used it to state a conjecture, more precise than earlier attempts by Langlands, on the structure of the points modulo a prime on a Shimura variety.

It is generally hoped that the standard cohomology functors to triangulated categories will factor through a triangulated category of motivic complexes with t-structure whose heart is (defined to be) the category of mixed motives (see, for example, Deligne 1994, §3). We show that, over a finite field, a triangulated category of motivic complexes exists with the expected properties if and only if the Tate conjecture holds and homological equivalence coincides with rational equivalence with $\mathbb{Q}$-coefficients (see Theorems 4.1 and 5.3 for more precise statements). Moreover, then a category of effective motivic complexes exists with the properties (A,B,C) of Beilinson 2002, and so there is a well-defined semisimple $\mathbb{Q}$-algebra of "correspondences at the generic point" attached to every variety over a finite field. We compute this $\mathbb{Q}$-algebra for smooth projective varieties (hence also for varieties birational to such a variety). As this requires the generalized Tate conjecture (in the sense of Grothendieck 1968, §10), we begin by giving an elementary proof that the generalized conjecture follows from the usual Tate conjecture.[1]

Notations

A variety is a geometrically-reduced separated scheme of finite type over a field. For a variety X over a perfect field k of characteristic $p \neq 0$ and algebraic closure $\overline{k}$, we set

$$H_l^i(X) = H_{\text{et}}^r(X_{\overline{k}}, \mathbb{Q}_l), \quad \text{if } l \neq p, \text{ and}$$
$$H_p^i(X) = H_{\text{crys}}^r(X/W) \otimes \mathbb{Q}, \quad W = W(k).$$

We use (r) to denote a Tate twist, and we write $\hom(l)$ for the equivalence relation on the space $Z^*(X)$ of algebraic cycles defined by H_l. Similarly, we write num and rat for numerical and rational equivalence. For an adequate equivalence relation $\sim$, $Z^i_\sim(X) = Z^i(X)/\sim$ and $Z^i_\sim(X)_\mathbb{Q} = Z^i_\sim(X) \otimes \mathbb{Q}$. For example, $Z^i_{\text{rat}}(X)$ is the Chow group $CH^i(X)$.

The symbol $\mathbb{F}$ denotes an algebraic closure of $\mathbb{F}_p$, and the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is denoted $\mathbb{Q}^{\text{al}}$.

A triangulated category with t-structure (Gelfand and Manin 1996, IV 4.2, p278) will be referred to simply as a t-category. All t-structures will be assumed to be bounded (i.e., $\bigcup_{n\geq 0} \mathcal{D}^{\leq n} = \mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}^{\geq -n}$) and nondegenerate (i.e., $\bigcap_{n\geq 0} \mathcal{D}^{\leq -n} = 0 = \bigcap_{n\geq 0} \mathcal{D}^{\geq n}$).

By a functor between additive categories, we mean an additive functor. A functor $F\colon \mathcal{C} \to \mathcal{C}'$ of triangulated categories together with an isomorphism of functors $F \circ T \simeq T' \circ F$ is said to be triangulated (formerly, exact) if it takes distinguished triangles to distinguished triangles. Isomorphisms are denoted $\approx$ and canonical isomorphisms $\simeq$.

1 The generalized Tate conjecture

In this section, k is the subfield $\mathbb{F}_q$ of $\mathbb{F}$, and $l \neq p$.

[1] Since this is apparently known to others, at least under additional hypotheses (see, for example, André 2004, 8.2), it should probably be considered folklore.

1.1 By the ***Tate conjecture*** for a smooth complete variety X over k and an $r \geq 0$, we mean the statement that the order of the pole of the zeta function $Z(X,t)$ at $t = q^{-r}$ is equal to the rank of the group of numerical equivalence classes of algebraic cycles of codimension r on X. If the Tate conjecture holds for X and r, then, for all $l \neq p$,

(a) the cycle class map $Z^r(X) \otimes \mathbb{Q}_l \to H_l^{2r}(X)(r)^{\mathrm{Gal}(\mathbb{F}/k)}$ is surjective, and

(b) the quotient map $Z^r_{\mathrm{hom}(l)}(X)_{\mathbb{Q}} \to Z^r_{\mathrm{num}}(X)_{\mathbb{Q}}$ is injective (i.e., $\mathrm{hom}(l)$ and num coincide with $\mathbb{Q}$-coefficients).

Conversely, if (a) and (b) hold for a single l, then the Tate conjecture holds for X and r (Tate 1994, §2). We refer to (a) as the ***weak Tate conjecture for*** l.

Statement of the generalized Tate conjecture

Define a ***Tate structure*** to be a finite-dimensional $\mathbb{Q}_l$-vector space with a linear (Frobenius) map ϖ whose characteristic polynomial lies in $\mathbb{Q}[T]$ and whose eigenvalues are Weil q-numbers, i.e., algebraic numbers α such that, for some integer m (called the weight of α), $|\rho(\alpha)| = q^{m/2}$ for every homomorphism $\rho\colon \mathbb{Q}[\alpha] \to \mathbb{C}$, and, for some integer n, $q^n\alpha$ is an algebraic integer. When the eigenvalues are all of weight m (resp. algebraic integers, resp. semisimple), we say that V is of ***weight*** m (resp. ***effective***, resp. ***semisimple***). For example, for any smooth complete variety X over k, $H_l^i(X)$ is an effective Tate structure of weight $i/2$ (Deligne 1980), which is semisimple if X is an abelian variety (Weil 1948, no. 70) or if the Tate conjecture holds for $X \times X$ (Milne 1986b, 8.6).

Let X be a smooth complete variety over k. For each r, let $F_a^r H_l^i(X) \subset H_l^i(X)$ denote the subspace of classes with support in codimension at least r, i.e.,

$$F_a^r H_l^i(X) = \bigcup\nolimits_U \mathrm{Ker}(H_l^i(X) \to H_l^i(U))$$

where U runs over the open subvarieties of X such that $X \smallsetminus U$ is of codimension at least r.

EXAMPLE 1.2 If Z is a *smooth* closed subvariety of X of codimension r, then there is an exact Gysin sequence

$$\cdots \to H_l^{i-2r}(Z)(-r) \to H_l^i(X) \to H_l^i(U) \to \cdots, \quad U = X \smallsetminus Z,$$

(e.g., Milne 1980, VI 5.4), and so the kernel of $H_l^i(X) \to H_l^i(U)$ is an effective Tate structure of weight i whose twist by $\mathbb{Q}_l(r)$ is still effective.

CONJECTURE 1.3 *(Generalized Tate conjecture; cf. Grothendieck 1968, 10.3.). For a smooth complete variety X over k, every Tate substructure $V \subset H_l^i(X)$ such that $V(r)$ is still effective is contained in $F_a^r H_l^i(X)$.*

REMARK 1.4 Let X be a smooth complete variety over k. For any i and r, the set of eigenvalues α of ϖ on $H_l^i(X)$ such that α/q^r is an algebraic integer is stable under Galois conjugation. It follows[2] that there exists a Tate substructure $F_b^r H_l^i(X)$ of $H_l^i(X)$ whose eigenvalues are exactly these αs. Clearly, it is the largest Tate substructure of $H_l^i(X)$ whose twist by $\mathbb{Q}_l(r)$ is still effective, and so the generalized Tate conjecture 1.3 asserts that $F_b^r H_l^i(X) \subset F_a^r H_l^i(X)$.

[2]Let V be a finite-dimensional vector space over a perfect field k equipped with a linear endomorphism α, and let k' be the splitting field of the characteristic polyonomial of α. Then $V_{k'}$ decomposes into a direct sum of generalized eigenspaces $V_{k'} = \bigoplus_{a \in I} V(a)$ for the set I of eigenvalues of α — here $V(a)$ is the subspace on which $\alpha - a$ is nilpotent. For any subset J of I stable under $\mathrm{Gal}(k'/k)$, the k'-space $V(J) \overset{\mathrm{def}}{=} \bigoplus_{a \in J} V(a)$ is also stable under $\mathrm{Gal}(k'/k)$, and so arises from k-subspace of V.

EXAMPLE 1.5 Let Z' be a closed irreducible subvariety of $X_{\mathbb{F}}$ of codimension r. Then

$$H^{2r}_{Z'}(X_{\mathbb{F}}, \mathbb{Q}_l(r)) \to H^{2r}(X_{\mathbb{F}}, \mathbb{Q}_l(r)) \to H^{2r}(X_{\mathbb{F}} \smallsetminus Z', \mathbb{Q}_l(r))$$

is exact, and $H^{2r}_{Z'}(X_{\mathbb{F}}, \mathbb{Q}_l(r)) \simeq \mathbb{Q}_l$; moreover, the image of 1 under the first map is the cohomology class of Z' (cf. Milne 1980, p269). For any open $U \subset X$, the kernel of

$$H^{2r}_l(X)(r) \to H^{2r}_l(U)(r)$$

is spanned by the cohomology classes of the irreducible components of $(X \smallsetminus U)_{\mathbb{F}}$ and some power of ω_X acts as 1 on it. On the other hand, $F^r_b H^{2r}_l(X)(r)$ is the subspace of $H^{2r}_l(X)(r)$ on which the eigenvalues of ϖ are roots of 1. Thus, the generalized Tate conjecture with $i = 2r$ states that this subspace is spanned by the classes of algebraic cycles of codimension r on $X_{\mathbb{F}}$. This is the weak Tate conjecture stated over $\mathbb{F}$ rather than $\mathbb{F}_q$.

The Tate conjecture implies the generalized Tate conjecture

Recall that, for a proper map $\pi\colon Y \to X$ of smooth varieties over an algebraically closed field, the Gysin map

$$\pi_*\colon H^i(Y, \mathbb{Q}_l) \to H^{i-2c}(X, \mathbb{Q}_l(-c)), \quad c = \dim Y - \dim X,$$

is defined to be the Poincaré dual of

$$\pi^*\colon H^{2d-i}_c(X, \mathbb{Q}_l(d)) \to H^{2d-i}_c(Y, \mathbb{Q}_l(d)), \quad d = \dim Y$$

(Milne 1980, VI 11.6). We shall need to know that these maps are compatible with restriction to open subvarieties.

