Asymptotic Analysis 88 (2014) 201–215
DOI 10.3233/ASY-141220
IOS Press

Smoluchowski–Kramers approximation and large deviations for infinite dimensional gradient systems

Sandra Cerrai * and Michael Salins
Department of Mathematics, University of Maryland, College Park, MD, USA

AUTHOR COPY

Abstract. In this paper, we explicitly calculate the quasi-potentials for the damped semilinear stochastic wave equation when the system is of gradient type. We show that in this case the infimum of the quasi-potential with respect to all possible velocities does not depend on the density of the mass and does coincide with the quasi-potential of the corresponding stochastic heat equation that one obtains from the zero mass limit. This shows in particular that the Smoluchowski–Kramers approximation can be used to approximate long time behavior in the zero noise limit, such as exit time and exit place from a basin of attraction.

Keywords: Smoluchowski–Kramers approximation, large deviations, exit problems, gradient systems

1. Introduction

In the present paper, we consider the following damped wave equation in a bounded regular domain $\mathcal{O} \subset \mathbb{R}^d$, perturbed by noise

$$\begin{cases} \mu \dfrac{\partial^2 u_\varepsilon^\mu}{\partial t^2}(t,\xi) = \Delta u_\varepsilon^\mu(t,\xi) - \dfrac{\partial u_\varepsilon^\mu}{\partial t}(t,\xi) + B\big(u_\varepsilon^\mu(t,\cdot)\big)(\xi) + \sqrt{\varepsilon}\,\dfrac{\partial w^Q}{\partial t}(t,\xi), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \xi \in \mathcal{O}, t > 0, \\ u_\varepsilon^\mu(0,\xi) = u_0(\xi), \qquad \dfrac{\partial u_\varepsilon^\mu}{\partial t}(0,\xi) = v_0(\xi), \quad \xi \in \mathcal{O}, u_\varepsilon^\mu(t,\xi) = 0,\ \xi \in \partial\mathcal{O} \end{cases} \tag{1.1}$$

for some parameters $0 < \varepsilon, \mu \ll 1$. Here, $\partial w^Q/\partial t$ is a cylindrical Wiener process, white in time and colored in space, with covariance operator Q^2, for some $Q \in \mathcal{L}(L^2(\mathcal{O}))$. Concerning the non-linearity B, we assume

$$B(x) = -Q^2 DF(x), \quad x \in L^2(\mathcal{O}),$$

for some $F : L^2(\mathcal{O}) \to \mathbb{R}$, satisfying suitable conditions. We also consider the semi-linear heat equation

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t}(t,\xi) = \Delta u_\varepsilon(t,\xi) + B\big(u_\varepsilon(t,\cdot)\big)(\xi) + \sqrt{\varepsilon}\,\dfrac{\partial w^Q}{\partial t}(t,\xi), \quad \xi \in \mathcal{O}, t > 0, \\ u_\varepsilon(0,\xi) = u_0(\xi), \qquad\qquad\qquad\qquad\qquad\qquad u_\varepsilon(t,\xi) = 0,\ \xi \in \partial\mathcal{O}. \end{cases} \tag{1.2}$$

*Corresponding author. E-mail: cerrai@math.umd.edu.

0921-7134/14/$27.50 © 2014 – IOS Press and the authors. All rights reserved

In [1], it has been proved that if $\varepsilon > 0$ is fixed and μ tends to zero, the solutions of (1.1) converge to the solution of (1.2), uniformly on compact intervals. More precisely, for any $\eta > 0$ and $T > 0$

$$\lim_{\mu\to 0} \mathbb{P}\Big(\sup_{t\in[0,T]} \big| u^\mu_\varepsilon(t) - u_\varepsilon(t)\big|_{L^2(\mathcal{O})} > \eta \Big) = 0.$$

Moreover, in the case $d = 1$ and $Q = I$, it has been proven that for any fixed $\varepsilon > 0$, the first marginal of the invariant measure of Eq. (1.1) coincides with the invariant measure of Eq. (1.2), for any $\mu > 0$.

In this paper, we are interested in comparing the small noise behavior of the two systems. More precisely, we keep $\mu > 0$ fixed and let ε tend to zero, to study some relevant quantities associated with the large deviation principle for these systems, as the quasi-potential that describes also the asymptotic behavior of the expected exit time from a domain and the corresponding exit places. Due to the gradient structure of (1.1), as in the finite dimensional case studied in [6], we are here able to calculate explicitly the quasi-potentials $V^\mu(x, y)$ for system (1.1) as

$$V^\mu(x,y) = \big|(-\Delta)^{1/2} Q^{-1} x\big|^2_{L^2(\mathcal{O})} + 2F(x) + \mu \big|Q^{-1} y\big|^2_{L^2(\mathcal{O})}. \tag{1.3}$$

Actually, we can prove that for any $\mu > 0$

$$V^\mu(x,y) = \inf\Big\{ I^\mu_{-\infty}(z)\colon\ z(0) = (x,y),\ \lim_{t\to-\infty} \big|C_\mu^{-1/2} z(t)\big|_H = 0\Big\}, \tag{1.4}$$

where

$$I^\mu_{-\infty}(z) = \frac{1}{2}\int_{-\infty}^0 \left| Q^{-1}\left(\mu \frac{\partial^2\varphi}{\partial t^2}(t) - \Delta\varphi(t) + \frac{\partial\varphi}{\partial t}(t) - B\big(\varphi(t)\big)\right)\right|^2_{L^2(\mathcal{O})} \mathrm{d}t$$

with $\varphi(t) = \Pi_1 z(t)$, and

$$C_\mu(u,v) = \frac{1}{2}\left((-\Delta)^{-1} Q^2 u, \frac{1}{\mu}(-\Delta)^{-1} Q^2 v\right), \quad (u,v) \in L^2(\mathcal{O}) \times H^{-1}(\mathcal{O}).$$

From (1.4), we obtain that

$$V^\mu(x,y) \geqslant \big|(-\Delta)^{1/2} Q^{-1} x\big|^2_{L^2(\mathcal{O})} + 2F(x) + \mu \big|Q^{-1} y\big|^2_{L^2(\mathcal{O})}.$$

Thus, we obtain the equality (1.3) by constructing a path which realizes the minimum. An immediate consequence of (1.3) is that for each $\mu > 0$

$$V_\mu(x) = \inf_{y\in H^{-1}(\mathcal{O})} V^\mu(x,y) = V^\mu(x,0) = V(x), \tag{1.5}$$

where $V(x)$ is the quasi-potential associated with Eq. (1.2).

