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Well-posedness of the martingale problem for some degenerate diffusion processes occurring in dynamics of populations ☆

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Abstract

In this paper we establish the well-posedness in $C([0,\infty);[0,1]^d)$, for each starting point $x \in [0,1]^d$, of the martingale problem associated with a class of degenerate elliptic operators which arise from the dynamics of populations as a generalization of the Fleming–Viot operator. In particular, we prove that such degenerate elliptic operators are closable in the space of continuous functions on $[0,1]^d$ and their closure is the generator of a strongly continuous semigroup of contractions.

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1. Introduction

Let $S_d := [0,1]^d$ and let $\mathcal{L}^+(\mathbb{R}^d)$ be the space of symmetric and non-negative definite $d \times d$-matrices. Moreover, let $A : S_d \to \mathcal{L}^+(\mathbb{R}^d)$ and $b : S_d \to \mathbb{R}^d$ be mappings of class C^2 such that

$$A\nu = 0, \langle b, \nu\rangle \geqslant 0, \quad \text{on } \partial S_d, \tag{1.1}$$

where ν is the unit inward normal at ∂S_d. We denote by $L_{A,b}$ the operator

$$L_{A,b} := \frac{1}{2}\operatorname{Tr}[AD^2] + \langle b, D\rangle, \quad D(L_{A,b}) = C^2(S_d). \tag{1.2}$$

In this paper we prove that the martingale problem associated with the operator $L_{A,b}$ is well posed on $C([0,\infty); S_d)$, in the classical sense of Stroock and Varadhan [15] (see also [10]). As well known, if for any $t \geqslant 0$ we define

$$x(t) : C\big([0,\infty); S_d\big) \to S_d, \quad \omega \mapsto x(t)(\omega) = \omega(t),$$

and if $\mathcal{F}_t$ and $\mathcal{F}$ are the σ-fields generated by $\{x(s),\ 0 \leqslant s \leqslant t\}$ and $\{x(s),\ s \geqslant 0\}$, respectively, then a probability $\mathbb{P}_x$ defined on $(C([0,\infty); S_d), \mathcal{F})$ is a solution of the martingale problem associated with the operator $L_{A,b}$ and starting from $x \in S_d$ at time $t = 0$, if $\mathbb{P}_x(x(0) = x) = 1$ and

$$\left\{ f\big(x(t)\big) - \int_0^t L_{A,b} f\big(x(s)\big)\, ds \right\}_{t \geqslant 0}$$

is a $\mathbb{P}_x$-martingale with respect to $\{\mathcal{F}_t\}_{t\geqslant 0}$.

Our main result is the following.

Theorem 1.1. *Assume that $A : S_d \to \mathcal{L}^+(\mathbb{R}^d)$ and $b : S_d \to \mathbb{R}^d$ are mappings of class C^2 which fulfill* (1.1) *and assume that for any $i, j = 1, \dots, d$*

$$A_{ij}(x) = A_{ij}(x_i, x_j), \quad x \in S_d.$$

Then, the operator $L_{A,b}$ defined in (1.2) *is closable in $C(S_d)$ and its closure is the generator of a C_0-semigroup of contractions in $C(S_d)$.*

This means in particular that the martingale problem associated with $L_{A,b}$ admits a unique martingale solution supported on $C([0,+\infty); S_d)$, for any starting point $x \in \mathbb{R}^d$.

Notice that, as a consequence of Lemma 2.6 below, we can generalize this result to operators

$$L := \frac{1}{2}\operatorname{Tr}[\gamma AD^2] + \langle b, D\rangle, \quad D(L) = C^2(S_d),$$

where A and b are as in the theorem above and γ is a strictly positive continuous function on S_d such that $b/\gamma \in C^2(S_d)$.

The existence of a solution for the martingale problem associated with $L_{A,b}$ is a well known consequence of the properties (1.1), which are necessary and sufficient conditions for stochastic invariance of the hypercube S_d (see [6] for the proof). Actually, if we set, as for example in [9],

$$\gamma(x) = y, \quad \text{if } |x-y| = \inf\{|x-z|, z \in S_d\},$$

we have that $A \circ \gamma : \mathbb{R}^d \to \mathcal{L}^+(\mathbb{R}^d)$ and $b \circ \gamma : \mathbb{R}^d \to \mathbb{R}^d$ are continuous and bounded extensions of A and b, so that the stochastic differential equation

$$dx(t) = (b \circ \gamma)\big(x(t)\big)\,dt + \sqrt{A \circ \gamma}\big(x(t)\big)\,dw_t, \quad x(0) = x,$$

admits a solution $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, w_t, x(t))$, where $x(t)$, $t \geqslant 0$, is a process having a.s. continuous trajectories. Moreover, such a solution lives in S_d. Indeed, if ν^i is the unit inward normal at $\partial S_d^i := \{x_i = 0\} \cup \{x_i = 1\}$, due to (1.1) we have

$$\big\langle b(x), \nu^i \big\rangle \geqslant 0, \quad \big|\sqrt{A}(x)\nu^i\big|^2_{\mathbb{R}^d} = \big\langle A(x)\nu^i, \nu^i \big\rangle = 0, \quad x \in \partial S_d^i. \tag{1.3}$$

Thus, since $\gamma(x) \in \partial S_d^i$, for any $x \in V_i := \{x \in \mathbb{R}^d \colon x_i \in [0,1]^c\}$, if we assume that $x(s) \in V_i$, for any $s \in [t_1, t_2]$, we have

$$\big\langle x(t_2) - x(t_1), \nu^i \big\rangle = \int_{t_1}^{t_2} \big\langle \sqrt{A \circ \gamma}\big(x(s)\big)\nu^i, dw_s \big\rangle + \int_{t_1}^{t_2} \big\langle (b \circ \gamma)\big(x(s)\big), \nu^i \big\rangle\, ds \geqslant 0.$$

This implies that

$$\mathbb{P}\Big(x(s) \in V_i, \text{ for } s \in [t_1, t_2], \text{ and } \big\langle x(t_2) - x(t_1), \nu^i \big\rangle < 0\Big) = 0,$$

so that, if we denote

$$V_{i,0} := \{x \in \mathbb{R}^d \colon x_i < 0\}, \qquad V_{i,1} := \{x \in \mathbb{R}^d \colon x_i > 1\},$$

we have

$$\mathbb{P}\bigg(\bigcup_{\substack{i=1,\dots,d,\, j=0,1 \\ t_2, t_1 \in \mathbb{Q}}} \Big\{x(s) \in V_{i,j}, \text{ for } s \in [t_1, t_2], \text{ and } \big\langle x(t_2) - x(t_1), \nu^{i,j} \big\rangle < 0\Big\}\bigg) = 0,$$

where $\nu^{i,j}$ is the inward normal at $\partial S_d^{i,j} := \{x_i = j\}$. Therefore, due to the continuity of the trajectories of $x(t)$, it is possible to conclude that

$$\mathbb{P}\big(x(s) \in S_d^c, \text{ for some } s > 0\big) = 0.$$

This means that the $\mathbb{P}$-distribution of $\{x(t), t \geqslant 0\}$ is supported on $C([0,\infty); S_d)$ and from the Itô's formula we have that it solves the martingale problem associated with $L_{A,b}$ and starting from $x \in S_d$ at time $t = 0$.

The new part concerns the uniqueness of the martingale solution. The degeneracy of diffusion coefficients implies that it is impossible to apply directly the classical results by Stroock and Varadhan (see [15]) in order to establish uniqueness of solutions for the martingale problem. Clearly, if one could prove pathwise uniqueness, then uniqueness in law would follow and this would be equivalent to uniqueness of the martingale problem. But here the square root of the diffusion matrix A is not assumed to be Lipschitz-continuous and nothing can be said in general about pathwise uniqueness.

To this purpose, in the present paper we prove uniqueness in law by showing that the operator $L_{A,b}$ is closable in the space of continuous functions $C(S_d)$ and its closure is the generator of a C_0-semigroup of contractions on $C(S_d)$. By general results (see for example [10, Theorem 4.4.1]), this implies that for any $x \in S_d$ the martingale problem associated with $L_{A,b}$ admits a unique solution supported on $C([0,\infty); S_d)$ and starting from x.

The study of such a class of degenerate problems in non-smooth domains arises in the theory of measure valued diffusion processes describing some dynamics of populations and in general in the theory of superprocesses (see [7] and the more recent [13] for an introduction to superprocesses). Actually, many discrete stochastic models in population genetics can be closely approximated by diffusions belonging to the class which we are considering. Probably, the most well known example is the Fleming–Viot process, whose generator, in the case of a population with a finite number of different types, is given by

$$Lu(x) = \frac{1}{2}\,\mathrm{Tr}\big[C(x)D^2u(x)\big] + \big\langle Qx, Du(x)\big\rangle, \quad x \in S, \tag{1.4}$$

where $S := \{x \in [0,1]^d\colon \sum_{i=1}^d x_i \leqslant 1\}$ is the symplex of probability measures on the finite set $E := \{0, 1, \dots, d\}$, C is the Fleming–Viot matrix

$$C_{ij}(x) = x_i(\delta_{ij} - x_j), \quad x \in S,$$

and Q is the infinitesimal matrix of a Markov process on E (for a review on Fleming–Viot processes, in the case of a finite or infinite number of types, see [10] and [11]).

In [8], by using some perturbation techniques in a infinite-dimensional setting, Dawson and March have considered the case of the Fleming–Viot diffusion term multiplied by a function γ which is very close to constant and have proved that also in this case uniqueness for the martingale problem holds. But it does not seem possible to adapt their arguments to more general situations.

In [9], Ethier has proved the existence and uniqueness of solutions for the martingale problem associated with a class of degenerate operators on S which generalize the classical Fleming–Viot operator (1.4). Namely, he has considered the following class of operators

$$Lu(x) = \frac{1}{2}\,\mathrm{Tr}\big[C(x)D^2u(x)\big] + \big\langle b(x), Du(x)\big\rangle, \quad x \in S, \tag{1.5}$$

for general vector fields b of class C^2, which satisfy the same invariance property which we are assuming in (1.1), that is

$$\big\langle b(x), \nu(x)\big\rangle \geqslant 0, \quad x \in \partial S.$$

In the present paper we go farther on generalizing and instead of C we consider a wider class of diffusion coefficients A which can be written as

$$A(x) = \gamma(x)\hat{A}(x), \quad x \in S_d,$$

for some strictly positive function γ of class C^2 and some matrix valued function $\hat{A}$ of class C^2 fulfilling the invariance property (1.1) and such that

$$\hat{A}_{ij}(x) = \hat{A}_{ij}(x_i, x_j), \quad x \in S_d.$$

It is important to stress that the step from the Fleming–Viot matrix C to more general diffusions coefficients A is not a straightforward generalization. Actually, the matrix C

has the important property that if $p(x)$ belongs to $\mathcal{P}_k(S)$, the subspace of polynomials of degree less or equal to k, then $\mathrm{Tr}[C(x)D^2p(x)] \in \mathcal{P}_k(S)$. Then, as the operator $\mathrm{Tr}[CD^2u]$ is dissipative in $C(S)$, for any $\lambda > 0$ and $k \in \mathbb{N}$ the mapping

$$\lambda I - \mathrm{Tr}[CD^2] : \mathcal{P}_k(S) \to \mathcal{P}_k(S),$$

is a bijection. This means that the image of $\lambda I - \mathrm{Tr}[CD^2]$ is dense in $C(S_d)$ and hence, by an operator sums argument, Ethier can conclude that the closure of the operator L defined in (1.5) generates a strongly continuous semigroup of contractions in $C(S)$.