LEMMA 1.6 *Let $\pi\colon Y \to X$ be a proper map of smooth complete varieties over an algebraically closed field, and let $j\colon U \hookrightarrow X$ an open immersion. Then the commutative diagram at left gives rise to the commutative diagram at right:*

$$\begin{array}{ccc} Y & \xleftarrow{j'} & \pi^{-1}(U) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ X & \xleftarrow{j} & U \end{array} \qquad \begin{array}{ccc} H^i(Y, \mathbb{Q}_l) & \xrightarrow{j'^*} & H^i(\pi^{-1}(U), \mathbb{Q}_l) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi'_*} \\ H^{i-2c}(X, \mathbb{Q}_l(-c)) & \xrightarrow{j^*} & H^{i-2c}(U, \mathbb{Q}_l(-c)) \end{array}$$

PROOF. Exercise for the reader.[3] □

PROPOSITION 1.7 *Every effective semisimple Tate structure is a Tate substructure of $H^*_l(A)$ for some abelian variety A over $\mathbb{F}_q$.*

PROOF. We may assume that the Tate structure V is simple (i.e., irreducible). Then V has weight m for some $m \geq 0$, and the characteristic polynomial $P(T)$ of ϖ is a monic irreducible polynomial with coefficients in $\mathbb{Z}$ whose roots all have real absolute value $q^{m/2}$. According to Honda's theorem (Honda 1968; Tate 1968), $P(T)$ is the characteristic polynomial of an abelian variety A over $\mathbb{F}_{q^m}$. Let B be the abelian variety over $\mathbb{F}_q$ obtained from A by restriction of the base field. The eigenvalues of the Frobenius map on $H^1_l(B)$ are the m^{th}-roots of the eigenvalues of the Frobenius map on $H^1_l(A)$, and it follows that V is a Tate substructure of $H^m_l(B)$. □

[3] "No mathematician ever writes these words unless he has no idea how to do it himself." (Linderholm 1972).

LEMMA 1.8 *Let z be an algebraic cycle of codimension $\dim T + r$ on the product $T \times X$ of two smooth complete varieties over k (i.e., z is an algebraic correspondence of degree r from T to X). Assume that the push-forward of z on X is nonzero. Then the image of the map*

$$z_* \colon H_l^{i-2r}(T)(-r) \to H_l^i(X)$$

defined by z is contained in $F_a^r H_l^i(X)$.

PROOF. Let p, q denote the projection maps $T \times X \rightrightarrows T, X$, and let $[z]$ denote the cohomology class of z in $H_l^{2d_T+2r}(T \times X)(d_T + r)$, $d_T = \dim T$. Then

$$z_*(a) = q_*([z] \cup p^*(a)), \quad a \in H_l^{i-2r}(T)(-r).$$

As the push-forward $q_*(z)$ of z is nonzero, its support Z has codimension r. Let $U = X \smallsetminus Z$. Then z has support in $T \times Z$, and so $[z]$ maps to zero in $H_l^{2d_T+2r}(T \times U)(d_T + r)$ (cf. 1.5). According to (1.6), the diagram

$$\begin{array}{ccc} H_l^{i+2d_T}(T \times X)(d_T) & \longrightarrow & H_l^{i+2d_T}(T \times U)(d_T) \\ \downarrow q_* & & \downarrow q_* \\ H_l^i(X) & \longrightarrow & H^i(U) \end{array}$$

commutes, which shows that $z_*(a)$ maps to zero in $H_l^i(U)$, and therefore lies in $F_a^r H_l^i(X)$.□

LEMMA 1.9 *Let X be a smooth complete variety over k and let $i, r \in \mathbb{N}$. If there exists a smooth complete variety T such that*

- *$H_l^{i-2r}(T)$ is a semisimple Tate structure,*
- *the weak Tate conjecture (1.1a) holds for $T \times X$ and l, and*
- *$F_b^r H_l^i(X)(r)$ is isomorphic to a Tate substructure of $H_l^{i-2r}(T)$*

then $F_b^r H_l^i(X) \subset F_a^r H_l^i(X)$.

PROOF. Let $d = \dim T$, and let V be a Tate substructure of $H_l^{i-2r}(T)$ for which there exists an isomorphism $f \colon V(-r) \to F_b^r H_l^i(X)$. Then

$$\begin{aligned} H_l^{2d+2r}(T \times X)(d+r) &\supset H_l^{2d+2r-i}(T)(d+r) \otimes H_l^i(X) \\ &\simeq \mathrm{Hom}(H_l^{i-2r}(T)(-r), H_l^i(X)) \\ &\supset \mathrm{Hom}(V(-r), F_b^r H_l^i(X)) \ni f. \end{aligned}$$

(The last inclusion depends on the choice of stable complement for V in $H_l^{i-2r}(T)$.) Obviously f is fixed by $\mathrm{Gal}(\mathbb{F}/k)$ and so can be approximated by the cohomology class of an algebraic correspondence z of degree r from T to X. Clearly, z can be chosen so that z_* is injective on V. Obviously z_* maps $H_l^{i-2r}(T)(-r)$ into $F_b^r H_l^i(X)$, and so

$$F_b^r H_l^i(X) \subset z_* V(-r) \overset{1.8}{\subset} F_a^r H_l^i(X).$$

□

THEOREM 1.10 *Let X be a smooth complete variety over k. If the weak Tate conjecture (1.1a) holds for all varieties of the form $A \times X$ with A an abelian variety (and some l), then the generalized Tate conjecture holds for X (and the same l).*

PROOF. As we noted above, $H_l^*(A)$ is a semisimple Tate structure when A is an abelian variety, and so this follows from (1.7) and 1.9. □

COROLLARY 1.11 *If the Tate conjecture holds for all abelian varieties over k (or all smooth complete varieties over k), then so does the generalized Tate conjecture.*

Complements

1.12 Let X be a smooth projective variety over k, and let $V = F_b^r H_l^i(X)$. We know that $V(-r) \subset H_l^{i-2r}(A)$ for some abelian variety A over k (see 1.7). If $\dim A = d > i - 2r$, then, according to the Lefschetz hypersurface-section theorem,[4] for any smooth hypersurface section Y of A (which exists by Gabber 2001), $V(-r) \subset H_l^{i-2r}(Y)$. Continuing in this fashion, we get that $V(-r) \subset H_l^{i-2r}(T)$ for some smooth projective T of dimension $i-2r$. Therefore, under the assumption of the Tate conjecture, there exists a smooth projective variety T of dimension at most $i-2r$ over k and an algebraic correspondence z from T to X of degree r such that $z_* H_l^{i-2r}(T)(r) = F_b^r H_l^i(X)$.

1.13 Deligne (1974b, 8.2.8) proves the following:

> Let X be a smooth complete variety over $\mathbb{C}$, and let Z be a closed subvariety of X of codimension r. For any desingularization $\tilde{Z} \to Z$ of Z, the sequence
>
> $$H^{i-2r}(\tilde{Z}, \mathbb{Q}(-r)) \to H^i(X, \mathbb{Q}) \to H^i(U, \mathbb{Q}), \quad U = X \setminus Z,$$
>
> is exact.

A similar argument[5] proves the following l-adic analogue:

> Let X be a smooth complete variety over a perfect field k, and let Z be a closed subvariety of X of codimension r. For any smooth alteration $\tilde{Z} \to Z$ of Z, the sequence
>
> $$H_l^{i-2r}(\tilde{Z})(-r) \to H_l^i(X) \to H_l^i(U), \quad U = X \setminus Z,$$
>
> is exact.

Since de Jong (1996, 3.1) shows that smooth alterations always exist, this implies that

$$F_a^r H_l^i(X) \subset F_b^r H_l^i(X).$$

The generalized Tate conjecture then states that

$$F_a^r H_l^i(X) = F_b^r H_l^i(X).$$

[4]Recall that the Lefschetz hypersurface-section theorem says that, for any hypersurface section Z of a projective variety X such that $U \stackrel{\text{def}}{=} X \setminus Z$ is smooth, $H_l^i(X) \to H_l^i(Z)$ is injective if $i = \dim X - 1$ and an isomorphism if $i < \dim X - 1$. To prove this, use the exact sequence

$$\cdots \to H_c^i(U, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l) \to H^i(Z, \mathbb{Q}_l) \to \cdots$$

(Milne 1980, III 1.30) noting that $H_c^i(U, \mathbb{Q}_l)$ is dual to $H^{2d-i}(U, \mathbb{Q}_l(d))$, $d = \dim X$, (ibid. VI 11.2), which is zero for $2d - i > d$, i.e., $i < d$ (ibid. VI 7.1).

[5]For any proper surjective morphism $f\colon Y \to Z$ from a smooth projective variety Y, we can find a smooth projective simplicial scheme $Y_\bullet$ with $Y_0 = Y$ that is a proper hypercovering of Z. The corresponding spectral sequence (l-adic analogue of the spectral sequence Deligne 1974b, 8.1.19.1) degenerates at E_2 with $\mathbb{Q}_l$-coefficients because of weight considerations, and gives an exact sequence

$$0 \to \frac{H_l^i(Z)}{W_{i-1} H_l^i(Z)} \to H_l^i(Y_0) \xrightarrow{\delta_0 - \delta_1} H_l^i(Y_1).$$

This implies that the image of $H_l^i(Z)$ in $H_l^i(Y)$ is the (largest) quotient of pure weight i of $H_l^i(Z)$. This implies the l-adic analogue of Deligne 1974b, 8.2.7, (the proof there works as the Gr_*^W functor is exact) and of ibid. 8.2.8.

