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Schauder estimates for a class of second order elliptic operators on a cube

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Abstract

We consider a class of second order elliptic operators on a d-dimensional cube S_d. We prove that if the coefficients are of class $C^{k+\delta}(S_d)$, with $k = 0, 1$ and $\delta \in (0, 1)$, then the corresponding elliptic problem admits a unique solution u belonging to $C^{k+2+\delta}(S_d)$ and satisfying non-standard boundary conditions involving only second order derivatives.

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1. Introduction

In this paper we are concerned with the solvability in Hölder spaces of the following second order elliptic equation

$$\begin{cases} \lambda u(x) - \Delta u(x) - Lu(x) = f(x), \quad x \in S_d, \\ D_i^2 u(x) = 0, \quad x \in S_d \cap \{x_i = 0, 1\},\ i = 1, \dots, d, \end{cases} \tag{1.1}$$

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where S_d is the d-dimensional hypercube $[0,1]^d$, λ is a positive constant and L is the second order operator defined by

$$Lu(x) := \frac{1}{2}\operatorname{Tr}\bigl[A(x)D^2u(x)\bigr] + \bigl\langle b(x), Du(x)\bigr\rangle, \quad x \in S_d.$$

Here we assume that $A(x)$ is a non-negative symmetric $d \times d$ matrix ($A(x) \in \mathcal{L}^+(\mathbb{R}^d)$) for each $x \in S_d$ and that $A : S_d \to \mathcal{L}^+(\mathbb{R}^d)$, $b : S_d \to \mathbb{R}^d$ satisfy the invariance condition for the stochastic flow associated with L (for definitions and more about stochastic invariance see e.g. [1]) Namely, we suppose that

$$A(x)\nu(x) = 0, \quad \bigl\langle b(x), \nu(x)\bigr\rangle \geqslant 0, \qquad x \in \partial S_d, \tag{1.2}$$

where $\nu(x)$ denotes the unit inward normal at point x of the boundary of S_d. Notice that in particular the operator L is not uniformly elliptic on S_d.

Our aim is showing that if the coefficients A and b and the right hand side f are of class $C^{k+\delta}(S_d)$, with $k = 0, 1$ and $\delta \in (0,1)$, then problem (1.1) admits a unique solution $u \in C^{2+k+\delta}(S_d)$, see Theorem 3.2.

Problem (1.1), when Δ is replaced by $\varepsilon\Delta$, can be considered as an approximate problem for the non-uniformly elliptic problem

$$\lambda u(x) - Lu(x) = f(x), \quad x \in S_d. \tag{1.3}$$

Such problems (which are degenerate problems in non-smooth domains) arise in the theory of measure valued diffusion processes describing some dynamics of populations. Actually, according to the classical theory of Stroock and Varadhan (see [11]), the study of existence and uniqueness of solutions for such problems is the key ingredient in the proof of well-posedness of the martingale problem associated with the operator L.

Probably the most well known example of such operators is the Fleming–Viot operator, that in the case of a population with a finite number of types is given by

$$Lu(x) = \frac{1}{2}\sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)D_{ij}u(x) + \sum_{i,j=1}^{d} q_{i,j}\, x_i D_i u(x), \quad x \in S,$$

where S is the simplex of probability measures on the finite set $E := \{1, \dots, d\}$, that is $S = \{x \in [0,1]^d\colon \sum_{i \in E} x_i \leqslant 1\}$, and $[q_{i,j}]_{i,j\in E}$ is the infinitesimal matrix of a Markov process on E (for a review on Fleming–Viot operators, in the case of a finite and of a infinite number of types, see [5,6]). The Fleming–Viot operator is clearly not uniformly elliptic and is defined on a domain with non-smooth boundary. Nevertheless, the degeneracy of the operator on the non-smooth boundary allows to prove its hypo-ellipticity (see [4], also for more general drift terms).

Our goal here is to extend the results obtained for the Fleming–Viot operator to more general degenerate operators on non-smooth domains, which only satisfy the stochastic invariance property (1.2). To this purpose we proceed in two steps. In the first one, which is given in the present article, we approximate Eq. (1.3) by introducing some boundary conditions on the second derivatives and by adding $\varepsilon\Delta$ to L, in order to have a uniformly elliptic operator for which it is possible to use several classical tools to prove existence and uniqueness of solutions in spaces of Hölder continuous functions (see [7] and [8]). In the second one, which is the argument of a forthcoming paper, we let ε go to zero and we get solutions for the original degenerate problem in an appropriate sense.

Due to the boundary conditions on the second derivatives, the main feature of the second order term

$$\Delta u + \frac{1}{2}\,\mathrm{Tr}[AD^2u] \tag{1.4}$$

is that it can be restricted to the boundary of S_d, and its restriction acts on functions defined on the boundary of S_d. This property of (1.4) is important, as in the proof of existence of solutions for problem (1.1) we can proceed by induction. By using a direct argument we solve problem (1.1) for $d = 1$. Then, due the inductive hypothesis, we first solve the problem on the boundary of S_d, which is a $(d-1)$-dimesional hypercube, and then, by a suitable extension result, we extend the solution to the whole set S_d and reduce to solve a problem with Dirichlet boundary conditions under suitable compatibility conditions on the right hand side f.

Finally, we would like to say that analogous results can also be proved for the problem

$$\begin{cases} \lambda u(x) - \Delta u(x) = f(x), & x \in S_d, \\ D_i^2 u(x) = 0, & x \in S_d \cap \{x_i = 0, 1\},\ i = 1, \dots, d, \end{cases} \tag{1.5}$$

by using the method of sums introduced by Da Prato and Grisvard (see [2]). However, we do not know how to apply this method in the case of a simplex. Moreover, we would like to recall that, as explained by Lamperti in [9, Section 7.8], problem (1.5) has a probabilistic interpretation. Actually, it corresponds to a Brownian motion that when reaches the boundary of S_d for the first time, instead of reflecting (as happens in the case of Neumann boundary conditions) or dying (as it happens in the case of Dirichlet boundary conditions), sticks at the boundary forever. This is the reason why such boundary conditions are known in the probabilistic literature as *sticking barrier* boundary conditions.

The paper is organized as follows. In Section 2 we introduce some notations about function spaces on S_d and on its boundary ∂S_d. Moreover, we prove an extension results for functions defined on ∂S_d. In Section 3 we prove the existence and uniqueness result. After giving hypotheses on the coefficients A and b, we state the main result. In Lemma 3.3 we prove that the maximum principle holds for problem (1.1), so that uniqueness follows. Due to uniqueness in Remark 3.4 we show how in the study of problem 1.1 it is possible to reduce to the case $b = 0$. In Proposition 3.5 we prove directly solvability of problem (1.1) in the case of space dimension $d = 1$. In Propositions 3.7 and 3.8, we show that for any $d \geqslant 1$ problem (1.1) is uniquely solvable in $C^{2+\delta}(S_d)$ and $C^{3+\delta}(S_d)$ respectively, under the extra condition that the right hand side f vanishes on the boundary of S_d. In Section 3.1 we prove the same result, in the case of general f.

2. Notations and preliminaries

Throughout the present paper we denote by S_d the set $[0,1]^d$. For any $i \in I := \{1, \dots, d\}$ and $j \in J := \{0, 1\}$ we set

$$\partial S_d^{i,j} := \{x \in S_d\colon\ x_i = j\}, \qquad \partial S_d^i := \partial S_d^{i,0} \cup \partial S_d^{i,1}.$$

Clearly for any $(i,j) \in I \times J$ the set $\partial S_d^{i,j}$ is a hypercube of dimension $d-1$ and

$$\bigcup_{i=1}^{d} \partial S_d^i = \partial S_d.$$

2.1. *Function spaces on S_d*

In what follows we shall denote by $C(S_d)$ the Banach space of continuous functions on S_d, endowed with the sup-norm $\|\cdot\|_{C(S_d)}$ and by $C_0(S_d)$ the subspace of continuous functions vanishing at the boundary. For any $\delta \in (0,1)$ we shall denote by $C^\delta(S_d)$ the Banach space of uniformly δ-Hölder continuous functions, endowed with the norm

$$\|u\|_{C^\delta(S_d)} := \|u\|_{C(S_d)} + \sup_{\substack{x,y\in S_d\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\delta} =: \|u\|_{C(S_d)} + [u]_{C^\delta(S_d)}.$$