Now, consider an open bounded domain $G \subset L^2(D)$, which is invariant for u^μ_0 and is attracted to the asymptotically stable equilibrium 0, and for any $x \in G$ let us define

$$\tau^\mu_\varepsilon := \inf\big\{ t \geqslant 0\colon\ u^\mu_\varepsilon(t) \in \partial G\big\} \quad \text{and} \quad \tau_\varepsilon = \inf\big\{ t \geqslant 0\colon\ u_\varepsilon(t) \in \partial G\big\}.$$

AUTHOR COPY

In a forthcoming paper we plan to prove

$$\lim_{\varepsilon\to 0}\varepsilon \log \mathbb{E}\tau_\varepsilon^\mu = \inf_{y\in\partial G} V_\mu(y). \tag{1.6}$$

As a consequence of (1.5), this would imply that, in the gradient case for any fixed $\mu > 0$ it holds

$$\lim_{\varepsilon\to 0}\varepsilon \log \mathbb{E}\tau_\varepsilon^\mu = \inf_{y\in\partial G} V(y) = \lim_{\varepsilon\to 0}\varepsilon \log \mathbb{E}\tau_\varepsilon.$$

2. Preliminaries and assumptions

Let $H = L^2(\mathcal{O})$ and let A be the realization of the Laplacian with Dirichlet boundary conditions in H. Let $\{e_k\}_{k\in\mathbb{N}}$ be the complete orthonormal basis of eigenvectors of A, and let $\{-\alpha_k\}_{k\in\mathbb{N}}$ be the corresponding sequence of eigenvalues, with $0 < \alpha_1 \leqslant \alpha_k \leqslant \alpha_{k+1}$, for any $k \in \mathbb{N}$.

The stochastic perturbation is given by a cylindrical Wiener process $w^Q(t,\xi)$, for $t \geqslant 0$ and $\xi \in \mathcal{O}$, which is assumed to be white in time and colored in space, in the case of space dimension $d > 1$. Formally, it is defined as the infinite sum

$$w^Q(t,\xi) = \sum_{k=1}^{+\infty} Qe_k(\xi)\beta_k(t), \tag{2.1}$$

where $\{e_k\}_{k\in\mathbb{N}}$ is the complete orthonormal basis in $L^2(\mathcal{O})$ which diagonalizes A and $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is a sequence of mutually independent standard Brownian motions defined on the same complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

Hypothesis 1. *The linear operator Q is bounded in H and diagonal with respect to the basis $\{e_k\}_{k\in\mathbb{N}}$ which diagonalizes A. Moreover, if $\{\lambda_k\}_{k\in\mathbb{N}}$ is the corresponding sequence of eigenvalues, we have*

$$\sum_{k=1}^{\infty}\frac{\lambda_k^2}{\alpha_k} < +\infty. \tag{2.2}$$

In particular, if $d = 1$ we can take $Q = I$, but if $d > 1$ the noise has to colored in space. Concerning the non-linearity B, we assume that it has the following gradient structure.

Hypothesis 2. *There exists $F : H \to \mathbb{R}$ of class C^1, with $F(0) = 0$, $F(x) \geqslant 0$ and $\langle DF(x), x\rangle \geqslant 0$ for all $x \in H$, such that*

$$B(x) = -Q^2 DF(x), \quad x \in H.$$

Moreover,

$$\big|DF(x) - DF(y)\big|_H \leqslant \kappa|x-y|_H, \quad x, y \in H. \tag{2.3}$$

Example 2.1.

(1) Assume $d = 1$ and take $Q = I$. Let $b : \mathbb{R} \to \mathbb{R}$ be a decreasing Lipschitz continuous function with $b(0) = 0$. Then the composition operator $B(x)(\xi) = b(x(\xi))$, $\xi \in \mathcal{O}$, is of gradient type. Actually,

AUTHOR COPY

if we set

$$F(x) = -\int_{\mathcal{O}} \int_0^{x(\xi)} b(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi, \quad x \in H,$$

we have

$$B(x) = -DF(x), \quad x \in H.$$

Moreover, it is clear that $F(0) = 0$, $F(x) \geqslant 0$ for all $x \in H$, and

$$\langle DF(x), x \rangle = -\int_{\mathcal{O}} b\big(x(\xi)\big)x(\xi)\,\mathrm{d}\xi \geqslant 0, \quad x \in H.$$

(2) Assume now $d \geqslant 1$, so that Q is a general bounded operator in H, satisfying Hypothesis 1. Let $b:\mathbb{R} \to \mathbb{R}$ be a function of class C^1, with Lipschitz-continuous first derivative, such that $b(0) = 0$ and $b(\eta) \geqslant 0$, for all $\eta \in \mathbb{R}$. Moreover, the only local minimum of b occurs at 0. Let

$$F(x) = \int_{\mathcal{O}} b\big(x(\xi)\big)\,\mathrm{d}\xi, \quad x \in H.$$

It is immediate to check that $F(0) = 0$ and $F(x) \geqslant 0$, for all $x \in H$. Furthermore, for any $x \in H$

$$DF(x)(\xi) = b'\big(x(\xi)\big), \quad \xi \in \mathcal{O}.$$

Therefore, the nonlinearity

$$B(x) = -Q^2 b'\big(x(\cdot)\big), \quad x \in H,$$

satisfies Hypothesis 2.

For any $\delta \in \mathbb{R}$, we denote by H^δ the completion of $C_0^\infty(\mathcal{O})$ in the norm

$$|u|_{H^\delta}^2 = \sum_{k=1}^{\infty} \alpha_k^\delta \langle u, e_k \rangle_H.$$

This is a Hilbert space, with the scalar product

$$\langle u, v \rangle_{H^\delta} = \sum_{k=1}^{\infty} \alpha_k^\delta \langle u, e_k \rangle_H \langle v, e_k \rangle_H.$$

In what follows, we shall define $\mathcal{H}_\delta = H^\delta \times H^{\delta-1}$ and we shall set $\mathcal{H} = \mathcal{H}_0$.

Next, for $\mu > 0$, we define $A_\mu : D(A_\mu) \subset \mathcal{H}_\delta \to \mathcal{H}_\delta$ by

$$A_\mu(u, v) = \left(v, \frac{1}{\mu}Au - \frac{1}{\mu}v\right), \quad (u, v) \in D(A_\mu) = \mathcal{H}_{\delta+1}, \tag{2.4}$$

and we denote by $S_\mu(t)$ the semigroup on $\mathcal{H}$ generated by A_μ.