1.14 The above statements hold *mutatis mutandis* for p. For a smooth complete variety X, $H_p^i(X)$ is an F-isocrystal, i.e., a finite-dimensional vector space over $B(\mathbb{F}_q) \overset{\text{def}}{=} W(\mathbb{F}_q) \otimes \mathbb{Q}$ equipped with a σ-linear bijection $F\colon H_p^i(X) \to H_p^i(X)$. The Tate conjecture for X and r is equivalent to

(a) the cycle class map $Z^r(X) \otimes \mathbb{Q}_p \to H_p^{2r}(X)(r)^{F=1}$ is surjective (***weak Tate conjecture for*** p), and

(b) the quotient map $Z^r_{\hom(p)}(X) \to Z^r_{\text{num}}(X)$ is injective.

Define

$$F_a^r H_p^i(X) = \bigcup\nolimits_Z \operatorname{Im}(H_p^{i-2r}(\tilde{Z})(-r) \to H_p^i(X))$$

where Z runs over the closed subvarieties of X such that Z is of codimension at least r and $\tilde{Z}$ is a smooth alteration of Z. If the Tate conjecture holds over k, then

$$F_a^r H_p^i(X) = F_b^r H_p^i(X)$$

where $F_b^r H_p^i(X) = H_p^i(X)_{[r,\infty)}$, the sub-isocrystal of $H_p^i(X)$ with slopes at least r. The proofs are similar to those in the case $l \neq p$ — the details are left as an exercise to the reader.

1.15 Similar arguments show that the generalized Tate conjecture over number fields follows from the usual Tate conjecture and an effective version of the Fontaine-Mazur conjecture (Fontaine and Mazur 1995, Conjecture 1, p44) that specifies which representations arise from effective motives.

NOTES It was known to Grothendieck that the generalized Hodge conjecture follows from the usual Hodge conjecture and the following weak analogue of (1.7),

> Let V be a simple Hodge substructure of the cohomology of a smooth complex projective variety; if its Tate twist $V(r)$ is still effective (i.e., has only nonnegative Hodge numbers), then $V(r)$ occurs in the cohomology of a smooth complex projective variety.

presumably by more-or-less the above argument. See Grothendieck 1969, top of p301 (also Abdulali 1997, §2 and Schoen 1989, §0).

2 The category of pure motives

In this section $k = \mathbb{F}_q$.

For any adequate equivalence relation $\sim$, Grothendieck's construction gives a rigid pseudo-abelian tensor $\mathbb{Q}$-category $\mathcal{M}_\sim(k)$ of pure motives (Saavedra Rivano 1972, VI 4.1.3.5) and a map h from the smooth projective varieties over k to $\mathcal{M}_\sim(k)$ that is natural for algebraic correspondences modulo $\sim$. Because rational equivalence is the finest adequate equivalence relation, h factors through a tensor functor $\mathcal{M}_{\text{rat}}(k) \to \mathcal{M}_\sim(k)$. Conversely, a tensor functor from $\mathcal{M}_{\text{rat}}(k)$ to an additive tensor category with $\operatorname{End}(\mathbb{1}) = \mathbb{Q}$ defines an adequate equivalence relation (cf. Jannsen 2000, 1.7). When $\sim$ is numerical equivalence, $\mathcal{M}_\sim(k)$ is a semisimple (Jannsen 1992).

For a smooth projective variety X over k, there are well-defined polynomials $P_{X,i}(T) \in \mathbb{Q}[T]$ such that $P_{X,i}(T) = \det(1 - \varpi_X T \mid H_l^i(X))$ for all l; moreover, $P_{X,i}$ has reciprocal roots of absolute value $q^{\frac{i}{2}}$ (Deligne 1974a). The $P_{X,i}(T)$ are relatively prime, and so there exist $P^i(T) \in \mathbb{Q}[T]$, well-defined up to a multiple of $\prod_i P_{X,i}(T)$, such that

$$P^i(T) \equiv \begin{cases} 1 & \mod P_{X,i}(T) \\ 0 & \mod P_{X,j}(T) \text{ for } j \neq i. \end{cases} \tag{1}$$

Because $\prod_i P_{X,i}(\varpi_X)$ acts as zero on $H_l^*(X)$, the graph p^i of $P^i(\varpi_X)$ is a well-defined element of $Z_{\text{hom}(l)}(X \times X)_{\mathbb{Q}}$ (or $Z_{\text{num}}(X \times X)_{\mathbb{Q}}$), and $\{p^0, \ldots, p^{2d}\}$ is a complete set of orthogonal idempotents. Let $hX = \bigoplus_i h^i X$ be the corresponding decomposition. When we use this decomposition to modify the commutativity constraint in $\mathcal{M}_{\text{num}}(k)$, the rank of each object of $\mathcal{M}_{\text{num}}(k)$ becomes a nonnegative integer, and so $\mathcal{M}_{\text{num}}(k)$ is a tannakian category (Deligne 1990, 7.1).

The category $\mathcal{M}_{\text{num}}(k)$ has a canonical (Frobenius) element $\varpi \in \text{Aut}^{\otimes}(\text{id}_{\mathcal{M}_{\text{num}}(k)})$ and a canonical (weight) $\mathbb{Z}$-gradation. An object M of $\mathcal{M}_{\text{num}}(k)$ is of pure weight m if and only if its Frobenius element ϖ_M has eigenvalues of absolute value $q^{m/2}$.

Recall (Deligne 1989, §6) that the fundamental group $\pi(\mathcal{T})$ of a Tannakian category is an affine group scheme in $\text{Ind}\,\mathcal{T}$ that acts on each object of $\mathcal{T}$ in such a way that these actions define an isomorphism

$$\omega(\pi(\mathcal{T})) \simeq \underline{\text{Aut}}^{\otimes}(\omega)$$

for each fibre functor ω. Any subgroup of the centre of $\pi(\mathcal{T})$ lies in $\text{Ind}\,\mathcal{T}^0$ where $\mathcal{T}^0$ is the full subcategory of trivial objects (those isomorphic to a multiple of $\mathbb{1}$). Since $\text{Hom}_{\mathcal{T}}(\mathbb{1}, -)$ defines an equivalence of $\mathcal{T}^0$ with the finite-dimensional vector spaces over the ground field, such a subgroup can be identified with an affine group scheme in the usual sense. For example, the centre of $\pi(\mathcal{T})$ is $\underline{\text{Aut}}^{\otimes}(\text{id}_{\mathcal{T}})$ (cf. Saavedra Rivano 1972, II 3.3.3.2).

Recall (e.g., Milne 1994, §2) that the Weil number group P is the affine group scheme of multiplicative type over $\mathbb{Q}$ whose character group consists of the Weil q-numbers in $\mathbb{Q}^{\text{al}}$. Define the Frobenius element ϖ_{univ} in $P(\mathbb{Q})$ to be that corresponding to $\alpha \mapsto \alpha$ under the bijection

$$P(\mathbb{Q}) \simeq \text{Hom}(X^*(P), \mathbb{Q}^{\text{al}})^{\text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})}.$$

Note that, for any smooth projective variety X over $\mathbb{F}_q$, the roots of $P_{X,i}(T)$ in $\mathbb{Q}^{\text{al}}$ are Weil q-integers of weight i (i.e., Weil q-numbers of weight i that are algebraic integers).

LEMMA 2.1 *The group of Weil q-numbers is generated by the Weil q-numbers of zero-dimensional varieties and abelian varieties over k.*

PROOF. Let α be a Weil q-number. It follows easily from the exact sequence in Milne 1994, 2.27 (see also the diagram in A.8 of Milne 2001) that some power of α is a product of powers of Weil q-integers of weight 1. On extracting roots, we find that α is a product of a root of 1 and powers of Weil q-integers of weight 1. All roots of 1 arise from varieties of dimension zero, and all Weil q-integers of weight 1 arise from abelian varieties (Honda's theorem; Tate 1968) □

PROPOSITION 2.2 *The affine subgroup scheme of $\pi(\mathcal{M}_{\text{num}}(k))$ generated by ϖ_{univ} is canonically isomorphic to P. It equals $\pi(\mathcal{M}_{\text{num}}(k))$ if and only if the Tate conjecture holds over k.*

PROOF. Let $Z = \underline{\text{Aut}}^{\otimes}(\omega)$ be the centre of $\pi(\mathcal{M}_{\text{num}}(k))$. Because $\mathcal{M}_{\text{num}}(k)$ is semi-simple, $\pi(\mathcal{M}_{\text{num}}(k))$ is a pro-reductive (not necessarily connected; cf. Deligne and Milne 1982, 2.23). Therefore Z is of multiplicative type, which implies that the subgroup scheme $\langle\varpi_{\text{univ}}\rangle$ generated by ϖ_{univ} is also of multiplicative type. The homomorphism of character groups $X^*(Z) \twoheadrightarrow X^*(\langle\varpi_{\text{univ}}\rangle)$ can be identified with $\chi \mapsto \chi(\varpi_{\text{univ}})$. The characters of $\langle\varpi_{\text{univ}}\rangle$ are the Weil q-numbers that occur as roots of the characteristic polynomial of ϖ_M

for some M in $\mathcal{M}_{\text{num}}(k)$. According to Lemma 2.1, these Weil q-numbers generate $X^*(P)$, and so $X^*(\langle\varpi\rangle) \simeq X^*(P)$. This proves that $\langle\varpi_{\text{univ}}\rangle \simeq P$.