Next, for any integer $k \geqslant 1$ and any $\delta \in [0,1)$ we shall denote by $C^{k+\delta}(S_d)$ the space of functions $u \in C(S_d) \cap C^k(S_d^\circ)$ such that

$$D^\alpha u \text{ is uniformly } \begin{cases} \text{continuous} & \text{if } \delta = 0,\\ \delta\text{-Hölder continuous} & \text{if } \delta \in (0,1) \end{cases} \quad \text{on } S_d^\circ,$$

for any multi-index α, with $|\alpha| = k$. In particular, for any $1 \leqslant |\alpha| \leqslant k$ the derivatives $D^\alpha u$ can be extended by continuity to uniformly δ-Hölder continuous functions defined up to the boundary of S_d. $C^{k+\delta}(S_d)$ is a Banach space endowed with the norm

$$\|u\|_{C^{k+\delta}(S_d)} := \sum_{\substack{0\leqslant h\leqslant k\\ |\alpha|=h}} \sup_{x\in S_d} \left|D^\alpha u(x)\right| + \sum_{|\alpha|=k} \sup_{\substack{x,y\in S_d\\ x\neq y}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\delta}$$
$$=: \sum_{h=0}^{k} \|u\|_{C^h(S_d)} + [u]_{C^{k+\delta}(S_d)}.$$

Notice that as a consequence of the Whitney extension theorem for any $u \in C(S_d)$ we have that $u \in C^{k+\delta}(S_d)$ if and only if there exists $v \in C^{k+\delta}(\mathbb{R}^d)$ such that $u = v_{|S_d}$ (for a proof see e.g. [5, Theorem 6.1 in Appendix 6]).

Finally, we define

$$C^2_\sharp(S_d) := \bigcap_{i=1}^{d} \left\{u \in C^2(S_d):\ D_i^2 u(x) = 0,\ x \in \partial S_d^i\right\}. \tag{2.1}$$

2.2. *Function spaces on ∂S_d*

Throughout this subsection we assume that $d \geqslant 2$.

As for S_d, we denote by $C(\partial S_d)$ the Banach space of continuous functions on ∂S_d, endowed with the sup-norm. For any $(i,j) \in I \times J$, we introduce the mapping

$$\rho_{i,j} : S_{d-1} \to \partial S_d^{i,j} \subset S_d,$$

by setting for any $y \in S_{d-1}$ and $k \in I$

$$\left[\rho_{i,j}(y)\right]_k := \begin{cases} j & \text{if } k = i, \\ y_{\langle k,i \rangle} & \text{if } k \neq i, \end{cases} \tag{2.2}$$

where

$$\langle k, i \rangle := \begin{cases} k & \text{if } k < i, \\ k-1 & \text{if } k > i. \end{cases}$$

Moreover, for any $\hat{u} \in C(\partial S_d)$ we set

$$\hat{u}_{i,j} := \hat{u} \circ \rho_{i,j} : S_{d-1} \to \mathbb{R}. \tag{2.3}$$

Hence, for any integer $k \geqslant 0$ and for any $\delta \in (0,1)$ we introduce the set

$$C^{k+\delta}(\partial S_d) := \bigcap_{(i,j) \in I \times J} \left\{ \hat{u} \in C(\partial S_d) \colon \hat{u}_{i,j} \in C^{k+\delta}(S_{d-1}) \right\}. \tag{2.4}$$

Clearly $C^{k+\delta}(\partial S_d)$ is the set of continuous functions $\hat{u}$ on ∂S_d such that their restrictions to the hypercubes $\partial S_d^{i,j}$ are of class $C^{k+\delta}$ as functions of $d-1$ variables.

Next lemma shows how to compute the derivatives of $\hat{u}$ in terms of the functions $\hat{u}_{i,j}$ and $\rho_{i,j}$.

Lemma 2.1. *Assume that $\hat{u} \in C^2(\partial S_d)$. Then for any $(i,j) \in I \times J$ and $h, k \in I$, with $h, k \neq i$, we have*

$$D_h \hat{u}(x) = D_{\langle h,i \rangle} \hat{u}_{i,j}\left(\rho_{i,j}^{-1}(x)\right), \quad x \in \partial S_d^{i,j}, \tag{2.5}$$

and

$$D_h D_k \hat{u}(x) = D_{\langle h,i \rangle}\, D_{\langle k,i \rangle} \hat{u}_{i,j}\left(\rho_{i,j}^{-1}(x)\right), \quad x \in \partial S_d^{i,j}. \tag{2.6}$$

Proof. Due to (2.3), for any $x \in \partial S_d^{i,j}$ we have

$$\hat{u}(x) = \hat{u}_{i,j}\left(\rho_{i,j}^{-1}(x)\right) = \hat{u}_{i,j}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d).$$

Then, as $\hat{u}_{i,j} \in C^1(S_{d-1})$ for any $\hat{u} \in C^1(\partial S_d)$, if $h \neq i$ we easily obtain

$$D_h \hat{u}(x) = \begin{cases} D_h \hat{u}_{i,j}(\rho_{i,j}^{-1}(x)) & \text{if } h < i, \\ D_{h-1} \hat{u}_{i,j}(\rho_{i,j}^{-1}(x)) & \text{if } h > i. \end{cases}$$

Recalling how $\langle h, i \rangle$ was defined, this yields (2.5).

Next, if $\hat{u} \in C^2(\partial S_d)$ we have $\hat{u}_{i,j} \in C^2(S_{d-1})$. Hence, with the same arguments used for the first derivative, we can differentiate once more in (2.5) and (2.6) follows. □

Now, for any $\hat{u} \in C(\partial S_d)$ and $(i,j) \in I \times J$, we set

$$v_{i,j} := \hat{u}_{i,j} \circ \pi_i = \hat{u} \circ \rho_{i,j} \circ \pi_i, \tag{2.7}$$

where

$$\pi_i : S_d \to S_{d-1}, \quad (x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d).$$

Since $\pi_i \in C^\infty(S_d; S_{d-1})$, from the definition of $C^{k+\delta}(\partial S_d)$ we have

$$\hat{u} \in C^{k+\delta}(\partial S_d) \Rightarrow v_{i,j} \in C^{k+\delta}(S_d). \tag{2.8}$$

Next lemma shows how to compute the derivatives of v_{ij}.

Lemma 2.2. *If $\hat{u} \in C^1(\partial S_d)$ then $v_{ij} \in C^1(S_d)$, for any $(i, j) \in I \times J$, and*

$$D_h v_{i,j}(x) = \begin{cases} D_{\langle h,i\rangle}\hat{u}_{i,j}(\pi_i(x)) & \text{if } h \neq i, \\ 0 & \text{if } h = i. \end{cases} \tag{2.9}$$

Moreover, if $\hat{u} \in C^2(\partial S_d)$ then $v_{ij} \in C^2(S_d)$ and

$$D_h D_k v_{i,j}(x) = \begin{cases} D_{\langle h,i\rangle} D_{\langle k,i\rangle}\hat{u}_{i,j}(\pi_i(x)) & \text{if } h \neq i \text{ and } k \neq i, \\ 0 & \text{if } h = i \text{ and/or } k = i. \end{cases} \tag{2.10}$$

Proof. If $\hat{u} \in C^1(\partial S_d)$, we have that $\hat{u}_{i,j} \in C^1(S_{d-1})$ and then, as $\pi_i \in C^\infty(S_d; S_{d-1})$, due to the chain rule we have

$$D_h v_{i,j}(x) = \sum_{l=1}^{d-1} D_l \hat{u}_{i,j}\big(\pi_i(x)\big)\, D_h \pi_i^l(x),$$

where $\pi_i^l(x) := [\pi_i(x)]_l$. Now, it is immediate to check that for any $l = 1, \ldots, d-1$

$$D_h \pi_i^l(x) = \begin{cases} \delta_{h,l} & \text{if } h < i \\ 0 & \text{if } h = i \\ \delta_{h,l+1} & \text{if } h > i, \end{cases}$$

and then (2.9) follows. Now (2.10) follows from analogous arguments, by applying again the chain rule, this time to $D_h v_{i,j}$.

