VALUES OF ZETA FUNCTIONS AT $s = \frac{1}{2}$

NIRANJAN RAMACHANDRAN

To PVR and SS, in gratitude

Abstract. We study the behaviour near $s = \frac{1}{2}$ of zeta functions of varieties over finite fields $\mathbb{F}_q$ with q a square. The main result is an Euler-characteristic formula for the square of the special value at $s = \frac{1}{2}$. The Euler-characteristic is constructed from the Weil-étale cohomology of a certain supersingular elliptic curve.

Introduction

Let V be a integral scheme of finite type over Spec $\mathbb{Z}$. The special values $a_V(n)$ at integers $s = n$ of the zeta function $\zeta(V, s)$ of V are conjecturally related to deep arithmetical invariants of V. One may ask if the special values of $\zeta(V, s)$ at non-integral values of s, e.g. $s = \frac{1}{2}$, also admit an arithmetical interpretation. The simplest example – and perhaps the most interesting – arises from the central value of the Riemann zeta function $\zeta(s) = \zeta(\text{Spec } \mathbb{Z}, s)$:

Is there a motivic interpretation of $\zeta(\frac{1}{2})$?

For instance, it is unknown if $\zeta(\frac{1}{2})$ is a period in the sense of [9]; however, for certain triple product L-functions, the work of M. Harris and S. Kudla [7] relates the central critical value (at $s = \frac{1}{2}$) to periods, cf. [22, Eq. (48), p.459].

The motivic philosophy indicates that $a_V(n)$ depends on the interaction of the motive $h(V)$ of V with $\mathbb{Z}(\pm n)$ (power of the Tate motive). This leads us to suspect that the special value at $s = \frac{1}{2}$ is governed by an exotic object (unknown to exist, as yet): a square root $\mathbb{Z}(\frac{1}{2})$ of the Tate motive $\mathbb{Z}(1)$ over Spec $\mathbb{Z}$.

It is clear that the investigation of special values at $s = \frac{1}{2}$ should begin with the important case of varieties over finite fields. Namely, consider the zeta function $\zeta(X, s)$ of a smooth projective variety X over a finite field $k = \mathbb{F}_q$ of characteristic $p > 0$. The function $Z(X, t)$, defined by $Z(X, q^{-s}) := \zeta(X, s)$, is a rational function of t with integer coefficients. For any integer $n \geq 0$, the order of the pole $\rho_n := -\text{ord}_{s=n}\zeta(X, s)$ at $s = n$ and the special value $a_X(n)$ of $Z(X, t)$ at $t = q^{-n}$ conjecturally admit a motivic interpretation.

The Tate conjecture (Conjecture 2) predicts that

$$\rho_n = \text{rank Hom}(\mathbb{Z}(-n), h^{2n}(X)),$$

in the category of motives and $h^{2n}(X)$ is part of the motive of X. A related variant is that ρ_n is the rank of the Chow group $CH^n(X)$ of algebraic cycles of codimension n on X.

The Lichtenbaum-Milne conjecture (Conjecture 3) expresses $a_X(n)$ as an Euler-characteristic of étale motivic cohomology $H^*(X, \mathbb{Z}_X(n))$; i.e., the cohomology of the (étale) motivic complexes $\mathbb{Z}_X(n)$ of S. Lichtenbaum; $\mathbb{Z}_X(0)$ is the constant sheaf $\mathbb{Z}$ and $\mathbb{Z}_X(1) = \mathbb{G}_m[-1]$ is the sheaf $\mathbb{G}_m$ in degree one. Their conjecture is known for $n = 0$ (unconditionally) and for $n = 1$ (modulo the Tate conjecture for

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divisors on X); cf. [11, 19]. For $n = 0, 1$, it takes the form

$$a_X(0) = \pm\chi(X, \mathbb{Z}), \qquad a_X(1) = \pm\frac{\chi(X, \mathbb{G}_a)}{\chi(X, \mathbb{G}_m)} = \pm\frac{q^{\chi(X,\mathcal{O}_X)}}{\chi(X, \mathbb{G}_m)}.$$

Lichtenbaum [12] has provided another elegant interpretation of $a_X(0)$ using his Weil-étale topology (cf. Theorem 5).

Let us now turn to the special value at $s = \frac{1}{2}$ or $t = 1/\sqrt{q}$. One can ask for a motivic description of

- the order of vanishing $\rho_X := \mathrm{ord}_{s=\frac{1}{2}}\zeta(X, s)$ and
- the corresponding special value c_X at $s = \frac{1}{2}$, viz., $c_X := \lim_{t\to 1/\sqrt{q}}(1 - \sqrt{q}t)^{-\rho_X} Z(X, t)$.

The main result of this paper provides such a description, under the condition that $q = p^{2f}$, i.e., that $\mathbb{F}_{p^2} \subset \mathbb{F}_q$. We note that c_X may not be rational, if the condition on q is dropped.

Our paper is an exploration, using the methods of [11, 19, 12, 21], of a beautiful *suggestion* of Yuri Manin that "a certain supersingular elliptic curve E might be useful in finding the expression for $\mathbb{Z}(\frac{1}{2})$", because its one-motive $h_1(E)$ is the square root of Tate's motive in the same sense as the Dirac operator is the square root of a Laplacian of a spin manifold. The rank of $h_1(E)$ is two whereas $\mathbb{Z}(1)$ has rank one. Spin structures on a curve involve a square root of the canonical bundle, the dualizing object or the orientation sheaf. The ℓ-adic analogue $\mathbb{Z}_\ell(1)$ of the dualizing sheaf, on a curve, comes from $\mathbb{Z}(1)$. This suggests that a motivic interpretation of $\zeta(\frac{1}{2})$ should involve a spin structure on the "curve" Spec $\mathbb{Z}$. We shall refer to $\mathbb{Z}(\frac{1}{2})$ as the Manin motive. Going to the double cover Spec $\mathbb{F}_{p^2}$ of Spec $\mathbb{F}_p$ seems essential in obtaining the square root $\mathbb{Z}(\frac{1}{2})$ of $\mathbb{Z}(1)$: the elliptic curve E exists only over $\mathbb{F}_{p^2}$.

Our starting point was the observation that E provides the required motivic interpretation of ρ_X :

$$2\rho_X = \text{rank } \mathrm{Hom}_{\mathbb{F}_q}(E, Pic(X)),$$

(Lemma 6) where the abelian variety $Pic(X)$ is the Picard variety of X. This suggests that the étale sheaf E on X may provide an interpretation of c_X.

Our main theorem is an analogue of Lichtenbaum's result (Theorem 5):

Theorem 1. *Let X be a smooth projective variety over $\mathbb{F}_q$ with $q = p^{2f}$ and E as before. Write θ for the generator of the Weil-étale cohomology group $H^1_W(\text{Spec } \mathbb{F}_q, \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$; cf.* [12, §8].

(i) *The Weil-étale cohomology groups $H^i_W(X, E)$ are finitely generated.*

(ii) $H^i_W(X, E) = 0$ *for $i > 1 + 2 \dim X$.*

(iii) *The alternating sum $\Sigma(-1)^i r_i$ of the ranks r_i of $H^i_W(X, E)$ is zero.*

(iv) *The secondary Euler-characteristic $\Sigma(-1)^i i r_i$ is equal to $-2\,\mathrm{ord}_{s=\frac{1}{2}}\zeta(X, s) = -2\rho_X$. In fact, $r_0 = r_1 = 2\rho_X$ and $r_i = 0$ for $i > 1$.*

(v) *The cohomology groups h^i of the complex*

$$H^0_W(X, E) \xrightarrow{\cup\theta} H^1_W(X, E) \xrightarrow{\cup\theta} H^2_W(X, E) \to \cdots$$

are finite; the special value at $s = \frac{1}{2}$, viz., $c_X := \lim_{t\to 1/\sqrt{q}}(1 - \sqrt{q}t)^{-\rho_X} Z(X, t)$ is given by

$$c_X^2 = \frac{q^{\chi(X,\mathcal{O}_X)}}{\chi(X, E)}$$

where

$$\chi(X, E) = \prod [h^i]^{(-1)^i}.$$

Tate's theorem [24] on abelian varieties is crucial here; since this result of Tate is a special case (of divisors on abelian varieties) of his conjecture, the case of $s = \frac{1}{2}$ is intermediate between the case of $s = 0$ and $s = 1$. As $h_1(E)$ has rank two, it is $2\rho_X$ and c_X^2 that arise rather than ρ_X and c_X. Note that the Weil-étale motivic cohomology groups $H^*_W(X, \mathbb{Z}_X(n))$ are known to be finitely generated only for $n = 0$ [12, §8]; the case $n \neq 0$ requires the Tate conjecture [5].