AUTHOR COPY

In [1, Proposition 2.4], it has been proven that $S_\mu(t)$ is a C_0-semigroup of negative type, namely, there exist $M_\mu > 0$ and $\omega_\mu > 0$ such that

$$\left\|S_\mu(t)\right\|_{L(\mathcal{H}_\delta)} \leqslant M_\mu \mathrm{e}^{-\omega_\mu t}, \quad t \geqslant 0. \tag{2.5}$$

Moreover, for any $\mu > 0$ we define the operator $Q_\mu : H^{\delta-1} \to \mathcal{H}_\delta$ by setting

$$Q_\mu v = \frac{1}{\mu}(0, Qv), \quad v \in H^{\delta-1}.$$

Therefore, if we define

$$\hat{F}(u,v) = F(u), \qquad \hat{Q}(u,v) = Qu, \quad (u,v) \in \mathcal{H},$$

Eq. (1.1) can be rewritten as the following abstract stochastic evolution equation in the space $\mathcal{H}$

$$\mathrm{d}z_\varepsilon^\mu(t) = \left[A_\mu z_\varepsilon^\mu(t) - Q_\mu \hat{Q} D\hat{F}\left(z_\varepsilon^\mu(t)\right)\right] \mathrm{d}t + \sqrt{\varepsilon} Q_\mu \,\mathrm{d}w(t), \quad z_\varepsilon^\mu(0) = (u_0, v_0). \tag{2.6}$$

Analogously, Eq. (1.2) can be rewritten as the following abstract stochastic evolution equation in H

$$\mathrm{d}u_\varepsilon(t) = \left[A u_\varepsilon(t) - Q^2 DF\left(u_\varepsilon(t)\right)\right] \mathrm{d}t + \sqrt{\varepsilon} Q \,\mathrm{d}w(t), \quad u_\varepsilon(0) = u_0. \tag{2.7}$$

Definition 2.2.

(1) A predictable process $z_\varepsilon^\mu \in L^2(\Omega, C([0,T]; \mathcal{H}))$ is a mild solution to (2.6) if

$$z_\varepsilon^\mu(t) = S_\mu(t)(u_0, v_0) - \int_0^t S_\mu(t-s) Q_\mu \hat{Q} D\hat{F}\left(z_\varepsilon^\mu(s)\right) \mathrm{d}s + \sqrt{\varepsilon} \int_0^t S_\mu(t) Q_\mu \,\mathrm{d}w(s).$$

(2) A predictable process $u^\varepsilon \in L^2(\Omega, C([0,T]; H))$ is a mild solution to (2.7) if

$$u^\varepsilon(t) = \mathrm{e}^{tA} u_0 - \int_0^t \mathrm{e}^{(t-s)A} Q^2 DF\left(u^\varepsilon(s)\right) \mathrm{d}s + \sqrt{\varepsilon} \int_0^t \mathrm{e}^{(t-s)A} Q \,\mathrm{d}w(s).$$

Remark 2.3. If we define

$$C_\mu = \int_0^{+\infty} S_\mu(t) Q_\mu Q_\mu^\star S_\mu^\star(t) \,\mathrm{d}t,$$

as shown in [1, Proposition 5.1] we have

$$C_\mu(u,v) = \frac{1}{2}\left((-A)^{-1} Q^2 u, \frac{1}{\mu}(-A)^{-1} Q^2 v\right), \quad (u,v) \in \mathcal{H}.$$

Therefore, we get

$$2A_\mu C_\mu D\hat{F}(u,v) = \left(0, -\frac{1}{\mu} Q^2 DF(u)\right) = -Q_\mu \hat{Q} D\hat{F}(u,v), \quad (u,v) \in \mathcal{H}.$$

AUTHOR COPY

This means that Eq. (2.6) can be rewritten as

$$\mathrm{d}z_\varepsilon^\mu(t) = \left[A_\mu z_\varepsilon^\mu(t) + 2A_\mu C_\mu D\hat{F}\left(z_\varepsilon^\mu(t)\right)\right]\mathrm{d}t + \sqrt{\varepsilon}Q_\mu\,\mathrm{d}w(t), \quad z_\varepsilon^\mu(0) = (u_0, v_0).$$

In the same way, Eq. (2.7) can be rewritten as

$$\mathrm{d}u_\varepsilon(t) = \left[Au_\varepsilon(t) + 2ACDF\left(u_\varepsilon(t)\right)\right]\mathrm{d}t + \sqrt{\varepsilon}Q\,\mathrm{d}w(t), \quad u_\varepsilon(0) = u_0,$$

where

$$C = \int_0^{+\infty} \mathrm{e}^{tA}QQ^\star\mathrm{e}^{tA^\star}\,\mathrm{d}t = \frac{1}{2}(-A)^{-1}Q^2.$$

In particular, both (2.6) and (2.7) are gradient systems (for more details see [7]).

As we are assuming that $DF : H \to H$ is Lipschitz continuous, we have that Eq. (2.6) has a unique mild solution $z_\varepsilon^\mu \in L^p(\Omega, C([0,T];\mathcal{H}))$ and Eq. (2.7) has a unique mild solution $u^\varepsilon \in L^p(\Omega, C([0,T];H))$, for any $p \geqslant 1$ and $T > 0$.

In [1, Theorem 4.6] it has been proved that in this case the so-called Smoluchowski–Kramers approximation holds. Namely, for any $\varepsilon, T > 0$ and $\eta > 0$

$$\lim_{\mu\to 0}\mathbb{P}\Big(\sup_{t\in[0,T]}\left|u_\varepsilon^\mu(u) - u_\varepsilon(t)\right|_H > \eta\Big) = 0,$$

where $u_\varepsilon^\mu(t) = \Pi_1 z_\varepsilon^\mu(t)$.