Finally, we define

$$C^2_\sharp(\partial S_d) := \bigcap_{\substack{i,i' \in I \\ i \neq i'}} \big\{\hat{u} \in C(\partial S_d)\colon\ D_i^2 \hat{u}(x) = D_{i'}^2 \hat{u}(x) = 0,\ x \in \partial S_d^i \cap \partial S_d^{i'}\big\}. \tag{2.11}$$

2.3. *An extension result*

In what follows we shall denote by $\gamma : C(S_d) \to C(\partial S_d)$ the usual restriction operator defined for any $u \in C(S_d)$ by

$$\gamma(u)(x) := u(x), \quad x \in \partial S_d.$$

It is immediate to check that for any integer $k \geqslant 0$ and for any $\delta \in (0, 1)$

$$\gamma\big(C^{k+\delta}(S_d)\big) \subseteq C^{k+\delta}(\partial S_d), \qquad \gamma\big(C^2_\sharp(S_d)\big) \subseteq C^2_\sharp(\partial S_d). \tag{2.12}$$

In next proposition we show that it is possible to construct an extension operator from $C(\partial S_d)$ to $C(S_d)$, extending also functions in $C^{k+\delta}(\partial S_d)$ to functions in $C^{k+\delta}(S_d)$ and functions in $C^2_\sharp(\partial S_d)$ to functions in $C^2_\sharp(S_d)$.

Proposition 2.3. *There exists an operator $E : C(\partial S_d) \to C(S_d)$ such that for any integer $k \geqslant 0$ and for any $\delta \in (0,1)$*

$$E\big(C^{k+\delta}(\partial S_d)\big) \subseteq C^{k+\delta}(S_d), \quad E\big(C^2_\sharp(\partial S_d)\big) \subseteq C^2_\sharp(S_d), \tag{2.13}$$

and such that $\gamma \circ E = \hat{I}$, where $\hat{I}$ denotes the identity operator on $C(\partial S_d)$.

Proof. *Step* 1. For any $i \in I$ and $\hat{u} \in C(\partial S_d)$ we define

$$G_i \hat{u}(x) := (1 - x_i) v_{i,0}(x) + x_i v_{i,1}(x), \quad x \in S_d, \tag{2.14}$$

where $v_{i,j}$ are the mappings introduced in (2.7). It is immediate to check that

$$\gamma \circ G_i \hat{u}(x) = \hat{u}(x), \quad x \in \partial S^i_d,$$

and all the operators G_i commute, in the sense that for any $i, h \in I$

$$G_h \circ \gamma \circ G_i = G_i \circ \gamma \circ G_h. \tag{2.15}$$

Further, due to (2.8) we have that $G_i(C^{k+\delta}(\partial S_d)) \subseteq C^{k+\delta}(S_d)$, for any integer $k \geqslant 0$ and any $\delta \in (0,1)$. Next we show that

$$G_i\big(C^2_\sharp(\partial S_d)\big) \subseteq C^2_\sharp(\partial S_d). \tag{2.16}$$

According to the definition of $C^2_\sharp(S_d)$, we have to show that for any $\hat{u} \in C^2_\sharp(\partial S_d)$ and $h \in I$

$$D^2_h(G_i \hat{u})(x) = 0, \quad x \in \partial S^h_d.$$

By differentiating twice both sides in (2.14) with respect to x_i, from the second identity in (2.9) we immediately have $D^2_i(G_i \hat{u})(x) = 0$, for any $x \in S_d$. Moreover, due to the first identity in (2.9) for any $h \neq i$ we have

$$\begin{aligned} D^2_h(G_i \hat{u})(x) &= (1 - x_i)\, D^2_h v_{i,0}(x) + x_i\, D^2_h v_{i,1}(x) \\ &= (1 - x_i)\, D^2_{\langle h,i\rangle} \hat{u}_{i,0}\big(\pi_i(x)\big) + x_i\, D^2_{\langle h,i\rangle} \hat{u}_{i,1}\big(\pi_i(x)\big). \end{aligned}$$

Defining for $j \in J$ and $x \in S_d$

$$x_{i,j}(x) := (x_1, \dots, x_{i-1}, j, x_{i+1}, \dots, x_d),$$

it is not difficult to see that if $x \in \partial S^{h,j'}_d$ then $x_{i,j}(x) \in \partial S^{h,j'}_d \cap \partial S^{i,j}_d$ and

$$\pi_i(x) = \rho^{-1}_{i,j}\big(x_{i,j}(x)\big).$$

Hence, thanks to (2.6) we have

$$D^2_{\langle h,i\rangle} \hat{u}_{i,j}\big(\pi_i(x)\big) = D^2_{\langle h,i\rangle} \hat{u}_{i,j}\big(\rho^{-1}_{i,j}\big(x_{i,j}(x)\big)\big) = D^2_h \hat{u}\big(x_{i,j}(x)\big).$$

Recalling how functions $\hat{u}$ in $C^2_\sharp(\partial S_d)$ are defined, as $x_{i,j}(x) \in \partial S^{h,j'}_d \cap \partial S^{i,j}_d$ we can conclude that $D^2_{\langle h,i\rangle} \hat{u}_{i,j}(\pi_i(x)) = 0$. This implies that $D^2_h G_i(\hat{u})(x) = 0$, for any $x \in \partial S^h_d$, and (2.16) follows.

Step 2. By recurrence we can define the operators E_i, by setting

$$E_1 := G_1, \qquad E_{i+1} := E_i + G_{i+1} \prod_{l=1}^{i} (\hat{I} - \hat{G}_l), \quad 1 \leqslant i \leqslant d - 1$$

where $\hat{I}$ is the identity operator on $C(\partial S_d)$ and $\hat{G}_l := \gamma \circ G_l$, for any $l \in I$. We show that $E := E_d$ extends functions in $C(\partial S_d)$ to functions in $C(S_d)$.

By using the definition of E_d, thanks to (2.15) it is easy to see that for any $\hat{u} \in C(\partial S_d)$

$$\hat{u} = (\gamma \circ E_d)\hat{u} + \prod_{l=1}^{d}(\hat{I} - \hat{G}_l)\hat{u}.$$

Now, as $\hat{G}_i\, \hat{u}(x) = \hat{u}(x)$, for any $x \in \partial S_d^i$, we have

$$\prod_{l=1}^{d}(\hat{I} - \hat{G}_l)\hat{u}(x) = 0, \quad x \in \partial S_d.$$

This implies that $\hat{u} = \gamma \circ E\hat{u}$ and hence E is an extension operator from $C(\partial S_d)$ to $C(S_d)$.

Step 3. We conclude by showing that inclusions (2.13) hold.

Proceeding by induction we verify that for any $i \in I$

$$E_i\big(C^{k+\delta}(\partial S_d)\big) \subseteq C^{k+\delta}(S_d), \qquad E_i\big(C^2_\sharp(\partial S_d)\big) \subseteq C^2_\sharp(S_d).$$

If $i = 1$ the two inclusions are true, as we have just proved the same inclusions for the operators G_i. Now, assume that they are true for some $i \in I$. Since

$$E_{i+1} = E_i + G_i \prod_{l=1}^{i}(\hat{I} - \hat{G}_l),$$

and both the operator E_i and the operator G_i fulfill the inclusions, we have only to prove that for any $1 \leqslant l \leqslant i$

$$(\hat{I} - \hat{G}_l)\big(C^{k+\delta}(\partial S_d)\big) \subseteq C^{k+\delta}(S_d), \qquad (\hat{I} - \hat{G}_l)\big(C^2_\sharp(\partial S_d)\big) \subseteq C^2_\sharp(S_d).$$

But these inclusions are completely trivial, as $\hat{G}_l = \gamma \circ G_l$ and the same inclusions hold both for G_l and for γ. □

3. Main result

In what follows we shall denote by ν the unit inward normal at ∂S_d, that is

$$\nu(x) := (-1)^j e_i, \quad x \in \partial S_d^{i,j}.$$

Definition 3.1.