Clearly these arguments are no more valid, as soon as one perturbs the matrix-valued mapping C, so that interesting discrete stochastic models in population genetics cannot be covered by the paper of Ethier [9]. The aim of our paper is precisely trying to cover this gap.

A similar aim has also motivated the papers [1] and [2], by Athreya, Barlow, Bass and Perkins and Bass and Perkins, respectively. In these two articles they prove the uniqueness of the martingale problem associated with some degenerate operators acting on C^2 functions defined on the non-negative orthant $\mathbb{R}^d_+$. In the first one [1] the following class of generators is studied

$$Lu(x) = \sum_{i=1}^{d} x_i \gamma_i(x) D_i^2 u(x) + \sum_{i=1}^{d} b_i(x) D_i u(x), \quad x \in \mathbb{R}^d_+,$$

where γ_i are continuous functions which are strictly positive on the whole orthant and b_i are continuous functions which have linear growth and are strictly positive on the boundary $\partial\mathbb{R}^d_+$. In the last section of the paper a one-dimensional counterexample is also given, which shows that the condition $b > 0$ cannot be weakened, if b is only continuous. To this purpose, notice that in our paper the drift b fulfills the more general stochastic invariance condition (1.1), but this does not contradict the counterexample in [1] as we are assuming A and b to be of class C^2. In the second paper [2] the following more general class is studied

$$Lu(x) = \sum_{i,j=1}^{d} \sqrt{x_i x_j}\, \gamma_{ij}(x) D_{ij}^2 u(x) + \sum_{i=1}^{d} b_i(x) D_i u(x), \quad x \in \mathbb{R}^d_+,$$

where the matrix $\gamma_{ij}(x)$ is positive definite and the off-diagonal terms are small for $x \in \partial\mathbb{R}^d_+$. Here, in contrast to previous work, the b_i need only be non-negative on the boundary, but both γ_{ij} and b_i have to be taken Hölder continuous.

In the present paper, our idea has been of approximating the problem

$$\lambda u(x) - L_{A,b}u(x) = f(x), \quad x \in S_d, \tag{1.6}$$

by the uniformly elliptic problems

$$\begin{cases} \lambda u(x) - \varepsilon \Delta u(x) - L_{A,b}u(x) = f(x), & x \in S_d, \\ D_i^2 u(x) = 0, & x \in \partial S_d^i,\ i = 1, \dots, d, \end{cases} \tag{1.7}$$

where ε is a positive constant. In [4] we have proved that if A and b fulfill the invariance property (1.1) and are of class $C^{k+\delta}$, with $k = 0, 1$ and $\delta \in (0,1)$, then for any $f \in C^{k+\delta}(S_d)$ there exists a unique solution $u_\varepsilon \in C^{2+k+\delta}(S_d)$ to problem (1.7).

In the present paper we prove that the operator $\varepsilon\Delta + L_{A,b}$, defined on suitable domains, is quasi-dissipative both in $C^1(S_d)$ and in $C^2(S_d)$, uniformly with respect to $\varepsilon > 0$. This provides uniform estimates in $C^1(S_d)$ and in $C^2(S_d)$ for the solutions u_ε of (1.7), and these uniform estimates allow in their turn to prove that the image of $\lambda I - L_{A,b}$ is dense in $C(S_d)$. Notice that if A, b and f are of class C^2, the solution u_ε to problem (1.7) belongs to $C^3(S_d)$ and then we can differentiate once both sides in (1.7). This allows to obtain the equation fulfilled by the first derivative of u_ε and to prove the quasi-dissipativity in $C^1(S_d)$. For the proof of quasi-dissipativity in $C^2(S_d)$ things are more difficult. Actually, as u_ε is only of class C^3, we cannot differentiate twice in (1.7) in order to have the equation fulfilled by the second derivative and this forces us to pass through a weak formulation of problems (1.6) and (1.7).

The paper is organized as follows. In Section 2 we introduce notations and assumptions on the coefficients A and b and some preliminary results on dissipative operators. In Section 3 we study a suitable generalization of the weak problem associated with equation (1.6). Namely, we show that a maximum principle for the weak problem can be proved also in this degenerate case. In Section 4 we prove the key results of the paper. Actually, we introduce the approximating problem (1.7) and we prove a priori estimates for the solutions u_ε, both in C^1 and in C^2, which are uniform with respect to $\varepsilon > 0$. This allows us to prove in Section 5 the full range condition for the closure of the operator $L_{A,b}$ in $C(S_d)$ and, as the main consequence, the uniqueness of the martingale problem associated with the operator $L_{A,b}$. Moreover, we obtain the existence and uniqueness of a C^1-solution for the weak version of problem (1.6).

2. Notations and preliminaries

Throughout the present paper, for any $d \geqslant 1$ we denote by S_d the set $[0,1]^d$. For any $i \in I := \{1, \dots, d\}$ and $j \in J := \{0, 1\}$ we set

$$\partial S_d^{i,j} := \{x \in S_d\colon\ x_i = j\}, \qquad \partial S_d^i := \partial S_d^{i,0} \cup \partial S_d^{i,1}.$$

It follows that for any $(i, j) \in I \times J$ the set $\partial S_d^{i,j}$ is a hypercube of dimension $d-1$ and

$$\bigcup_{i=1}^{d} \partial S_d^i = \partial S_d.$$

2.1. *Function spaces on S_d*

In what follows we shall denote by $C(S_d)$ the Banach space of continuous functions on S_d, endowed with the supremum norm $\|\cdot\|_{C(S_d)}$, and by $C_0(S_d)$ the subspace of continuous functions which vanish at the boundary. For any $\delta \in (0,1)$ we shall denote by $C^\delta(S_d)$ the Banach space of uniformly δ-Hölder continuous functions, endowed with the norm

$$\|u\|_{C^\delta(S_d)} := \|u\|_{C(S_d)} + \sup_{\substack{x,y \in S_d \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\delta} =: \|u\|_{C(S_d)} + [u]_{C^\delta(S_d)}.$$

Next, for any integer $k \geqslant 1$ and any $\delta \in [0,1)$ we shall denote by $C^{k+\delta}(S_d)$ the space of functions $u \in C(S_d) \cap C^k(S_d^\circ)$ such that

$$D^\alpha u \text{ is uniformly } \begin{cases} \text{continuous} & \text{if } \delta = 0, \\ \delta\text{-Hölder continuous} & \text{if } \delta \in (0,1) \end{cases} \quad \text{on } S_d^\circ,$$

for any multi-index α, with $|\alpha| = k$. In particular, for any α, with $1 \leqslant |\alpha| \leqslant k$, the derivatives $D^\alpha u$ can be extended by continuity to uniformly δ-Hölder continuous functions defined up to the boundary of S_d. $C^{k+\delta}(S_d)$ is a Banach space endowed with the norm

$$\|u\|_{C^k(S_d)} := \|u\|_{C(S_d)} + \sum_{|\alpha|=k} \sup_{x \in S_d} \left|D^\alpha u(x)\right|, \quad \text{if } \delta = 0,$$

and

$$\|u\|_{C^{k+\delta}(S_d)} := \|u\|_{C^k(S_d)} + \sum_{|\alpha|=k} \sup_{\substack{x,y \in S_d \\ x \neq y}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\delta}, \quad \text{if } \delta > 0.$$

Notice that, as a consequence of the Whitney extension theorem, for any $u \in C(S_d)$ we have that $u \in C^{k+\delta}(S_d)$ if and only if there exists $v \in C^{k+\delta}(\mathbb{R}^d)$ such that $u = v_{|S_d}$ (for a proof see e.g. [10, Theorem 6.1 in Appendix 6]).

Finally, we define

$$C^2_\sharp(S_d) := \bigcap_{i=1}^d \left\{u \in C^2(S_d)\colon\ D_i^2 u(x) = 0,\, x \in \partial S_d^i\right\}. \tag{2.1}$$

2.2. Dissipative operators

We recall here some basic facts about dissipative operators which will be used in what follows (for all details and proofs see e.g. [5, Section 3.3] and [14]).

Let E be a real Banach space, endowed with the norm $\|\cdot\|_E$ and the duality $\langle\cdot,\cdot\rangle_E$ between E and $E^\star$. A linear operator $M : D(M) \subseteq E \to E$ is *dissipative* if for any $f \in D(M)$ and $\lambda > 0$

$$\|f\|_E \leqslant \frac{1}{\lambda}\|\lambda f - Mf\|_E \tag{2.2}$$

or, equivalently, if for any $f \in D(M)$ there exists

$$f^\star \in \partial\|f\|_E := \left\{g^\star \in E^\star\colon\ \|g^\star\|_{E^\star} = 1, \langle f, g^\star\rangle_E = \|f\|_E\right\}$$

such that

$$\langle Af, f^\star\rangle_E \leqslant 0. \tag{2.3}$$

Moreover, M is *quasi-dissipative* if there exists some $\lambda_0 > 0$ such that $M - \lambda_0$ is dissipative. Finally, M is *strongly dissipative* if (2.3) holds for any $f \in D(M)$ and $f^\star \in \partial\|f\|_E$.

Clearly a strongly dissipative operator is a dissipative operator, which in turn is a quasi-dissipative operator.

Theorem 2.1 [5, Theorem 3.14]. *Let $M : D(M) \subseteq E \to E$ and $N : D(N) \subseteq E \to E$ be two linear operators.*

(1) *If M is dissipative and $\overline{D(M)} \supseteq \mathrm{Range}(M)$, then M is strongly dissipative.*
(2) *If M is strongly dissipative and N is dissipative, then $M + N : D(M) \cap D(N) \to E$ is dissipative.*

In particular, from the previous theorem we immediately have that if M is dissipative and densely defined and N is dissipative, then the sum $M + N$, defined on $D(M + N) = D(M) \cap D(N)$, is dissipative.

