Chapter 1
Geometrization of Hitchin representations in $\text{PSL}(4, \mathbb{R})$

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Let $\Sigma$ be a closed oriented surface of genus $g > 1$ and $G$ a real split simple Lie group (for example $\text{PSL}(n, \mathbb{R})$, $\text{PSO}(n, n)$, $\text{PSO}(n, n+1)$, $\text{PSp}(2n, \mathbb{R})$). It follows from the work of Kostant [?] that there exists a unique (up to conjugation) irreducible morphism

$$\rho_G : \text{PSL}(2, \mathbb{R}) \to G.$$  

This morphism provides a preferred embedding $\iota_G$ of the Teichmüller space $\mathcal{T}(\Sigma)$ into the moduli space $R(\Sigma, G) := \text{Hom}^\text{ss}(\pi_1 \Sigma, G)/G$ of conjugacy classes of semi-simple morphisms from the fundamental group of $\Sigma$ into $G$.

Using the theory of Higgs bundles, N. Hitchin proved the following result [?]

**Theorem 1 (Hitchin 1992).** The connected component of $R(\Sigma, G)$ containing $\iota_G(\mathcal{T}(\Sigma))$ is homeomorphic to a ball of dimension $\dim(G)(2g - 2)$.

This preferred component is now called the **Hitchin component** and will be denoted $\mathcal{H}(\Sigma, G)$. However, as pointed out by N. Hitchin [?]: “Unfortunately, the analytical point of view used for the proofs gives no indication of the geometrical significance of the Teichmüller component.”

The goal is to interpret the representations in the Hitchin component for $\text{PSL}(4, \mathbb{R})$ as holonomy of projective structures.

**Definition 1.** A projective structure on a $n$-dimensional manifold $M$ is an atlas of charts taking value in open set of $\mathbb{R}P^n$ such that the transition functions are restriction of elements in $\text{PGL}(n+1, \mathbb{R})$. Two projective structures on $M$ are equivalent if there exists a homeomorphism $h : M \to M$ isotopic to the identity whose expression in local charts is given by projective transformations.

We denote by $\mathcal{P}(M)$ the space of equivalence classes of projective structures on $M$.

Given an equivalence class of projective structures on $M$, one can associate a developing pair $(\text{dev}, \text{hol})$ where
\[ \begin{align*}
\{ \text{dev} : \widetilde{M} & \rightarrow \mathbb{RP}^n \\
\text{hol} : \pi_1 M & \rightarrow \text{PGL}(n+1, \mathbb{R}) \}
\end{align*} \]

and \( \text{dev} \) is hol-equivariant and locally injective. Moreover, two developing pairs \((\text{dev}_1, \text{hol}_1)\) and \((\text{dev}_2, \text{hol}_2)\) correspond to the same projective structure if and only if there exists an element \( g \in \text{PGL}(n+1, \mathbb{R}) \) and a homeomorphism \( h : M \rightarrow M \) isotopic to the identity such that

\[ \begin{align*}
\text{dev}_1 \circ \widetilde{h} &= g^{-1} \circ \text{dev}_2 \\
\text{hol}_2(\gamma) &= g \circ \text{hol}_1 \circ g^{-1}, \quad \forall \gamma \in \pi_1 M
\end{align*} \]

In particular, we get a well-defined map

\[ \text{hol} : \mathcal{P}(M) \rightarrow \text{Hom}(\pi_1 M, \text{PGL}(n+1, \mathbb{R})/\sim). \]

From now on, \( M = T^1 \Sigma \), the unit tangent bundle of \( \Sigma \). In these notes we will explain the following result [2]:

**Theorem 2 (Guichard, Wienhard).** \( \mathcal{H}(\Sigma, \text{PSL}(4, \mathbb{R})) \) naturally parametrizes the space of properly convex foliated (pcf) projective structures on \( M \). More precisely, the holonomy of a pcf projective structure on \( M \) factors through a Hitchin representation \( \rho : \pi_1 \Sigma \rightarrow \text{PSL}(4, \mathbb{R}) \). Moreover, each Hitchin representation is the holonomy of a unique pcf-projective structure on \( M \).

In the first section, we describe the geometry of \( M \) and define the moduli space \( \mathcal{P}_{\text{pcf}}(M) \) of pcf-projective structures on \( M \). In section 2, we give some examples of projective structures on \( M \). In Section 3, we associate a pcf-projective structure on \( M \) associated to a Hitchin representation. Finally in Section 4, we explain the geometric structures associated to Hitchin representations in \( \text{PSp}(4, \mathbb{R}) \).