(1) A mapping $A = [a_{h,k}] : S_d \to \mathcal{L}^+(\mathbb{R}^d)$ belongs to $\mathcal{H}_A(S_d)$ if is continuous and

$$A(x)\, \nu(x) = 0, \quad x \in \partial S_d.$$

(2) A mapping $b : S_d \to \mathbb{R}^d$ belongs to $\mathcal{H}_b(S_d)$ if is continuous and

$$\big\langle (x), \nu(x)\big\rangle \geqslant 0, \quad x \in \partial S_d.$$

In particular, if $A \in \mathcal{H}_A(S_d)$ then for any $i, h \in I$

$$a_{h,i}(x) = a_{i,h}(x) = 0, \quad x \in \partial S_d^i,$$

and if $b \in \mathcal{H}_b(S_d)$ then for any $(i, j) \in I \times J$

$$(-1)^j b_i(x) \geqslant 0, \quad x \in \partial S_d^{i,j}.$$

Now, given $A \in \mathcal{H}_A(S_d)$ and $b \in \mathcal{H}_b(S_d)$ we define

$$L_{A,b} u := \frac{1}{2} \operatorname{Tr}[A D^2 u] + \langle b, Du \rangle, \quad u \in C^2(S_d). \tag{3.1}$$

Next, for any $\hat{u} \in C^2(\partial S_d)$ we define

$$\Delta|_{\partial S_d} \hat{u}(x) := \sum_{\substack{k=1,\dots,d \\ k \neq i}} D_k^2 \hat{u}(x), \quad x \in \partial S_d^i. \tag{3.2}$$

Analogously, the restriction of the operator $L_{A,b}$ to ∂S_d is defined by

$$\begin{aligned} L_{A,b}|_{\partial S_d} \hat{u}(x) := \frac{1}{2} &\sum_{\substack{h,k=1,\dots,d \\ h,k \neq i}} a_{h,k}(x) D_h D_k \hat{u}(x) \\ &+ \sum_{\substack{h=1,\dots,d \\ h \neq i}} b_h(x) D_h \hat{u}(x), \quad x \in \partial S_d^i, \end{aligned} \tag{3.3}$$

for any $\hat{u} \in C^2(\partial S_d)$. It is immediate to check that if $i, h, k \in I$, with $h, k \neq i$, then for any $v \in C^2(S_d)$

$$D_k(\gamma \circ v)(x) = D_k v(x), \qquad D_h D_k(\gamma \circ v)(x) = D_h D_k v(x), \qquad x \in \partial S_d^i. \tag{3.4}$$

Hence, if $v \in C^2_\sharp(S_d)$ we clearly have

$$\Delta|_{\partial S_d}(\gamma \circ v)(x) = \sum_{\substack{k=1,\dots,d \\ k \neq i}} D_k^2(\gamma \circ v)(x) = \sum_{\substack{k=1,\dots,d \\ k \neq i}} D_k^2 v(x) = \Delta v(x), \quad x \in \partial S_d^i,$$

so that

$$v \in C^2_\sharp(S_d) \Rightarrow \Delta|_{\partial S_d}(\gamma \circ v) = \gamma \circ \Delta v. \tag{3.5}$$

Analogously, since $a_{i,h}(x) = 0$ on ∂S_d^i, for any $v \in C^2(S_d)$ and $x \in \partial S_d^i$ we have

$$\begin{aligned} 2L_{A,0}|_{\partial S_d}(\gamma \circ v)(x) &= \sum_{\substack{h,k=1,\dots,d \\ h,k \neq i}} a_{h,k}(x) D_h D_k (\gamma \circ v)(x) \\ &= \sum_{h,k=1,\dots,d} a_{h,k}(x) D_h D_k v(x), \end{aligned}$$

so that

$$v \in C^2(S_d) \Rightarrow L_{A,0}|_{\partial S_d}(\gamma \circ v) = \gamma \circ L_{A,0} v. \tag{3.6}$$

Now we are ready to state the main result of this paper.

Theorem 3.2. *Let us fix $A \in \mathcal{H}_A(S_d)$ and $b \in \mathcal{H}_b(S_d)$ such that $A \in C^{k+\delta}(S_d; \mathcal{L}^+(\mathbb{R}^d))$ and $b \in C^{k+\delta}(S_d; \mathbb{R}^d)$, for $k = 0, 1$ and $\delta \in (0, 1)$. Then for any $\lambda > 0$ and $f \in C^{k+\delta}(S_d)$ there exists a unique $u \in C^{2+k+\delta}(S_d) \cap C^2_\sharp(S_d)$ such that*

$$\lambda u = \Delta u + L_{A,b}u + f. \tag{3.7}$$

The first preliminary result in the proof of previous theorem is the following *maximum principle*, which assures the uniqueness of solutions for problem (3.7).

Lemma 3.3. *Assume that $u \in C^2_\sharp(S_d)$ is such that*

$$\lambda u(x) - \Delta u(x) - L_{A,b}u(x) \geqslant 0, \quad x \in S_d,$$

for some $A \in \mathcal{H}_A(S_d)$, $b \in \mathcal{H}_b(S_d)$ and $\lambda > 0$. Then $u(x) \geqslant 0$, for any $x \in S_d$.

Proof. We prove that if $\bar{x} \in S_d$ is a point where u achieves its minimum, then

$$\Delta u(\bar{x}) \geqslant 0, \quad L_{A,0}u(\bar{x}) \geqslant 0, \quad \big\langle b(\bar{x}), Du(\bar{x})\big\rangle \geqslant 0. \tag{3.8}$$

This immediately implies the lemma.

In the proof we proceed as in [3] by induction on the space dimension d. Assume that $d = 1$. If the minimum is achieved at some point $\bar{x}$ in $(0, 1)$, then (3.8) is clearly verified. Thus, assume that $\bar{x} = 0$ (the case $\bar{x} = 1$ can be treated analogously). As $A \in \mathcal{H}_A(S_d)$, we have that $A(0) = 0$ and then $L_{A,0}\,u(0) = 0$. Moreover, as $b \in \mathcal{H}_b(S_d)$, we have that $b(0) \geqslant 0$ and then, since $Du(0) \geqslant 0$ we obtain $b(0)\,Du(0) \geqslant 0$. Finally, as $u \in C^2_\sharp([0, 1])$ we have $u''(0) = 0$ and then we can conclude that (3.8) holds when $d = 1$.

Next, assume that (3.8) holds for some $d - 1 \geqslant 1$. We show that it holds also for d. If the minimum is attained at some $\bar{x}$ in S°_d, then (3.8) is fulfilled. Otherwise, assume that $\bar{x} \in \partial S^{i,j}_d$, for some $(i, j) \in I \times J$.

As $b \in \mathcal{H}_b(S_d)$, we have

$$b_i(x)(-1)^j \geqslant 0, \quad x \in \partial S^{i,j}_d.$$

Moreover, since $\bar{x}$ is a minimum point for u, then $t = j$ is a minimum point for the mapping

$$\varphi : [0, 1] \to \mathbb{R}, \quad t \mapsto \varphi(t) := u\big(\bar{x} + (t - j)e_i\big),$$

and then

$$D_i u(\bar{x}) = \big\langle Du(\bar{x}), e_i\big\rangle = \varphi'(j) \begin{cases} \geqslant 0 & \text{if } j = 0, \\ \leqslant 0 & \text{if } j = 1. \end{cases}$$

In particular

$$b_i(x)D_i u(x) = b_i(x)(-1)^j(-1)^j\varphi'(j) \geqslant 0, \quad x \in \partial S^{i,j}_d,$$

and hence

$$\begin{aligned} \big\langle b(x), Du(x)\big\rangle &= b_i(x)D_i u(x) + \sum_{\substack{h=1,\dots,d \\ h \neq i}} b_h(x)D_h u(x) \\ &\geqslant \sum_{\substack{h=1,\dots,d \\ h \neq i}} b_h(x)D_h u(x), \quad x \in \partial S^{i,j}_d. \end{aligned}$$

Due to (3.4) we have

$$\sum_{\substack{h=1,\dots,d\\ h\neq i}} b_h(x)D_hu(x) = \sum_{\substack{h=1,\dots,d\\ h\neq i}} b_h(x)D_h(\gamma\circ u)(x)$$
$$=: \big\langle b(x)|_{\partial S_d}, D(\gamma\circ u)(x)\big\rangle,\quad x\in\partial S_d^{i,j},$$

and then we obtain

$$\big\langle b(x), Du(x)\big\rangle \geqslant \big\langle b(x)|_{\partial S_d}, D(\gamma\circ u)(x)\big\rangle,\quad x\in\partial S_d^{i,j}. \tag{3.9}$$

Moreover, from to (3.5) and (3.6) we have

$$\Delta u(x) = \Delta|_{\partial S_d}(\gamma\circ u)(x),\quad x\in\partial S_d^{i,j} \tag{3.10}$$

and

$$L_{A,0}u(x) = L_{A,0}|_{\partial S_d}(\gamma\circ u)(x),\quad x\in\partial S_d^{i,j}. \tag{3.11}$$