Proposition 2.2 [5, Proposition 3.7]. *Let $M : D(M) \subseteq E \to E$ be a dissipative operator.*

(1) *If $\overline{D(M)} \supseteq \mathrm{Range}(M)$, in particular if M is densely defined, then M is closable.*
(2) *If M is closable, then its closure is also dissipative.*

Proposition 2.3 [5, Proposition 3.10]. *Let $M : D(M) \subseteq E \to E$ be a dissipative operator with dense domain. Then the following assertions are equivalent.*

(1) $\mathrm{Range}(\lambda - M)$ *is dense in E, for any $\lambda > 0$.*
(2) $\mathrm{Range}(\lambda_0 - M)$ *is dense in E, for some $\lambda_0 > 0$.*
(3) *M is essentially m-dissipative, that is its closure $\overline{M}$ is m-dissipative. More precisely, $\overline{M}$ is dissipative and* $\mathrm{Range}(\lambda_0 - \overline{M}) = E$, *for some $\lambda_0 > 0$ (and hence for all $\lambda > 0$).*

Finally, we recall the Lumer–Phillips theorem on the generation of C_0-semigroups of contractions by dissipative operators.

Theorem 2.4 [10, Theorem 1.2.12]. *A linear operator $M : D(M) \subseteq E \to E$ is closable and its closure $\overline{M}$ is the generator of a strongly continuous contractions semigroup on E if and only if $D(M)$ is dense in E, M is dissipative and* $\mathrm{Range}(\lambda_0 - M)$ *is dense in E, for some $\lambda_0 > 0$.*

Remark 2.5.

(1) As proved e.g. in [5, Proposition 3.8], a m-dissipative operator M is *maximal* dissipative, i.e. if N is dissipative and $M \subseteq N$, then $M = N$. In particular, this means that if M is an operator fulfilling the conditions of Theorem 2.4, then its closure is its only dissipative extension and there is uniqueness of the generated semigroup.
(2) Theorems 2.1 and 2.4 and Propositions 2.2 and 2.3 are still valid if we replace dissipativity with quasi-dissipativity. The only difference is that the conditions on the range holds only for $\lambda > \lambda_0$, for some $\lambda_0 > 0$, and the Lumer–Philips theorem provides a C_0-semigroup which is not of contraction.

We conclude this subsection recalling a useful result whose proof can be found in [3] (see Lemma 2.2).

Lemma 2.6. *Let $C(K)$ be the Banach space of continuous functions defined on the compact set K, equipped with the supremum norm, and let A be a m-dissipative operator on $C(K)$. Moreover, let B the operator defined by*

$$D(B) = D(A), \quad B = GA,$$

where for any $u : K \to \mathbb{R}$

$$(Gu)(x) = \gamma(x)u(x), \quad x \in K,$$

for some continuous and strictly positive function γ defined on K.

Then, if B is dissipative we have that B is m-dissipative.

2.3. Assumptions on A and b

We conclude this section describing the conditions imposed on the coefficients A and b of the operator

$$L_{A,b}u := \frac{1}{2}\operatorname{Tr}[AD^2u] + \langle b, Du\rangle, \quad u \in D(L_{A,b}) := C^2(S_d).$$

Throughout the paper for any $i \in I$ we shall denote by $\nu^i(x)$ the unit inward normal at the point $x \in \partial S_d^i$, that is

$$\nu^i(x) := (-1)^j e_i, \quad x \in \partial S_d^{i,j}.$$

Definition 2.7.

(1) A mapping $A = [a_{h,k}] : S_d \to \mathcal{L}^+(\mathbb{R}^d)$ belongs to $\mathcal{H}_A(S_d)$ if it is continuous and for any $i \in I$

$$A(x)\nu^i(x) = 0, \quad x \in \partial S_d^i.$$

(2) A mapping $A = [a_{h,k}] : S_d \to \mathcal{L}^+(\mathbb{R}^d)$ belongs to $\mathcal{K}_A(S_d)$ if it is continuous and for any $h, k \in I$

$$a_{h,k}(x) = a_{h,k}(x_h, x_k), \quad x \in S_d.$$

(3) A mapping $b : S_d \to \mathbb{R}^d$ belongs to $\mathcal{H}_b(S_d)$ if is continuous and for any $i \in I$

$$\big\langle b(x), \nu^i(x)\big\rangle \geqslant 0, \quad x \in \partial S_d^i.$$

In particular, $A \in \mathcal{H}_A(S_d)$ if and only if for any $i \in I$

$$a_{h,i}(x) = a_{i,h}(x) = 0, \quad x \in \partial S_d^i,\ h \in I, \tag{2.4}$$

and $A \in \mathcal{K}_A(S_d) \cap C^r(S_d; \mathcal{L}^+(\mathbb{R}^d))$ if and only if for any $h, k \in I$

$$i_1, \dots, i_r \notin \{h, k\} \quad \Rightarrow \quad D_{i_1,\dots,i_r} a_{h,k}(x) = 0, \quad x \in S_d. \tag{2.5}$$

Moreover, $b \in \mathcal{H}_b(S_d)$ if and only if for any $(i, j) \in I \times J$

$$(-1)^j b_i(x) \geqslant 0, \quad x \in \partial S_d^{i,j}. \tag{2.6}$$

From (2.6) we get the following equivalent definition of $\mathcal{H}_b(S_d)$.

Lemma 2.8. *A continuous mapping $b : S_d \to \mathbb{R}^d$ belongs to $\mathcal{H}_b(S_d)$ if and only if for any $x \in \partial S_d$ there exists $\rho_x > 0$ such that*

$$x + \rho b(x) \in S_d, \quad 0 \leqslant \rho \leqslant \rho_x. \tag{2.7}$$

Proof. Assume first that $b \in \mathcal{H}_b(S_d)$ and fix $x \in \partial S_d$. We have to show that there exists $\rho_x > 0$ such that for any $i \in I$ and $0 \leqslant \rho \leqslant \rho_x$

$$x_i + \rho b_i(x) \in [0, 1]. \tag{2.8}$$

If $x_i \in (0, 1)$, there clearly exists $\rho_x^i > 0$ such that (2.8) holds for any $0 \leqslant \rho \leqslant \rho_x^i$. If $x_i = 0$, then due to (2.6) we have that $b_i(x) \geqslant 0$ and then we can find $\rho_x^i > 0$ such that $x_i + \rho b_i(x) = \rho b_i(x) \in [0, 1]$, for any $0 \leqslant \rho \leqslant \rho_x^i$. Analogously, if $x_i = 1$ due to (2.6) we have $b_i(x) \leqslant 0$ and then we can find $\rho_x^i > 0$ such that $x_i + \rho b_i(x) = 1 + \rho b_i(x) \in [0, 1]$, for any $0 \leqslant \rho \leqslant \rho_x^i$. By taking $\rho_x := \min_{i \in I} \rho_x^i$ we obtain (2.7).

Viceversa, assume that (2.7) holds. If $x \in \partial S_d^{i,j}$ there exists $\rho_x > 0$ such that

$$j + \rho b_i(x) \in [0, 1], \quad 0 \leqslant \rho \leqslant \rho_x.$$

In particular, if $j = 0$ we have $\rho b_i(x) \geqslant 0$, so that $(-1)^j b_i(x) = b_i(x) \geqslant 0$, and if $j = 1$ we have $1 + \rho b_i(x) \leqslant 1$, so that $(-1)^j b_i(x) = -b_i(x) \geqslant 0$. Due to (2.6) this implies that $b \in \mathcal{H}_b(S_d)$. □

3. The weak formulation

We start with the following integration by parts formula.

Lemma 3.1. *Assume that $A \in \mathcal{H}_A(S_d) \cap C^1(S_d; \mathcal{L}^+(\mathbb{R}^d))$. Then for any $u \in C^2(S_d)$ and $\varphi \in W^{1,\infty}(S_d)$ we have*

$$\int_{S_d} (L_{A,0}u)\varphi \, dx = -\frac{1}{2} \int_{S_d} \big[\langle ADu, D\varphi\rangle + \langle DA, Du\rangle\varphi\big] \, dx, \tag{3.1}$$

where the vector field DA is defined by

$$\big[DA(x)\big]_i := \sum_{j=1}^{d} D_j a_{i,j}(x), \quad x \in S_d, \ i \in I.$$

Proof. For any $i, j \in I$ we have

$$D_i(a_{i,j}\varphi D_j u) = a_{i,j} D_i\varphi D_j u + a_{i,j}\varphi D_{ij} u + D_i a_{i,j}\varphi D_j u.$$

Then, summing over $i, j \in I$, we get

$$\sum_{i=1}^{d} D_i\left(\sum_{j=1}^{d} a_{i,j}\varphi D_j u\right) = \langle ADu, D\varphi\rangle + 2(L_{A,0}u)\varphi + \langle DA, Du\rangle\varphi. \tag{3.2}$$

Thanks to (2.4) we easily have

$$\int_{S_d} D_i\left(\sum_{j=1}^d a_{i,j}\varphi D_j u\right)dx = \int_{S_{d-1}}\int_0^1 D_i\left(\sum_{j=1}^d a_{i,j}\varphi D_j u\right)dx_i\,dy = 0,$$

and then, integrating on S_d both sides in (3.2) we obtain (3.1). □

Remark 3.2. With the same arguments used above we have that formula (3.1) is still valid if we do not assume that $A \in \mathcal{H}_A(S_d)$, but we take $\varphi \in W^{1,\infty}(S_d) \cap C_0(S_d)$.

The previous lemma leads us to the following definition.

Definition 3.3. A function $u \in C^1(S_d)$ is a weak solution of problem (1.6) if for any $\varphi \in W^{1,\infty}(S_d)$

$$\int_{S_d}\left[\lambda u + \frac{1}{2}\langle ADu, D\varphi\rangle + \frac{1}{2}\langle DA, Du\rangle\varphi - \langle b, Du\rangle\varphi\right]dx = \int_{S_d} f\varphi\,dx. \tag{3.3}$$

Due to (3.1), if $A \in \mathcal{H}_A(S_d) \cap C^1(S_d; \mathcal{L}^+(\mathbb{R}^d))$ a $C^2(S_d)$ solution is clearly a weak solution. On the other hand, if u is a weak solution and belongs to $C^2(S_d)$ it is immediate to check that it is a classical solution.

In what follows we shall prove that a maximum principle holds for the weak problem.