The proof of the main theorem in §5 depends on the results in §4. As usual, the main difficulties lie in the p-part. This involves the nontrivial computation of the cohomology of the flat group scheme ${}_{p^n}E$ on X; we do this by using the de Rham-Witt complex of X; our approach was suggested by [17, §13]. The special values at $s = \frac{1}{2}$ for the L-functions of curves and motives are treated first in §2, 3, using the work of Milne [14, 15, 21], after some preliminaries in §1. The reader may be amused by the results in §2.2: the square root of the order of a Tate-Shafarevich group turns up in the description of the special value at $s = \frac{1}{2}$.

Even though $\zeta(X, 1/3)$ will be a nonzero rational number when $q = p^{3f}$, one does not have $\mathbb{Z}(\frac{1}{3})$ over such a field; similar comments apply to $1/4, 1/5, \cdots$. The case of $\mathbb{Z}(\frac{1}{2})$ is special and its existence ultimately has its origins in the structure of Weil q-numbers [20]. We refer to C. Deninger [3, §7] and Manin [13] for ideas on the exotic Tate motives $\mathbb{C}(s)$ for $s \in \mathbb{C} - \mathbb{Z}$.

Supersingular elliptic curves are important examples of motives [20, Theorem 2.41]. For instance, as indicated by J.-P. Serre [6], their existence implies the non-neutrality of the Tannakian category $\mathcal{M}$ of motives over a finite field (i.e. that $\mathcal{M}$ does not have a fibre functor to $\mathbb{Q}$-vector spaces).

Finally, elliptic curves or abelian varieties cannot provide $\mathbb{Z}(\frac{1}{2})$ over Spec $\mathbb{Z}$: the Hodge numbers are incompatible. Since elliptic curves with noncommutative endomorphism rings provide $\mathbb{Z}(\frac{1}{2})$ over a finite field, it may be that the arithmetic theory of non-commutative tori (initiated by Manin) holds the key to the definition of $\mathbb{Z}(\frac{1}{2})$ over Spec $\mathbb{Z}$.

Notations. Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$ and let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_q$. For any scheme V over $\mathbb{F}_q$, we write $\bar{V}$ for its base change to $\mathbb{F}$. Write Γ for the Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ and $\Gamma_0 \cong \mathbb{Z}$ for the subgroup (the Weil group) generated by the Frobenius automorphism γ: $x \mapsto x^q$. The additive and multiplicative valuations of $\mathbb{Q}$ are normalized so that $|p|_p = 1/p$ and $\mathrm{ord}_p(p) = 1$. We write H (resp. H_{fl}, H_W) for the étale (resp. flat, Weil-étale [12]) cohomology groups. Finally, $[S]$ denotes the order of a finite set S.

From §2 onwards, we assume that q is a square, i.e., that $\mathbb{F}_{p^2} \subset \mathbb{F}_q$.

1. Preliminaries

Here we recall some relevant facts and conjectures. We begin with an example.

1.1. **Example.** Let us consider the scheme $X = \mathrm{Spec}\ \mathbb{F}_q$. The value $(1-q^n)^{-1}$ of $\zeta(X, s) = (1-q^{-s})^{-1}$ at any negative integer $-n$ has many interpretations:

$$-\frac{1}{\zeta(X, -n)} = (q^n - 1) = [K_{2n-1}(\mathbb{F}_q)] = [\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{Z}, \mathbb{Z}(n))] = [\mathbb{G}_m(\mathbb{F}_{q^n})] = [T_n(\mathbb{F}_q)]$$

where $K_{2n-1}(\mathbb{F}_q)$ is the Quillen K-group of $\mathbb{F}_q$, $\mathcal{A}$ is the category of effective integral motives over $\mathbb{F}_q$ of [21], T_n is the torus over $\mathbb{F}_q$ obtained from $\mathbb{G}_m$ over $\mathbb{F}_{q^n}$ via Weil restriction of scalars.

The value of $\zeta(X, s)$ at $s = -\frac{1}{2}$ is $(1 - q^{\frac{1}{2}})^{-1}$ which is not rational unless q is an even power of p. So assume that $q = p^{2f}$. Even then, one cannot interpret $p^f - 1$ as the order of $\mathbb{F}_q$-points of any group (for varying f). But all is not lost. Consider $\zeta(X, -\frac{1}{2})^{-2} = (p^f - 1)^2$ which could be the order of $\mathbb{F}_q$-rational points of an elliptic curve E. In fact, one has

$$\frac{1}{\zeta(X, -\frac{1}{2})^2} = (p^f - 1)^2 = [E(\mathbb{F}_q)] = [\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{Z}, h_1(E))]$$

for an elliptic curve E with Frobenius eigenvalues (p^f, p^f). Such an E is supersingular.

As for $s = \frac{1}{2}$, we find that

$$\zeta(X, \frac{1}{2})^2 = \frac{q}{(1 - p^f)^2} = \frac{q^{\chi(\mathcal{O}_X)}}{[E(\mathbb{F}_q)]}$$

where $\chi(\mathcal{O}_X)$ is the Euler-characteristic of $\mathcal{O}_X$. Note that the numerator q^m of $\zeta(X, m) = \frac{q^m}{q^m - 1}$ for any positive integer m is an Euler-characteristic by Conjecture 3; cf. [19, Thm. 7.2]. □

1.2. **Supersingular elliptic curves and $\mathbb{Z}(\frac{1}{2})$.** We indicate the relations of supersingular elliptic curves to the motive $\mathbb{Z}(\frac{1}{2})$ by quoting the email message (dated 10 August 2004) of Milne:

"A. Grothendieck (and P. Deligne) knew already in the 1960's that a supersingular elliptic curve E over $\mathbb{F}$ provides a square root of the Tate motive in the following precise sense: after extending the coefficient field to a field $k \supset \mathbb{Q}$ splitting $\mathrm{End}(E)$, $h_1(E)$ decomposes into $\mathbb{Q}(\frac{1}{2}) \oplus \mathbb{Q}(\frac{1}{2})$ where $\mathbb{Q}(\frac{1}{2})^{\otimes 2} = \mathbb{Q}(1)$. This is exploited throughout [20]. For example, as for any category of motives, there is an exact sequence

$$0 \to \mathbb{G}_m \xrightarrow{\omega} P \xrightarrow{t} \mathbb{G}_m \to 0$$

such that $t \circ \omega = 2$ where P is the kernel of the Tannakian groupoid. The map ω is given by the weight gradation and t by the Tate motive. From E one gets a homomorphism $P \xrightarrow{e} \mathbb{G}_m$ such that $e \circ \omega = 1$, and so P decomposes into $P = P^0 \times w(\mathbb{G}_m)$; cf. [20, 2.41].

Over a finite field $\mathbb{F}_q$, supersingular elliptic curves come in three types:

(a) eigenvalues $\pm\sqrt{-q}$;
(b) $q = p^{2f}$; eigenvalues $-p^f, -p^f$;
(c) $q = p^{2f}$; eigenvalues p^f, p^f.

This follows from Tate's theorem [24, Thm. 1(a)] but was probably known to Deuring. The ones in (c) play the same role over $\mathbb{F}_{p^{2f}}$ as E over $\mathbb{F}$."

Remark. The elliptic curves in (c) are all $\mathbb{F}_q$-isogenous. Let E be an elliptic curve over $\mathbb{F}_{p^{2f}}$ of type (c). It is actually defined over $\mathbb{F}_{p^2}$ and its endomorphism ring $\mathrm{End}_{\mathbb{F}_q}(E) = \mathrm{End}_{\mathbb{F}}(E)$ is a maximal order in a quaternion algebra over $\mathbb{Q}$, ramified only at p and ∞; cf. p. 528 and Theorems 4.1, 4.2 of [25].