Therefore we can conclude. Actually, thanks to (3.4) we have that $\gamma\circ u\in C^2_\sharp(\partial S_d)$. Then, as the operators $\Delta|_{\partial S_d}$ and $L_{A,0}|_{\partial S_d}$ and the vector $b|_{\partial S_d}$ fulfill the same hypotheses of the present lemma in the hypercube $\partial S_d^{i,j}$ and $\bar{x}$ is a minimum point for $\gamma\circ u$ on $\partial S_d^{i,j}$, according to the inductive hypothesis

$$\Delta|_{\partial S_d}(\gamma\circ u)(\bar{x})\geqslant 0,\quad L_{A,0}|_{\partial S_d}(\gamma\circ u)(\bar{x})\geqslant 0,\quad \big\langle b|_{\partial S_d}(\bar{x}), D(\gamma\circ u)(\bar{x})\big\rangle\geqslant 0.$$

By (3.9), (3.10) and (3.11) this implies that the maximum principle holds also on S_d. □

Remark 3.4. Due to the lemma above, in the proof of Theorem 3.2 it is sufficient to restrict our attention to the case of $b=0$. Actually, if Theorem 3.2 holds for $L_{A,0}$ we have that the mapping

$$\lambda I-\Delta-L_{A,0} : C^{2+k+\delta}(S_d)\cap C^2_\sharp(S_d)\to C^{k+\delta}(S_d)$$

is continuous and surjective. Thus it is an isomorphism from $C^{2+k+\delta}(S_d)\cap C^2_\sharp(S_d)$ onto $C^{k+\delta}(S_d)$. This means that Eq. (3.7) is equivalent to the equation

$$u = (\lambda I-\Delta-L_{A,0})^{-1}\big(\langle b, Du\rangle + f\big) =: Ku + (\lambda I-\Delta-L_{A,0})^{-1}f. \tag{3.12}$$

Since the operator K is compact on $C^{2+k+\delta}(S_d)\cap C^2_\sharp(S_d)$, due to the Fredholm's alternative theorem Eq. (3.12) admits a unique solution $u\in C^{2+k+\delta}(S_d)\cap C^2_\sharp(S_d)$ if and only if $N(I-K)=\{0\}$. But this is equivalent to the fact that if we take $f=0$ in (3.7), then $u=0$, which follows from the maximum principle proved in Lemma 3.3. Hence, from now on we shall consider the problems

$$\lambda u = \Delta u + L_{A,0}u + f, \tag{3.13}$$

for $A\in\mathcal{H}_A(S_d)$.

Next proposition shows how in the case of space dimension $d=1$ Theorem 3.2 can be proved by constructing explicitly the unique solution $u\in C^{2+k+\delta}[0,1]$.

Proposition 3.5. *Fix $a \in C[0,1]$ non-negative. Then for any $\lambda > 0$ and $f \in C[0,1]$ there exists a unique $u \in C^2[0,1]$ such that*

$$\begin{cases} \lambda u(x) = u''(x) + \frac{1}{2}\,a(x)u''(x) + f(x), & x \in [0,1],\\ u''(0) = u''(1) = 0. \end{cases}$$

Moreover if $a, f \in C^{k+\delta}[0,1]$, for $\delta \in (0,1)$ and $k \in \{0,1\}$, then $u \in C^{2+k+\delta}[0,1]$.

Proof. For any $x \in [0,1]$ we define

$$f_1(x) := (1-x)f(0) + xf(1), \qquad f_2(x) := f(x) - f_1(x).$$

Due to the Liouville theorem the problem

$$\begin{cases} \lambda v(x) = v''(x) + \frac{1}{2}a(x)v''(x) + f_2(x), & x \in [0,1]\\ v(0) = v(1) = 0 \end{cases}$$

admits a unique solution $v \in C^2[0,1]$ and such solution belongs to $C^{2+k+\delta}[0,1]$ if $a, f \in C^{k+\delta}[0,1]$. Moreover, as $f_2(0) = f_2(1) = 0$, it is immediate to check that $v''(0) = v''(1) = 0$.

Now, if we define

$$u(x) := v(x) + \frac{1}{\lambda} f_1(x), \quad x \in [0,1], \tag{3.14}$$

we have that

$$\lambda u(x) = \lambda v(x) + f_1(x) = \left(1 + \frac{1}{2}a(x)\right)v''(x) + f_2(x) + f_1(x), \quad x \in [0,1].$$

Then, as $v''(x) = u''(x)$ this implies that

$$\begin{cases} \lambda u(x) = u''(x) + \frac{1}{2}a(x)u''(x) + f(x), & x \in [0,1],\\ u''(0) = u''(1) = 0. \end{cases}$$

Finally, since u can be represented as in (3.14) and $f_1 \in C^\infty[0,1]$ we can conclude that if $a, f \in C^{k+\delta}[0,1]$ then $u \in C^{2+k+\delta}[0,1]$. □

In the previous proposition we have seen that in the case of space dimension $d = 1$ problem (3.13) (and hence problem (3.7)) is solvable in $C^{2+k+\delta}[0,1]$, with $k = 0, 1$. Our aim now is showing that the same is true also in the case of arbitrary space dimension $d \geqslant 1$, under the further condition that f vanishes on ∂S_d.

To this purpose, we first prove an extension result for A and f on the set $T_d := [-1,1]^d$.

Lemma 3.6. *For any $f \in C^\delta(S_d) \cap C_0(S_d)$ there exists $\bar f \in C^\delta(T_d) \cap C_0(T_d)$ such that $\bar f \equiv f$ on S_d.*

Moreover, for any $A \in \mathcal{H}_A(S_d) \cap C^\delta(S_d; \mathcal{L}^+(\mathbb{R}^d))$ there exists $\bar B \in C^\delta(T_d; \mathcal{L}^+(\mathbb{R}^d))$ such that $\bar B \equiv I + 1/2\, A$ on S_d and

$$\bar B_{i,j}(\eta_k(x)) = (-1)^\alpha \bar B_{i,j}(x), \quad i,j,k \in I,\ x \in T_d, \tag{3.15}$$

where

$$\alpha = \alpha(i,j,k) := \begin{cases} -1 & \text{if } i = k \text{ and } j \neq k,\\ 1 & \text{otherwise}, \end{cases} \tag{3.16}$$

and for each $k \in I$ and $x \in T_d$

$$\eta_k(x_1, \dots, x_d) := (x_1, \dots, x_{k-1}, -x_k, x_{k+1}, \dots, x_d). \tag{3.17}$$

Proof. Since $f \in C^\delta(S_d)$ and vanishes at ∂S_d, we can extend it by oddness in all variables. More precisely, we can find a unique function $\bar{f} \in C^\delta(T_d) \cap C_0(T_d)$, with $T_d := [-1, 1]^d$, such that

$$\bar{f}(x) = f(x), \quad x \in S_d, \qquad \bar{f}\big(\eta_i(x)\big) = -\bar{f}(x), \quad x \in T_d,\ i \in I.$$

Next, we extend the mapping

$$B := I + \frac{1}{2} A : S_d \to \mathcal{L}^+(\mathbb{R}^d)$$

to the hypercube T_d. To this purpose we proceed by induction on $d \geqslant 1$. We set

$$T_d^0 := [0, 1]^d, \qquad T_d^l := [0, 1]^{d-l} \times [-1, 1]^l, \quad 1 \leqslant l < d, \qquad T_d^d := T_d.$$

The operator B is clearly defined on T_d^0. Now we extend it to an operator valued function $B^{(1)}$ defined on T_d^1, by setting $B^{(1)}(x) := B(x)$, for any $x \in S_d$, and

$$\begin{array}{ll} B^{(1)}_{d,d}(x) := B_{d,d}\big(\eta_d(x)\big), & B^{(1)}_{i,j}(x) := B_{i,j}\big(\eta_d(x)\big), \quad 1 \leqslant i, j < d, \\ B^{(1)}_{d,j}(x) := -B_{d,j}\big(\eta_d(x)\big), & B^{(1)}_{j,d}(x) := -B_{j,d}\big(\eta_d(x)\big), \quad 1 \leqslant j < d, \end{array} \tag{3.18}$$

for any $x \in [0, 1]^{d-1} \times [-1, 0)$.