3. A characterization of the quasi-potential

For any $\mu > 0$ and $t_1 < t_2$, and for any $z \in C([t_1,t_2];\mathcal{H})$ and $z_0 \in \mathcal{H}$, we define

$$I_{t_1,t_2}^\mu(z) = \frac{1}{2}\inf\left\{|\psi|^2_{L^2((t_1,t_2);H)}\colon z = z_{z_0,\psi}^\mu\right\}, \tag{3.1}$$

where $z_{z_0,\psi}^\mu$ solves the following skeleton equation associated with (2.6)

$$z_{z_0,\psi}^\mu(t) = S_\mu(t-t_1)z_0 + \int_{t_1}^t S_\mu(t-s)A_\mu C_\mu D\hat{F}_\mu\left(z_{z_0,\psi}^\mu(s)\right)\mathrm{d}s + \int_{t_1}^t S_\mu(t-s)Q_\mu\psi(s)\,\mathrm{d}s. \tag{3.2}$$

Analogously, for $t_1 < t_2$, and for any $\varphi \in C([t_1,t_2];H)$ and $u_0 \in H$, we define

$$I_{t_1,t_2}(\varphi) = \frac{1}{2}\inf\left\{|\psi|^2_{L^2((t_1,t_2);H)}\colon \varphi = \varphi_{\psi,u_0}\right\}, \tag{3.3}$$

where $\varphi_{u_0,\psi}$ solves the problem

$$\varphi_{u_0,\psi}(t) = \mathrm{e}^{(t-t_1)A}u_0 - \int_{t_1}^t \mathrm{e}^{(t-s)A}Q^2DF\left(\varphi_{u_0,\psi}(s)\right)\mathrm{d}s + \int_{t_1}^t \mathrm{e}^{(t-s)A}Q\psi(s)\,\mathrm{d}s. \tag{3.4}$$

AUTHOR COPY

In what follows, we shall also denote

$$I^{\mu}_{-\infty}(z) = \sup_{t<0} I^{\mu}_{t,0}(z), \qquad I_{-\infty}(\varphi) = \sup_{t<0} I_{t,0}(\varphi).$$

Since (2.6) and (2.7) have additive noise, as a consequence of the contraction lemma we have that the family $\{z^{\mu}_{\varepsilon}\}_{\varepsilon>0}$ satisfies the large deviations principle in $C([0,T];\mathcal{H})$, with respect to the rate function $I^{\mu}_{0,T}$ and the family $\{u^{\varepsilon}\}_{\varepsilon>0}$ satisfies the large deviations principle in $C([0,T];H)$, with respect to the rate function $I_{0,T}$.

In what follows, for any fixed $\mu > 0$ we shall denote by V^{μ} the quasi-potential associated with system (2.6), namely

$$V^{\mu}(x,y) = \inf\bigl\{I^{\mu}_{0,T}(z)\colon\ z(0) = 0, z(T) = (x,y), T > 0\bigr\}.$$

Analogously, we shall denote by V the quasi-potential associated with Eq. (2.7), that is

$$V(x) = \inf\bigl\{I_{0,T}(\varphi)\colon\ \varphi(0) = 0, \varphi(T) = x, T > 0\bigr\}.$$

Moreover, for any $\mu > 0$ we shall define

$$V_{\mu}(x) = \inf_{y\in H^{-1}(\mathcal{O})} V^{\mu}(x,y) = \inf\bigl\{I^{\mu}_{0,T}(z)\colon\ z(0) = 0, \Pi_1 z(T) = x, T > 0\bigr\}.$$

In [3, Proposition 5.4], it has been proven that $V(x)$ can be represented as

$$V(x) = \inf\Bigl\{I_{-\infty}(\varphi)\colon\ \varphi(0) = x, \lim_{t\to-\infty}\bigl|\varphi(t)\bigr|_H = 0\Bigr\}.$$

Here, we want to prove a similar representations for $V^{\mu}(x,y)$, for any fixed $\mu > 0$. To this purpose, we first introduce the operator $L^{\mu}_{t_1,t_2} : L^2((t_1,t_2)\colon\ H) \to \mathcal{H}$, defined as

$$L^{\mu}_{t_1,t_2}\psi = \int_{t_1}^{t_2} S_{\mu}(t_2 - s)Q_{\mu}\psi(s)\,\mathrm{d}s. \tag{3.5}$$

Theorem 3.1. *For any $\mu > 0$ and $(x,y) \in \mathcal{H}$ we have*

$$V^{\mu}(x,y) = \inf\Bigl\{I^{\mu}_{-\infty}(z)\colon\ z(0) = (x,y), \lim_{t\to-\infty}\bigl|C_{\mu}^{-1/2}z(t)\bigr|_H = 0\Bigr\}. \tag{3.6}$$

Proof. First we observe that by the definitions of $I^{\mu}_{t_1,t_2}$ and $V^{\mu}(x,y)$,

$$V^{\mu}(x,y) = \inf\bigl\{I^{\mu}_{-T,0}(z)\colon\ z(-T) = 0, z(0) = (x,y), T > 0\bigr\}.$$

Now, for any $\mu > 0$ and $(x,y) \in \mathcal{H}$, we define

$$M^{\mu}(x,y) := \inf\Bigl\{I^{\mu}_{-\infty}(z)\colon\ z(0) = (x,y), \lim_{t\to-\infty}\bigl|C_{\mu}^{-1/2}z(t)\bigr|_{\mathcal{H}} = 0\Bigr\}.$$

Clearly, we want to prove that $M^{\mu}(x,y) = V^{\mu}(x,y)$, for all $(x,y) \in \mathcal{H}$.

AUTHOR COPY

If z is a continuous path with $z(-T) = 0$ and $z(0) = (x, y)$, we can extend it in $C((-\infty, 0); \mathcal{H})$, by defining $z(t) = 0$, for $t < -T$. Then, since $DF(0) = 0$, we see that

$$I^\mu_{-\infty}(z) = I^\mu_{-T,0}(z),$$

so that $M^\mu(x, y) \leqslant V^\mu(x, y)$.

Now, let us prove that the opposite inequality holds. If $M^\mu(x, y) = +\infty$, there is nothing else to prove. Thus, we assume that $M^\mu(x, y) < +\infty$. This means that for any $\varepsilon > 0$ there must be some $z^\mu_\varepsilon \in C((-\infty, 0); \mathcal{H})$, with the properties that $z^\mu_\varepsilon(0) = (x, y)$ and

$$\lim_{t\to-\infty} \bigl|C_\mu^{-1/2} z^\mu_\varepsilon(t)\bigr|_{\mathcal{H}} = 0, \qquad I^\mu_{-\infty}\bigl(z^\mu_\varepsilon\bigr) \leqslant M^\mu(x, y) + \varepsilon.$$

In what follows we shall prove that the following auxiliary result holds.