Lemma 3.4. *Let us fix any $B \in C^1(S_d; \mathcal{L}^+(\mathbb{R}^d))$, $b \in \mathcal{H}_b(S_d)$, $c \in C(S_d)$ and assume that $u \in C^1(S_d)$ satisfies*

$$\int_{S_d}\left[cu\varphi + \langle BDu, D\varphi\rangle + \langle DB - b, Du\rangle\varphi\right]dx \geqslant \int_{S_d} f\varphi\,dx,$$

for any $\varphi \in W^{1,\infty}(S_d) \cap C_0(S_d)$, with $\varphi \geqslant 0$ on S_d. Moreover, assume that there exists $x_0 \in S_d^\circ$ such that

$$\min_{x\in S_d} u(x) = u(x_0), \qquad f(x_0) \geqslant 0.$$

Then

$$\min_{x\in S_d} c(x) > 0 \quad\Rightarrow\quad \min_{x\in S_d} u(x) \geqslant 0.$$

Proof. For any $\rho > 0$ we define

$$u_\rho(x) := u(x) + \rho|x - x_0|^2 =: u(x) + \rho h(x), \qquad x \in S_d.$$

Notice that $u_\rho(x) \geqslant u(x)$, for any $x \in S_d$, and hence

$$u(x_0) = u_\rho(x_0) = \min_{x\in S_d} u_\rho(x).$$

As $h \in C^2(\mathbb{R}^d)$, due to (3.1) and to Remark 3.2 for any $\varphi \in W^{1,\infty}(S_d) \cap C_0(S_d)$ we have

$$\int_{S_d} (L_{2B,0}h)\varphi\,dx = -\int_{S_d} \big[\langle BDh, D\varphi\rangle + \langle DB, Dh\rangle\varphi\big]\,dx.$$

This means that

$$\int_{S_d} \big[cu_\rho\varphi + \langle BDu_\rho, D\varphi\rangle + \langle DB - b, Du_\rho\rangle\varphi\big]\,dx \geqslant \int_{S_d} \big[f + \rho(ch - L_{2B,b}h)\big]\varphi\,dx.$$

Thus, setting

$$m := \min_{x \in S_d}\big[c(x)h(x) - L_{2B,b}h(x)\big],$$

for any $\varphi \geqslant 0$ we obtain

$$\int_{S_d} \big[cu_\rho\varphi + \langle BDu_\rho, D\varphi\rangle + \langle DB - b, Du_\rho\rangle\varphi\big]\,dx \geqslant \int_{S_d} f\varphi\,dx + m\rho\int_{S_d}\varphi\,dx. \tag{3.4}$$

Next, for any $\rho > 0$ and $n \in \mathbb{N}$ we define

$$\psi_{\rho,n}(x) := \left[u_\rho(x_0) + \frac{1}{n} - u_\rho(x)\right]^+, \quad x \in S_d.$$

It is immediate to check that $\psi_{\rho,n} \in \mathrm{Lip}(S_d)$ and

$$|\psi_{\rho,n}|_{\mathrm{Lip}(S_d)} \leqslant |u_\rho|_{\mathrm{Lip}(S_d)}. \tag{3.5}$$

Moreover, as proved e.g. in [12, Lemma 7.6], if we define

$$\Lambda_{\rho,n} := \left\{x \in S_d\colon\ u_\rho(x) < u_\rho(x_0) + \frac{1}{n}\right\}, \quad n \in \mathbb{N},\ \rho > 0,$$

we have

$$D\psi_{\rho,n}(x) = \begin{cases} -Du_\rho(x) & x \in \Lambda_{\rho,n}, \\ 0 & x \notin \Lambda_{\rho,n}. \end{cases}$$

Now, as $\psi_{\rho,n}(x_0) = 1/n > 0$ and $\psi_{\rho,n} \geqslant 0$, we can define

$$\varphi_{\rho,n}(x) := \frac{1}{z_{\rho,n}}\psi_{\rho,n}(x), \quad x \in S_d,$$

where

$$z_{\rho,n} := \int_{S_d} \psi_{\rho,n}\,dx.$$

Clearly we have $\varphi_{\rho,n} \in W^{1,\infty}(S_d)$, $\varphi_{\rho,n} \geqslant 0$ and

$$\int_{S_d} \varphi_{\rho,n}\, dx = 1.$$

Moreover it is easy to see that

$$\operatorname{supp} \varphi_{\rho,n} = \operatorname{supp} \psi_{\rho,n} = \overline{\Lambda}_{\rho,n} \subseteq \overline{B}\left(x_0, \frac{1}{\sqrt{\rho n}}\right), \quad \rho > 0,\ n \in \mathbb{N},$$

and

$$D\varphi_{\rho,n}(x) = \begin{cases} -\frac{1}{z_{\rho,n}} Du_\rho(x) & x \in \Lambda_{\rho,n} \\ 0 & x \notin \Lambda_{\rho,n}. \end{cases} \tag{3.6}$$

In particular, since $x_0 \in S_d^\circ$ we have that for any $\rho > 0$ there exists some $n_0 = n(\rho)$ such that

$$\operatorname{supp} \varphi_{\rho,n} \subseteq \overline{B}\left(x_0, \frac{1}{\sqrt{\rho n}}\right) \Subset S_d^\circ, \quad n \geqslant n_0,$$

so that $\varphi_{\rho,n} \in C_0(S_d)$.

Next, by adding and subtracting terms, for any $n \geqslant n_0$ we obtain

$$\begin{aligned} u_\rho(x_0) \int_{S_d} c\varphi_{\rho,n}\, dx &= \int_{S_d} c\big(u_\rho(x_0) - u_\rho\big)\varphi_{\rho,n}\, dx \\ &\quad + \int_{S_d} \big[cu_\rho\varphi_{\rho,n} + \langle BDu_\rho, D\varphi_{\rho,n}\rangle + \langle DB - b, Du_\rho\rangle\varphi_{\rho,n}\big]\, dx \\ &\quad - \int_{S_d} \langle BDu_\rho, D\varphi_{\rho,n}\rangle\, dx - \int_{S_d} \langle DB - b, Du_\rho\rangle\varphi_{\rho,n}\, dx \\ &=: I_1^n + I_2^n + I_3^n + I_4^n. \end{aligned}$$

Our aim now is estimating separately each term. We have

$$I_1^n \geqslant -\|c\|_{C(S_d)} |u_\rho|_{\mathrm{Lip}(S_d)} \int_{\overline{B}(x_0, 1/\sqrt{\rho n})} |x_0 - x| \varphi_{\rho,n}\, dx,$$

so that

$$\liminf_{n\to+\infty} I_1^n \geqslant 0. \tag{3.7}$$

Moreover, as $\varphi_{\rho,n} \geqslant 0$, in view of (3.4) we have

$$I_2^n \geqslant \int_{S_d} f\varphi_{\rho,n}\, dx + m\rho \int_{S_d} \varphi_{\rho,n}\, dx, \quad n \geqslant n_0,$$

and then

$$\liminf_{n\to+\infty} I_2^n \geqslant f(x_0) + m\rho. \tag{3.8}$$

According to (3.6) we have

$$I_3^n = -\int\limits_{\Lambda_{\rho,n}} \langle BDu_\rho, D\varphi_{\rho,n}\rangle\,dx = \frac{1}{z_{\rho,n}}\int\limits_{\Lambda_{\rho,n}} \langle BDu_\rho, Du_\rho\rangle\,dx \geqslant 0. \tag{3.9}$$

Finally,

$$I_4^n = -\int\limits_{S_d} \big\langle (DB-b)(x_0), Du_\rho\big\rangle \varphi_{\rho,n}\,dx + \int\limits_{S_d} \big\langle (DB-b)(x_0) - (DB-b), Du_\rho\big\rangle \varphi_{\rho,n}\,dx.$$

Concerning the first term in I_4^n, we note that due to (3.6) for any $\xi \in \mathbb{R}^d$ we have

$$\langle \xi, Du_\rho\rangle\varphi_{\rho,n} = -z_{\rho,n}\langle \xi, D\varphi_{\rho,n}\rangle\varphi_{\rho,n} = -\frac{z_{\rho,n}}{2}\langle \xi, D\varphi_{\rho,n}^2\rangle.$$

Then, as $\varphi_{\rho,n} \in C_0(S_d)$, integrating we get

$$\int\limits_{S_d} \langle \xi, Du_\rho\rangle\varphi_{\rho,n}\,dx = -\frac{z_{\rho,n}}{2}\int\limits_{S_d} \langle \xi, D\varphi_{\rho,n}^2\rangle\,dx = 0,$$

so that

$$I_4^n \geqslant -|u_\rho|_{\mathrm{Lip}(S_d)} \int\limits_{\overline{B}(x_0, 1/\sqrt{\rho n})} \big|(DB-b)(x_0) - (DB-b)\big|\varphi_{\rho,n}\,dx.$$

As $DB - b$ is continuous at x_0, this allows us to conclude that

$$\liminf_{n\to+\infty} I_4^n \geqslant 0 \tag{3.10}$$

and collecting together (3.7), (3.8), (3.9) and (3.10) we get

$$f(x_0) + m\rho \leqslant \liminf_{n\to+\infty} u_\rho(x_0)\int\limits_{S_d} c\varphi_{\rho,n}\,dx \leqslant u_\rho(x_0)\|c\|_{C(S_d)}.$$

Thus, as $u(x_0) = u_\rho(x_0)$ and $f(x_0) \geqslant 0$, we obtain

$$u(x_0) \geqslant \|c\|_{C(S_d)}^{-1}\big(f(x_0) + m\rho\big) \geqslant \|c\|_{C(S_d)}^{-1} m\rho,$$

and hence, by taking the limit as ρ tends to zero, we get $u(x_0) \geqslant 0$. □

Remark 3.5. Let $u \in C^2(S_d)$ be a solution of the problem

$$\lambda u = \varepsilon\Delta u + \frac{1}{2}\mathrm{Tr}[AD^2u] + \langle b, Du\rangle + cu + f,$$

for some $\lambda > 0$, $\varepsilon \geqslant 0$, $A \in C(S^d; \mathcal{L}(\mathbb{R}^d))$, $b \in C(S_d; \mathbb{R}^d)$ and $c, f \in C(S_d)$.

If we set

$$u_\pi(x_1, \dots, x_d) := u(x_{\pi(1)}, \dots, x_{\pi(d)}), \quad x \in S_d,$$

where π is a permutation of the set I, then clearly $u_\pi \in C^2(S_d)$ and solves the problem

$$\lambda u = \varepsilon \Delta u + \frac{1}{2}\operatorname{Tr}[A_\pi D^2 u] + \langle b_\pi, Du\rangle + c_\pi u + f_\pi,$$

where the coefficients A_π, b_π and c_π and the datum f_π are obtained as u_π by permuting the coordinates.