If the Tate conjecture holds, then, for any fibre functor ω over $\mathbb{Q}^{\text{al}}$ and smooth projective variety X, the $\mathbb{Q}^{\text{al}}$-span of the algebraic cycles in $\omega(h^{2i}(X)(i))$ consists of the tensors fixed by ϖ_{univ}. Therefore, the inclusion $\langle\varpi_{\text{univ}}\rangle \hookrightarrow \underline{\text{Aut}}^{\otimes}(\omega)$ is an isomorphism, i.e., $\omega(P) \hookrightarrow \omega(\pi(\mathcal{M}_{\text{num}}(k))$ is an isomorphism, which implies that $P \hookrightarrow \pi(\mathcal{M}_{\text{num}}(k))$ is an isomorphism. The converse can be proved by the same argument as in the proof of Milne 1999, Proposition 7.4. □

If num and $\text{hom}(l)$ coincide with $\mathbb{Q}$-coefficients, then H_l defines a fibre functor ω_l on $\mathcal{M}_{\text{num}}(k)$. Without any assumptions, it is known that there exists a polarizable semisimple tannakian category with fundamental group P and with fibre functors ω_l for all l. Moreover, any two such systems are equivalent (Langlands and Rapoport 1987; Milne 2003, §6). However, it has not been shown that there exists a natural functor from $\mathcal{M}_{\text{rat}}(k)$ to such category. In fact, we have the following:

PROPOSITION 2.3 *If there exists a full tensor functor r preserving Frobenius elements from $\mathcal{M}_{\text{rat}}(k)$ to a tannakian category $\mathcal{M}$ with fundamental group P, then the Tate conjecture holds over k, and r defines an equivalence of tensor categories $\mathcal{M}_{\text{num}}(k) \to \mathcal{M}$.*

PROOF. Such a functor r defines an adequate equivalence relation $\sim$ (see above) such that r factors into

$$\mathcal{M}_{\text{rat}}(k) \to \mathcal{M}_{\sim}(k) \xrightarrow{\bar{r}} \mathcal{M}$$

with $\bar{r}$ a fully faithful tensor functor. Because P is a pro-reductive (not necessarily connected), $\mathcal{M}$ is semisimple (cf. Deligne and Milne 1982, 2.23). It follows that $\mathcal{M}_{\sim}(k)$ is semisimple (apply the criterion in Jannsen 1992, Lemma 2), and so $\sim$ is numerical equivalence (ibid. Theorem 1). The simple objects of $\mathcal{M}$ are classified by the orbits of $\text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ acting on $X^*(P)$, i.e., by the conjugacy classes of Weil q-numbers, and so Lemma 2.1 shows that $\mathcal{M}$ is generated as a tensor category by the images of Artin motives and abelian varieties. Therefore, $\bar{r}$ is a tensor equivalence, and so defines an isomorphism of P with $\pi(\mathcal{M}_{\text{num}}(k))$. We can now apply Proposition 2.2. □

When we drop the condition that r is full, we obtain a conjecture that is much weaker than the Tate conjecture, but which has many of the same consequences.

CONJECTURE 2.4 *There exists a $\mathbb{Q}$-linear tannakian category $\mathcal{M}$ with fundamental group P and a $\mathbb{Q}$-linear tensor functor $r\colon \mathcal{M}_{\text{rat}}(k) \to \mathcal{M}$ preserving Frobenius elements such that, for each prime l (including p), there is a fibre functor ω_l on $\mathcal{M}$ whose composite with r is H_l.*

2.5 There are canonical homomorphisms

$$\mathbb{G}_m \xrightarrow{w} P \xrightarrow{t} \mathbb{G}_m$$

such that $t(w(x)) = x^{-2}$ and $X^*(w)(\varpi) = \text{weight}(\varpi)$. Therefore, on any $\mathbb{Q}$-linear Tannakian category with fundamental group P, there is a natural weight gradation for which an object M is of pure weight m if and only if its Frobenius element ϖ_M has eigenvalues of absolute value $q^{m/2}$, and there is a Tate object of weight -2. Assume that there exists an

r as in the conjecture, and set $h_r^i(X) = (rh(X))^i$ for a smooth projective variety X. Then, $\omega_l(h_r^i(X)) = H_l^i(X)$ and the natural map

$$\mathrm{Hom}_{\mathcal{M}}(\mathbb{1}, h_r^{2i}(X)(i)) \otimes \mathbb{Q}_l \longrightarrow \mathrm{Hom}_{\mathbb{Q}_l\text{-linear}}(\mathbb{Q}_l, H_l^{2i}(X)(i)) = H_l^{2i}(X)(i) \tag{2}$$

is injective (Deligne 1990, 2.13) with image the $\mathbb{Q}_l$-space of Tate classes in $H_l^{2i}(X)(i)$ (i.e., the subspace conjectured by Tate to be the $\mathbb{Q}_l$-span of the algebraic classes). In other words, (2) is a $\mathbb{Q}$-structure on the space of l-adic Tate classes, and so the elements of

$$T^i(X) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{M}}(\mathbb{1}, h_r^{2i}(X)(i))$$

deserve to be called rational Tate classes. In essence, the conjecture says that there is a "good" theory of rational Tate classes.

2.6 One may even hope that r factors through $\mathcal{M}_{\text{num}}(k)$. The theory of quotients of tannakian categories (Milne 2005) gives the following description of the exact $\mathbb{Q}$-linear tensor functors $r\colon \mathcal{M}_{\text{num}}(k) \to \mathcal{M}$ identifying $\pi(\mathcal{M})$ with the subgroup P of $\pi(\mathcal{M}_{\text{num}}(k))$.

Let $\mathcal{M}_{\text{num}}(k)^{\langle\varpi_{\text{univ}}\rangle}$ be the full subcategory of $\mathcal{M}_{\text{num}}(k)$ of objects with trivial Frobenius element. Assume $\mathcal{M}_{\text{num}}(k)^{\langle\varpi_{\text{univ}}\rangle}$ is neutral, and choose a $\mathbb{Q}$-valued functor ω_0 on $\mathcal{M}(k)^{\langle\varpi_{\text{univ}}\rangle}$. Let $(\mathcal{M}_{\text{num}}(k)/\omega_0)'$ be the category with one object $\overline{X}$ for each object X of $\mathcal{M}_{\text{num}}(k)$, and with

$$\mathrm{Hom}_{(\mathcal{M}_{\text{num}}(k)/\omega_0)'}(\overline{X}, \overline{Y}) = \omega_0(\underline{\mathrm{Hom}}(\overline{X}, \overline{Y})^{\langle\omega_{\text{univ}}\rangle}).$$

There is a unique tensor structure on $(\mathcal{M}_{\text{num}}(k)/\omega_0)'$ for which

$$q\colon \mathcal{M}_{\text{num}}(k) \to (\mathcal{M}_{\text{num}}(k)/\omega_0)'$$

is a tensor functor. With this structure, $(\mathcal{M}_{\text{num}}(k)/\omega_0)'$ is rigid, and we define $\mathcal{M}_{\text{num}}(k)/\omega_0$ to be its pseudo-abelian hull. Then $\mathcal{M}_{\text{num}}(k)/\omega_0$ is a tannakian category with fundamental group P, and every "quotient" of $\mathcal{M}_{\text{num}}(k)$ with fundamental group P arises in this way. In particular, such a quotient exists if and only if $\mathcal{M}_{\text{num}}(k)^{\langle\varpi_{\text{univ}}\rangle}$ is neutral (which will be so, for example, if its fundamental group satisfies a Hasse principle for H^1).

2.7 For a smooth projective variety X over k, let $H_f^*(X)$ be the restricted topological product of the cohomology algebras $H_l^*(X)$ (see Milne and Ramachandran 2004, §2). Call a class α in $H_f^*(X)$ strongly motivated if $\eta^r \cup \alpha$ is algebraic for some ample divisor class η on X and $r \geq 0$, and motivated if it is of the form $p_{X*}(\beta \cup \gamma)$ with β and γ algebraic and strongly motivated classes respectively on $X \times Y$ for some smooth projective variety Y. Then the motivated classes on X form a graded $\mathbb{Q}$-subalgebra of $H_f^{2*}(X)(*)$, and, for every regular map γ, the maps γ^* and γ_* send motivated classes to motivated classes; moreover, the motivated classes form the smallest collection containing the algebraic classes and satisfying these conditions and the Lefschetz standard conjecture (André 1996, §2). Assume:

> for all motivated classes α, β of complementary dimension on a smooth projective variety over k, $\langle \alpha \cup \beta \rangle \in \mathbb{Q}$ (inside $\mathbb{A}_f$).

and define $\mathcal{M}'(k)$ using motivated classes modulo numerical equivalence instead of algebraic classes. There is a canonical $\mathbb{Q}$-linear tensor functor $\mathcal{M}_{\text{rat}}(k) \to \mathcal{M}'(k)$, and one hopes that the fundamental group of $\mathcal{M}'(k)$ is P. The arguments of Milne 1999 (with "algebraic class" replaced by "motivated class") show that this is true of the subcategory generated by abelian varieties and Artin motives,[6] and so P is a direct factor of $\pi(\mathcal{M}'(k))$.

[6]This uses the Abdulali-André theorem (André 1996, 0.6.2) that all Hodge classes on complex abelian varieties are motivated, and assumes the statement that motivated classes specialize to motivated classes (see André 2004, Chap. 10 and the reference [A04a] there).

3 The category of motives

The next observation goes back to Grothendieck.

PROPOSITION 3.1 *Let $\mathcal{MM}(\mathbb{F}_q)$ be a pseudo-abelian category containing $\mathcal{M}_{num}(\mathbb{F}_q)$ as a full subcategory. Assume*

(a) *each object M of $\mathcal{MM}(\mathbb{F}_q)$ has a (weight) filtration*

$$\cdots \subset W_{i-1}M \subset W_i M \subset \cdots$$

such that $W_iM/W_{i-1}M$ is a pure motive of weight i;

(b) *the Frobenius element extends to $\mathcal{MM}(\mathbb{F}_q)$ and preserves the weight filtrations.*

Then[7] $\mathcal{MM}(\mathbb{F}_q) = \mathcal{M}_{num}(\mathbb{F}_q)$.