### 1.1 The geometry of \( M \)

Denote by \( \Gamma := \pi_1 \Sigma \), and \( \overline{\Gamma} = \pi_1 M \). We have the following exact sequence:

\[ 0 \rightarrow \mathbb{Z} \rightarrow \Gamma \rightarrow \overline{\Gamma} \rightarrow 1, \]

associated to the covering

\[ \widetilde{M} \xrightarrow{\tau} M := T^1 \Sigma \xrightarrow{\Gamma} M. \]

Set \( \tau := t + 1 \in \overline{T} \). We have the following presentations

\[ \Gamma = \langle a_1, b_1, ..., a_g, b_g \mid [a_1, g_1] ... [a_g, b_g] = 1 \rangle \]

\[ \overline{T} = \langle a_1, b_1, ..., a_g, b_g, \tau \mid [a_1, g_1] ... [a_g, b_g] = \tau^2 \rangle. \]
Fix a hyperbolic metric $h$ on $\Sigma$. Such a metric provides an equivariant identification between $\tilde{\Sigma}$ and the hyperbolic disk $\mathbb{H}^2$ and an identification between $\overline{M}$ and $T^1\mathbb{H}^2$.

The $\text{PSL}(2, \mathbb{R})$ model: The group $\text{PSL}(2, \mathbb{R})$ acts freely and transitively on $T^1\mathbb{H}^2$, so fixing a unit tangent vector $u \in T^1\mathbb{H}^2$ provides an identification between $\text{PSL}(2, \mathbb{R})$ and $T^1\mathbb{H}^2$ (given by the orbit map).

The triple of points model: A point $(p, u) \in T^1\mathbb{H}^2$ defines canonically a triple of positively oriented points $(x_-, x_t, x_+)$ on the boundary $\partial\mathbb{H}^2$ in the following way: the geodesic $\gamma \subset \mathbb{H}^2$ tangents to $v$ intersects $\partial\mathbb{H}^2$ in $x_-$ and $x_+$. Moreover, there exists a unique geodesic intersecting $\gamma$ orthogonally at $p$; set $x_t$ the intersection of this geodesic with $\partial\mathbb{H}^2$ such that $(x_-, x_t, x_+)$ is positively oriented (see Figure 1.1). We get a canonical identification between $\overline{M}$ and the space $\partial\mathbb{H}^2(3)$ of positively oriented pairwise distinct triple of points on $\partial\mathbb{H}^2$.

![Fig. 1.1 The unit tangent bundle to $\mathbb{H}^2$.](image)

It is well-known (see for instance [?]) that the geodesic flow $\varphi$ on $\overline{M}$ is Anosov as so defines two foliations:

- A codimension 2 foliation by geodesic leaves, where the geodesic leaf $\varphi(x)$ passing through $x \in \overline{M}$ is
  $$\varphi(x) = \{ \varphi_t(x), \ t \in \mathbb{R} \}.$$
  We denote by $\mathcal{G}$ (respectively $\mathcal{\tilde{G}}$) the space of geodesic leaves on $M$ (respectively the space of lift of geodesic leaves on $\tilde{M}$).

- A codimension 1 foliation by (weakly) stable leaves, where the stable leaf $\overline{f}(x)$ passing through $x \in \overline{M}$ is
  $$\overline{f}(x) = \{ y \in \overline{M}, \ (d\overline{M}(\varphi_t(x), \varphi_t(y)))_{t>0} \text{ bounded} \}.$$
We denote by $\mathcal{F}$ (respectively $\widetilde{\mathcal{F}}$) the space of stable leaves on $\mathcal{M}$ (respectively $\mathcal{M}$).

In the triple of points model of $\mathcal{M}$, given $p = (x_-, x_t, x_+) \in \mathcal{M}$, one easily checks that

$$\mathcal{G}(x) = \{(y_-, y_t, y_+) \in \mathcal{M}, y_- = x_-, y_+ = x_+\},$$
$$\mathcal{F}(p) = \{(y_-, y_t, y_+) \in \mathcal{M}, y_+ = x_+\}.$$

It follows that $\mathcal{F}$ identifies with the space of pair of distinct points in $\partial \mathbb{H}^2$ and $\widetilde{\mathcal{F}}$ identifies with $\partial \mathbb{H}^2$.