It is immediate to check that $B^{(1)}(x) = (B^{(1)}(x))^t$, for any $x \in T_d^1$. Moreover, as $B_{d,j}(x) = B_{j,d}(x) = 0$, for any $1 \leqslant j < d$ and $x \in \partial S_d^{d,0}$, we clearly have

$$B^{(1)} \in C^\delta\big(T_d^1; \mathcal{L}(\mathbb{R}^d)\big).$$

Next, due to the Sylvester theorem on positive definite matrices, as $\det B(x) > 0$, for any $x \in S_d$, all principal minors of $B(x)$ are positive and then all principal minors of $B^{(1)}(x)$ of order $1 \leqslant l < d$ are positive as well. Thus, if we show that the $\det B^{(1)}(x) > 0$, by using again the Sylvester theorem we can conclude that $B^{(1)}(x)$ is positive definite for any $x \in T_d^1$, so that $B^{(1)} \in C^\delta(T_d^1; \mathcal{L}^+(\mathbb{R}^d))$. If we develop the determinant of $B^{(1)}(x)$ along the last row, for any $x \in [0, 1]^{d-1} \times [-1, 0)$ we have

$$\det\big[B^{(1)}(x)\big] = (-1)^{d+d} B^{(1)}_{d,d}(x) \det \hat{B}^{(1)}_{d,d}(x) + \sum_{j=1}^{d-1} (-1)^{j+d} B^{(1)}_{d,j}(x) \det \hat{B}^{(1)}_{d,j}(x),$$

where $\hat{B}^{(1)}_{i,j}(x)$ denotes the matrix obtained from $B^{(1)}(x)$ eliminating the ith row and the jth column. Then, as

$$\det \hat{B}^{(1)}_{d,j}(x) = -\det \hat{B}_{d,j}\big(\eta_d(x)\big), \qquad \det \hat{B}^{(1)}_{d,d}(x) = \det \hat{B}_{d,d}\big(\eta_d(x)\big)$$

from (3.18) we can conclude that

$$\begin{aligned} \det\big[B^{(1)}(x)\big] &= (-1)^{d+d} B_{d,d}\big(\eta_d(x)\big) \det \hat{B}_{d,d}\big(\eta_d(x)\big) \\ &\quad + \sum_{j=1}^{d-1} (-1)^{j+d} B_{d,j}\big(\eta_d(x)\big) \det \hat{B}_{d,j}\big(\eta_d(x)\big) \\ &= \det\big[B\big(\eta_d(x)\big)\big] > 0. \end{aligned}$$

Next, we extend $B^{(1)}$ to T_d^2 in a similar fashion, where the role of d and $\eta_d(x)$ is played respectively by $d-1$ and $\eta_{d-1}(x)$. In this way we obtain $B^{(2)}$ on T_d^2 which is an extension of $B^{(1)}$ (and hence of B) satisfying $B^{(2)} \in C^\delta(T_d^2; \mathcal{L}^+(\mathbb{R}^d))$.

By induction we obtain an extension $\bar{B} := B^{(d)}$ on $T_d^d = T_d$ satisfying the same properties and in particular satisfying (3.15). □

Due to this extension result for the operator A and the function f, we can solve (3.13) in $C^{2+\delta}(S_d)$.

Proposition 3.7. *Assume that* $A \in \mathcal{H}_A(S_d) \cap C^\delta(S_d; \mathcal{L}^+(\mathbb{R}^d))$, *for some* $\delta \in (0,1)$. *Then for any* $\lambda > 0$ *and* $f \in C^\delta(S_d) \cap C_0(S_d)$ *there exists a unique* $u \in C^{2+\delta}(S_d) \cap C^2_\sharp(S_d)$ *which is solution of* (3.13).

Proof. We notice that the domain S_d is bounded and satisfies an exterior sphere condition at every $x \in \partial S$, the operator $\Delta + L_{A,0}$ is uniformly elliptic and its coefficients belong to $C^\delta(S_d)$. Then, due to [7, Theorem 6.13] problem (3.13) admits a unique solution

$$u \in C(S_d) \cap C^{2+\delta}(S_d^\circ).$$

Our aim is proving that such solution u can be extended up to the boundary of S_d as a $C^{2+\delta}(S_d)$ function.

Since $u \in C(S_d) \cap C^{2+\delta}(S_d^\circ)$, in view of the compactness of S_d and ∂S_d it is sufficient to prove that for any $x \in \partial S_d$ there exists $r^x > 0$ and a function $v^x \in C^{2+\delta}(B(x, r^x))$ such that

$$u(y) = v^x(y), \quad y \in B(x, r^x) \cap S_d$$

(here $B(z,r)$ denotes the ball $\{x \in \mathbb{R}^d;\ |x - z| < r\}$).

We consider the Dirichlet problem

$$\begin{cases} \lambda u(x) = \mathrm{Tr}[\bar{B}(x) D^2 u(x)] + \bar{f}(x), & x \in T_d^\circ, \\ u(x) = 0, & x \in \partial T_d, \end{cases} \tag{3.19}$$

where $\bar{B}$ and $\bar{f}$ are respectively the extensions of $I + 1/2\, A$ and f to the hypercube $T_d := [-1,1]^d$, introduced in Lemma 3.6.

In view of [7, Theorem 6.13] such a problem admits a unique solution $\bar{u} \in C(T_d) \cap C^{2+\delta}(T_d^\circ)$.

Now, for any $v : T_d \to \mathbb{R}$ and $k \in I$ we define

$$R_k v(x) := v\big(\eta_k(x)\big), \quad x \in T_d.$$

Since for any $i, j, k \in I$

$$D_i D_j (R_k u)(x) = (-1)^\alpha D_i D_j u\big(\eta_k(x)\big), \quad x \in T_d,$$

where $\alpha = \alpha(i, j, k)$ is defined as in (3.16), due to (3.15) it is not difficult to check that for any $x \in T_d$

$$R_k(\bar{B}_{i,j} D_i D_j \bar{u})(x) = \bar{B}_{i,j}\big(\eta_k(x)\big) D_i D_j \bar{u}\big(\eta_k(x)\big) = \bar{B}_{i,j}(x) D_i D_j (R_k \bar{u})(x),$$

so that

$$R_k\big(\mathrm{Tr}[\bar B D^2\bar u]\big)(x)=\mathrm{Tr}\big[\bar B(x)D^2(R_k\bar u)(x)\big],\quad x\in T_d. \tag{3.20}$$

Therefore, as $R_k\bar f=-\bar f$, by applying R_k to both sides in (3.19) we obtain

$$\begin{cases}\lambda R_k\bar u(x)=\mathrm{Tr}[\bar B(x)D^2(R_k\bar u)(x)]-\bar f(x), & x\in T_d^\circ,\\ R_k\bar u(x)=0, \quad x\in\partial T_d.\end{cases}$$

This means that $R_k\bar u$ solves the problem

$$\begin{cases}\lambda z(x)=\mathrm{Tr}[\bar B(x)D^2 z(x)]-\bar f(x), & x\in T_d^\circ,\\ z(x)=0, \quad x\in\partial T_d,\end{cases}$$

where $z\in C(T_d)\cap C^{2+\delta}(T_d^\circ)$. From the uniqueness part in [7, Theorem 6.13] we have $\bar u=-R_k\bar u$ in T_d, that is

$$\bar u(x)=-\bar u\big(\eta_k(x)\big),\quad x\in T_d.$$

In particular for any $k\in I$

$$\bar u(x)=0,\quad x\in\partial S_d^{k,0}.$$

Hence, as $\bar f\equiv f$ and $\bar B\equiv I+1/2\,A$ on S_d, the restriction of $\bar u$ to S_d solves the problem

$$\begin{cases}\lambda u(x)=\Delta u(x)+\frac12\mathrm{Tr}[A(x)D^2u(x)]+f(x), & x\in S_d^\circ,\\ u(x)=0, \quad x\in\partial S_d,\end{cases}$$

and since $\bar u\in C(S_d)\cap C^{2+\delta}(S_d^\circ)$, by using the uniqueness result of [7, Theorem 6.13] in S_d, we can conclude that $\bar u\equiv u$ on S_d.

