Poincare-Pontrjagin Duality for Differential Characters

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Abstract

In 1973 Jeff Cheeger and Jim Simons introduced the theory of differential characters into geometry. This theory, which included the basic Chern-Simons invariant, had a wealth of applications. It provided new invariants for flat vector bundles. It gave a unified derivation of the Bott-Godbillon-Vey classes for foliations. It was used to prove new non-conformal-immersability theorems in riemannian geometry. A basic feature of the theory is that it provides an important refinement of the classical Chern-Weil Theory of characteristic classes.

In recent years differential characters have appeared in Deligne cohomology and also in mirror symmetry (as "gerbes with connection"). A holomorphic analogue of differential characters occurs in Arakelov theory – in the Arithmetic Riemann-Roch Theorem of Bismut-Gillet-Soulé.

In my lecture I will sketch a derivation of the basic theory from a de Rham-Federer point of view. I will show how the rings of differential characters can be defined in terms of differential forms and currents, in the spirit of the classical theorems of de Rham and Federer for cohomology.

The basic objects in the theory are **sparks**. These are currents T on a manifold with the property that

$$dT = \phi - R$$

where ϕ is a smooth differential form and R is a rectifiable current (e.g. a manifold). Such objects occur quite naturally. For example if σ is a holomorphic section of a hermitian line bundle, then setting $T = \frac{1}{2\pi i} d^C \log \|\sigma\|$ we have that

$$dT = c_1(\Omega) - Div(\sigma)$$

where $c_1(\Omega)$ is the first Chern form of the line bundle and $Div(\sigma)$ is the divisor of the section.

I will then show that differential characters obey Poincaré-Pontrjagin duality. That is, on a compact oriented n-manifold X the pairing

$$\widehat{\mathbb{H}}^k(X) \times \widehat{\mathbb{H}}^{n-k-1}(X) \ \to S^1$$

$$(a, b) \ \mapsto \ a * b([X])$$

given by Cheeger product and evaluation on the fundamental cycle, gives an isomorphism of the group of differential characters of degree k with the smooth Pontryagin dual to characters in degree $n-k-1$.