We fix an elliptic curve of type (c) over $\mathbb{F}_{p^2}$ and we refer to it throughout as E.

Remark. An abelian variety A over $\mathbb{F}_{p^{2f}}$ whose Frobenius spectrum is pure with support p^f is, by Tate's theorem [24, Thm. 1(a)] (cf. [25, pp.526-27]), isogenous to a power of E.

1.3. **Conjectures of Tate and Lichtenbaum-Milne.** Let X be a smooth projective (geometrically connected) variety over $\mathbb{F}_q$. Let $\zeta(X,s)$ be its zeta function. The associated function $Z(X,t)$, defined by $\zeta(X,s) = Z(X,q^{-s})$, is known to be a rational function of t with integer coefficients. The special value of $\zeta(X,s)$ at $s = n$ is the special value $a_X(n)$ of $Z(X,t)$ at $t = q^{-n}$, up to powers of $\log(q)$.

A motivic description of $a_X(n)$ requires a knowledge of the poles of $\zeta(X,s)$ given by

Conjecture 2. (Tate) $T(X,r)$: *For any integer $r \geq 0$, the dimension of the subspace of $H^{2r}(\bar{X}, \mathbb{Q}_\ell(r))$ generated by algebraic cycles is equal to the order ρ_r of the pole of $\zeta(X,s)$ at $s = r$.*

This conjecture provides a motivic description of ρ_r, i.e., $\rho_r = \mathrm{rank}\ \mathrm{Hom}(\mathbb{Z}(-r), h^{2r}(X))$ in the category of motives over $\mathbb{F}_q$. While Conjecture 2 is unknown in general, even for divisors $r = 1$, it does hold for divisors on abelian varieties [24, Thm. 1(a)].

In the case of a curve X, it is well known that $a_X(0), a_X(1)$ are related to the class number h of X [11, p. 191].

The description of $a_X(1)$ for a surface X can be considered, following M. Artin and J. Tate [23], as a geometric analogue of the Birch-Swinnerton-Dyer conjecture. Inspired by [23], Lichtenbaum [11] and Milne [19] have conjectured (Conjecture 3) a complete description (including the p-part) of $a_X(n)$ for any X as an Euler-characteristic in étale motivic cohomology $H^*(X, \mathbb{Z}_X(n))$. Namely, Lichtenbaum (resp. Milne) has defined a non-zero rational number $\chi(X, \mathbb{Z}(r))$ (resp. an integer $\chi(X, \mathcal{O}, r)$) via motivic cohomology $H^*(X, \mathbb{Z}_X(r))$ (resp. $\Sigma_{i=0}^{i=r}(r-i)\chi(X, \Omega^i)$) of X; we refer to [19, §0] for the precise definitions. These are related to $a_X(n)$ via the

Conjecture 3. (Lichtenbaum-Milne) [19, 0.1] *The special value $a_X(r)$ of $Z(X,t)$ at $t = q^{-r}$ can be given as an Euler-characteristic of motivic cohomology. Namely, one has*

$$\text{(LM(X,r))} \qquad Z(X,t) \sim \pm\chi(X, \mathbb{Z}(r)).q^{\chi(X,\mathcal{O},r)}(1 - q^r t)^{-\rho_r}, \quad \text{as } t \to q^{-r}$$

where the terms on the right are defined in [19, Conj. 0.1]

This conjecture generalizes [11] that of Artin-Tate [23] for the case of $r = 1$ and X a surface. We recall one result about Conjecture 3 and refer to [11, 18, 19, 21] for other results.

Theorem 4. [11, 18] (a) *Conjecture 3 is true for* $r = 0$.
(b) *If* $T(X,1)$ *holds, then the terms in* (LM(X,1)) *are finite and* (LM(X,1)) *is true.*

Part (b) generalises the result of Artin-Tate [23] for surfaces, and includes the p-part [16].

1.4. **Weil-étale cohomology.** As before, X is a smooth projective geometrically connected variety. Lichtenbaum [12] has given a beautiful interpretation, via his Weil-étale cohomology groups $H^*_W(X,\mathbb{Z})$, of the behaviour of $Z(X,t)$ at $t = 1$. Namely, cup-product with the generator θ of $H^1_W(\mathrm{Spec}\ \mathbb{F}_q,\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}$ gives a complex $(H^\bullet_W(X,\mathbb{Z}),\theta)$ of finitely generated abelian groups whose cohomology groups $h^*_W(X)$ are finite; write $\chi_W(X) = \frac{[h^0_W(X)]\cdots}{[h^1_W(X)]\cdots}$ for the alternating product of the orders $[h^*_W(X)]$ of $h^*_W(X)$. Then, Lichtenbaum's result[1] on the behaviour of $Z(X,t)$ at $t = 1$ is as follows:

Theorem 5. (Lichtenbaum)
(i) *the usual Euler characteristic* $\Sigma(-1)^i r_i$ *of* X *is zero; here* $r_i = \mathrm{rank}\ H^i_W(X,\mathbb{Z})$.
(ii) *the order of the zero of* $Z(X,t)$ *at* $t = 1$ *is given by the* secondary Euler characteristic $\Sigma(-1)^i i.r_i$.
(In our case, this order is minus one.)
(iii) *the special value* $a_X(0) := \lim_{t\to 1} Z(X,t)(1-t)$ *is the* multiplicative *Euler characteristic:*

$$a_X(0) = \pm\chi_W(X).$$

1.5. **Behaviour at $s = \frac{1}{2}$: basic invariants.** Our aim in this paper is a motivic description, when $q = p^{2f}$, of the following arithmetic invariants of X:

- the integer $\rho_X :=$ the order of the zero of $\zeta(X,s)$ at $s = \frac{1}{2}$.
- the special value at $t = q^{-\frac{1}{2}}$ of $Z(X,t)$, viz., $c_X := \lim_{t\to 1/\sqrt{q}}(1-\sqrt{q}t)^{-\rho_X} Z(X,t)$.

As before, c_X differs from the special value at $s = \frac{1}{2}$ of $\zeta(X,s)$ by factors of *log* p. As $q = p^{2f}$, one has $c_X \in \mathbb{Q}^*$. Unlike ρ_r, the integer ρ_X has an unconditional motivic description, viz.,

Lemma 6. *Let* $q = p^{2f}$ *and let* $Pic(X)$ *be the Picard variety of* X. *One has*

$$2\rho_X = \mathrm{rank}\ \mathrm{Hom}_{\mathbb{F}_q}(E, Pic(X)).$$

For instance, $\rho_E = 2$ and rank $\mathrm{End}(E) = 4$. The RHS is always divisible by 4 because $\mathrm{Hom}_{\mathbb{F}_q}(E, Pic(X))$ is a module over $\mathrm{End}(E)$. Similarly, the LHS is divisible by 4: the integer ρ_X is even because the roots $\alpha \neq \sqrt{q}$ of the factor $P_1(t)$ of $Z(X,t)$ come in pairs $(\alpha, q/\alpha)$ and the degree of $P_1(t)$ is even.

Proof. This follows from Tate's theorem [24, Thm. 1(a)]: by the Weil conjectures, ρ_X is the multiplicity of the root $\sqrt{q}$ in the factor $P_1(t)$ of $Z(X,t)$ which is the characteristic polynomial of Frobenius on $Pic(X)$. For $A = E$ and $B = Pic(X)$, the integer $r(f_A, f_B)$ defined by Tate [24, (4), p. 138] is $2\rho_X$. □

Thus, the vanishing of $\zeta(X,s)$ at $s = \frac{1}{2}$ is controlled by the (non)-ordinarity of $Pic(X)$. The study of c_X occupies the rest of the article.

2. Curves

In this section, the work of Milne [14, 15] is used to provide motivic interpretations of c_X in the case of curves. Let X be a curve of genus g over $\mathbb{F}_q$ with $q = p^{2f}$, i.e., $\mathbb{F}_{p^2} \subset \mathbb{F}_q$. Write J for its Jacobian.