In the $\text{PSL}(2, \mathbb{R})$ model of $\mathcal{M}$, a geodesic leaf is conjugate to the Cartan subgroup

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \in \mathbb{R} \right\}.$$

A stable leaf is conjugated to the parabolic subgroup

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}.$$

Remark 1. One can define these foliations topologically. To do so, one has to use the boundary $\partial \infty \Gamma$ of $\Gamma$. It gives an identification between $\mathcal{M}$ and $\partial \infty \Gamma^{(3)}$.

We are now able to define the pcf projective structures:

Definition 2. • A projective structure on $\mathcal{M}$ is foliated if the developing map sends each geodesic leaf $\mathcal{G} \in \mathcal{F}$ into a projective line and each stable leaf $\mathcal{F} \in \widetilde{\mathcal{F}}$ into a projective plane.

• A foliated projective structure on $\mathcal{M}$ is properly convex (pcf) if the image by the developing map of each stable leaf is a convex proper.

• Two pcf projective structures on $\mathcal{M}$ are equivalent if there exists a projective homeomorphism $h : \mathcal{M} \to \mathcal{M}$ isotopic to the identity which respect the foliations.

Finally, we denote by $\mathcal{P}_{\text{pcf}}(\mathcal{M})$ the set of equivalence class of pcf projective structures on $\mathcal{M}$.

1.2 Examples of projective structures on $\mathcal{M}$

One can easily construct projective structures on $\mathcal{M}$ whose holonomy factor through a Fuchsian representation. To do so, let $\iota : \Gamma \to \text{PSL}_2(\mathbb{R})$ be a Fuchsian representation, $\rho : \text{PSL}_2(\mathbb{R}) \to \text{PSL}(4, \mathbb{R})$ be a morphism and $x \in \mathbb{R}P^3$ a point such that

$$\text{Stab}_{\text{PSL}_2} := \{g \in \text{PSL}_2(\mathbb{R}), \rho(g)x = x\}$$
is finite.

Then the map
defines a projective structure on $M$ whose holonomy factors through $\rho \circ \iota$.

Moreover, the developing map sends geodesic leaves (respectively stable leaves) into projective lines (respectively projective planes) if and only if $\text{dev}(A)$ (respectively $\text{dev}(P)$) is contained in a projective line (resp. projective planes).

### 1.2.1 Diagonal example

Consider
\[
\rho : \text{PSL}_2(\mathbb{R}) \longrightarrow \text{PSL}(4, \mathbb{R})
\]
\[
g \mapsto \text{diag}(g, g)
\]

and set $x := [1:0:0:1] \in \mathbb{RP}^3$. One easily checks that
- $\text{Stab}_{\text{PSL}_2} = \{\text{Id}\}$.
- $\text{dev}(A) = \{[u:0:0:1], u > 0\} \subset \mathbb{RP}^1$.
- $\text{dev}(P) = \{[1:0:u:v], u \in \mathbb{R}, v > 0\} \subset \mathbb{RP}^2$.

However, the set $\text{dev}(P)$ is not proper, so the associated projective structure fails to be properly convex.

### 1.2.2 Non-convex irreducible example

Identify $\mathbb{R}^4$ with the space of degree 3 homogeneous polynomials in 2 variables. It provides a canonical action of $\text{PSL}_2(\mathbb{R})$ on $\mathbb{R}^4$ by pre-composing a polynomial with an element of $\text{PSL}_2(\mathbb{R})$. This action defines the irreducible representation $\rho_4 : \text{SL}_2(\mathbb{R}) \longrightarrow \text{PSL}(4, \mathbb{R})$.

Taking $x = [Q] = [XY(2X+Y)]$ (so $Q$ has three distinct real roots), one gets a projective structure whose holonomy lies in the Hitchin component but which is not convex.

### 1.2.3 Convex irreducible example

Consider the same $\rho_4$ as previously but set $x = [R] = [X(X^2+Y^2)] = [1:0:1:0] \in \mathbb{RP}^3$. One obtains:
- $\text{dev}(A) = \{[e^{3t}:0:e^{-t}:0], t \in \mathbb{R}\} = \{[0:1:0:u], u > 0\} \subset \mathbb{RP}^1$.
- $\text{dev}(P) = \{[a^3+b^2a:2b:a^{-1}:0], t \in \mathbb{R}\}$ which is the projectivization of the convex cone $\{(\alpha, \beta, \gamma, 0) \in \mathbb{R}^4, \beta^2 - 4\alpha\beta < 0\}$. 
It follows that this defines a properly convex projective structure on $M$.