Notice that all assumptions of the proposition are also fulfilled after having performed a permutation of the coordinates

$$S_d\ni(x_1,\dots,x_d)\mapsto(x_{\pi(1)},\dots,x_{\pi(d)})\in S_d,$$

(here π denotes a permutation of the set I). Hence, it is sufficient to study the regularity of u around boundary points x of the following type

$$\begin{cases}x\in\bigcap_{i=l}^d\partial S_d^{i,0}\quad\text{and}\quad x\notin\bigcup_{i=1}^{l-1}\partial S_d^{i,1}, & l=2,\dots,d\\ x=(0,\dots,0).\end{cases} \tag{3.21}$$

As all these points x are in T_d°, for all of them there exists a ball $\overline{B(x,r^x)}\subset T_d^\circ$ such that $\bar u\in C^{2+\delta}(\overline{B(x,r^x)})$. Thus, as $u\equiv\bar u$ on $\overline{B(x,r^x)}\cap S_d$, we can conclude that $u\in C^{2+\delta}(S_d)$.

Finally, we show that $u\in C^2_\sharp(S_d)$. As $u\equiv 0$ on ∂S_d^i, for $h,k\neq i$ we have $D_hD_ku\equiv 0$ on ∂S_d^i. Hence, since $A(x)e_i=0$ for $x\in\partial S_d^i$ (and in particular $a_{h,i}(x)=0$, for any $h\in I$), this implies that

$$L_{A,0}u(x)=\frac12\sum_{\substack{h,k=1\\ h,h\neq i}}^d a_{h,k}(x)D_hD_ku(x)+\sum_{h=1}^d a_{h,i}(x)D_iD_hu(x)=0,\quad x\in\partial S_d^i.$$

Therefore, as $f\equiv 0$ on ∂S_d, Eq. (3.13) reduces to $D_i^2u(x)=0$, for $x\in\partial S_d^i$, which means that $u\in C^2_\sharp(S_d)$. □

Next proposition shows that if A and f are of class $C^{1+\delta}$, then the solution of problem (3.13) belongs to $C^{3+\delta}(S_d)$. We emphasize that this case and the case of Proposition 3.7 are not considered in [10].

Proposition 3.8. *If $A \in \mathcal{H}_A(S_d) \cap C^{1+\delta}(S_d; \mathcal{L}^+(\mathbb{R}^d))$ and $f \in C^{1+\delta}(S_d) \cap C_0(S_d)$, for some $\delta \in (0,1)$, then the unique solution u to problem* (3.13) *belongs to $C^{3+\delta}(S_d)$.*

Proof. Let u be the solution of Eq. (3.13). We show that $v_k := D_k u$ belongs to $C^{2+\delta}(S_d)$, for any $k \in I$.

According to [7, Theorem 6.17], if $A \in C^{1+\delta}(S_d; \mathcal{L}^+(\mathbb{R}^d))$ and $f \in C^{1+\delta}(S_d)$, we have that $u \in C^{3+\delta}(S_d^\circ)$. Then we can differentiate both sides in (3.13) with respect to x_k and we obtain

$$\lambda\, v_k(x) - \Delta\, v_k(x) - L_{A,0} v_k(x) = F_k(x), \quad x \in S_d^\circ,$$

where

$$F_k(x) := D_k f(x) + L_{D_k A,0} u(x)$$

and

$$\big[D_k A(x)\big]_{i,j} = D_k a_{i,j}(x).$$

As $u \equiv 0$ on ∂S_d, we have that $v_k(x) = 0$, for any $x \in \partial S_d^i$ with $i \neq k$. Moreover, as $u \in C^2_\sharp(S_d)$, we have that $D_k v_k(x) = 0$, for any $x \in \partial S_d^k$. This means that v_k solves the problem

$$\begin{cases} \lambda v(x) - \Delta v(x) - L_{A,0} v(x) = F_k(x), & x \in S_d^\circ, \\ v(x) = 0, & x \in \partial S_d^i,\ i \neq k, \\ \frac{\partial v}{\partial \nu}(x) = 0, & x \in \partial S_d^k \end{cases} \tag{3.22}$$

(which is the same problem solved by u, with a different right hand side and different boundary conditions).

First of all we notice that $F_k(x) = 0$, for any $x \in \partial S_d^i$, with $i \neq k$. Actually, since $u \in C_0(S_d)$, we have that $D_h D_l u(x) = 0$, for any $x \in \partial S_d^i$, with $h \neq i$ and/or $l \neq i$, so that

$$L_{D_k A,0}\, u(x) = \frac{1}{2} D_k a_{i,i}(x) D_i^2 u(x), \quad x \in \partial S_d^i.$$

Hence, as $u \in C^2_\sharp(S_d)$ we have $D_i^2 u(x) = 0$, for $x \in \partial S_d^i$, and this yields

$$L_{D_k A,0} u(x) = 0, \quad x \in \partial S_d^i.$$

Therefore, as $f \in C_0(S_d)$ we have $D_k f(x) = 0$ for any $x \in \partial S_d^i$ and $i \neq k$, so that we can conclude that $F_k(x) = 0$, for any $x \in \partial S_d^i$ and $i \neq k$. As $F_k \in C^\delta(S_d)$, this implies that we can extend it by eveness in the x_k variable and by oddness in the remaining x_i's variables to a function $\bar{F}_k \in C^\delta(T_d)$ such that

$$\begin{aligned} &\bar{F}_k(x) = F_k(x), \quad x \in S_d, \qquad \bar{F}_k\big(\eta_k(x)\big) = \bar{F}_k(x), \\ &\bar{F}_k\big(\eta_i(x)\big) = -\bar{F}_k(x), \quad i \neq k. \end{aligned} \tag{3.23}$$

Now, we consider the problem

$$\begin{cases} \lambda v(x) = \mathrm{Tr}[\bar{B}(x)\, D^2 v(x)] + \bar{F}_k(x), & x \in T_d, \\ v(x) = 0, & x \in \partial T_d, \end{cases} \tag{3.24}$$

where $\bar{B}$ is the extension of the matrix-valued function $B = I + 1/2\,A$ constructed in Lemma 3.6. Due to [7, Theorem 6.13], such a problem admits a unique solution $\bar{v} \in C(T_d) \cap C^{2+\delta}(T_d^\circ)$. If we define as in the proof of Proposition 3.7

$$R_i v(x) := v\big(\eta_i(x)\big), \quad x \in T_d,$$

for any $v : T_d \to \mathbb{R}$ and $i \in I$, thanks to (3.23) we have that

$$R_k \bar{F}_k = \bar{F}_k, \qquad R_i \bar{F}_k = -\bar{F}_k, \quad i \neq k.$$

Then, by applying R_i to both sides in (3.24), due to (3.20) we obtain

$$\begin{cases} \lambda, R_i v(x) = \mathrm{Tr}[\bar{B}(x) D^2 (R_i v)(x)] - \bar{F}_k(x), & x \in T_d^\circ, \\ R_i v(x) = 0, & x \in \partial T_d, \end{cases}$$

if $i \neq k$, and

$$\begin{cases} \lambda R_k v(x) = \mathrm{Tr}[\bar{B}(x) D^2 (R_k v)(x)] + \bar{F}_k(x), & x \in T_d^\circ, \\ R_k v(x) = 0, & x \in \partial T_d, \end{cases}$$

if $i = k$. Hence, from the uniqueness part in [7, Theorem 6.13] we have $R_i \bar{v} = -\bar{v}$, if $i \neq k$, and $R_k \bar{v} = \bar{v}$, so that

$$\bar{v}(x) = -\bar{v}\big(\eta_i(x)\big), \quad i \neq k, \qquad \bar{v}(x) = \bar{v}\big(\eta_k(x)\big), \quad x \in T_d^\circ.$$

This means that

$$\bar{v}(x) = 0, \quad x \in \partial S_d^{i,0},\ i \neq k$$

and

$$\frac{\partial \bar{v}}{\partial \nu}(x), \quad x \in \partial S_d^{k,0}.$$

Hence, as $\bar{F}_k \equiv F_k$ and $\bar{B} \equiv B$ on S_d, we can conclude that $\bar{v}$ solves problem (3.22). Now, if we show that the solution of problem (3.22) is unique we have that $v_k = \bar{v}$ on S^d and then, proceeding as in the proof of Proposition 3.7 we can conclude that $v_k \in C^{2+\delta}(S_d)$.