[1]His results include the case of arbitrary curves and smooth surfaces.

2.1. **Extensions and c_X.**

Theorem 7. *(i)* $\rho_X = \frac{1}{2}\text{rank } \mathrm{Hom}_{\mathbb{F}_q}(E,J)$.

(ii) the special value at $s=\frac{1}{2}$ of $\zeta(X,s)$ is given by

$$c_X^2 = q^{\chi(\mathcal{O}_X)}.\frac{[\mathrm{Ext}^1_{\mathbb{F}_q}(J,E)]}{[E(\mathbb{F}_q)]^2}.D$$

where $\chi(\mathcal{O}_X) = 1-g$ is the Euler-characteristic of $\mathcal{O}_X$, Ext^1 is computed in the category of group schemes over Spec $\mathbb{F}_q$*, and D is the discriminant of the pairing*

$$\mathrm{Hom}_{\mathbb{F}_q}(E,J) \times \mathrm{Hom}_{\mathbb{F}_q}(J,E) \xrightarrow{\circ} \mathrm{End}_{\mathbb{F}_q}(E) \xrightarrow{trace} \mathbb{Z}.$$

(iii) one has

$$c_X^2 = q^{\chi(\mathcal{O}_X)}.\frac{[H^1(X,E)]}{[H^0(X,E)_{tors}]^2}.D$$

where $H^*(X,E)$ is the étale cohomology of the sheaf defined by E and D is as in (ii).

Proof. (i) was proved earlier.

(ii) When $g=0$, one easily verifies that

$$c_{\mathbb{P}^1}^2 = \frac{q}{[E(\mathbb{F}_q)]^2}.$$

When $g \neq 0$, $Z(X,t) = P(t).Z(\mathbb{P}^1,t)$ where $P(t) = \prod(1-\alpha_i t)$, the numerator of $Z(X,t)$, is the characteristic polynomial of Frobenius on J. The result now follows from Lemma (6) and [14, Thm. 3].

(iii) As $\mathrm{Ext}^1_{\mathbb{F}_q}(J,E) \cong H^1(X,E)$ [15, Cor. 3] and $E(\mathbb{F}_q) = H^0(X,E)_{tors}$ [15, p. 120], this is clear. □

Remark. If X is ordinary, i.e., its Hasse-Witt matrix is invertible or the abelian variety J is ordinary, then the formula simplifies to

$$\zeta(X,\frac{1}{2})^2 = c_X^2 = q^{1-g}\frac{[\mathrm{Ext}^1_{\mathbb{F}_q}(J,E)]}{[E(\mathbb{F}_q)]^2}.$$

Since J is ordinary, the roots of P satisfy: $\alpha_i \neq \sqrt{q}$; thus $\zeta(X,\frac{1}{2}) \neq 0$. For an ordinary elliptic curve X, a simple computation yields

$$\pm c_X = 1 - \frac{[X(\mathbb{F}_q)]}{[E(\mathbb{F}_q)]}.$$

2.2. **BSD and c_X.** These formulas for c_X^2 are very reminiscent of the Birch-Swinnerton-Dyer (BSD) conjecture. In fact, Milne [15] has shown $\mathrm{Ext}^1_{\mathbb{F}_q}(J,E) \cong Ш(E/K)$ of the constant elliptic curve E over the function field $K = \mathbb{F}_q(X)$. Since the order of $Ш(E/K)$ is a square, $[\mathrm{Ext}^1_{\mathbb{F}_q}(J,E)] = m^2$ for a positive integer m. This gives a motivic interpretation of c_X for X ordinary:

$$c_X = \pm p^{(1-g)f}\frac{m}{[E(\mathbb{F}_q)]}.$$

Thus, the square root of the order of Ш is related to special values at $s=\frac{1}{2}$.

Theorem 8. *The special value at $s=1$ of the L-function $L(E/K,s)$ of E over K is:*

$$\lim_{s\to 1} L(E/K,s).(s-1)^{-2\rho_X} = c_X^2.(\log\, q)^{\rho}$$

where $\rho = \text{rank } E(K) = 2\rho_X = \text{rank } \mathrm{Hom}_{\mathbb{F}_q}(J,E)$.

Proof. This follows from [15, Thm. 3] and the first equation on page 102 of [15]. □

3. Integral motives

Some of the results in the previous section generalize to effective integral motives [21].

Recall the category $\mathcal{M}^+(\mathbb{F}_q;\mathbb{Z})$ of effective integral motives over $\mathbb{F}_q$ and the category $\mathcal{M}(\mathbb{F}_q;\mathbb{Z})$ of integral motives (see [21]). Assume that $q = p^{2f}$. The elliptic curve E defines an effective integral motive [21, 5.16] via its h^1 which is still denoted E. Its dual $E^\vee = h_1(E)$ is just $E \otimes \mathbb{Z}(1)$. The special value at $s = \frac{1}{2}$ of the L-function $L(M,s)$ of any effective integral motive M is related to the order of Ext-groups in $\mathcal{M}^+(\mathbb{F}_q;\mathbb{Z})$:

Proposition 9. *Let $q = p^{2f}$ and let r be the rank of M and $\rho_M := \mathrm{ord}_{s=\frac{1}{2}} L(M,s)$. One has*

(i) $2\rho_M = \mathrm{rank}\ \mathrm{Hom}(E,M)$.

(ii)

$$q^r L(M, s+\frac{1}{2})^2 \sim \frac{[\mathrm{Ext}^1(E,M)].D(M)}{[\mathrm{Hom}(E,M)_{tors}].[\mathrm{Hom}(M,E)_{tors}]}.(1-q^{-s})^\rho \quad \text{as} \quad s \to 0,$$

where $D(M)$ is the discriminant of the pairing

$$\mathrm{Hom}(E,M) \times \mathrm{Hom}(M,E) \xrightarrow{\circ} \mathrm{End}(E) \xrightarrow{trace} \mathbb{Z}.$$

(iii) *One has $L(M, s+\frac{1}{2})^2 = L(M \otimes E^\vee, s)$.*

Proof. If $L(M,s) = \prod(1 - a_i q^{-s})$, then by [21, 9.2(d)],

$$L(M \otimes E^\vee, s) = \prod_{i,j}(1 - \frac{a_i}{b_j} q^{-s}) = \prod_i (1 - a_i q^{-s-\frac{1}{2}})^2$$

because $L(E,s) = \prod(1 - b_j q^{-s}) = (1 - q^{\frac{1}{2}} q^{-s})^2$. This proves (iii).

Now it is clear that (i) and (ii) follow from [21, Thm 10.1]. □

It is possible to prove a Weil-étale variant [12] of the above result using [21, Thm. 10.5].

Remark. (i) One has

$$L(\mathbb{Z}, \frac{1}{2})^2 = (1 - q^{-\frac{1}{2}})^2 = \frac{[E(\mathbb{F}_q)]}{q}$$

By the proposition, $qL(\mathbb{Z}, \frac{1}{2})^2$ is $[\mathrm{Ext}^1(E,\mathbb{Z})]$ which, by [21, 8.7], is $[E(\mathbb{F}_q)]$.

(ii) Let A be an ordinary abelian variety of dimension d. One computes $L(h_1(A), \frac{1}{2}) \neq 0$ to be [14]

$$L(h_1(A), \frac{1}{2})^2 = q^{-d}[\mathrm{Ext}^1(A,E)]$$

where the Ext-group is computed in the category of group schemes over $\mathbb{F}_q$.

(iii) For any motive M over a global (or finite) field and for any $n \in \mathbb{Z}$, we write $M(n)$ for its (Tate) twist by $\mathbb{Z}(n)$. On the level of L-functions, twisting corresponds to translations on the s-axis: $L(M, s+n) = L(M(n), s)$. The above proposition computes special values of $L(M,s)$ at other half-integers as well. □

4. The p-adic Tate module of E

As before, X is a smooth projective geometrically connected variety over Spec $\mathbb{F}_q$ with $q = p^{2f}$, and E is our fixed elliptic curve.