In order to generalize this construction to all Hitchin representations, one wants to describe this example as a map
\[ \text{dev} : \partial \Gamma(3) \to \mathbb{RP}^3. \]

To do so, identify $\mathbb{R}^2$ with the set of degree 1 homogeneous polynomials in 2 variables and consider the Veronese embedding
\[ \xi_1 : \mathbb{RP}^1 \to \mathbb{RP}^3 \]
\[ [S] \mapsto [S^3] \]
which is $\rho_4$-equivariant (so in particular $(\rho_4 \circ \iota)$-equivariant). The Veronese embedding lifts to a flag curve
\[ \xi = (\xi_1, \xi_2, \xi_3) : \mathbb{RP}^4 \to \text{Flag}(\mathbb{R}^4) \]
where $\text{Flag}(\mathbb{R}^4)$ is the space of complete flags in $\mathbb{R}^4$ and
- $\xi_1(S)$ is the line of polynomials divisible by $S^3$.
- $\xi_2(S)$ is the plane of polynomials divisible by $S^2$.
- $\xi_3(S)$ is the hyperplane of polynomials divisible by $S$.

Moreover, the orbits of $\text{PSL}_2(\mathbb{R})$ in $\mathbb{RP}^3$ are:
- One open orbit consisting of polynomials with three distinct real roots (hence the set of points in $\mathbb{RP}^3$ contained in exactly 3 pairwise distinct $\xi_3(t)$).
- One open orbit consisting of polynomials with one real root and two complex conjugates (hence the set of points in $\mathbb{RP}^3$ contained in exactly one $\xi_3(t)$ and no $\xi_2(t)$).
- One relatively closed orbit $\bigcup_{t \in \partial \Gamma} \xi_2(t) \setminus \xi_1(t)$.
- The closed orbit $\xi_1(\partial \Gamma)$.

Note that the two open orbits are connected components of the complementary of the discriminant (ruled) surface $\bigcup_{t \in \partial \Gamma} \xi_1(t)$.

With this picture, the irreducible non-convex example is easy to recover. In fact, it is given by
\[ \text{dev}' : \partial \Gamma(3) \to \mathbb{RP}^3 \]
\[ (x_-, x_0, x_+) \mapsto \xi_3(x_-) \cap \xi_3(x_0) \cap \xi_3(x_+) \]
For the convex irreducible example, the situation is more complicated.
1.2.4 Convex irreducible example revisited

Fix \((x_-, x_t, x_+) \in \partial_\omega \Gamma^{(3)}\). Using the identification \(\partial_\omega \Gamma \cong \mathbb{R}P^3\), one can see the flag curve \(\xi\) as a map from \(\partial_\omega \Gamma\) to Flag(\(\mathbb{R}^4\)).

Moreover, the image by \(\xi\) of distinct points giving transverse flags, for each \(t \neq t' \in \partial_\omega \Gamma\), \(\xi_3(t) \cap \xi_3(t')\) gives a single point in \(\mathbb{R}P^3\). Here, we identify linear subspaces of \(\mathbb{R}^4\) with their image in \(\mathbb{R}P^3\).

Consider the curve \(\mathcal{D} := \{\xi_2(t) \cap \xi_2(\mathbb{x}_+), t \in \partial_\omega \Gamma, t \neq \mathbb{x}_+\}\)
\[= \left( \bigcup_{t \in \partial_\omega \Gamma, t \neq \mathbb{x}_+} \xi_2(t) \cap \xi_2(\mathbb{x}_+) \right) \subset \mathbb{R}P^3\]

Adding the point \(\xi_1(\mathbb{x}_+) \in \mathbb{R}P^3\) to \(\mathcal{D}\), one gets a curve which bound a strictly convex proper set in \((\xi_1(\mathbb{x})\} \cong \mathbb{R}P^3\).

The tangent line to \(\mathcal{D}\) passing through \(\xi_2(\mathbb{x}_-) \cap \xi_2(\mathbb{x}_+) \in \mathcal{D}\) intersects \(\xi_2(\mathbb{x}_+)\) in a point \(p = \xi_3(\mathbb{x}_-) \cap \xi_2(\mathbb{x}_+) \in \mathbb{R}P^3\).