Lemma 3.2. *For any $\mu > 0$, there exists $T_\mu > 0$ such that for any $t_1 < t_2 - T_\mu$ we have* $\mathrm{Im}(L^\mu_{t_1,t_2}) = D(C_\mu^{-1/2})$ *and*

$$\bigl|\bigl(L^\mu_{t_1,t_2}\bigr)^{-1} z\bigr|_{L^2((t_1,t_2);H)} \leqslant c(\mu, t_2 - t_1)\bigl|C_\mu^{-1/2} z\bigr|_{\mathcal{H}}, \quad z \in \mathrm{Im}\bigl(L^\mu_{t_1,t_2}\bigr), \tag{3.7}$$

where

$$C_\mu(x, y) = \left((-A)^{-1}Q^2 x, \frac{1}{\mu}(-A)^{-1}Q^2 y\right).$$

Thus, let T_μ be the constant from Lemma 3.2 and let $t_\varepsilon < 0$ be such that

$$\bigl|C_\mu^{-1/2} z^\mu_\varepsilon(t_\varepsilon)\bigr|_{\mathcal{H}} < \varepsilon.$$

Moreover, let $\psi^\mu_\varepsilon := (L^\mu_{t_\varepsilon - T_\mu, t_\varepsilon})^{-1} z^\mu_\varepsilon(t_\varepsilon)$. By Lemma 3.2 we have

$$\bigl|\psi^\mu_\varepsilon\bigr|_{L^2(t_\varepsilon - T_\mu, t_\varepsilon; H)} \leqslant c_\mu \varepsilon.$$

Next, we define

$$\hat{z}^\mu_\varepsilon(t) = \int_{t_\varepsilon - T_\mu}^t S_\mu(t - s) Q_\mu \psi^\mu_\varepsilon(s)\,\mathrm{d}s.$$

Thanks to (2.5), we have

$$\begin{aligned}
\int_{t_\varepsilon - T_\mu}^{t_\varepsilon} \bigl|\hat{z}^\mu_\varepsilon(t)\bigr|^2_{\mathcal{H}}\,\mathrm{d}t &\leqslant \int_{t_\varepsilon - T_\mu}^{t_\varepsilon} \left(\int_{t_\varepsilon - T_\mu}^t \frac{M_\mu}{\mu} \mathrm{e}^{-\omega_\mu(t-s)} \bigl|Q\psi^\mu_\varepsilon(s)\bigr|_{H^{-1}}\,\mathrm{d}s\right)^2 \mathrm{d}t \\
&\leqslant \frac{M_\mu^2}{2\mu^2\omega_\mu} \int_{t_\varepsilon - T_\mu}^{t_\varepsilon} \bigl|Q\psi^\mu_\varepsilon\bigr|^2_{L^2((t_\varepsilon - T_\mu, t\varepsilon); H^{-1})}\,\mathrm{d}t \\
&\leqslant T_\mu \frac{M_\mu^2}{2\mu^2\omega_\mu} \bigl|Q\psi^\mu_\varepsilon\bigr|^2_{L^2((t_\varepsilon - T_\mu, t\varepsilon); H^{-1})} \leqslant c_\mu \varepsilon^2.
\end{aligned}$$

AUTHOR COPY

Furthermore, $\hat{z}^\mu_\varepsilon(t_\varepsilon - T_\mu) = 0$ and $\hat{z}^\mu_\varepsilon(t_\varepsilon) = z(t_\varepsilon)$. Finally, we notice that

$$\hat{z}^\mu_\varepsilon(t) = -\int_{t_\varepsilon - T_\mu}^t S_\mu(t-s)Q_\mu QDF\big(\hat{z}^\mu_\varepsilon(s)\big)\,\mathrm{d}s + \int_{t_\varepsilon - T_\mu}^t S_\mu(t-s)Q_\mu\big(\psi^\mu_\varepsilon(s) + QDF\big(\hat{z}^\mu_\varepsilon(s)\big)\big)\,\mathrm{d}s,$$

so that

$$I^\mu_{t_\varepsilon - T_\mu, t_\varepsilon}\big(\hat{z}^\mu_\varepsilon\big) = \frac{1}{2}\big|\psi^\mu_\varepsilon + QDF\big(\hat{z}^\mu_\varepsilon(s)\big)\big|^2_{L^2((t_\varepsilon - T_\mu, t_\varepsilon);H)} \leqslant c_\mu \varepsilon^2.$$

Now if we define

$$\tilde{z}^\mu_\varepsilon(t) = \begin{cases} \hat{z}^\mu_\varepsilon(t) & \text{if } t_\varepsilon - T_\mu \leqslant t < t_\varepsilon, \\ z^\mu_\varepsilon(t) & \text{if } t_\varepsilon \leqslant t \leqslant 0, \end{cases}$$

we see that $\tilde{z}^\mu_\varepsilon \in C((t_\varepsilon - T_\mu, 0); \mathcal{H})$ and

$$V^\mu(x,y) \leqslant I^\mu_{t_\varepsilon - T_\mu, 0}\big(\tilde{z}^\mu_\varepsilon\big) = I^\mu_{t_\varepsilon - T_\mu, t_\varepsilon}\big(\hat{z}^\mu_\varepsilon\big) + I^\mu_{t_\varepsilon, 0}\big(z^\mu_\varepsilon\big) \leqslant c_\mu \varepsilon^2 + M^\mu(x,y) + \varepsilon.$$

Due to the arbitrariness of $\varepsilon > 0$, we can conclude. □

Proof of Lemma 3.2. It is immediate to check that

$$\big|\big(L^\mu_{t_1,t_2}\big)^\star z\big|^2_{L^2((t_1,t_2);H)} = \frac{1}{\mu}\int_0^{t_2 - t_1} \big|Q^\star_\mu S^\star_\mu(s) z\big|^2_H\,\mathrm{d}s$$

therefore, since

$$Q^\star_\mu(u,v) = \frac{1}{\mu}(-A)^{-1}Qv, \quad (u,v) \in \mathcal{H}$$

(see [1, Section 5] for a proof), we can conclude

$$\big|\big(L^\mu_{t_1,t_2}\big)^\star z\big|^2_{L^2((t_1,t_2);H)} = \frac{1}{\mu^2}\int_0^{t_2 - t_1} \big|Q(-A)^{-1}\Pi_2 S^\star_\mu(s) z\big|^2_H\,\mathrm{d}s.$$