The main result (Theorem 13) of this section relates c_X to the cohomology groups $H^*_{fl}(X, {}_{p^n}E)$ of the finite flat group scheme ${}_{p^n}E$ on X. This result is the technical core of the paper.

We shall freely use the results and methods of [18]. We recall from [18, §1] the category Pf of perfect affine schemes over $\mathbb{F}_q$, endowed with the étale topology. The computation of $H^*_{fl}(\bar{X}, {}_{p^n}E)$ here was inspired by [17, §13 (a)].

For any perfect field k of positive characteristic, we write $W(k)$ for the Witt ring of k. Recall the Dieudonne ring $A = W(\mathbb{F}_q)_\sigma[F,V]$; put $\bar{A} = A \otimes_{W(\mathbb{F}_q)} W(\mathbb{F})$. Note that the relation $FV = p = VF$ holds in A and $\bar{A}$.

Lemma 10. *(i) For $n \geq 1$ and $i \geq 0$, one has an exact sequence*

$$\cdots H^i_{fl}(\bar{X}, {}_{p^n}E) \to H^i(\bar{X}, W_{2n}\mathcal{O}) \xrightarrow{F-V} H^i(\bar{X}, W_{2n}\mathcal{O}) \to \cdots,$$

where $W_{2n}\mathcal{O}$ is the Witt vector sheaf of order $2n$ on $\bar{X}$.

(ii) The presheaf $T \mapsto H^i(X_T, {}_{p^n}E)$ on Pf is represented by an affine perfect group scheme [18, pp.303-4].

Proof. (i) This follows from the exact sequence of flat sheaves

$$0 \to {}_{p^n}E \to W_{2n}\mathcal{O} \xrightarrow{F-V} W_{2n}\mathcal{O} \to 0$$

on $\bar{X}$, a consequence of the fact that the Dieudonne module of the p-divisible group $E(p^\infty)$ (resp. W_{2n}) is $E' := \bar{A}/(F-V)$ (resp. $\bar{A}/(V^{2n})$). The cohomology of $W_{2n}\mathcal{O}$ is the same in the étale and flat topologies.

(ii) Follows from (i) in a standard manner, cf. [18, Lemma 1.8], because the presheaf $T \mapsto H^i(X_T, W_{2n}\mathcal{O})$ is representable. $\square$

As in [18, p. 322], the long exact sequence of Lemma 10 (i) can be regarded as one of affine perfect group schemes. Write $\mathcal{H}^i(X, W\mathcal{O}) := \varprojlim \mathcal{H}^i(X, W_{2n}\mathcal{O})$ and $\mathcal{H}^i(X, {}_{p^n}E)$ for the perfect group scheme in (ii). Write b_i for the dimension of the perfect pro-algebraic group scheme $\mathcal{H}^i(X, T_pE) := \varprojlim \mathcal{H}^i(X, {}_{p^n}E)$. Thus, $b_i = \infty$ unless the neutral component $\mathcal{H}^i(X, T_pE)^0$ of $\mathcal{H}^i(X, T_pE)$ is algebraic, in which case b_i is the number of copies of $\mathbb{G}_a^{pf}$ occuring as quotients in a composition series for $\mathcal{H}^i(X, T_pE)^0$.

Write W for $W(\mathbb{F})$. There is an exact sequence [18, p. 321]

$$0 \to H^i(\bar{X}, W\mathcal{O})_t \to H^i(\bar{X}, W\mathcal{O}) \to B_i \to 0, \tag{1}$$

of $W[[V]]$-modules in which B_i is a free W-module of finite rank and $H^i(\bar{X}, W\mathcal{O})_t$ is a finitely generated $(W/p^nW)[[V]]$-module for some n. Write d_i for the length of $H^i(\bar{X}, W\mathcal{O})_t \otimes_{W[[V]]} W((V))$ as a $W((V))$-module. The integer d_i is equal to the number of copies of $\mathbb{F}[[V]]$ occuring in a composition series for $H^i(\bar{X}, W\mathcal{O})_t$.

Proposition 11. *With notations as above, b_i is finite for all i and $\Sigma(-1)^i b_i = -\Sigma(-1)^i d_i$.*

Proof. We follow the proof of [18, Prop. 3.1] to which we refer for the properties of the integer $\chi(\alpha) := \dim \mathrm{Ker}(\alpha) - \dim \mathrm{Coker}(\alpha)$ attached to a morphism α of group schemes.

We first compute $\chi(F-V)$ on $H^i(\bar{X}, W\mathcal{O})$, in the terminology of [18, Prop. 3.1]. By [18, 3.2 (b)], it is possible to neglect finitely generated torsion W-modules which arise as subquotients of $H^i(\bar{X}, W\mathcal{O})$. As $F = 0$ on $\mathbb{F}[[V]]$, one has $\chi(F-V|\mathbb{F}[[V]]) = \chi(-V|\mathbb{F}[[V]]) = -1$. So $\chi(F-V|H^i(\bar{X}, W\mathcal{O})_t) = -d_i$.

We claim that $\chi(F-V|H^i(\bar{X}, W\mathcal{O})) = \chi(F-V|H^i(\bar{X}, W\mathcal{O})_t)$. Namely, we claim that $\chi(F-V|B_i) = 0$. This is a consequence of the semisimplicity of the category of isocrystals over $\mathbb{F}$: namely, the kernel (resp. cokernel) of $F-V$ on B_i is $\mathrm{Hom}_{\mathbb{F}}(E', B_i)$ (resp. $\mathrm{Ext}^1_{\mathbb{F}}(E', B_i)$) in the category of Dieudonne modules over $\mathbb{F}$. Here $E' := \bar{A}/(F-V)$ denotes the Dieudonne module of the p-divisible group $E(p)$. As $\mathrm{Hom}_{\mathbb{F}}(E', B_i)$ is a finitely generated free $\mathbb{Z}_p$-module and $\mathrm{Ext}^1_{\mathbb{F}}(E', B_i)$ a p-primary torsion group whose p-torsion is finite, one has $\chi(F-V|B_i) = 0$.

Writing K_i and C_i for the kernel and cokernel of $F-V$ on $\mathcal{H}^i(\bar{X}, W\mathcal{O})$, one has an exact sequence

$$0 \to C_{i-1} \to \mathcal{H}^i(\bar{X}, T_pE) \to K_i \to 0. \tag{2}$$

This, as in [18, p. 323], proves the proposition. $\square$

Any group G in (2) is an extension of an étale group G^{et} by a connected group G^0. Modulo finite groups, one has $K_i^{et}(\mathbb{F}) = \mathrm{Hom}_{\mathbb{F}}(E', B_i)$ and $C_i^{et}(\mathbb{F}) = \mathrm{Ext}^1_{\mathbb{F}}(E', B_i)$, and an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{F}}(E', B_{i-1}) \to \mathcal{H}^i(\bar{X}, T_pE)^{et}(\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(E', B_i) \to 0.$$

Note $H^i(\bar{X}, T_pE) = \mathcal{H}^i(\bar{X}, T_pE)(\mathbb{F})$. Set $H^*(X, T_pE) := \varprojlim H^i_{fl}(X, {}_{p^n}E)$.

Lemma 12. *There is an exact sequence*

$$0 \to H^{i-1}(\bar{X}, T_pE)_\Gamma \to H^i(X, T_pE) \to H^i(\bar{X}, T_pE)^\Gamma \to 0.$$

Proof. Easy adaptation of the proof of [19, Lemma 3.4]. □

4.1. **The action of γ on $H^*(\bar{X}, T_pE)$.** We now come to the main step in relating the p-adic Tate module of E to the special value c_X.