Finally we define \(\text{dev}(x_-, x_t, x_+)\) as the intersection of the projective line \(L_3\) passing through \(p\) and \(\xi_2(\mathbb{x}_-) \cap \xi_3(\mathbb{x}_+) \in \mathcal{D}\) and the projective line \(L_2\) passing through \(\xi_2(\mathbb{x}_-) \cap \xi_3(\mathbb{x}_+) \in \mathcal{D}\) and \(\xi_1(\mathbb{x}_+)\) (see Figure 1.2). More explicitly, we have

\[\text{dev}(x_-, x_t, x_+) = \left(\left(\xi_2(\mathbb{x}_-) \cap \xi_3(\mathbb{x}_+)\right) \oplus \xi_1(\mathbb{x}_+)\right) \cap \left(\left(\xi_3(\mathbb{x}_-) \cap \xi_2(\mathbb{x}_+)\right) \oplus \left(\xi_2(\mathbb{x}_-) \cap \xi_3(\mathbb{x}_+)\right)\right)\]

Note that with this description, one easily checks that the geodesic leaf passing through \((x_-, x_t, x_+)\) is sent into \(L_2\) and the stable leaf passing through \((x_-, x_t, x_+)\) is sent into the properly convex set bounded by \(\mathcal{D}\).

1.3 From \(\mathcal{H}(\Sigma, \text{PSL}(4, \mathbb{R}))\) to \(\mathcal{D}_{\text{pcf}}(M)\)

Extending the previous construction to the whole Hitchin component is now very easy. In fact, it follows from a result of Labourie [2] that given a Hitchin representation \(\rho \in \mathcal{H}(\Sigma, \text{PSL}(4, \mathbb{R}))\), there exists a unique hyperconvex curve

\[\xi_1 : \partial_\omega \Gamma \rightarrow \mathbb{R}P^3\]

which lifts to a flag curve

\[\xi : \partial_\omega \Gamma \rightarrow \text{Flag}(\mathbb{R}^4)\].

Moreover, this curve satisfies the transverse condition (namely that each pair of distinct points \(t, t' \in \partial_\omega \Gamma\) are sent into transverse flags). So one can define the following developing map:
Fig. 1.2 The developing map.

\[ \text{dev}(x_-, x_t, x_+) = \left( (\xi_2(x_-) \cap \xi_3(x_+)) \oplus \xi_1(x_+) \right) \cap \left( (\xi_3(x_-) \cap \xi_2(x_+)) \oplus (\xi_2(x_t) \cap \xi_3(x_+)) \right). \]

This map is \( \rho \)-equivariant because the flag curve is. Moreover it is clear that it is foliated (for the same arguments as previously).

So one gets a map

\[ \mathcal{H}(\Sigma, \text{PSL}(4, \mathbb{R})) \rightarrow \mathcal{P}_{\text{pcf}}(M). \]

It is proved in [?] that this map is one-to-one. It means in particular that the holonomy of a pcf projective structure on \( M \) factors through a Hitchin representation.

### 1.4 The PSp\((4, \mathbb{R})\) case

Let \((\mathbb{R}^4, \omega)\) be the 4-dimensional real space endowed with a symplectic form \( \omega \). The subgroup of PSL\((4, \mathbb{R})\) preserving \( \omega \) is the symplectic group PSp\((4, \mathbb{R})\).
Given a line $L \subset \mathbb{R}^4$ (necessarily isotropic), its orthogonal complement with respect to $\omega$ is a 3-dimensional subspace $L^\perp \subset \mathbb{R}^4$. It implies that each point $x \in \mathbb{RP}^3$ defines a projective plane $x^\perp \cong \mathbb{RP}^2$ containing $x$.

By considering the tangent space of $x^\perp$ inside the tangent space of $\mathbb{RP}^3$ at $x$, one can define a hyperplane distribution in $T\mathbb{RP}^3$. This distribution corresponds to a contact structure on $\mathbb{RP}^3$ that we call contact projective structure.

Given a representation $\rho \in \mathcal{H}(\Sigma, \text{PSp}(4, \mathbb{R}))$, the associated flag curve $\xi$ takes value in the complete flag manifold associated to $\text{PSp}(4, \mathbb{R})$. More precisely, $\xi$ associates to each point of $\partial_\infty \Gamma$ a complete $\text{PSp}(4, \mathbb{R})$-flag $(l, \mathcal{L}, l^\perp)$ where $l \subset \mathbb{R}^4$ is a line, $\mathcal{L}$ is a Lagrangian plane and $l^\perp$ is the orthogonal of $l$.

The same construction as previously gives rise to a pcf contact projective structure on $M$ where the geodesic leaves are sent into (projection of) Lagrangian planes.