Now, if we expand $S^\star_\mu(t)$ into Fourier coefficients, by [1, Proposition 2.3], we get

$$S^\star_\mu(u,v) = \sum_{k=1}^\infty \big(\hat{f}^\mu_k(t)e_k, \hat{g}^\mu_k(t)e_k\big),$$

where

$$\begin{cases} \mu \dfrac{\mathrm{d}\hat{f}^\mu_k}{\mathrm{d}t}(t) = -\hat{g}^\mu_k(t), & \hat{f}^\mu_k(0) = u_k = \langle u, e_k\rangle_H, \\ \mu \dfrac{\mathrm{d}\hat{g}^\mu_k}{\mathrm{d}t}(t) = \mu\alpha_k \hat{f}^\mu_k(t) - \hat{g}^\mu_k(t), & \hat{g}^\mu_k(0) = v_k = \langle v, e_k\rangle_H. \end{cases}$$

AUTHOR COPY

From these equations we see that

$$\left|\hat{g}_k^\mu(t)\right|^2 = -\frac{\mu^2\alpha_k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\hat{f}_k^\mu(t)\right|^2 - \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\hat{g}_k^\mu(t)\right|^2.$$

This means that

$$\begin{aligned}
&\left|\left(L_{t_1,t_2}^\mu\right)^\star z\right|^2_{L^2((t_1,t_2);H)} \\
&\quad = \frac{1}{2}\sum_{k=1}^\infty\left(\frac{\lambda_k^2}{\alpha_k}|u_k|^2 + \frac{\lambda_k^2}{\mu\alpha_k^2}|v_k|^2 - \frac{\lambda_k^2}{\alpha_k}\left|\hat{f}_k^\mu(t_2-t_1)\right|^2 - \frac{\lambda_k^2}{\mu\alpha_k^2}\left|\hat{g}_k^\mu(t_2-t_1)\right|^2\right) \\
&\quad = \frac{1}{2}\left(\left|C_\mu^{1/2}z\right|^2_{\mathcal{H}} - \left|C_\mu^{1/2}S_\mu^\star(t_2-t_1)z\right|^2_{\mathcal{H}}\right).
\end{aligned} \tag{3.8}$$

Notice that

$$(1\wedge\sqrt{\mu})\left|C_\mu^{1/2}z\right|_{\mathcal{H}} \leqslant \left|C_1^{1/2}z\right|_{\mathcal{H}} \leqslant (1+\sqrt{\mu})\left|C_\mu^{1/2}z\right|_{\mathcal{H}}$$

and that $C_1^{1/2}$ commutes with $S_\mu^\star(t)$. Therefore, by (2.5)

$$\left|C_\mu^{1/2}S_\mu^\star(t)z\right|_{\mathcal{H}} \leqslant \frac{1}{1\wedge\sqrt{\mu}}\left|S_\mu^\star(t)C_1^{1/2}z\right|_{\mathcal{H}} \leqslant \frac{1+\sqrt{\mu}}{1\wedge\sqrt{\mu}}M_\mu \mathrm{e}^{-\omega_\mu t}\left|C_\mu^{1/2}z\right|_{\mathcal{H}}.$$

According to (3.8), this implies

$$\left|\left(L_{t_1,t_2}^\mu\right)^\star z\right|^2_{L^2((t_1,t_2);H)} \geqslant \frac{1}{2}\left(1-\left(\frac{1+\sqrt{\mu}}{1\wedge\sqrt{\mu}}\right)^2 M_\mu^2\mathrm{e}^{-2\omega_\mu(t_2-t_1)}\right)\left|C_\mu^{1/2}z\right|^2_{\mathcal{H}},$$

so that, if we take $T_\mu > 0$ such that

$$\left(\frac{1+\sqrt{\mu}}{1\wedge\sqrt{\mu}}\right)^2 M_\mu^2\mathrm{e}^{-2\omega_\mu T_\mu} < 1$$

we can conclude. □

4. The main result

If $z \in C((-\infty,0];\mathcal{H})$ is such that $I^\mu_{-\infty,0}(z) < +\infty$, then we have

$$I^\mu_{-\infty}(z) = \frac{1}{2}\int_{-\infty}^0\left|Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t) + \frac{\partial\varphi}{\partial t}(t) - A\varphi(t) + Q^2DF\big(\varphi(t)\big)\right)\right|^2_H \mathrm{d}t. \tag{4.1}$$

AUTHOR COPY

Actually, if $I^{\mu}_{-\infty,0}(z) < +\infty$, then there exists $\psi \in L^2((-\infty,0);H)$ such that $\varphi = \Pi_1 z$ is a weak solution to

$$\mu \frac{\partial^2\varphi}{\partial t^2}(t) = A\varphi(t) - \frac{\partial\varphi}{\partial t}(t) - Q^2 DF\big(\varphi(t)\big) + Q\psi.$$

This means that

$$\psi(t) = Q^{-1}\left(\mu \frac{\partial^2\varphi}{\partial t^2}(t) + \frac{\partial\varphi}{\partial t}(t) - A\varphi(t) + Q^2 DF\big(\varphi(t)\big)\right)$$

and (4.1) follows.

By the same argument, if $I_{-\infty,0}(\varphi) < +\infty$, then it follows that

$$I_{-\infty}(\varphi) = \int_{-\infty}^{0} \left| Q^{-1}\left(\frac{\partial\varphi}{\partial t}(t) - A\varphi(t) + Q^2 DF\big(\varphi(t)\big)\right)\right|_H^2 \mathrm{d}t. \tag{4.2}$$

Theorem 4.1. *For any fixed* $\mu > 0$ *and* $(x,y) \in D((-A)^{1/2}Q^{-1}) \times D(Q^{-1})$ *it holds*

$$V^{\mu}(x,y) = \big|(-A)^{1/2}Q^{-1}x\big|_H^2 + 2F(x) + \mu\big|Q^{-1}y\big|_H^2. \tag{4.3}$$