Consider the map $\alpha_i : H^i(X, T_pE) \to H^{i+1}(X, T_pE)$ defined by the following commutative diagram

$$\begin{array}{ccc} H^i(X, T_pE) & \xrightarrow{\alpha_i} & H^{i+1}(X, T_pE) \\ \downarrow & & \uparrow \\ H^i(\bar{X}, T_pE)^\Gamma & \longrightarrow & H^i(\bar{X}, T_pE)_\Gamma; \end{array}$$

the vertical maps arise from the previous lemma. As the lower map is cup-product with the canonical generator $\theta_p \in H^1(\Gamma, W)$ and $\theta_p^2 = 0$,

$$M^\bullet : \quad \cdots \to H^i(X, T_pE) \xrightarrow{\alpha_i} H^{i+1}(X, T_pE) \to \cdots$$

is a complex; cf. [21, pp. 544-545] for details. Define

$$z = \prod [H^i(M^\bullet)]^{(-1)^i}$$

when these numbers are finite. Write, as usual, $Z(X, t) = \prod P_i(t)^{(-1)^{i+1}}$ and $P_i(t) = \prod_j (1 - \omega_{ij} t)$ the characteristic polynomial of γ on ℓ-adic cohomology $H^i(\bar{X}, \mathbb{Z}_\ell)$. Our next result is a variant of [21, Theorem 9.6] (cf. §1.4).

Theorem 13. *(i) $H^*(X, T_pE)$ are finitely generated $\mathbb{Z}_p$-modules.*

(ii) the usual Euler characteristic $\Sigma(-1)^i$ rank $H^i(X, T_pE)$ is zero.

(iii) the secondary Euler characteristic $\Sigma(-1)^i i.$rank $H^i(X, T_pE)$ is $2\rho_X$.

(iv) the cohomology groups $H^i(M^\bullet)$ are finite.

(v) the value of z is given by

$$z = \prod_i ([H^i(M^\bullet)])^{(-1)^i} = \Big| \frac{q^{\chi(X, \mathcal{O}_X)}}{c_X^2} \Big|_p.$$

(vi) $H^i(X, T_pE)$ is finite for $i \neq 1, 2$.

Proof. One can write z as a product $z = z_0 z_{et}$ corresponding to the decomposition

$$0 \to \mathcal{H}^i(\bar{X}, T_pE)^0 \to \mathcal{H}^i(\bar{X}, T_pE) \to \mathcal{H}^i(\bar{X}, T_pE)^{et} \to 0.$$

of $\mathcal{H}^i(\bar{X}, T_pE)$ into its connected and étale parts. For each of the groups in the exact sequence

$$0 \to C^0_{i-1} \to \mathcal{H}^i(\bar{X}, T_pE)^0 \to K_i^0 \to 0$$

of connected group schemes, the map $\gamma - 1$ on the $\mathbb{F}$-points is surjective because it is an étale endomorphism of a connected group. Therefore, we obtain that $\mathcal{H}^i(\bar{X}, T_pE)^0_\Gamma = 0$, $[(\mathcal{H}^i(\bar{X}, T_pE)^0)^\Gamma] = q^{b_i}$, and

$$z_0 = q^{\Sigma(-1)^i b_i} = \Big| q^{-\Sigma(-1)^i b_i} \Big|_p.$$

The finite W-torsion in $H^*(\bar{X}, W\mathcal{O})$ does not contribute to z [14, pp. 80-81]. Thus, we may assume that $K_i^{et}(\mathbb{F}) = \mathrm{Hom}_{\mathbb{F}}(E', B_i)$ and $C_i^{et}(\mathbb{F}) = \mathrm{Ext}^1_{\mathbb{F}}(E', B_i)$, and that there is an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{F}_q}(E', B_{i-1}) \to H^i(X, T_pE)^{et} \to \mathrm{Hom}_{\mathbb{F}_q}(E', B_i) \to 0.$$

Now the first term is finite and the last term is a finitely generated $\mathbb{Z}_p$-module. As $H^i(X, T_pE)^{et}$ and $H^i(X, T_pE)$ differ only by a finite group of order q^{b_i}, this proves (i).

The étale part of $M^\bullet$ sits in the commutative diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathbb{F}_q}(E', B_1) \arrow[r, "\beta_1"] & \mathrm{Ext}^1_{\mathbb{F}_q}(E', B_1) \arrow[d] & \mathrm{Hom}_{\mathbb{F}_q}(E', B_3) \cdots \\
H^0(X, T_pE)^{et} \arrow[r, "\alpha_0"] \arrow[d] & H^1(X, T_pE)^{et} \arrow[r, "\alpha_1"] \arrow[u] & H^2(X, T_pE)^{et} \arrow[r, "\alpha_2"] \arrow[d] & H^3(X, T_pE)^{et} \cdots \arrow[u] \\
\mathrm{Hom}_{\mathbb{F}_q}(E', B_0) \arrow[r, "\beta_0"] & \mathrm{Ext}^1_{\mathbb{F}_q}(E', B_0) \arrow[u] & \mathrm{Hom}_{\mathbb{F}_q}(E', B_2) \arrow[r, "\beta_2"] & \mathrm{Ext}^1_{\mathbb{F}_q}(E', B_2) \cdots \arrow[u]
\end{tikzcd}$$

where the maps β_i are the ones in [14, Lemma 4] for the p-divisible groups G_i (whose dimension we denote by g_i) whose *dual* corresponds to B_i, and $H = E(p)$. We shall prove that z_{et} is defined and calculate its value by appealing to [14, Lemma 4]. Let us recall two relevant facts for this purpose.

• [18, Remark 5.5] the characteristic polynomial $P_i(t) = \prod_j (1 - \omega_{ij} t) := \det(1 - \gamma t)$ of γ on $H^i(\bar{X}, \mathbb{Q}_\ell)$ is the same as the characteristic polynomial of γ on the crystalline cohomology $H^i_{crys}(\bar{X}) \otimes \mathbb{Q}_p$ of $\bar{X}$.

• [8, 3.5.3, p. 616] the characteristic polynomial of γ on $H^i(\bar{X}, W\mathcal{O}) \otimes \mathbb{Q}_p$ (by (1), this is the same as on B_i) is $\prod_j (1 - \omega_{ij})$ where the product is over all ω_{ij} with $\mathrm{ord}_q(\omega_{ij}) < 1$.

We can now apply [14, Lemma 4] to obtain that z_{et} is defined (which proves (iv)) and given by:[2]

$$z_{et} = \left| q^{\Sigma(-1)^i g_i} \prod_i \Big(\prod_{\omega_{ij} \neq \sqrt{q}} (1 - \frac{\omega_{ij}}{\sqrt{q}}) \Big)^{(-1)^i} \right|_p^2$$

where the product is over all ω_{ij} with $\mathrm{ord}_q(\omega_{ij}) < 1$. Using

$$\left| 1 - \frac{\omega_{ij}}{\sqrt{q}} \right|_p = 1 \quad \text{if} \quad \mathrm{ord}_q(\omega_{ij}) > \frac{1}{2},$$

this condition $\mathrm{ord}_q(\omega_{ij}) < 1$ may be disregarded in z_{et}. This proves (iv).

We now complete the proof of (v); given the formulas for z_{et} and z_0, it remains to show that $\sum (-)^i (g_i - b_i) = \chi(X, \mathcal{O}_X)$. As this will use the case $r = 1$ of [18, Proposition 4.1], we translate that result into our context. First, we observe that our $\chi(X, \mathcal{O}_X)$ is $\chi(X, \mathcal{O}_X, 1)$ of [18, Proposition 4.1] – the definition of the latter is on top of page 325 of [18]. Next, our d_i (defined just before Proposition 11) is Milne's $d^i(0)$ (defined on [18, p. 321], just before Proposition 3.1).