Similarly, if we set for $1 \leqslant l \leqslant d$

$$u_l(x_1, \dots, x_d) = u(1 - x_1, \dots, 1 - x_l, x_{l+1}, \dots, x_d), \quad x \in S_d,$$

we have that $u_l \in C^2(S_d)$ solves the problem

$$\lambda u = \varepsilon \Delta u + \frac{1}{2}\operatorname{Tr}[A_l D^2 u] + \langle b_l, Du\rangle + c_l u + f_l,$$

where the coefficients A_l and c_l and the datum f_l are obtained as u_l and

$$\begin{cases} (b_l)_k(x_1, \dots, x_d) = -b_k(1 - x_1, \dots, 1 - x_l, x_{l+1}, \dots, x_d), \quad 1 \leqslant k \leqslant l, \\ (b_l)_k(x_1, \dots, x_d) = b_k(1 - x_1, \dots, 1 - x_l, x_{l+1}, \dots, x_d), \\ \quad l \leqslant d-1,\ l+1 \leqslant k \leqslant d. \end{cases}$$

It is immediate to check that if $A \in \mathcal{H}_A(S_d)$ (respectively $\mathcal{K}_A(S_d)$), then $A_\pi, A_l \in \mathcal{H}_A(S_d)$ (respectively $\mathcal{K}_A(S_d)$) and if $b \in \mathcal{H}_b(S_d)$, then $b_\pi, b_l \in \mathcal{H}_b(S_d)$. Moreover, if $u \in C^2_\sharp(S_d)$, then $u_\pi, u_l \in C^2_\sharp(S_d)$.

Therefore, without loss of generality if $x \in \partial S_d$ in what follows we can always make a permutation π and reflections $x_k \mapsto 1 - x_k$, $k \in I$, in such a way that either $x = 0$ or

$$x \in \bigcap_{k=l+1}^{d} \partial S_d^{k,0} \quad \text{and} \quad x \notin \bigcup_{\substack{k=1,\dots,l \\ j=0,1}} \partial S_d^{k,j}, \quad 1 \leqslant l \leqslant d-1.$$

This means that in what follows we can always assume that any $x \in \partial S_d$ is of the following type

$$\begin{cases} x = (y, 0, \dots, 0), \quad y \in S_l^\circ,\ 1 \leqslant l \leqslant d-1, \\ x = (0, \dots, 0). \end{cases} \tag{3.11}$$

Theorem 3.6. *Let $A \in C^1(S_d; \mathcal{L}^+(\mathbb{R}^d)) \cap \mathcal{H}_A(S_d)$, $b \in \mathcal{H}_b(S_d)$ and $c \in C(S_d)$. Moreover let $u \in C^1(S_d)$ satisfies*

$$\int_{S_d} \left[cu\varphi + \frac{1}{2}\langle ADu, D\varphi\rangle + \left\langle \frac{1}{2} DA - b, Du \right\rangle \varphi \right] dx \geqslant 0,$$

for any $\varphi \in W^{1,\infty}(S_d)$, with $\varphi \geqslant 0$. Then

$$\min_{x \in S_d} c(x) > 0 \quad \Rightarrow \quad \min_{x \in S_d} u(x) \geqslant 0.$$

Proof. Let $x_0 \in S_d$ such that

$$u(x_0) = \min_{x \in S_d} u(x).$$

In view of Lemma 3.4, if $x_0 \in S_d^\circ$ we have that $u(x) \geqslant 0$, for any $x \in S_d$ and we are done. Thus we can assume that $x_0 \in \partial S_d$ and, according to Remark 3.5, we can assume that either $x_0 = (y_0, 0, \ldots, 0)$, for some $y_0 \in S_l^\circ$, with $1 \leqslant l \leqslant d-1$ (case $l \geqslant 1$), or $x_0 = 0$ (case $l = 0$).

For any $n \in \mathbb{N}$ we define

$$\psi_n(t) := \begin{cases} 2n(1-nt) & 0 \leqslant t \leqslant \frac{1}{n}, \\ 0 & \frac{1}{n} < t \leqslant 1. \end{cases} \tag{3.12}$$

In the case $l \geqslant 1$ we fix any function $\varphi \in W^{1,\infty}(S_l)$, with $\varphi \geqslant 0$ on S_l, and we define

$$\varphi_n(x) := \varphi(x_1, \ldots, x_l) \prod_{k=l+1}^{d} \psi_n(x_k), \quad x \in S_d.$$

Clearly $\varphi_n \in W^{1,\infty}(S_d)$ and $\varphi_n \geqslant 0$ on S_d, so that

$$\int_{S_d} \left[cu\varphi_n + \frac{1}{2}\langle ADu, D\varphi_n\rangle + \left\langle \frac{1}{2}DA - b, Du \right\rangle \varphi_n \right] dx \geqslant 0. \tag{3.13}$$

Our aim is showing that by taking the limit above as n goes to infinity we get

$$\int_{S_l} \left[\hat{c}\hat{u}\varphi + \frac{1}{2}\langle \hat{A}_l D\hat{u}, D\varphi\rangle + \left\langle \frac{1}{2}D\hat{A}_l - \hat{b}_l, D\hat{u} \right\rangle \varphi \right] dy$$
$$\geqslant \int_{S_l} \sum_{i=l+1}^{d} b_i(y, 0, \ldots, 0) D_i u(y, 0, \ldots, 0)\, dy, \tag{3.14}$$

where

$$\hat{u}(y) := u(y, 0, \ldots, 0), \quad y \in S_l$$

(and analogously for $\hat{c}$ and any other function defined on S_d) and $\hat{A}_l : S_l \to \mathcal{L}^+(\mathbb{R}^l)$ and where $\hat{b}_l : S_l \to \mathbb{R}^l$ are defined respectively by

$$(\hat{A}_l)_{i,j}(y) := a_{i,j}(y, 0, \ldots, 0), \quad 1 \leqslant i, j \leqslant l, \tag{3.15}$$

and

$$(\hat{b}_l)_i(y) := b_i(y, 0, \ldots, 0), \quad 1 \leqslant i \leqslant l.$$

Note that if $A \in \mathcal{H}_A(S_d)$ then $\hat{A}_l \in \mathcal{H}_A(S_l)$ and if $b \in \mathcal{H}_b(S_d)$ then $\hat{b}_l \in \mathcal{H}_b(S_l)$. Moreover, due to (2.6) we have that $b_i(y_0, 0, \ldots, 0) \geqslant 0$ for any $i \geqslant l+1$ and, as $(y_0, 0, \ldots, 0)$ is a minimum point for u, we have that $D_i u(y_0, 0, \ldots, 0) \geqslant 0$ for any $i \in I$. This means that

$$\sum_{i=l+1}^{d} b_i(y_0, 0, \ldots, 0) D_i u(y_0, 0, \ldots, 0) \geqslant 0,$$

and then, since

$$\hat{u}(y_0) = u(y_0, 0, \ldots, 0) = u(x_0) = \min_{x \in S_d} u(x) \leqslant \min_{y \in S_l} \hat{u}(y),$$

$y_0 \in S_l^\circ$ and $\varphi \in W^{1,\infty}(S_l)$, with $\varphi \geqslant 0$, if (3.14) is true we can apply Lemma 3.4 in S_l (with $B = 1/2\hat{A}_l$ and $b = \hat{b}_l$) and we can conclude that

$$\min_{x \in S_d} u = u(x_0) = \hat{u}(y_0) \geqslant 0.$$

Hence, in order to conclude the proof in the case $l \geqslant 1$ we have to prove (3.14). For any $n \in \mathbb{N}$ we have

$$\int_{S_{d-l}} \prod_{k=l+1}^{d} \psi_n(z_k)\, dz = \prod_{k=l+1}^{d} \int_{[0,1/n]} 2n(1 - nz_k)\, dz_k = 1,$$

then it is immediate to check that for any $v \in C(S_d)$ and $\psi \in L^\infty(S_l)$

$$\lim_{n\to+\infty} \int_{S_d} v(y,z)\psi(y) \prod_{k=l+1}^{d} \psi_n(z_k)\, dy\, dz = \int_{S_l} \hat{v}(y)\psi(y)\, dy. \tag{3.16}$$

Thus, since u, Du, DA, c and b are continuous on S_d, we obtain

$$\begin{aligned} &\lim_{n\to+\infty} \int_{S_d} \left[cu\varphi_n + \left\langle \frac{1}{2} DA - b, Du \right\rangle \varphi_n \right] dx \\ &\quad = \int_{S_l} \left[\hat{c}\hat{u}\varphi + \left\langle \frac{1}{2} \widehat{DA} - \hat{b}, \widehat{Du} \right\rangle \varphi \right] dy. \end{aligned} \tag{3.17}$$

Next, we have to take the limit of the other term

$$\frac{1}{2} \int_{S_d} \langle ADu, D\varphi_n \rangle\, dx =: I_n^1 + I_n^2,$$

where

$$I_n^1 := \frac{1}{2} \int_{S_{d-l}} \left(\int_{S_l} \sum_{i=1}^{d} \sum_{j=1}^{l} a_{i,j}(y,z) D_i u(y,z) D_j\varphi(y)\, dy \right) \prod_{k=l+1}^{d} \psi_n(z_k)\, dz,$$

and

$$I_n^2 := \frac{1}{2} \int_{S_{d-l}} \sum_{j=l+1}^{d} \left(\int_{S_l} \sum_{i=1}^{d} a_{i,j}(y,z) D_i u(y,z) \varphi(y)\, dy \right) \psi_n'(z_j) \prod_{\substack{k=l+1 \\ k\neq j}}^{d} \psi_n(z_k)\, dz.$$

From (3.16), applied to $\psi = D_j\varphi$, we have

$$\lim_{n\to+\infty} I_n^1 = \frac{1}{2} \int_{S_l} \sum_{i=1}^{d} \sum_{j=1}^{l} a_{i,j}(y,0) D_i u(y,0) D_j\varphi(y)\, dy.$$

Then, as $a_{i,j}(y,0) = 0$, for any $i \geqslant l+1$ (see (2.4)), we conclude that

$$\lim_{n\to+\infty} I_n^1 = \frac{1}{2} \int_{S_l} \sum_{i=1}^{l} \sum_{j=1}^{l} \hat{a}_{i,j} \widehat{D_i u} D_j\varphi\, dy = \frac{1}{2} \int_{S_l} \langle \hat{A}_l D\hat{u}, D\varphi \rangle\, dy, \tag{3.18}$$

where $\hat{A}_l : S_l \to \mathcal{L}^+(\mathbb{R}^d)$ is the matrix-valued function defined in (3.15). Now, if we show that

$$\lim_{n\to+\infty} I_2^n = -\frac{1}{2}\int_{S_l}\sum_{i=1}^d\sum_{j=l+1}^d D_j a_{i,j}(y,0,\dots,0)D_i u(y,0,\dots,0)\varphi(y)\,dy, \tag{3.19}$$

since $D_j a_{i,j}(y,0,\dots,0)=0$ for any $i\geqslant l+1$ and $j\leqslant l$, it is immediate to see that

$$\int_{S_l}\frac{1}{2}\langle\widehat{DA},\widehat{Du}\rangle\varphi\,dy+\lim_{n\to+\infty} I_2^n=\int_{S_l}\frac{1}{2}\langle D\hat{A}_l, D\hat{u}\rangle\varphi\,dy,$$

so that from (3.17) and (3.18) we easily get (3.14).