Moreover,

$$V(x) = \big|(-A)^{1/2}Q^{-1}x\big|_H^2 + 2F(x). \tag{4.4}$$

In particular, for any $\mu > 0$,

$$V_\mu(x) := \inf_{y \in H^{-1}} V^{\mu}(x,y) = V^{\mu}(x,0) = V(x).$$

Proof. First, we observe that if $\varphi(t) = \Pi_1 z(t)$, then

$$\begin{aligned} I^{\mu}_{-\infty}(z) = {} & \frac{1}{2}\int_{-\infty}^{0} \left| Q^{-1}\left(\mu \frac{\partial\varphi}{\partial t}(t) - \frac{\partial\varphi}{\partial t}(t) - A\varphi(t) + Q^2 DF\big(\varphi(t)\big)\right)\right|_H^2 \mathrm{d}t \\ & + 2\int_{-\infty}^{0} \left\langle Q^{-1}\frac{\partial\varphi}{\partial t}(t), Q^{-1}\left(\mu \frac{\partial^2\varphi}{\partial t^2}(t) - A\varphi(t)\right) + QDF\big(\varphi(t)\big)\right\rangle_H \mathrm{d}t. \end{aligned} \tag{4.5}$$

Now, if

$$\lim_{t \to -\infty} \big|C_\mu^{-1/2} z(t)\big|_{\mathcal{H}} = 0,$$

then

$$\lim_{t \to -\infty} \big|(-A)^{1/2}Q^{-1}\varphi(t)\big|_H + \left|Q^{-1}\frac{\partial\varphi}{\partial t}(t)\right|_H = 0,$$

AUTHOR COPY

so that

$$2\int_{-\infty}^{0}\left\langle Q^{-1}\frac{\partial\varphi}{\partial t}(t), Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t)-A\varphi(t)\right)+QDF\big(\varphi(t)\big)\right\rangle_H \mathrm{d}t$$
$$=\left|(-A)^{1/2}Q^{-1}\varphi(0)\right|_H^2+2F\big(\varphi(0)\big)+\mu\left|Q^{-1}\frac{\partial\varphi}{\partial t}(0)\right|_H^2.$$

This yields

$$V^\mu(x,y)\geqslant\left|(-A)^{1/2}Q^{-1}x\right|_H^2+2F(x)+\mu\left|Q^{-1}y\right|_H^2.$$

Now, let $\tilde z(t)$ be a mild solution of the problem

$$\tilde z(t)=S_\mu(t)(x,-y)-\int_0^t S_\mu(t-s)Q_\mu QDF\big(\tilde z(s)\big)\,\mathrm{d}s,$$

and let $(x,y)\in D(C_\mu^{-1/2})$. Then $\tilde\varphi(t)=\Pi_1\tilde z(t)$ is a weak solution of the problem

$$\mu\frac{\partial^2\tilde\varphi}{\partial t^2}(t)=A\tilde\varphi(t)-\frac{\partial\tilde\varphi}{\partial t}(t)-Q^2DF\big(\tilde\varphi(t)\big),\quad \tilde\varphi(0)=x,\ \frac{\tilde\varphi}{\partial t}(0)=-y.$$

Moreover, as proven in Lemma 4.2,

$$\lim_{t\to-\infty}\left|C_\mu^{-1/2}\tilde z(t)\right|_{\mathcal H}=0.$$

Then, if we define $\hat\varphi(t)=\tilde\varphi(-t)$ for $t\leqslant 0$, we see that $\hat\varphi(t)$ solves

$$\mu\frac{\partial^2\hat\varphi}{\partial t^2}(t)=A\hat\varphi(t)+\frac{\partial\hat\varphi}{\partial t}(t)-Q^2DF\big(\hat\varphi(t)\big),\quad \hat\varphi(0)=x,\ \frac{\partial\hat\varphi}{\partial t}(0)=y.$$

Thanks to (4.5) this yields

$$I^\mu_{-\infty}(\hat\varphi)=\left|(-A)^{1/2}Q^{-1}x\right|_H^2+2F(x)+\mu\left|Q^{-1}y\right|_H^2$$

and then

$$V^\mu(x,y)=\left|(-A)^{1/2}Q^{-1}x\right|_H^2+2F(x)+\mu\left|Q^{-1}y\right|_H^2.$$

As known, an analogous result holds for $V(x)$. In what follows, for completeness, we give a proof. We have

$$I_{-\infty}(\varphi)=\frac{1}{2}\int_{-\infty}^{0}\left|Q^{-1}\left(\frac{\partial\varphi}{\partial t}(t)+A\varphi(t)-Q^2DF\big(\varphi(t)\big)\right)\right|_H^2\mathrm{d}t$$
$$+2\int_{-\infty}^{0}\left\langle Q^{-1}\frac{\partial\varphi}{\partial t}(t), Q^{-1}\big(-A\varphi(t)+Q^2DF\big(\varphi(t)\big)\big)\right\rangle_H\mathrm{d}t. \tag{4.6}$$

AUTHOR COPY

From this we see that

$$V(x) \geqslant \left|(-A)^{1/2}Q^{-1}x\right|_H^2 + 2F(x).$$

Just as for the wave equation, for $x \in D((-A)^{1/2}Q^{-1})$, we define $\tilde{\varphi}$ to be the solution of

$$\tilde{\varphi}(t) = \mathrm{e}^{tA}x - \int_0^t \mathrm{e}^{(t-s)A}Q^2 DF\big(\tilde{\varphi}(s)\big)\,\mathrm{d}s.$$

We have

$$\lim_{t\to+\infty}\left|(-A)^{1/2}Q^{-1}\tilde{\varphi}(t)\right|_H = 0.$$

Then, if we define $\hat{\varphi}(t) = \tilde{\varphi}(-t)$ we get

$$\frac{\partial\hat{\varphi}}{\partial t}(t) = -A\hat{\varphi}(t) + Q^2 DF\big(\hat{\varphi}(t)\big),$$

so that

$$I_{-\infty}(\hat{\varphi}) = \left|(-A)^{1/2}Q^{-1}x\right|_H^2 + 2F(x)$$

and

$$V(x) = \left|(-A)^{1/2}Q^{-1}x\right|_H^2 + 2F(x). \qquad \square$$

Now, in order to conclude the proof of Theorem 4.1, we have to prove the following result.