Now, it remains to interpret the two terms on the right hand side of [18, Proposition 4.1] with $r = 1$. The second term $\sum_i (-1)^i d^i(0)$, which is our $\sum_i (-1)^i d_i$, can be replaced by $-\sum (-1)^i b_i$ by Proposition 11. For the first term, we recall that for any p-divisible group A over $\mathbb{F}_q$, the dimension of the dual group A^t is given by the well known (see, for example, [18, p. 81]; it also follows from [18, Thm. 1 (e)])

$$\dim(A^t) = \mathrm{height}(A) - \dim(A).$$

Now consider the first term in the right hand side of [18, Proposition 4.1] with $r = 1$. The numbers λ_{ij} relevant here are those which satisfy $\lambda_{ij} \leq 1$; by the result [8, 3.5.3, p. 616] also mentioned earlier, these are exactly the slopes of the roots of the characteristic polynomial of γ acting on $H^*(\bar{X}, W\mathcal{O}) \otimes \mathbb{Q}_p$ or

[2] In the formula for $z(g)$ in [14, Lemma 4], the exponent of q should read $d(G^t).d(H)$ - in his notation, this is $n_2 m_1$, as follows from an inspection of the calculation at the bottom of p. 81 of [14].

on the associated p-divisible groups $B_1, B_2, \cdots$ defined earlier. Milne's λ_{ij} are our $\mathrm{ord}_q(\omega_{ij})$. The first term can therefore be written as

$$\sum_{\mathrm{ord}_q(\omega_{ij}) \leq 1} (-1)^i m_{ij}(1 - \mathrm{ord}_q(\omega_{ij})),$$

where m_{ij} is the multiplicity of $\mathrm{ord}_q(\omega_{ij})$. It is now an easy exercise to see that this sum is $\sum_i (-1)^i \dim(B_i^t)$. But g_i is the dimension of the p-divisible group whose *dual* corresponds to B_i. We summarise our discussion

$$\begin{aligned}
\chi(X, \mathcal{O}_X) &= \chi(X, \mathcal{O}_X, 1) \\
&= \sum_{\mathrm{ord}_q(\omega_{ij}) \leq 1} (-1)^i m_{ij}(1 - \mathrm{ord}_q(\omega_{ij})) + \sum_i (-1)^i d^i(0) \\
&= \sum_i (-1)^i (\dim(B_i^t) + d^i(0)) \\
&= \sum_i (-1)^i g_i + \sum_i (-1)^i d_i \\
&= \sum_i (-1)^i (g_i - b_i).
\end{aligned}$$

This proves (v).

We note that the $\mathbb{Z}_p$-ranks of $\mathrm{Hom}_{\mathbb{F}_q}(E', B_i)$ and $\mathrm{Ext}^1_{\mathbb{F}_q}(E', B_i)$ are equal (the ranks are unchanged by an isogeny and one reduces to the cases treated in [14, pp.81-83]). While the first contributes to $H^i(X, T_pE)^{et}$, the second contributes to $H^{i+1}(X, T_pE)^{et}$, which proves (ii).

Now, the rank of $\mathrm{Hom}_{\mathbb{F}_q}(E', B_i)$ is non-zero only if $\sqrt{q}$ – a Weil number of weight one – is a root of the minimal polynomial of Frobenius on B_i [14, p. 81]. But a root ω_{ij} of Frobenius on B_i (or $H^i(X, W\mathcal{O})$) is a Weil number of weight i. Thus, the rank is zero for $i \neq 1$. This proves (vi). The $\mathbb{Z}_p$-rank of $\mathrm{Hom}_{\mathbb{F}_q}(E', B_1)$ is, by Tate's theorem [2], equal to the $\mathbb{Z}$-rank of $\mathrm{Hom}_{\mathbb{F}_q}(E, Pic(X))$ which is $2\rho_X$. As $(-1)^0 0 + (-1)1 \times 2\rho_X + (-1)^2 2 \times 2\rho_X = 2\rho_X$, this proves (iii). □

5. The Weil-étale cohomology of an elliptic curve

As before, $q = p^{2f}$, X is a smooth projective geometrically connected variety over Spec $\mathbb{F}_q$, and E is our fixed elliptic curve. In this section, we compute $H^*(X, E)$ and the Weil-étale [12]) cohomology groups $H^*_W(X, E)$ and prove Theorem 1. We write A for the Albanese variety of X.

5.1. **Cohomology of the elliptic curve** E.

The basic properties of $H^*(X, E)$ are given the following result whose formulation was inspired by [11, Proposition 2.1].

Theorem 14. *(a) The étale cohomology of E is given as follows:*

(i) $H^0(X, E)$ is finitely generated and

$$\mathrm{rank}\ H^0(X, E) = \mathrm{rank}\ \mathrm{Hom}_{\mathbb{F}_q}(A, E) = \mathrm{rank}\ \mathrm{Hom}_{\mathbb{F}_q}(E, Pic(X)) = 2\rho_X.$$

(ii) $H^j(X, E)$ is torsion for $j > 0$ and zero for $j > 1 + 2 \dim X$.

(iii) $H^j(X, E)$ is finite for $j \geq 3$.

(iv) $H^1(X, E)$ is finite.

(v) $H^2(X, E)$ is co-finite of corank $2\rho_X$.

(vi) The $\hat{\mathbb{Z}}$-modules $H^i(X, TE) = \prod_l H^i(X, T_lE)$ (all primes l including $l = p$) are finite for $i \neq 1, 2$.

(b) The Weil-étale cohomology groups $H^i_W(X, E)$ of E are finitely generated. The rank of $H^i_W(X, E)$ is zero for $i \neq 0, 1$ and $2\rho_X$ for $i = 0, 1$.

Remark. If the Néron-Severi group $NS(X)$ is torsion-free and the Picard scheme Pic_X is smooth, then $H^1(X, E)$ is isomorphic to the finite group $\mathrm{Ext}^1_{\mathbb{F}_q}(A, E)$ [15, Thm. 1]. □

Proof. Part (b) will be proved later (as part of Theorem 1). We prove part (a).

(i) This is well known (Mordell-Weil theorem); cf. for instance, [10].

(ii) It is straightforward that $H^j(X, E)$ and $H^j(\bar{X}, E) = H^j_W(\bar{X}, E)$ are torsion for $j > 0$ using the fact that E over X is the Neron model of E over $\mathbb{F}_q(X)$; cf. [1, p. 41]. Consider the Kummer sequence

$$0 \to {}_nE \to E \xrightarrow{n} E \to 0, \quad (n > 0) \tag{3}$$

which is exact in the étale (for n coprime to p) and flat topologies. When $(n, p) = 1$, the identity ${}_nE = (\mathbb{Z}/n\mathbb{Z})^2$ over $\mathbb{F}$ tells us that $H^j(\bar{X}, {}_nE)$ are finite (zero for $j > 2 \dim X$). This implies (ii) up to p-torsion using the spectral sequence

$$H^r(\Gamma_0, H^s(\bar{X}, E)) \Rightarrow H^{r+s}(X, E). \tag{4}$$

By [15, VI, Rmk. 1.5], $H^j_{fl}(X, {}_pE) = 0$ for $j > 2 + \dim X$. As E is smooth, $H^j_{fl}(X, E) = H^j(X, E)$ [15, p. 92] and (ii) follows.

(iii) It suffices to consider the non-p torsion. The p-part can be treated similarly, given the sequence

$$0 \to H^{j-1}(X, E) \otimes \mathbb{Z}_p \to H^j(X, T_pE) \to T_pH^j(X, E) \to 0$$

and the results of the previous section on $H^*(X, T_pE)$. Though the proof of the non-p-part is standard (cf. [18, Cor. 6.4]), nevertheless we recall it for the convenience of the reader.

Let γ be the Frobenius automorphism on the Tate module $T_\ell E$ of E; here ℓ is a prime distinct from p. By the Riemann hypothesis, $(1 - \gamma)$ is a quasi-isomorphism of $H^j(\bar{X}, T_\ell E) = H^j(\bar{X}, \mathbb{Z}_\ell) \otimes T_\ell E$ for $j \neq 1$; thus, if $j \neq 1$, then $H^j(\bar{X}, T_\ell E)^\Gamma$ and $H^j(\bar{X}, T_\ell E)_\Gamma$ are finite. From

$$0 \to H^{j-1}(\bar{X}, T_\ell E)_\Gamma \to H^j(X, T_\ell E) \to H^j(\bar{X}, T_\ell E)^\Gamma \to 0,$$

we obtain that $H^j(X, T_\ell E)$ is finite for $j \neq 1, 2$. Now, (3) provides the exact sequence

$$0 \to H^{j-1}(X, E) \otimes \mathbb{Z}_\ell \to H^j(X, T_\ell E) \to T_\ell H^j(X, E) \to 0, \tag{5}$$

whereby, for $j > 2$, $H^j(X, T_\ell E)$ is finite and hence $T_\ell H^j(X, E)$, being torsion-free, is zero. As the ℓ-torsion of $H^j(X, E)$ is finite, the ℓ-primary subgroup $H^j(X, E)(\ell)$ itself is finite for $j > 2$ and isomorphic to $H^{j+1}(X, T_\ell E)$. For the finiteness of non-p-part of $H^j(X, E)$ for $j \geq 3$, we need the following lemma whose formulation and proof are as in [11, Lem. 1.1, pp.176-7].