Hence, it remains to prove (3.19). If we assume $j\geqslant l+1$, we have that $a_{i,j}(y,z)=0$ if $z_j=0$, so that if we set

$$\hat{z}^j := (z_{l+1},\dots,z_{j-1},0,z_{j+1},\dots,z_d),$$

we have

$$a_{i,j}(y,z)=a_{i,j}(y,z)-a_{i,j}(y,\hat{z}^j)=D_j a_{i,j}(y,\hat{z}^j)z_j+\sigma_j(y,z)z_j,$$

for some function $\sigma_j(y,z)$ such that

$$\lim_{z_j\to 0}\sigma_j(y,z)=0,$$

uniformly with respect to the other variables. Thus, we can rewrite

$$I_n^2=\frac{1}{2}\int_{S_{d-l}}\sum_{j=l+1}^d\left(\int_{S_l}\sum_{i=1}^d\big(D_j a_{i,j}(y,\hat{z}^j)+\sigma_j(y,z)\big)D_i u(y,z)\varphi(y)\,dy\right)$$
$$\times\,\psi_n'(z_j)z_j\prod_{\substack{k=l+1\\k\neq j}}^d\psi_n(z_k)\,dz,$$

and since

$$\int_0^1\psi_n'(t)t\,dt=-2n^2\int_0^{1/n}t\,dt=-1,$$

by arguing as in (3.16) (with $\psi=\varphi$) we can conclude that (3.19) holds.

Finally, if $x_0=0$ (case $l=0$) we define

$$\varphi_n(x):=\prod_{k=1}^d\psi_n(x_k),\qquad x\in S_d.$$

In this case, by reasoning as above we can take the limit in (3.13) as n goes to infinity and we obtain

$$
\begin{aligned}
0 &\leqslant c(0)u(0) - \frac{1}{2}\sum_{i,j=1}^{d} D_j a_{i,j}(0) D_j u(0) - \big\langle b(0), Du(0)\big\rangle \\
&= c(0)u(0) - \big\langle (0), Du(0)\big\rangle.
\end{aligned}
$$

According to (2.6) we have $b_i(0) \geqslant 0$, for any $i \leqslant d$. Moreover, since $x_0 = 0$ is a minimum point for u we have $D_i u(0) \geqslant 0$, for any $i \leqslant d$. This means that $\langle b(0), Du(0)\rangle \geqslant 0$ and then, as $c > 0$ on S_d we can conclude that $u(0) \geqslant 0$. □

In what follows we shall use the following easy consequence of the previous theorem.

Corollary 3.7. *Let $f \in C(S_d)$ and let $u \in C^1(S_d)$ such that*

$$
\int_{S_d} \left[(\lambda - c)u\varphi + \frac{1}{2}\langle ADu, D\varphi\rangle + \left\langle \frac{1}{2}DA - b, Du \right\rangle \varphi \right] dx = \int_{S_d} f\varphi\, dx, \tag{3.20}
$$

for any $\varphi \in W^{1,\infty}(S_d)$. Then, under the same conditions of Theorem 3.6 *we have*

$$
\|u\|_{C(S_d)} \leqslant \frac{1}{\lambda - \lambda_0}\|f\|_{C(S_d)}, \tag{3.21}
$$

for any $\lambda > \lambda_0 := \max_{x \in S_d} c(x)$.

Proof. If we take $\lambda > \lambda_0$ and define

$$
\tilde{u} := u + \frac{1}{\lambda - \lambda_0}\|f\|_{C(S_d)},
$$

we have that $\tilde{u}$ verifies (3.20) with f replaced by

$$
\tilde{f} := f + \frac{\lambda - c}{\lambda - \lambda_0}\|f\|_{C(S_d)}.
$$

Now, replacing if necessary u by $-u$, we can assume without loss of generality that there exists $x_0 \in S_d$ such that

$$
u(x_0) = \min_{x \in S_d} u(x) = -\|u\|_{C(S_d)}.
$$

As

$$
\tilde{u}(x_0) = u(x_0) + \frac{1}{\lambda - \lambda_0}\|f\|_{C(S_d)} = \min_{x \in S_d} u(x) + \frac{1}{\lambda - \lambda_0}\|f\|_{C(S_d)} = \min_{x \in S_d} \tilde{u}
$$

and $\tilde{f} \geqslant 0$, due to Theorem 3.6 we have that $\tilde{u}(x_0) \geqslant 0$. This means that

$$
-\|u\|_{C(S_d)} = u(x_0) \geqslant -\frac{1}{\lambda - \lambda_0}\|f\|_{C(S_d)},
$$

which implies (3.21). □

Remark 3.8. We have seen that if $u \in C^2(S_d)$ and $f := \lambda u - L_{A,b}u$, for some $\lambda > 0$, then

$$
\int_{S_d} \left[\lambda u\varphi + \left\langle \frac{1}{2}ADu, D\varphi \right\rangle + \left\langle \frac{1}{2}DA - b, Du \right\rangle \varphi \right] dx = \int_{S_d} f\varphi\, dx,
$$

for any $\varphi \in W^{1,\infty}(S_d)$. Hence, according to Corollary 3.7 we have

$$\|u\|_{C(S_d)} \leqslant \frac{1}{\lambda}\|f\|_{C(S_d)},$$

so that the operator $L_{A,b}$ defined on the domain $D(L_{A,b}) = C^2(S_d)$ is dissipative in $C(S_d)$.

4. The approximating problem

In this section we consider the approximating problem

$$\begin{cases} \lambda u(x) - \varepsilon \Delta u(x) - L_{A,b}u(x) = f(x), & x \in S_d, \\ D_i^2 u(x) = 0, \quad x \in \partial S_d^i, \ i = 1, \dots, d, \end{cases} \tag{4.1}$$

where λ and ε are positive constants and $f \in C(S_d)$.

In [4, Theorem 3.2] we have shown that if the coefficients A and b and the datum f are sufficiently regular, then problem (4.1) is solvable in Hölder spaces of functions. Namely we have proved the following result.

Theorem 4.1. *Assume $A \in \mathcal{H}_A(S_d) \cap C^\delta(S_d; \mathcal{L}^+(\mathbb{R}^d))$ and $b \in \mathcal{H}_b(S_d) \cap C^\delta(S_d; \mathbb{R}^d)$, for some $\delta \in (0,1)$. Then for any $\lambda, \varepsilon > 0$ and $f \in C^\delta(S_d)$ there exists a unique $u_\varepsilon \in C^{2+\delta}(S_d) \cap C^2_\sharp(S_d)$ which solves problem* (4.1).

Moreover, if we also assume that $A \in C^{1+\delta}(S_d; \mathcal{L}^+(\mathbb{R}^d))$, $b \in C^{1+\delta}(S_d; \mathbb{R}^d)$ and $f \in C^{1+\delta}(S_d)$, then $u_\varepsilon \in C^{3+\delta}(S_d)$.

In [4, Lemma 3.3] it has also been proved that for the operator $\varepsilon\Delta + L_{A,b}$ the following maximum principle holds.

Lemma 4.2. *Assume that $u \in C^2_\sharp(S_d)$ satisfies*

$$\lambda u(x) - \varepsilon \Delta u(x) - L_{A,b}u(x) \geqslant 0, \quad x \in S_d,$$

for some $A \in \mathcal{H}_A(S_d)$, $b \in \mathcal{H}_b(S_d)$ and $\lambda, \varepsilon > 0$. Then $u(x) \geqslant 0$, for any $x \in S_d$.

In particular, as the constant function 1 verifies

$$(\varepsilon\Delta + L_{A,b})1 = 0,$$

by standard arguments we can conclude that the operator $\varepsilon\Delta + L_{A,b}$ is dissipative in $C(S_d)$, so that, if u_ε denotes the solution of (4.1), we have

$$\|u_\varepsilon\|_{C(S_d)} \leqslant \frac{1}{\lambda}\|f\|_{C(S_d)}, \quad \varepsilon > 0. \tag{4.2}$$

In the next theorem we shall prove that the operator $\varepsilon\Delta + L_{A,b}$, defined on suitable domains, is quasi-dissipative both in $C^1(S_d)$ and in $C^2(S_d)$.

Theorem 4.3.

1. *If $A \in \mathcal{H}_A(S_d) \cap \mathcal{K}_A(S_d) \cap C^1(S_d; \mathcal{L}^+(S_d))$ and $b \in \mathcal{H}_b(S_d) \cap C^1(S_d; \mathbb{R}^d)$, then the operator $\varepsilon\Delta + L_{A,b}$, defined on the domain*
$$D(\varepsilon\Delta + L_{A,b}) := C^3(S_d) \cap C^2_\sharp(S_d),$$
is quasi-dissipative in $C^1(S_d)$.
2. *If $A \in \mathcal{H}_A(S_d) \cap \mathcal{K}_A(S_d) \cap C^2(S_d; \mathcal{L}^+(S_d))$ and $b \in \mathcal{H}_b(S_d) \cap C^2(S_d; \mathbb{R}^d)$, then the operator $\varepsilon\Delta + L_{A,b}$, defined on the domain*
$$D(\varepsilon\Delta + L_{A,b}) := \big\{u \in C^3(S_d) \cap C^2_\sharp(S_d):\ (\varepsilon\Delta + L_{A,0})u \in C^2(S_d)\big\}, \tag{4.3}$$
is quasi-dissipative in $C^2(S_d)$.

In particular, for each $i = 1, 2$ there exists $\mu_i > 0$ such that for any $\lambda > \mu_i$
$$\|u\|_{C^i(S_d)} \leqslant \frac{1}{\lambda - \mu_i}\big\|\lambda u - (\varepsilon\Delta + L_{A,b})u\big\|_{C^i(S_d)}, \quad \varepsilon > 0. \tag{4.4}$$

Proof. The operator $\varepsilon\Delta + L_{A,b}$ with domain $C^3(S_d) \cap C^2_\sharp(S_d)$ can be written as
$$\varepsilon\Delta + L_{A,b} = (\varepsilon\Delta + L_{A,0}) + L_{0,b},$$
where $D(\varepsilon\Delta + L_{A,0}) = C^3(S_d) \cap C^2_\sharp(S_d)$ and $D(L_{0,b}) = C^2(S_d)$. Thus, as the domain of $L_{0,b}$ is dense in $C^1(S_d)$, in view of Theorem 2.1 and Remark 2.5 the quasi-dissipativity in $C^1(S_d)$ of $L_{A,b}$ follows once we prove in the next subsections that both $\varepsilon\Delta + L_{A,0}$ and $L_{0,b}$, defined on their respective domains, are quasi-dissipative in $C^1(S_d)$.