Lemma 4.2. *Let* $(x,y) \in D((-A)^{1/2}Q^{-1}) \times D(Q^{-1})$ *and let* φ *solve the problem*

$$\mu\,\frac{\partial^2\varphi}{\partial t^2}(t) = A\varphi(t) - \frac{\partial\varphi}{\partial t}(t) - Q^2 DF\big(\varphi(t)\big), \quad \varphi(0) = x,\ \frac{\partial\varphi}{\partial t}(0) = y. \tag{4.7}$$

Then

$$z(t) = \left(\varphi(t), \frac{\partial\varphi}{\partial t}(t)\right) \in D\big((-A)^{1/2}\big) \times D\big(Q^{-1}\big), \quad t \geqslant 0$$

and

$$\lim_{t\to+\infty}\left|C_1^{-1/2}z(t)\right|_{\mathcal{H}} = 0. \tag{4.8}$$

Proof. If in (4.7) we take the inner product with $2Q^{-2}\,\partial\varphi/\partial t(t)$, we have

$$2\left|Q^{-1}\,\frac{\partial\varphi}{\partial t}(t)\right|_H^2 = -\frac{\mathrm{d}}{\mathrm{d}t}\big(\mu\left|Q^{-1}\varphi(t)\right|_H^2 + \left|(-A)^{1/2}Q^{-1}\varphi(t)\right|_H^2 + 2F\big(\varphi(t)\big)\big). \tag{4.9}$$

AUTHOR COPY

Therefore, if we define

$$\Phi_\mu(x,y) = \left|(-A)^{1/2}Q^{-1}x\right|_H^2 + \mu\left|Q^{-1}y\right|_H^2 + 2F(x),$$

as a consequence of (4.9) we get

$$\Phi_\mu\big(z(t)\big) \leqslant \Phi_\mu(u,v). \tag{4.10}$$

Next, by (4.7) and the assumption that $\langle DF(x),x\rangle \geqslant 0$, we calculate that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left|Q^{-1}\left(\mu\,\frac{\partial\varphi}{\partial t}(t)+\varphi(t)\right)\right|_H^2 \\
&\quad = 2\left\langle Q^{-1}\left(\mu\,\frac{\partial\varphi}{\partial t}(t)+\varphi(t)\right), Q^{-1}A\varphi(t) - QDF\big(\varphi(t)\big)\right\rangle_H \\
&\quad \leqslant -2\left|Q^{-1}(-A)^{1/2}\varphi(t)\right|_H^2 - \mu\,\frac{\mathrm{d}}{\mathrm{d}t}\left|Q^{-1}(-A)^{1/2}\varphi(t)\right|_H^2 - 2\mu\,\frac{\mathrm{d}}{\mathrm{d}t}F\big(\varphi(t)\big).
\end{aligned}$$

A consequence of this is that

$$2\int_0^\infty \left|Q^{-1}(-A)^{1/2}\varphi(t)\right|_H^2\,\mathrm{d}t \leqslant \left|\mu Q^{-1}y + Q^{-1}x\right|_H^2 + \mu\left|Q^{-1}(-A)^{1/2}x\right|_H^2 + 2\mu F(x). \tag{4.11}$$

Now, if $z(t) = (\varphi(t), \frac{\partial\varphi}{\partial t}(t))$, for any $t, T > 0$ we have

$$z(T+t) = S_\mu(t)z(T) - \int_T^{T+t} S_\mu(T+t-s)Q_\mu QDF\big(\varphi(s)\big)\,\mathrm{d}s.$$

By (2.5) and (2.3) we have

$$\begin{aligned}
&\left|C_1^{-1/2}\int_T^{T+t} S_\mu(t+T-s)Q_\mu QDF\big(\varphi(s)\big)\,\mathrm{d}s\right|_{\mathcal{H}} \\
&\quad \leqslant \int_T^{T+t}\left|S_\mu(t+T-s)Q_\mu(-A)^{1/2}DF\big(\varphi(s)\big)\right|_{\mathcal{H}}\mathrm{d}s \\
&\quad \leqslant c\int_T^{T+t}\mathrm{e}^{-\omega_\mu(t+T-s)}\left|(-A)^{1/2}QDF\big(\varphi(s)\big)\right|_{H^{-1}}\mathrm{d}s \\
&\quad \leqslant c\int_T^{T+t}\mathrm{e}^{-\omega_\mu(t+T-s)}\left|\varphi(s)\right|_H\mathrm{d}s \leqslant c|\varphi|_{L^2((T,T+t);H)}.
\end{aligned}$$

Therefore, by (4.11), for any $\varepsilon > 0$ we can pick $T_\varepsilon > 0$ large enough so that for all $t > 0$

$$\left|C_1^{-1/2}\int_{T_\varepsilon}^{T_\varepsilon+t} S_\mu(t+T_\varepsilon-s)Q_\mu QDF\big(\varphi(s)\big)\,\mathrm{d}s\right|_{\mathcal{H}} < \frac{\varepsilon}{2}.$$

AUTHOR COPY

Next, by (2.5),

$$\left|C_1^{-1/2} S_\mu(t) z(T)\right|_{\mathcal{H}} \leqslant M_\mu \mathrm{e}^{-\omega_\mu t} \left|C_1^{-1/2} z(T)\right|_{\mathcal{H}}.$$

Then, as

$$\left|C_1^{-1/2} z\right|_{\mathcal{H}} \leqslant c\Phi_\mu(z), \quad z \in \mathcal{H},$$

by (4.10) we can find a t_ε large enough so that for all $T > 0$ and $t > t_\varepsilon$

$$\left|C_1^{-1/2} S_\mu(t) z(T)\right|_{\mathcal{H}} < \frac{\varepsilon}{2}.$$

Then for $t > T_\varepsilon + t_\varepsilon$

$$\left|C_1^{-1/2} z(t)\right|_{\mathcal{H}} < \varepsilon$$

which is what we were trying to prove. □

References

[1] S. Cerrai and M.I. Freidlin, On the Smoluchowski–Kramers approximation for a system with an infinite number of degrees of freedom, *Probability Theory and Related Fields* **135** (2006), 363–394.
[2] S. Cerrai and M.I. Freidlin, Smoluchowski–Kramers approximation for a general class of SPDE's, *Journal of Evolution Equations* **6** (2006), 657–689.
[3] S. Cerrai and M. Röckner, Large deviations for invariant measures of stochastic reaction–diffusion systems with multiplicative noise and non-Lipschitz reaction term, *Annales de l'Institut Henri Poincaré (Probabilités et Statistiques)* **41** (2005), 69–105.
[4] Z. Chen and M. Freidlin, Smoluchowski–Kramers approximation and exit problems, *Stoch. Dyn.* **5** (2005), 569–585.
[5] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge Univ. Press, Cambridge, 1992.
[6] M. Freidlin, Some remarks on the Smoluchowski–Kramers approximation, *J. Statist. Phys.* **117** (2004), 617–634.
[7] M. Fuhrman, Analyticity of transition semigroups and closability of bilinear forms in Hilbert spaces, *Studia Mathematica* **115** (1995), 53–71.

AUTHOR COPY