In order to prove uniqueness for problem (3.22) it is clearly enough to show that if in (3.22) we assume $F_k \geqslant 0$ on S_d°, then $v \geqslant 0$ on S_d. If there exists $x_0 \in S_d^\circ$ such that $v(x_0) = \min_{S_d} u$, then, by using the classical strong maximum principle for elliptic operators in connected open bounded sets, we have that v is constant within S_d° and then, as $v \in C(S_d)$ and $v = 0$ on ∂S_d^j, for any $j \neq k$, we can conclude that $v = 0$ on S_d. Thus, we can assume that v attains its minimum at some point $x_0 \in \partial S_d$. If $x_0 \in \partial S_d^j$, for some $j \neq k$, we have that $v(x_0) = 0$ and then $v(x) \geqslant 0$, for any $x \in S_d$. Finally, if $x_0 \in \partial S_d^k \setminus \bigcup_{j \neq k} \partial S_d^j$, we clearly have that S_d satisfies the interior ball condition at x_0, that is there exists an open ball $B \subset S_d$ with $x_0 \in \partial B$. Then, if $v(x_0) \leqslant 0$ and $v(x) < v(x_0)$ for any $x \in S_d^\circ$, due to the Hopf's lemma we can say that $\partial v/\partial \nu(x_0) > 0$, which is not possible, as $\partial v/\partial \nu = 0$ on ∂S_d^k. Therefore we have that $v(x_0) \geqslant 0$. □

3.1. Proof of Theorem 3.2

As explained in Remark 3.4, it is sufficient to assume $b = 0$ and prove the theorem for operators $L_{A,0}$, with $A \in \mathcal{H}_A(S_d)$.

In the proof we proceed by induction on the space dimension d. If $d = 1$ the theorem follows from Proposition 3.5. Hence, assume that the theorem is true for $d - 1 \geqslant 1$. We prove that then it is true for d, as well.

Step 1. We show that if $\hat{f} \in C^{k+\delta}(\partial S_d)$, then there exists a unique

$$\hat{u} \in C^{2+k+\delta}(\partial S_d) \cap C^2_\sharp(\partial S_d)$$

such that

$$\lambda \hat{u} = \Delta|_{\partial S_d} \hat{u} + L_{A,0}|_{\partial S_d} \hat{u} + \hat{f}.$$

We first consider the case $d > 2$. For any $(i, j) \in I \times J$, let $\hat{f}_{i,j}$ be the restriction of $\hat{f}$ to $\partial S^{i,j}_d$. As $\hat{f}_{i,j} \in C^{k+\delta}(\partial S^{i,j}_d)$, $\partial S^{i,j}_d$ is an hypercube of dimension $d - 1$ and the operators $\Delta|_{\partial S^{i,j}_d}$ and $L_{A,0}|_{\partial S^{i,j}_d}$ defined respectively in (3.2) and (3.3) satisfy the hypotheses of the theorem, by the inductive assumption we have that there exists a unique

$$\hat{u}_{i,j} \in C^{2+k+\delta}\big(\partial S^{i,j}_d\big) \cap C^2_\sharp\big(\partial S^{i,j}_d\big)$$

such that

$$\lambda \hat{u}_{i,j} = \Delta|_{\partial S^{i,j}_d} \hat{u}_{i,j} + L_{A,0}|_{\partial S^{i,j}_d} \hat{u}_{i,j} + \hat{f}_{i,j}, \quad \text{in } C^{k+\delta}\big(\partial S^{i,j}_d\big).$$

Now, let $(i, j), (i', j') \in I \times J$ with $i \neq i'$ be such that

$$\partial S^{i,j}_d \cap \partial S^{i',j'}_d \neq \emptyset.$$

In this case $\partial S^{i,j}_d \cap \partial S^{i',j'}_d$ is an hypercube of dimension $d - 2 \geqslant 1$ and on this hypercube $\hat{u}_{i,j}$ and $\hat{u}_{i',j'}$ satisfy the same equation. Hence, by uniqueness

$$\hat{u}_{i,j}|_{\partial S^{i,j}_d \cap \partial S^{i',j'}_d} = \hat{u}_{i',j'}|_{\partial S^{i,j}_d \cap \partial S^{i',j'}_d}.$$

This allows us to define the function $\hat{u} \in C(\partial S_d)$ by setting

$$\hat{u}(x) := \hat{u}_{i,j}(x), \quad x \in \partial S^{i,j}_d.$$

Concerning the function $\hat{u}$, clearly it belongs to $C^{2+k+\delta}(\partial S_d)$. It remains to show that it belongs also to $C^2_\sharp(\partial S_d)$. To this purpose we have to show that

$$D^2_i \hat{u}(x) = D^2_{i'} \hat{u}(x) = 0, \quad x \in \partial S^i_d \cap \partial S^{i'}_d.$$

But this follows from the fact that $\hat{u}_{i,j} \in C^2_\sharp(\partial S^{i,j}_d)$ and $\hat{u}_{i',j'} \in C^2_\sharp(\partial S^{i',j'}_d)$ and these two functions coincide on $S^{i,j}_d \cap S^{i',j'}_d$.

Finally, we consider the case $d = 2$. As above, given $\hat{f} \in C^{k+\delta}(S_2)$ for any $i \in \{1, 2\}$ and $j \in \{0, 1\}$ the functions $\hat{u}_{i,j}$ are well defined in $C^{2+k+\delta}(\partial S^{i,j}_2)$ and satisfy

$$\lambda \hat{u}_{i,j} = \Delta|_{\partial S^{i,j}_2} \hat{u}_{i,j} + L_{A,0}|_{\partial S^{i,j}_2} \hat{u}_{i,j} + \hat{f}_{i,j}.$$

Thus, if we take for example $i = 1$ we obtain

$$\lambda\hat{u}_{1,j}(x) = \left(1 + \frac{1}{2}a_{2,2}(x)\right)D_2^2\hat{u}_{1,j}(x) + \hat{f}_{1,j}(x), \quad x \in \partial S_2^{1,j}.$$

Since $\hat{u}_{1,j} \in C_\sharp^2(\partial S_2)$ we have

$$D_2^2\hat{u}_{1,j}(j,0) = D_2^2\hat{u}_{1,j}(j,1) = 0$$

and hence

$$\lambda\hat{u}_{1,j}(j,j') = \hat{f}_{1,j}(j,j'), \quad j' \in \{0,1\}.$$

As an analogous argument can be used for $i = 2$, this implies that if $\partial S_2^{i,j} \cap \partial S_2^{i',j'} \neq \emptyset$, then $\hat{u}_{i,j}$ and $\hat{u}_{i',j'}$ coincide on this intersection, so that $\hat{u} \in C^{2+k+\delta}(\partial S_2) \cap C_\sharp^2(\partial S_2)$ can be defined as above.

Step 2. If $f \in C^{k+\delta}(S_d)$, let $\hat{f}$ be its restriction to ∂S_d. According to what we have proved in the first step, there exists a unique $\hat{u} \in C^{2+k+\delta}(\partial S_d) \cap C_\sharp^2(\partial S_d)$ such that

$$\lambda\hat{u} = \Delta|_{\partial S_d}\hat{u} + L_{A,0}|_{\partial S_d}\hat{u} + \hat{f}.$$

Now, due to Proposition 2.3 there exists $v \in C^{2+k+\delta}(S_d) \cap C_\sharp^2(S_d)$ such that $\gamma \circ v = \hat{u}$. Thus, if we define

$$g := \lambda v - \Delta v - L_{A,0}v \in C^{k+\delta}(S_d)$$

and apply γ to both sides, from (3.5) and (3.6) we get

$$\gamma \circ g = \lambda(\gamma \circ v) - \gamma \circ (\Delta v - L_{A,0}v) = \lambda\hat{u} - \Delta|_{\partial S_d}\hat{u} + L_{A,0}|_{\partial S_d}\hat{u} = \hat{f} = \gamma \circ f.$$

This means that $\gamma \circ (f - g) = 0$, so that $f - g \in C^{k+\delta}(S_d) \cap C_0(S_d)$. By Propositions 3.7 and 3.8 this implies that there exists a unique $w \in C^{2+k+\delta}(S_d) \cap C_0(S_d) \cap C_\sharp^2(S_d)$ such that

$$\lambda w = \Delta w - L_{A,0}w + (f - g).$$

Therefore the function $u := w + v$ belongs to $C^{2+k+\delta}(S_d) \cap C_\sharp^2(S_d)$ and solves (3.13). Uniqueness follows from Lemma 3.3.

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