PROOF. For X in $\mathcal{MM}(\mathbb{F}_q)$, let $P_i(T)$ be the characteristic polynomial of $\varpi_{W_iM/W_{i-1}M}$, and define $P^i(T)$ to satisfy (1). Let $p^i = P^i(\varpi_M)$. Then the p^i form a complete set of orthogonal idempotents in $\mathrm{End}(M)$ that decompose M into a direct sum isomorphic to $\bigoplus_i W_iM/W_{i-1}M$. □

4 Triangulated motivic categories

By a ***triangulated motivic category*** over a field k, we mean a triangulated rigid tensor category[8] $\mathcal{D}$ together with a covariant functor

$$R\colon \mathcal{M}_{\mathrm{rat}}(k) \to \mathcal{D}$$

and isomorphisms for all smooth projective varieties X and all $i, j \in \mathbb{Z}$

$$K_{2j-i}(X)^{(j)} \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, R(hX)(j)[i]) \tag{3}$$

that are natural for the maps defined by algebraic correspondences and reduce to the identity map when X is a point and $i = j = 0$ (cf. Jannsen 2000, §7, p257). Here $K_i(X)^{(j)}$ is the subspace of $K_i(X) \otimes \mathbb{Q}$ on which each Adams operator ψ^m acts as m^j. According to Jannsen 2000, p257, over any field k that admits resolution of singularities, triangulated motivic categories have been constructed (independently) by Hanamura (1995, 1999, 2004), Levine (1998), and Voevodsky (2000). When $k = \mathbb{F}_q$, ψ^q acts as ϖ_X (Hiller 1981, §5; Soulé 1985, 8.1), and so[9] $K_i(X)^{(j)}$ is the subspace on which ϖ_X acts as q^j.

Let $\mathcal{D}$ be a triangulated motivic category. As we noted in the introduction, for the "true" triangulated motivic category, there should be a t-structure on $\mathcal{D}(k)$ whose heart $\mathcal{MM}(k) \overset{\text{def}}{=} \mathcal{D}(k)^{\heartsuit}$ is the category of mixed motives. As Jannsen (2000, §7, p257) explains, there should be the following compatibilities between R and the t-structure:

[7]More pedantically, the inclusion $\mathcal{M}_{\mathrm{num}}(\mathbb{F}_q) \to \mathcal{MM}(\mathbb{F}_q)$ is an equivalence of categories.

[8]Not being able to find a definition in the literature (see Tips #2), we suggest one. A ***triangulated rigid tensor category*** is a category $\mathcal{C}$ with a rigid tensor structure (in the sense of Deligne and Milne 1982) and a triangulated structure (in the sense of Verdier 1977) satisfying the following compatibility conditions:

– for each object A of $\mathcal{C}$, the functor $C \mapsto A \otimes C\colon \mathcal{C} \to \mathcal{C}$ is triangulated;
– the functor $C \mapsto C^\vee : \mathcal{C}^{\mathrm{opp}} \to \mathcal{C}$ is triangulated.

[9]Because the k^i-eigenspace of ψ^k is independent of k (Seiler 1988, Theorem 1).

(a) for each standard Weil cohomology, the composite

$$\begin{array}{ccccc} \mathcal{M}_{\text{rat}}(k) & \xrightarrow{R} & \mathcal{D} & \xrightarrow{\bigoplus_i H^i} & \mathcal{MM}(k) \\ & & K & \mapsto & \bigoplus_i H^i(K) \end{array}$$

factors through $\mathcal{M}_{\text{hom}}(k)$, and defines a fully faithful functor $\overline{R}\colon \mathcal{M}_{\text{hom}}(k) \to \mathcal{MM}(k)$ (here $H^i(K) = \tau_{\leq 0}\tau_{\geq 0}(K[-i])$);

(b) for each smooth projective variety X, $\bigoplus_i H^i(R(hX))$ is the weight gradation of hX.

Evidently, there should also be the following compatibility between the tensor structures and the t-structure:

(c) the rigid tensor structure on $\mathcal{D}$ induces a rigid tensor structure on its heart $\mathcal{MM}(k)$ for which $\mathcal{M}_{\text{rat}}(k) \to \mathcal{MM}(k)$ is a tensor functor.

When k is finite, condition (b) says that $H^i(R(hX)) = \overline{R}(h^i(X))$. A t-structure satisfying these conditions (a,b,c) will be said to be ***compatible.***

THEOREM 4.1 *Let k be a finite field. If there exists a triangulated motivic category $\mathcal{D}$ over k and a compatible t-structure on $\mathcal{D}$ such that*

– *the heart of $\mathcal{D}$ is a tannakian category $\mathcal{M}$ with fundamental group P, and*
– *the functor $\mathcal{M}_{\text{rat}}(k) \to \mathcal{M}$ in (a) preserves Frobenius elements,*

then

(a) *the Tate conjecture holds for all smooth projective varieties over k;*
(b) *for each l, the functor $R_l\colon \mathcal{M}_{\text{hom}(l)}(k) \to \mathcal{M}$ defined by R is an equivalence of abelian categories;*
(c) *rational equivalence equals numerical equivalence ($\mathbb{Q}$-coefficients);*
(d) *for all M, N in $\mathcal{M}(k)$ and $i \neq 0$, $\mathrm{Hom}_{\mathcal{D}}(M, N[i]) = 0$.*

PROOF. Proposition 2.3 shows that the Tate conjecture holds and that R_l is essentially surjective (hence an equivalence). Moreover, it allows us to identify $\mathcal{M}$ with $\mathcal{M}_{\text{num}}(k)$.

Because $\mathcal{M}$ is a rigid subcategory of $\mathcal{D}$, for M, N in $\mathcal{M}$ there exists an object $\underline{\mathrm{Hom}}(M,N)$ in $\mathcal{M}$ such that $\mathrm{Hom}_{\mathcal{D}}(T \otimes M, N) \simeq \mathrm{Hom}_{\mathcal{D}}(T, \underline{\mathrm{Hom}}(M,N))$ for all T in $\mathcal{D}$. In particular,

$$\mathrm{Hom}_{\mathcal{D}}(M, N[i]) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, \underline{\mathrm{Hom}}(M,N)[i]).$$

Therefore, because every object of $\mathcal{M}$ is a direct summand of $R(hX)(j)$ for some smooth projective variety X and integer j, it suffices to prove (d) with $M = \mathbb{1}$ and $N = R(hX)(j)$.

In the case $i = 2j$, formula (3) becomes

$$K_0(X)^{(j)} \simeq \mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, R(hX)(j)[2j]). \tag{4}$$

As we noted above, ϖ_X acts on $K_0(X)^{(j)}$ as q^j. According to[10] Jannsen 2000, p257, for any smooth projective variety X, there is a noncanonical isomorphism

$$R(hX)(j)[2j] \approx \bigoplus\nolimits_s h^s(X)(j)[2j-s]. \tag{5}$$

The characteristic polynomial $P_{X,s}$ of ϖ_X on h^sX has roots of absolute value $q^{s/2}$, and $P_{X,s}(\varpi_X)$ acts as zero on $h^s(X)$ and hence on $\mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, h^s(X)(j)[2j-s])$. But we know

[10] Since this reference offers no proof, we note that it possible avoid using the decomposition (5) by exploiting the truncation functors instead.

from (4) that it acts as $P_{X,s}(q^j)$. Therefore, $\mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, h^s(X)(j)[2j-s]) = 0$ unless $s = 2j$, and so (4) becomes

$$K_0(X)^{(j)} \simeq \mathrm{Hom}_{\mathcal{M}_{\mathrm{num}}(k)}(\mathbb{1}, h^{2j}(X)(j)).$$

Under Grothendieck's isomorphism $K_0(X)_{\mathbb{Q}} \simeq CH^*(X)_{\mathbb{Q}}$, the factors $K_0(X)^{(j)}$ and $CH^j(X)_{\mathbb{Q}}$ correspond (this is obvious over a finite field), and (by definition)

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{num}}(k)}(\mathbb{1}, h^{2j}(X)(j)) = Z^j_{\mathrm{num}}(X)_{\mathbb{Q}}.$$

Moreover, our conditions imply that the map

$$CH^j(X) \simeq Z^j_{\mathrm{num}}(X)_{\mathbb{Q}}$$

obtained from these isomorphisms is the canonical one.[11] Hence, we have proved (c), and we have shown that

$$\mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, R(hX)(j)[i]) = 0 \qquad (6)$$

when $i = 2j \neq 0$.

To complete the proof of (d), we show that (6) holds also whenever $i \neq 2j$. Because of (3), it suffices to show that (a) and (c) imply that $K_i(X)_{\mathbb{Q}} = 0$ whenever $i \neq 0$. This is done in Geisser 1998, 3.3. We recall the proof. The functors $K_i(X) \otimes \mathbb{Q}$ factor through $\mathcal{M}_{\mathrm{rat}}(k)$ (Soulé 1984), and hence (because of (c)) through $\mathcal{M}_{\mathrm{num}}(k)$. Therefore, it suffices to prove that $K_i(M) \otimes \mathbb{Q} = 0$ ($i \neq 0$) for M a simple motive in $\mathcal{M}_{\mathrm{num}}(k)$. If $M = \mathbb{L}^j$, then $K_i(\mathbb{L}^j)$ is a direct factor of $K_i(\mathbb{P}^j)$, which is torsion (Quillen 1973). If $M \neq \mathbb{L}^j$, then $P_M(T)$ does not have q^j as a root (Milne 1994, 2.6). As $P_M(\varpi_X)$ acts as the nonzero rational number $P_X(q^j)$ on $K_i(M)^{(j)}$, and also as zero, the group $K_i(M)^{(j)}$ must be zero. □

COROLLARY 4.2 *Let $\mathcal{D}$ be as in the theorem, and let $\mathcal{M}$ be its heart. If the inclusion $\mathcal{M} \to \mathcal{D}$ extends to a functor $\mathcal{D}^b(\mathcal{M}) \to \mathcal{D}$, then that functor is an equivalence.*

PROOF. It suffices to show that $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{M})}(M, N[i]) \to \mathrm{Hom}_{\mathcal{D}}(M, N[i])$ is an isomorphism for all M, N in $\mathcal{M}$ and all i (see 4.5b below). For $i = 0$ this is automatic, and for $i \neq 0$, both groups are zero (recall that $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{M})}(M, N[i]) \simeq \mathrm{Ext}^i_{\mathcal{M}}(M, N)$, and that $\mathcal{M}$ is semisimple). □

REMARK 4.3 The existence of a compatible t-structure on a triangulated motivic category $\mathcal{D}$ implies the existence of a Bloch-Beilinson filtration on the Chow groups of smooth projective varieties for which

$$Gr^s(CH^j(X)) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, h^{2j-s}(X)(j)[s]) \qquad (7)$$

(Jannsen 2000, p258, 4.3). For a finite field, the existence of a Bloch-Beilinson filtration implies that rational equivalence equals numerical equivalence ($\mathbb{Q}$-coefficients) (ibid., 4.17).