Analogously, if we take $D(\varepsilon\Delta + L_{A,0})$ defined as in (4.3) (with $b = 0$) and $D(L_{0,b}) = C^3(S_d)$, we have that the quasi-dissipativity of $\varepsilon\Delta + L_{A,b}$ in $C^2(S_d)$ follows once we prove that both $\varepsilon\Delta + L_{A,0}$ and $L_{0,b}$, defined on their respective domains, are quasi-dissipative in $C^2(S_d)$. □

4.1. Quasi-dissipativity of $L_{0,b}$ in $C^k(S_d)$, for $k \geqslant 1$

Theorem 4.4. *If $b \in \mathcal{H}_b(S_d)$ then the operator $L_{0,b}$ defined on $D(L_{0,b}) = C^1(S_d)$ is dissipative in $C(S_d)$. Moreover, if $b \in \mathcal{H}_b(S_d) \cap C^k(S_d; \mathbb{R}^d)$ for some integer $k \geqslant 1$, then the operator $L_{0,b}$ with domain $D(L_{0,b}) = C^{k+1}(S_d)$ is quasi-dissipative in $C^k(S_d)$.*

Proof. We first prove that $L_{0,b}$ is dissipative in $C(S_d)$. Let $u \in C^1(S_d) \setminus \{0\}$, with $x_0 \in S_d$ such that
$$\|u\|_{C(S_d)} = \big|u(x_0)\big|.$$
Without loss of generality we can assume that $u(x_0) < 0$ (otherwise we take $-u$). If $x_0 \in S_d^\circ$, we have $Du(x_0) = 0$ and then
$$\lambda\|u\|_{C(S_d)} = -\lambda u(x_0) = -\big(\lambda u(x_0) - \big\langle b(x_0), Du(x_0)\big\rangle\big) \leqslant \big\|\lambda u - \langle b, Du\rangle\big\|_{C(S_d)}.$$

If $x_0 \in \partial S_d$, due to Lemma 2.8 there exists $\rho_0 > 0$ such that $x_0 + \rho_0 b(x_0) \in S_d$. Then the function

$$\varphi : [0,1] \to \mathbb{R}, \quad t \mapsto \varphi(t) := u\big((1-t)\big(x_0 + \rho_0 b(x_0)\big) + t x_0\big),$$

is well defined, of class C^1 and attains its minimun at $t = 1$. Then

$$\big\langle Du(x_0), -\rho_0 b(x_0)\big\rangle = \varphi'(1) \leqslant 0,$$

so that

$$\lambda \|u\|_{C(S_d)} = -\lambda u(x_0) \leqslant -\big(\lambda u(x_0) - \big\langle b(x_0), Du(x_0)\big\rangle\big) \leqslant \big\|\lambda u - \langle b, Du\rangle\big\|_{C(S_d)}.$$

Next, we show that $L_{0,b}$ is quasi-dissipative in $C^k(S_d)$. If we assume that $b \in C^k(S_d; \mathbb{R}^d)$, for any multi-index α, with $|\alpha| = k$, we have

$$\begin{aligned} D^\alpha(L_{0,b}u) &= \sum_{j=1}^d D^\alpha(b_j D_j u) = \sum_{j=1}^d \sum_{0 \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta b_j D^{\alpha-\beta} D_j u \\ &= L_{0,b}(D^\alpha u) + \sum_{\substack{0 \leqslant \beta \leqslant \alpha \\ |\beta| > 0}} \binom{\alpha}{\beta} \sum_{j=1}^d D^\beta b_j D^{\alpha-\beta+e_j} u. \end{aligned}$$

Thus, by setting $f := \lambda u - L_{0,b}u$, we get

$$\lambda D^\alpha u = L_{0,b}(D^\alpha u) + D^\alpha f + \sum_{\substack{0 \leqslant \beta \leqslant \alpha \\ |\beta| > 0}} \binom{\alpha}{\beta} \sum_{j=1}^d D^\beta b_j D^{\alpha-\beta+e_j} u.$$

As $L_{0,b}$ is dissipative in $C(S_d)$, by interpolation this easily implies

$$\begin{aligned} \lambda \|D^\alpha u\|_{C(S_d)} &\leqslant \|D^\alpha f\|_{C(S_d)} + \sum_{\substack{0 \leqslant \beta \leqslant \alpha \\ |\beta| > 0}} \binom{\alpha}{\beta} \sum_{j=1}^d \|D^\beta b_j D^{\alpha-\beta+e_j} u\|_{C(S_d)} \\ &\leqslant \|D^\alpha f\|_{C(S_d)} + c(\alpha, b, k)\|u\|_{C^k(S_d)}, \end{aligned}$$

for some constant $c(\alpha, b, k)$ depending on α, b and k. Summing on all multi-indices α, with $|\alpha| = k$, by using again the dissipativity of $L_{0,b}$ in $C(S_d)$ we have

$$\begin{aligned} \|u\|_{C^k(S_d)} &= \|u\|_{C(S_d)} + \sum_{|\alpha|=k} \|D^\alpha u\|_{C(S_d)} \\ &\leqslant \frac{1}{\lambda}\|f\|_{C(S_d)} + \frac{1}{\lambda} \sum_{|\alpha|=k} \|D^\alpha f\|_{C(S_d)} + \frac{1}{\lambda} \sum_{|\alpha|=k} c(\alpha, b, k)\|u\|_{C^k(S_d)} \\ &= \frac{1}{\lambda}\|f\|_{C^k(S_d)} + \frac{c(b,k)}{\lambda}\|u\|_{c^k(S_d)}. \end{aligned}$$

Therefore, if we take $\lambda > c(b, k)$, we obtain

$$\|u\|_{C^k(S_d)} \leqslant \frac{1}{\lambda - c(b,k)} \|f\|_{C^k(S_d)},$$

and this allows to conclude that $L_{A,b}$ is quasi-dissipative in $C^k(S_d)$. $\square$

4.2. Quasi-dissipativity of $\varepsilon\Delta + L_{A,0}$ in $C^1(S_d)$ and $C^2(S_d)$

In the previous subsection we have shown that if $b \in C^k(S_d)$ then the operator $L_{0,b}$ is quasi-dissipative in $C^k(S_d)$, for any $k \geqslant 1$. Hence, in order to prove that $L_{A,b}$ is quasi-dissipative both in $C^1(S_d)$ and in $C^2(S_d)$, in view of what we have said in the proof of Theorem 4.3 it remains to prove that $\varepsilon\Delta + L_{A,0}$ is quasi-dissipative both in $C^1(S_d)$ and in $C^2(S_d)$. To this purpose we need some preliminary results.

Lemma 4.5. *Let $A \in \mathcal{K}_A(S_d) \cap C^1(S_d; \mathcal{L}^+(\mathbb{R}^d))$ and let $u \in C^3(S_d)$. Then for any $i \in I$ we have*

$$D_i(L_{A,0}u) = L_{A,0}(D_i u) + \langle b^i_A, D(D_i u)\rangle, \tag{4.5}$$

where the vector b^i_A is defined by

$$(b^i_A)_h := \left(1 - \frac{\delta_{ih}}{2}\right) D_i a_{i,h}, \quad h \in I. \tag{4.6}$$

Moreover, if $A \in \mathcal{K}_A(S_d) \cap C^2(S_d; \mathcal{L}^+(\mathbb{R}^d))$ and $u \in C^4(S_d)$, for any $i \in I$ we have

$$D^2_i(L_{A,0}u) = L_{A,0}(D^2_i u) + 2\langle b^i_A, D(D^2_i)u\rangle + \langle \gamma^i_A, D(D_i u)\rangle, \tag{4.7}$$

where

$$(\gamma^i_A)_k := \left(1 - \frac{\delta_{ik}}{2}\right) D^2_i a_{i,k}, \quad k = 1, \dots, d,$$

and for any $i, j \in I$, with $i \neq j$, we have

$$D_{ij}(L_{A,0}u) = L_{A,0}(D_{ij}u) + \langle b^i_A + b^j_A, D(D_{ij}u)\rangle + c^{i,j}_A D_{ij}u, \tag{4.8}$$

where

$$c^{i,j}_A := D_{ij} a_{i,j}.$$

Proof. We first prove (4.5). For any $i \in I$ we have

$$\begin{aligned} D_i(L_{A,0}u) &= D_i\left(\frac{1}{2}\sum_{k,h=1}^d a_{h,k} D_{hk}u\right) \\ &= \frac{1}{2}\sum_{k,h=1}^d a_{h,k} D_{hk} D_i u + \frac{1}{2}\sum_{k,h=1}^d D_i a_{h,k} D_{hk} u \\ &= L_{A,0}(D_i u) + \frac{1}{2}\sum_{\substack{k=1\\k\neq i}}^d \sum_{h=1}^d D_i a_{h,k} D_{hk}u + \frac{1}{2}\sum_{h=1}^d D_i a_{h,i} D_{hi} u. \end{aligned}$$

Then, as $A \in \mathcal{K}_A(S_d)$, due to (2.5) we have

$$\begin{aligned}D_i(L_{A,0}u) &= L_{A,0}(D_iu) + \frac{1}{2}\sum_{\substack{k=1\\k\neq i}}^{d} D_i a_{i,k} D_{ik}u + \frac{1}{2}\sum_{h=1}^{d} D_i a_{h,i} D_{hi}u\\ &= L_{A,0}(D_iu) + \sum_{\substack{k=1\\k\neq i}}^{d} D_i a_{i,k} D_{ik}u + \frac{1}{2} D_i a_{i,i} D_i^2 u\end{aligned}$$

and this clearly yields (4.5).

Next, if $A \in \mathcal{K}_A(S_d) \cap C^2(S_d; \mathcal{L}^+(\mathbb{R}^d))$, by using again (2.5) it is immediate to check that for any $i, j \in I$

$$D_j(b_A^i)_k = \delta_{ij}\left(1 - \frac{\delta_{ik}}{2}\right) D_i^2 a_{i,k} + (1-\delta_{ij})\delta_{jk} D_{ij} a_{i,j}, \quad k \in I. \tag{4.9}$$

Thus, thanks to (4.5) we have

$$\begin{aligned}D_{ij}(L_{A,0}u) &= D_j\left(L_{A,0}(D_iu) + \left\langle b_A^i, D(D_iu)\right\rangle\right)\\ &= L_{A,0}(D_{ij}u) + \left\langle b_A^i + b_A^j, D(D_{ij}u)\right\rangle + \left\langle D_j(b_A^i), D(D_iu)\right\rangle,\end{aligned}$$

and by using (4.9) we obtain (4.7) and (4.8). □

Now we show that the vector field b_A^i introduced in the previous lemma belongs to $\mathcal{H}_b(S_d)$.