Lemma 15. *Write $P_j(t) = det(1 - \gamma t)$ for the characteristic polynomial of γ on $H^j(\bar{X}, \mathbb{Z}_\ell)$. Similarly, we define $Q_j(t)$ for $H^j(\bar{X}, T_\ell E)$. For all $j \neq 0, 1$, we have*

$$[H^j(\bar{X}, E(\ell))^\Gamma] = [H^{j+1}(\bar{X}, T_\ell E)^\Gamma].|P_j(q^{-\frac{1}{2}})|_\ell^{-2},$$

where $|-|_\ell$ is the absolute value normalized so that $|\ell|_\ell = \ell^{-1}$.

One has $|P_j(q^{-\frac{1}{2}}t)|_\ell^2 = |Q_j(t)|_\ell$.

As in [11, (c), pp.181], for $j > 2$, $H^j(X, E)(\ell)$ is trivial unless ℓ divides $P_j(q^{-\frac{1}{2}})$ or $H^*(\bar{X}, \mathbb{Z}_\ell)$ has torsion. By O. Gabber [4], $H^j(X, E)(\ell)$ is nontrivial only for a finite number of primes ℓ thereby showing the finiteness of the non-p part of $H^j(X, E)$ for $j > 2$.

(iv) Recall that the Hochschild-Serre spectral sequence

$$H^i(\Gamma, H^j(\bar{X}, T_\ell E)) \Rightarrow H^{i+j}(X, T_\ell E)$$

yields the sequence

$$0 \to H^0(\bar{X}, T_\ell E)_\Gamma \to H^1(X, T_\ell E) \to H^1(\bar{X}, T_\ell E)^\Gamma \to 0.$$

But

$$H^1(\bar{X}, T_\ell E)^\Gamma = \mathrm{Hom}(T_\ell A, T_\ell E)^\Gamma \xleftarrow{\cong} \mathrm{Hom}_{\mathbb{F}_q}(A, E) \otimes \mathbb{Z}_\ell.$$

As $H^0(\bar{X}, T_\ell E)_\Gamma$ is finite, the rank of $H^1(X, T_\ell E)$ is $2\rho_X = \text{rank } H^1(\bar{X}, T_\ell E)^\Gamma$. The sequence

$$0 \to H^0(X, E) \otimes \mathbb{Z}_\ell \to H^1(X, T_\ell E) \to T_\ell H^1(X, E) \to 0,$$

given (i), shows that $T_\ell H^1(X,E)$ has rank zero and, being torsion-free, is zero. This proves (iv).

(v) A similar argument as in (iv) using that the rank of $H^1(\bar{X}, T_\ell E)_\Gamma$ is $2\rho_X$ shows that the rank of $T_\ell H^2(X,E)$ is $2\rho_X$ thereby proving (v).

(vi) was proved in the course of the proof of (iii). □

We can now formulate an adelic version of Theorem 13.

Theorem 16. *The cohomology groups $H^i(N^\bullet)$ of the complex*

$$N^\bullet: \quad \cdots H^i(X,TE) \xrightarrow{\alpha_i} H^{i+1}(X,TE) \to \cdots$$

are finite; the maps α_i are induced by cup-product with the generator of $H^1(\mathrm{Spec}\,\mathbb{F}_q, \hat{\mathbb{Z}})$. One has

$$c_X^2 = q^{\chi(X,\mathcal{O}_X)} \prod([H^i(N^\bullet)])^{(-1)^i}.$$

Proof. The first statement needs verification only in the cases $i = 1, 2$, given (vi) of Theorem 14. We need to check that the kernel and cokernel of $H^1(X,TE) \to H^2(X,TE)$ are finite. The p-part is proved in Theorem 13. The non-p part is straightforward, given the previous lemma and the obvious variant of the non-p-part of [18, Lemma 6.2] (or the étale case of [14, Lemma 4]); see the proof of [18, Theorem 0.1]. □

5.2. **Proof of Theorem 1.** Recall [12, §8] the generator θ of $H^1_W(\mathrm{Spec}\,\mathbb{F}_q, \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}$. It is clear that $H^0_W(X,E) = H^0(X,E)$ is finitely generated.

The groups $H^i(\bar{X},E)$ are torsion for $i > 0$ [1, p. 41]. By Theorem 14 and [12, Prop. 2.3], $H^i(X,E) \cong H^i_W(X,E)$ for $i \geq 3$ and thus proves (ii). To prove (i), it remains to show that $H^1_W(X,E)$ and $H^2_W(X,E)$ are finitely generated. Now, [12, Prop. 2.3] provides a spectral sequence

$$H^r(\Gamma_0, H^s_W(\bar{X},E)) \Rightarrow H^{r+s}_W(X,E)$$

with a map from (4) of spectral sequences. This gives exact sequences

$$0 \to H^2(\Gamma, H^0(\bar{X},E)) \to H^2(X,E) \to H^2_W(X,E) \to 0,$$

and

$$0 \to H^1(X,E) \to H^1_W(X,E) \to \frac{H^1(\Gamma_0, H^0(\bar{X},E))}{H^1(\Gamma, H^0(\bar{X},E))} \to 0.$$

As $H^0(\bar{X},E)$ differs from $Y := \mathrm{Hom}_{\mathbb{F}}(\bar{A},\bar{E})$ by torsion, we have

$$H^2(\Gamma, H^0(\bar{X},E)) \cong H^1(\Gamma_0, H^0(\bar{X},E)) \otimes \frac{\mathbb{Q}}{\mathbb{Z}} \cong H^1(\Gamma_0, Y) \otimes \frac{\mathbb{Q}}{\mathbb{Z}},$$

using [12, Lemma 1.2]. The sequence of Γ_0-modules $0 \to Y \xrightarrow{n} Y \to Y/nY \to 0$ gives

$$H^1(\Gamma_0, Y) \otimes \frac{\mathbb{Q}}{\mathbb{Z}} \cong H^1(\Gamma_0, Y \otimes \frac{\mathbb{Q}}{\mathbb{Z}}).$$

and, by Tate's theorem [24], the corank of $H^1(\Gamma_0, Y \otimes \frac{\mathbb{Q}}{\mathbb{Z}})$ is $2\rho_X$. This proves that $H^2_W(X,E)$ is finite, the rank of $H^1_W(X,E)$ is $2\rho_X$ thereby finishing the proof of (i), (iii) and (iv).

Part (v) follows from Theorem 16 and the isomorphisms $H^i_W(X,E) \otimes \hat{\mathbb{Z}} \cong H^{i+1}(X,TE)$ (see (5); the nontrivial cases are $i = 0, 1$) which are compatible with the maps $\cup\theta$ and α_i. In more detail, combining $TH^1(X,E) = 0$ (Theorem 14 (iv)), the isomorphism $H^0(X,E) = H^0_W(X,E)$, and (5) proves the case $i = 0$. The exactness of (3) in the Weil-étale or Weil-flat topology gives an exact sequence

$$0 \to H^i_W(X,E) \otimes \hat{\mathbb{Z}} \to H^{i+1}_W(X,TE) \to TH^{i+1}_W(X,E) \to 0,$$

where the first map is an isomorphism because $H^*_W(X,E)$ are finitely generated abelian groups and so $TH^*_W(X,E) = 0$. As $H^*(X,TE) \xrightarrow{\sim} H^*_W(X,TE)$, this finishes the proof of Theorem 1. □

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N. Ramachandran: Department of Mathematics, University of Maryland, College Park, MD 20912 USA

Institut des Hautes Études Scientifiques, Route de Chartres, F-91440, Bures-sur-Yvette, France
E-mail address: atma@math.umd.edu