[11] Let p and q be the projection maps

$$X \xleftarrow{p} X \times \mathrm{pt} \xrightarrow{q} \mathrm{pt}.$$

Let $\gamma \in CH^j(X)$, and let f be the map $CH^*(\mathrm{pt}) \to CH^*(X)$ defined by the correspondence $p^*(\gamma)$. Then

$$f(1_{\mathrm{pt}}) \overset{\mathrm{def}}{=} p_*(p^*(\gamma) \cup q^*(1_{\mathrm{pt}})) = \gamma \cup p_* q^*(1_{\mathrm{pt}}) = \gamma \cup 1_X = \gamma.$$

REMARK 4.4 Beilinson has conjectured that, for a smooth projective variety X,

$$Gr^s(CH^j(X)) = \mathrm{Ext}^s_{\mathcal{MM}_{\mathrm{num}}(k)}(\mathbb{1}, h^{2j-s}(X)(j)).$$

This is compatible with (7) only if $\mathcal{D} = \mathcal{D}^b(\mathcal{MM}_{\mathrm{num}}(k))$ (see the next remark).

REMARK 4.5 (a) Let $\mathcal{D}$ be a t-category with heart $\mathcal{C}$. Then $D^b(\mathcal{C})$ is also a t-category with heart $\mathcal{C}$, but in general there is no obvious relation between $D^b(\mathcal{C})$ and $\mathcal{D}$ (cf. Gelfand and Manin 1996, IV 4.13, p285). In particular, there will be no obvious functor $r\colon D^b(\mathcal{C}) \to \mathcal{D}$ extending the inclusion of $\mathcal{C}$ into $\mathcal{D}$ unless $\mathcal{D}$ is endowed with an additional structure. Beilinson (1987) defines the notion of a filtered triangulated category, and states[12] that such a category over a t-category $\mathcal{D}$ gives rise to a well-defined t-exact functor $r\colon D^b(\mathcal{C}) \to \mathcal{D}$ inducing the identity functor on $\mathcal{C}$ (ibid. A.6). The usual triangulated categories are endowed with filtered triangulated categories over them (ibid. A.2; Beilinson et al. 1982, 3.1).

(b) Let $\mathcal{D}$ be a t-category with heart $\mathcal{C}$. A t-exact functor $r\colon D^b(\mathcal{C}) \to \mathcal{D}$ inducing the identity functor on $\mathcal{C}$ need not be an equivalence even when $\mathcal{C}$ is semisimple (Deligne 1994, 3.1). We need the following well-known criterion:

> Let $r\colon D^b(\mathcal{C}) \to \mathcal{D}$ be a t-exact functor inducing the identity functor on $\mathcal{C}$; then r is an equivalence of t-categories if and only if the maps $\mathrm{Hom}_{D^b(\mathcal{C})}(M, N[i]) \to \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{C}}}(M, N[i])$ it defines are isomorphisms for all M, N in $\mathcal{C}$ and all i.

For M, N in $\mathcal{C}$, let $\mathrm{Ext}^i_{\mathcal{C}}(M, N)$ denote the Yoneda Ext-group, and for M, N in the heart of $\mathcal{D}$, let

$$\mathrm{Ext}^i_{\mathcal{D}}(M, N) = \mathrm{Hom}_{\mathcal{D}}(M, N[i]).$$

Since $\mathrm{Ext}^i_{\mathcal{C}}(M, N) \simeq \mathrm{Hom}_{D^b(\mathcal{C})}(M, N[i])$ (Verdier 1996, III.3.2.12), the criterion states that $r\colon D^b(\mathcal{C}) \to \mathcal{D}$ is an equivalence of t-categories if and only if the maps $\mathrm{Ext}^i_{\mathcal{C}}(M, N) \to \mathrm{Ext}^i_{\mathcal{D}}(M, N)$ it defines are isomorphisms for all M, N, and i.

5 The motivic t-category

Throughout this section, $k = \mathbb{F}_q$.

If we want the category of motives to have the Weil-number group P as its fundamental group, then Corollary 4.2 shows that $\mathcal{D}^b(\mathcal{M}_{\mathrm{num}}(k))$ is essentially the only candidate for a triangulated motivic category, and that it will have a compatible t-structure only if the Tate conjecture holds over k and rational equivalence equals numerical equivalence ($\mathbb{Q}$-coefficients). In this section, we prove that, when we assume these two conjectures, $\mathcal{D}^b(\mathcal{M}_{\mathrm{num}}(k))$ does have the hoped for properties.

PROPOSITION 5.1 *Let $\mathcal{D} = D^b(\mathcal{M}_{num}(k))$. Then $\mathcal{D}$ is a triangulated rigid tensor category with t-structure, and there exists a functor*

$$R\colon \mathcal{M}_{rat}(k) \to \mathcal{D},$$

unique up to a unique isomorphism, such that $H^i(RX) = h^i(X)[-i]$ for all i.

[12] Without proof; cf. the discussion Beilinson et al. 1982, 3.1, which, however, states that (at that time) the situation had not been axiomatised.

PROOF. Let $C^b(\mathcal{M}_{\text{num}}(k))$ be the category of bounded complexes of objects in $\mathcal{M}_{\text{num}}(k)$, and let $C_0^b(\mathcal{M}_{\text{num}}(k))$ be the full subcategory of bounded complexes whose differentials are zero. Because $\mathcal{M}_{\text{num}}(k)$ is semisimple, the functor $D^b(\mathcal{M}(k)) \to C_0^b(\mathcal{M}_{\text{num}}(k))$ sending A to

$$\bigoplus_r H^r(A)[-r] = \cdots \to H^{r-1}(A) \xrightarrow{0} H^r(A) \to \cdots$$

is an equivalence of categories (Gelfand and Manin 1996, p146) quasi-inverse to the inclusion functor. Since $C_0^b(\mathcal{M}_{\text{num}}(k))$ is a direct sum of copies of $\mathcal{M}_{\text{num}}(k)$, and $\mathcal{M}_{\text{num}}(k)$ is tannakian, it follows that $\mathcal{D}$ is a rigid tensor category. Define R to be

$$X \mapsto (\cdots \to h^{r-1}(X) \xrightarrow{0} h^r(X) \to \cdots). \qquad \square$$

REMARK 5.2 Deligne (1968, 1.11, 1.13)[13] proves the following:

> Let $\mathcal{A}$ be an abelian category, and suppose that an object C of $D^b(\mathcal{A})$ admits endomorphisms $p_i\colon C \to C$ such that $H^j(p_i) = \delta_{ij}$ and the p_i are orthogonal idempotents; then there is a unique isomorphism $C \simeq \bigoplus_i H^i(C)[-i]$ inducing the identity map on cohomology and such that p_i is the i^{th} projection map.

Let R' be a functor $\mathcal{M}_{\text{rat}}(k) \to \mathcal{D}^b(\mathcal{M}_{\text{num}}(k))$. Then Deligne's result shows that, for any smooth projective variety X over k, there is a unique isomorphism $R'(X) \simeq R(X)$ inducing the identity on cohomology and such that $P^i(\varpi_X)$ is the projection from $R'(X)$ onto $h^i(X)[-i]$. Here P^i is as in (1).

THEOREM 5.3 *Assume that the Tate conjecture holds over k and that numerical equivalence coincides with rational equivalence (with $\mathbb{Q}$-coefficients).*

(a) *$D^b(\mathcal{M}_{num}(k))$ has a natural structure of a triangulated motivic category.*
(b) *The standard t-structure on $D^b(\mathcal{M}_{num}(k))$ is compatible, and it is the unique compatible t-structure with heart $\mathcal{M}_{num}(k)$.*
(c) *The functor $X \mapsto RX$ sending a smooth projective variety over k to its motivic complex (see 5.1) has a unique extension to all varieties over k.*
(d) *For each l (including p) there is a t-exact functor R_l from $D^b(\mathcal{M}_{num}(k))$ to a t-category $\mathcal{D}_l$ such that $X \mapsto R_l(RX)$ is the functor giving rise to the absolute l-adic cohomology.*

In the remainder of this section, we explain these statements in more detail and prove them.

Statement (a). In computing the right hand side of (3), we can replace $\mathcal{D}^b(\mathcal{M}_{\text{num}}(k))$ with the equivalent category $C_0^b(\mathcal{M}_{\text{num}}(k)) \simeq \bigoplus_r \mathcal{M}_{\text{num}}(k)[r]$. Therefore,

$$\mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, R(X)(j)[i]) = \bigoplus_s \mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, h^s(X)(j)[i-s]),$$

and

$$\mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, h^s(X)(j)[i-s]) = \mathrm{Ext}^{i-s}_{\mathcal{M}}(\mathbb{1}, h^s(X)(j)),$$

which is zero for $i \neq s$ because $\mathcal{M}$ is semisimple, and for $i = s$, $s \neq 2j$, because $\mathbb{1}$ and $h^s(X)(j)$ will then have different weights. It is immediate from the definition of $\mathcal{M}_{\text{num}}(k)$, that

$$\mathrm{Hom}(\mathbb{1}, h^{2j}(X)(j)) \simeq Z^j_{\text{num}}(X)_{\mathbb{Q}}.$$

[13]This also applies to t-categories. To check this, one only has to check that the spectral sequence in Deligne's proof exists for t-categories (for which there exist references).

On the other hand, $K_i(X)_{\mathbb{Q}} = 0$ for $i \neq 0$ (see the proof 4.1), and $K_0(X)^{(j)} \simeq CH^j(X)_{\mathbb{Q}}$. Therefore, we can define (3) to be the natural map

$$CH^j(X)_{\mathbb{Q}} \to Z^j_{\text{num}}(X)_{\mathbb{Q}}$$

when $i = 2j$ and zero otherwise.

Statement (b). By hypothesis, rational, l-homological, and numerical equivalence coincide ($\mathbb{Q}$-coefficients), and so the standard t-structure is obviously compatible. It is the unique t-structure with heart $\mathcal{M}(k)$ because the heart determines the t-structure (Beilinson et al. 1982, 1.2, 1.3).

Statement (c). We only sketch the argument, deferring a more detailed proof to Milne and Ramachandran nd. The key point is that de Jong's theorem (de Jong 1996, 3.1) allows one to define a simplicial resolution

$$V \xleftarrow{f} U_\bullet \xrightarrow{j} X_\bullet$$

of any variety V over k in which j is simplicial strict compactification and f is a proper hypercovering of V by a split simplicial smooth variety (cf. Berthelot 1997, 6.3). One first extends R to the category of strict compactifications, and then to the simplicial objects in the category of strict compactifications. Then one defines $RV = R(U_\bullet \to X_\bullet)$, and verifies that it is independent of the choice of the simplicial resolution (up to a well-defined isomorphism).

Statement (d), $l \neq p$. For $l \neq p$, let $\mathcal{D}(k, \mathbb{Z}_l)$ be the category $D^b_c(k, \mathbb{Z}_l)$ defined in Deligne 1980, 1.1.2. It is a t-category whose heart is $\mathcal{R}(k, \mathbb{Z}_l)$, the category of finitely generated $\mathbb{Z}_l$-modules endowed with a continuous action of $\mathrm{Gal}(\mathbb{F}/k)$. Each variety X over k defines an object $R\varGamma X$ in $\mathcal{D}(k; \mathbb{Z}_l)$ such that $H^i(R\varGamma V) \simeq H^i_{\text{et}}(V, \mathbb{Z}_l)$ (as an object of $\mathcal{R}(k; \mathbb{Z}_l)$). In (Milne and Ramachandran nd), it is shown that $\mathcal{D}(k, \mathbb{Z}_l) \simeq D^b(\mathcal{R}(k, \mathbb{Z}_l))$. Now quotient out by the torsion objects to obtain $\mathbb{Q}_l$-linear categories $\mathcal{D}(k, \mathbb{Q}_l) \simeq D^b(\mathcal{R}(k, \mathbb{Q}_l))$. We define R_l to be the derived functor of $\omega_l \colon \mathcal{M}(k, \mathbb{Q}) \to \mathcal{R}(k, \mathbb{Q}_l)$ (which obviously exists). Applying Deligne 1968, 1.11, 1.13 (cf. 5.2), we see that, for each smooth projective variety X over k, there is a unique isomorphism $R_l(X) \simeq \bigoplus_i H^i_l(X)[-i]$ inducing the identity map on cohomology and such that $P^i(\omega_X)$ is the i^{th} projection map. Here P^i is the polynomial in (1).

Statement (d), $l = p$. Let R be the Raynaud ring, and $D(R)$ the derived category of the category of graded R-modules (Illusie 1983, 2.1). For a smooth projective variety X over k, let $W\Omega^\bullet_X$ be the de Rham-Witt complex on X, and let $R\varGamma(W\Omega^\bullet_X)$ be its image under the derived functor of $\varGamma = \varGamma(X, -)$. Then $R\varGamma(W\Omega^\bullet_X)$ lies in the full subcategory $D^b_c(R)$ of $D(R)$ consisting of bounded R-complexes whose cohomology modules are coherent (Illusie and Raynaud 1983, II 2.2), and $H^i(R\varGamma(W\Omega^\bullet_X) \simeq H^i_{\text{crys}}(X/W)$. When we endow $D^b_c(R)$ with Ekedahl's t-structure (Illusie 1983, 2.4.8) and quotient out by torsion objects, we obtain a $\mathbb{Q}_p$-linear t-category $\mathcal{D}(k, \mathbb{Q}_p)$ whose heart is $\mathcal{R}(k, \mathbb{Q}_p)$, the category of F-isocrystals. Moreover, $\mathcal{D}(k, \mathbb{Q}_p) \simeq D^b(\mathcal{R}(k, \mathbb{Q}_p))$ (Milne and Ramachandran nd). Define $R_p \colon \mathcal{M}(k; \mathbb{Q}) \to \mathcal{D}(k; \mathbb{Q}_p)$ in the obvious way. Applying Deligne 1968, 1.11, 1.13 again, we see that, for each smooth projective variety X over k, there is a unique isomorphism

$R_p(X) \simeq \bigoplus_i H_p^i(X)[-i]$ inducing the identity map on cohomology and such that $P^i(\omega_X)$ is the i^{th} projection map.

REMARK 5.4 Statement (c) and (d) of the theorem are very strong. Consider, for example, a closed subvariety Z of codimension r in a smooth projective variety X and a smooth alteration $\tilde{Z} \to Z$. Then the theorem says that there is an exact sequence

$$h^{i-2r}(\tilde{Z})(r) \to h^i(X) \to h^i(U), \quad U = X \smallsetminus Z,$$

whose l-adic realization is the sequence in (1.13) for $l \neq p$.

Application.

Using (c) and (d), we can extend the definition of $\mathbb{Q}_p$ cohomology (Milne 1986a, p309) from smooth projective varieties to all varieties, namely, for any variety X over k, define

$$H^i(X, \mathbb{Q}_p(r)) = \mathrm{Hom}_{\mathcal{D}(k;\mathbb{Q}_p)}(\mathbb{1}, R_p(RX)(r)[i]).$$

The main theorem of Milne and Ramachandran 2005 shows that this agrees with the original definition when X is smooth and projective.

6 The $\mathbb{Q}$-algebra of correspondences at the generic point

In this section, $k = \mathbb{F}_q$ and we assume that the Tate conjecture holds over k and that numerical equivalence equals rational equivalence ($\mathbb{Q}$-coefficients). We allow $l = p$.

Effective motives

Let $\mathcal{M}^{\text{eff}}(k)$ be the category of effective motives given by Grothendieck's construction using algebraic classes modulo numerical equivalence as correspondences. It is an abelian nonrigid tensor category, and we let $\mathcal{D}^{\text{eff}}(k) = D^b(\mathcal{M}^{\text{eff}}(k))$. Much of Theorem 5.3 continues to hold. In particular, attached to a smooth projective variety X and an open subvariety U, there is a well-defined restriction map $h^i(X) \to h^i(U)$ whose l-adic realization is $H_l^i(X) \to H_l^i(U)$ (cf. 5.4). We define

$$F_a^r h^i(X) = \bigcup\nolimits_U \mathrm{Ker}(h^i(X) \to h^i(U))$$

where U runs over the open subvarieties of X such that $X \smallsetminus U$ is of codimension at least r.

PROPOSITION 6.1 *For all l (including $l = p$)*

$$R_l(F_a^r h^i(X)) = F_b^r H_l^i(X).$$

PROOF. The functor R_l is exact, and so

$$R_l(F_a^r h^i(X)) = F_a^r H_l^i(X).$$

Therefore, the statement follows from the generalized Tate conjecture (1.10, 1.12, 1.15). □

Definition of the $\mathbb{Q}$-algebra of correspondences at the generic point

In this subsection, we translate some definitions and results of Beilinson 2002 into our context. Let X be a connected algebraic variety of dimension n over a finite field k, and let η be its generic point. Define

$$CH^n(\eta \times \eta) = \varinjlim CH^n(U \times U),$$

where U runs over the open subvarieties of X. Following Beilinson 2002, 1.4, we define

$$A(X) = CH^n(\eta \times \eta) \otimes \mathbb{Q}.$$

Composition of correspondences makes $A(X)$ into an associative $\mathbb{Q}$-algebra, called the ***$\mathbb{Q}$-algebra of correspondences at the generic point.***

Denote by $\overline{h}^n(X)$ the image of the canonical map $h^n(X) \to h^n(\eta)$ (ind object of $\mathcal{M}^{\text{eff}}(k)$).

THEOREM 6.2 *For any connected smooth projective varieties X, X' of dimension n over k, the map*

$$CH^n(\eta' \times \eta) \otimes \mathbb{Q} \to \operatorname{Hom}(\overline{h}^n(\eta), \overline{h}^n(\eta'))$$

is an isomorphism.

PROOF. Beilinson's proof (2002, 4.9) applies in our context. We leave it as an exercise to the reader to verify this. □

COROLLARY 6.3 *For any connected smooth projective variety X of dimension n over k, there is a canonical isomorphism of $\mathbb{Q}$-algebras*

$$A(X) \simeq \operatorname{End}(\overline{h}^n(X)).$$

PROOF. It is only necessary to observe that composition of correspondences corresponds to composition of endomorphisms (Beilinson 2002, 4.10). □

COROLLARY 6.4 *The $\mathbb{Q}$-algebra $A(X)$ is finite-dimensional and semisimple.*

PROOF. Immediate from (6.3) because $\mathcal{M}^{\text{eff}}(k)$ is a semisimple category over $\mathbb{Q}$ with finite-dimensional Homs. □

Calculation of the $\mathbb{Q}$-algebra of correspondences at the generic point

PROPOSITION 6.5 *For a connected curve X over k,*

$$A(X) \simeq \operatorname{End}(J) \otimes \mathbb{Q}$$

where J is the Jacobian of a smooth complete model of X.

PROOF. As X is geometrically reduced, its smooth locus X' can be embedded in a smooth projective curve Y, and $X' \hookrightarrow Y$ is uniquely determined up to a unique isomorphism. As

$$A(X) \simeq A(X') \simeq A(Y)$$

we may as well assume that X itself is smooth and projective. For any nonempty open U, the map $h^1(X) \to h^1(U)$ is injective because $H^1_l(X) \to H^1_l(U)$ is injective, and so $\overline{h}^1(X) = h^1(X)$. Therefore, $A(X) \simeq \mathrm{End}(h^1(X))$, and it follows from an isomorphism (of Weil)

$$CH^1(X \times X) \simeq CH^1(X) \oplus CH^1(X) \oplus \mathrm{End}(J)$$

that

$$\mathrm{End}(h^1(X)) \simeq \mathrm{End}(J) \otimes \mathbb{Q}$$

(see Scholl 1994, 3.3). □

For a connected smooth projective variety X of dimension n over k, define[14]

$$\overline{H}^n_l(X) = H^n_l(X)/F^0_b H^n_l(X).$$

For $l \neq p$, the quotient map $H^n_l(X) \to \overline{H}^n_l(X)$ defines an isomorphism of $\overline{H}^n_l(X)$ with the Tate substructure of $H^n_l(X)$ whose Frobenius eigenvalues α are such that a/q is not an algebraic integer. The quotient map $H^n_p(X) \to \overline{H}^n_p(X)$ can be identified with the map

$$H^n(X, W\Omega^\bullet)_\mathbb{Q} \to H^n(X, W\mathcal{O}_X)_\mathbb{Q} \simeq H^n_p(X)_{[0,1[}$$

(Illusie 1979, II 3.5.3, p616).

PROPOSITION 6.6 *For all primes l (including $l = p$),*

$$R_l(\overline{h}^n(X)) \simeq \overline{H}^n_l(X).$$

PROOF. Clearly,

$$0 \to F^0_a h^n(X) \to h^n(X) \to \overline{h}^n(X) \to 0$$

is exact. On applying the exact functor R_l, this gives an exact sequence

$$0 \to F^0_b H^n_l(X) \to H^n_l(X) \to \overline{H}^n_l(X) \to 0$$

by (6.3). □

THEOREM 6.7 *For all primes l (including $l = p$),*

$$A(X) \otimes \mathbb{Q}_l \simeq \mathrm{End}(\overline{H}^n_l(X))$$

(endomorphisms of $\overline{H}^n_l(X)$ as a Tate structure when $l \neq p$; endomorphisms of $\overline{H}^n_p(X)$ as an F-isocrystal when $l = p$).

PROOF. Follows from Proposition 6.6 and the fact that R_l defines isomorphisms

$$\mathrm{Hom}(M, N) \otimes_\mathbb{Q} \mathbb{Q}_l \simeq \mathrm{Hom}(R_l M, R_l N).$$ □

EXAMPLE 6.8 If $H^n(X, W\mathcal{O}_X)$ is torsion, then $A(X) = 0$. This is the case, for example, if X is a supersingular abelian surface, a supersingular $K3$ surface, or an Enriques surface (Illusie 1979, 7.1, 7.2, 7.3).

[14] For $l \neq p$, this is $Gr^0 H^n_l(X)$, the “composante pure de niveau n” of $H^n_l(X)$, of Grothendieck (1968, p162).

REMARK 6.9 It is possible to recover the rank of a motive M from its endomorphism algebra $\mathrm{End}(M)$: if

$$\mathrm{End}(M)=\prod\nolimits_j M_{r_j}(D_j)$$

is the decomposition of $\mathrm{End}(M)$ into a product of simple $\mathbb{Q}$-algebras, so each D_j is a division algebra over $\mathbb{Q}$, and Z_j is the centre of D_j, then

$$\mathrm{rank}(M)=\sum\nolimits_j r_j\cdot[Z_j\colon\mathbb{Q}]\cdot[D_j\colon Z_j]^{1/2}.$$

Since $A(X)$ is a birational invariant, (6.3) shows that the rank of $\overline{h}^n(X)$ is a birational invariant of connected smooth projective varieties. Hence the same is true of its p-adic realization, i.e.,

$$\dim H^n(X,W\mathcal{O}_X)=\dim H^n(X,W\Omega_X)$$

is a birational invariant of connected smooth projective varieties over a finite field.

Explicit description of $A(X)$

6.10 Let X be a smooth projective variety over k, and let $(\alpha_i)_{1\leq i\leq\beta_n}$ be the family of eigenvalues of ϖ_X on $H_l^n(X)$. Then the family $S(X)$ of eigenvalues of $\varpi_{\overline{h}^n(X)}$ consists of the α_i for which α_i/q is not an algebraic integer. Therefore, by Milne 1994, 2.14–2.15, the semisimple $\mathbb{Q}$-algebra $A(X)=\mathrm{End}(\overline{h}(X))$ has the following description. Let $o_1,\ldots,o_s$ be the distinct orbits for the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ on $S(X)$ and let r_j be the multiplicity of o_j:

$$F(X)=\coprod\nolimits_j r_j o_j.$$

Then

$$\overline{h}^n(X)=\sum\nolimits_j r_j N_j$$

where N_j is a simple motive with Frobenius eigenvalues the elements of o_j, and

$$A(X)\simeq\prod\nolimits_j M_{r_j}(\mathrm{End}(N_j)).$$

Let $\alpha\in o_j$. Then $\mathrm{End}(N_j)$ is isomorphic to a central simple algebra D_j over $Z_j=\mathbb{Q}[\alpha]$ with invariants (at the primes v of $\mathbb{Q}[\alpha]$)

$$\mathrm{inv}_v(D_j)=\left\{\begin{array}{ll}\frac{1}{2} & \text{if } v \text{ is real and } n \text{ is odd}\\ \frac{\mathrm{ord}_v(\alpha)}{\mathrm{ord}_v(q)}\cdot[\mathbb{Q}[\alpha]_v:\mathbb{Q}_p] & \text{if } v|p\\ 0 & \text{otherwise.}\end{array}\right\}$$

Therefore, the degree $[\mathbb{Q}[\alpha]\colon\mathbb{Q}]$ is the order of o_i, and the degree $[D_j\colon\mathbb{Q}[\alpha]]=e^2$ where e is the least common denominator of the numbers $\mathrm{inv}_v(D_j)$.

Following Beilinson (2002, p37), the gloomy will be tempted to look for counterexamples to this above calculations in order to ruin the conjectures.

7 Base fields algebraic over a finite field

Let k be a subfield of $\mathbb{F}$, and assume that the Tate conjecture holds and numerical equivalence equals rational equivalence ($\mathbb{Q}$-coefficients) for finite subfields of k.

7.1 We define $\mathcal{M}(k)$, $\mathcal{D}(k)$, etc. as the 2-category direct limits of the categories $\mathcal{M}(k')$, $\mathcal{D}(k')$, etc. for k' running over the finite subfields of k. These categories inherit the properties of $\mathcal{M}(k')$, $\mathcal{D}(k')$, etc.. We leave the details as an exercise to the reader.

A Solutions to the exercises

Exercise 1.14

To be added.

Exercise 1.6

It suffices to prove this with $\mathbb{Q}_\ell$ replaced by a finite ring Λ such that $\ell^m\Lambda = \Lambda$ for some m. Let $j\colon U \hookrightarrow X$ be an open immersion with X complete. Then (by definition), $H^i(U,\Lambda) = H^i(X, j_!\Lambda)$. There is a canonical map $j_!\Lambda \hookrightarrow \Lambda$, and hence a map $H^i_c(U,\Lambda) \to H^i(X,\Lambda)$ which we denote j_* (cf. Milne 1980, II 3.13).

A.1 *With the notations of (1.6), the following diagram commutes:*

$$\begin{array}{ccc} H^i(Y,\Lambda) & \xleftarrow{j'_*} & H^i_c(\pi^{-1}U,\Lambda) \\ \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi'^*} \\ H^i(X,\Lambda) & \xrightarrow{j_*} & H^i_c(U,\Lambda) \end{array}$$

PROOF. Choose compatible injective resolutions of $j_!\Lambda_X$ and Λ_X:

$$\begin{array}{ccc} j_!\Lambda_X & \longrightarrow & I_1^\bullet \\ \downarrow & & \downarrow \\ \Lambda_X & \longrightarrow & I_2^\bullet. \end{array}$$

Now pull-back by π^* to get the middle square of the following commutative diagram:

$$\begin{array}{ccccccc} j'_!\Lambda_Y & \xrightarrow{\simeq} & \pi^*(j_!\Lambda_X) & \xrightarrow{\sim} & \pi^*I_1^\bullet & \xrightarrow{\sim} & J_1^\bullet \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Lambda_Y & \xrightarrow{\simeq} & \pi^*\Lambda_X & \xrightarrow{\sim} & \pi^*I_2^\bullet & \xrightarrow{\sim} & J_2^\bullet. \end{array}$$

Here $\simeq$ denotes an isomorphism and $\sim$ a quasi-isomorphism; $J_1^\bullet$ and $J_2^\bullet$ are complexes of injectives. On applying $\Gamma(Y,-)$ we get the right hand square of the next commutative diagram:

$$\begin{array}{ccccc} \Gamma(X,I_1^\bullet) & \longrightarrow & \Gamma(Y,\pi^*I_1^\bullet) & \longrightarrow & \Gamma(Y,J_1^\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(X,I_2^\bullet) & \longrightarrow & \Gamma(Y,\pi^*I_2^\bullet) & \longrightarrow & \Gamma(Y,J_2^\bullet). \end{array}$$

On omitting the middle column, and taking cohomology, we get

$$\begin{array}{ccc} H^i_c(U,\Lambda) & \longrightarrow & H^i_c(\pi^{-1}U,\Lambda) \\ \downarrow & & \downarrow \\ H^i(X,\Lambda) & \longrightarrow & H^i(Y,\Lambda) \end{array}$$

After a reflection, this becomes the required diagram. □

A.2 *The Poincaré dual of the diagram in (A.1) (with i replaced by $2\dim Y - i$) is the diagram in (1.6).*

PROOF. For π_* and π'_*, this is the definition. For j_* (and j'_*), it is the commutativity of the diagram

$$\begin{array}{ccccccc} H^i(U,\Lambda) & \times & H_c^{2\dim X-i}(U,\Lambda(\dim X) & \to & H_c^{2\dim X}(U,\Lambda(\dim X)) & \simeq & \mathbb{Q}_l \\ \uparrow j^* & & \downarrow j_* & & & & || \\ H^i(X,\Lambda) & \times & H^{2\dim X-i}(X,\Lambda(\dim X)) & \to & H^{2d}(X,\Lambda(\dim X) & \simeq & \mathbb{Q}_l \end{array}$$

□

Exercise 6.2.

To be added.

Exercise 7.1

To be added, and add the main statements to the text.

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