Lemma 4.6. *Assume that $A \in \mathcal{H}_A(S_d) \cap \mathcal{K}_A(S_d) \cap C^1(S_d; \mathcal{L}^+(\mathbb{R}^d))$. Then for any $i \in I$ the mapping b_A^i defined in* (4.6) *belongs to $\mathcal{H}_b(S_d)$.*

Proof. According to (2.6) we have to prove that for any $i, k \in I$ and $j \in J$

$$(-1)^j (b_A^i)_k(x) \geqslant 0, \quad x \in \partial S_d^{k,j},$$

that is

$$(-1)^j \left(1 - \frac{\delta_{ik}}{2}\right) D_i a_{i,k}(x) \geqslant 0, \quad x \in \partial S_d^{k,j}. \tag{4.10}$$

From (2.4) we have that $a_{k,i}(x) = 0$, for any $x \in \partial S_d^{k,j}$. Thus, if $k \neq i$ we have

$$D_i a_{i,k}(x) = 0, \quad x \in \partial S_d^{k,j},$$

and (4.10) follows when $k \neq i$.

Next, since A is non-negative definite, for any $x \in S_d$ we have $a_{i,i}(x) = \langle Ae_i, e_i\rangle \geqslant 0$. Then, as $a_{i,i}(x) = 0$, for $x \in \partial S_d^{i,j}$, we have that for any $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \in S_{d-1}$ the mapping

$$t \in [0,1] \mapsto \varphi(t) := a_{i,i}(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_d) \in \mathbb{R}_+,$$

has a minimum point at $t = j$. Hence

$$(-1)^j D_i a_{i,i}(x) = (-1)^j \varphi'(t)_{|_{t=j}} \geqslant 0, \quad x \in \partial S_d^{i,j},$$

so that (4.10) holds also when $k = i$. □

Now, we can show that the operator $\varepsilon\Delta + L_{A,0}$ is dissipative in $C^1(S_d)$.

Theorem 4.7. *Assume that $A \in \mathcal{H}_A(S_d) \cap \mathcal{K}_A(S_d) \cap C^2(S_d; \mathcal{L}^+(\mathbb{R}^d))$. Then for any $\varepsilon > 0$ the operator $\varepsilon\Delta + L_{A,0}$ defined on $D(\varepsilon\Delta + L_{A,0}) = C^3(S_d) \cap C^2_\sharp(S_d)$ is dissipative in $C^1(S_d)$, that is for any $u \in C^3(S_d) \cap C^2_\sharp(S_d)$ and $\lambda > 0$*

$$\|u\|_{C^1(S_d)} \leqslant \frac{1}{\lambda}\left\|\lambda u - (\varepsilon\Delta + L_{A,0})u\right\|_{C^1(S_d)}. \tag{4.11}$$

Proof. If we fix $u \in C^3(S_d) \cap C^2_\sharp(S_d)$ and $\lambda > 0$ and define $f := \lambda u - (\varepsilon\Delta + L_{A,0})u$, differentiating with respect to x_i and setting $v := D_i u$, from (4.5) we get

$$\lambda v - (\varepsilon\Delta + L_{A,0})v - \langle b^i_A, Dv\rangle = D_i f.$$

Since $u \in C^2_\sharp(S_d)$, it is immediate to check that v fulfills the following boundary conditions

$$\begin{cases} D^2_k v(x) = 0 & k \neq i,\ x \in \partial S^k_d, \\ D_i v(x) = 0 & x \in \partial S^i_d. \end{cases}$$

We show that this allows to conclude that if $x_0 \in S_d$ is a point where v achieves its minimum, then

$$\Delta v(x_0) \geqslant 0, \quad L_{A,0}v(x_0) \geqslant 0, \quad \left\langle b^i_A(x_0), Dv(x_0)\right\rangle \geqslant 0. \tag{4.12}$$

In particular

$$D_i f(x) \geqslant 0, \quad x \in S_d \quad \Rightarrow \quad v(x) \geqslant 0,\ x \in S_d,$$

which implies

$$\|D_i u\|_{C(S_d)} \leqslant \frac{1}{\lambda}\|D_i f\|_{C(S_d)}$$

(see the proof of Corollary 3.7). Hence, since $\varepsilon\Delta + L_{A,0}$ is dissipative in $C(S_d)$ (see Lemma 4.2), we obtain (4.11).

As in the proof of [4, Lemma 3.3], in order to prove (4.12) we proceed by induction on the dimension d. When $d = 1$ the function v solves the problem

$$\lambda v - \left(\varepsilon + \frac{1}{2}A\right)v'' - b^1_A v' = f', \quad v'(0) = v'(1) = 0.$$

If the minimum point x_0 is in S°_d then (4.12) is clearly satisfied. If $x_0 = 0$ (the case $x_0 = 1$ can be treated in the same way), since $A \in \mathcal{H}_A(S_d)$, we have $A(0)v''(0) = 0$. Moreover, since 0 is a minimum point and $v'(0) = 0$, we have $v''(0) \geqslant 0$, so that (4.12) follows.

Next, assume that (4.12) holds for some $d - 1 \geqslant 1$. We show that it holds also for d. As above we can assume that $x_0 \in \partial S_d$. If $x_0 \in \partial S^{k,j}_d$, for some $k \neq i$ and $j = 0, 1$, recalling that $b^i_A \in \mathcal{H}_b(S_d)$ (see Lemma 4.6) we have

$$(-1)^j (b^i_A)_k(x_0) \geqslant 0.$$

Moreover, since x_0 is a minimum point for v, we have $(-1)^j D_k v(x_0) \geqslant 0$ and then

$$\begin{aligned}\langle b_A^i(x_0), Dv(x_0)\rangle &= (b_A^i)_k(x_0) D_k v(x_0) + \sum_{\substack{h=1\\h\neq k}}^d (b_A^i)_h(x_0) D_h v(x_0)\\ &\geqslant \sum_{\substack{h=1\\h\neq k}}^d (b_A^i)_h(x_0) D_h v(x_0) =: \langle (b_A^i)|_{\partial S_d^k}(x_0), D(\gamma_k \circ v)(x_0)\rangle, \end{aligned}\tag{4.13}$$

where $\gamma_k \circ v$ is the restriction of v to ∂S_d^k. Moreover, since $A \in \mathcal{H}_A(S_d)$ we have that $a_{k,h}(x) = 0$ on ∂S_d^k, for any $h \in I$ and then

$$L_{A,0} v(x_0) = \frac{1}{2} \sum_{\substack{h,l=1\\h,l\neq k}}^d a_{h,l}(x_0) D_{hl} v(x_0) =: L_{A,0}|_{\partial S_d^k}(\gamma_k \circ v)(x_0). \tag{4.14}$$

Analogously, since $D_k^2 v(x_0) = 0$, we have

$$\Delta v(x_0) = \sum_{\substack{h=1\\h\neq k}}^d D_h^2 v(x_0) =: \Delta|_{\partial S_d^k}(\gamma_k \circ v)(x_0). \tag{4.15}$$

Now, we notice that if x_0 is a minimum for v it is also a minimum for the function $\gamma_k \circ v$ defined on the $(d-1)$-dimensional cube $\partial S_d^{k,j}$. Furthermore $\gamma_k \circ v$ fulfills the boundary conditions

$$\begin{cases} D_h^2(\gamma_k \circ v)(x) = 0, & h \neq k, i,\ x \in \partial S_d^h \cap \partial S_d^{k,j},\\ D_i(\gamma_k \circ v)(x) = 0, & x \in \partial S_d^i \cap \partial S_d^{k,j}. \end{cases}$$

Then, as $\Delta|_{\partial S_d^k}$, $L_{A,0}|_{\partial S_d^k}$ and $(b_A^i)|_{\partial S_d^k}$ satisfy the same hypotheses as Δ, $L_{A,0}$ and b_A^i in the $(d-1)$-dimensional cube $\partial S_d^{k,j}$, according to (4.13), (4.14) and (4.15) and to the inductive assumption, we can conclude that (4.12) holds.

Finally, if

$$x_0 \in \partial S_d^i \cap \Bigg(\bigcup_{\substack{h=1\\h\neq i}}^d \partial S_d^h\Bigg)^c,$$

we have that $((x_0)_1, \ldots, (x_0)_{i-1}, (x_0)_{i+1}, \ldots, (x_0)_d)$ is an interior minimum point for the function

$$v_i : S_{d-1} \to \mathbb{R}, \quad (y_1, \ldots, y_{d-1}) \mapsto v_i(y_1, \ldots, y_{i-1}, (x_0)_i, y_{i+1}, \ldots, y_{d-1}).$$

This implies that $D_h v(x_0) = 0$, for any $h \neq i$, so that, recalling that $D_i v(x_0) = 0$, we have

$$\langle b_A^i(x_0), Dv(x_0)\rangle = 0.$$

In a similar way, we have

$$\Delta v(x_0) = \sum_{\substack{h=1 \\ h \neq i}}^{d} D_h^2 v(x_0) + D_i^2 v(x_0) \geqslant D_i^2 v(x_0)$$

and recalling that $A \in \mathcal{H}_A(S_d)$

$$L_{A,0} v(x_0) = \frac{1}{2} \sum_{\substack{h,l=1 \\ h,l \neq i}}^{d} a_{h,l}(x_0) D_{hl} v(x_0) \geqslant 0.$$

Therefore, since $D_i^2 v(x_0) \geqslant 0$ (x_0 is a minimum point for v and $D_i v(x_0) = 0$) we can conclude that (4.12) holds. □

Finally, we prove the quasi-dissipativity of $\varepsilon\Delta + L_{A,0}$ in $C^2(S_d)$.

Theorem 4.8. *Assume that $A \in \mathcal{H}_A(S_d) \cap \mathcal{K}_A(S_d) \cap C^2(S_d; \mathcal{L}^+(\mathbb{R}^d))$. Then for any $\varepsilon > 0$ the operator $\varepsilon\Delta + L_{A,0}$ defined on the domain*

$$D(\varepsilon\Delta + L_{A,0}) := \left\{ u \in C^3(S_d) \cap C^2_\sharp(S_d) \colon (\varepsilon\Delta + L_{A,0})u \in C^2(S_d) \right\},$$

is quasi-dissipative in $C^2(S_d)$, uniformly with respect to $\varepsilon > 0$. That is, there exists $\lambda_0 \geqslant 0$ such that for any $\lambda > \lambda_0$ and for any $u \in D(\varepsilon\Delta + L_{A,0})$

$$\|u\|_{C^2(S_d)} \leqslant \frac{1}{\lambda - \lambda_0} \left\| \lambda u - (\varepsilon\Delta + L_{A,0})u \right\|_{C^2(S_d)}, \quad \varepsilon > 0.$$

Proof. Thanks to Lemma 4.2 (note that $0 \in \mathcal{H}_b(S_d)$) the operator $\varepsilon\Delta + L_{A,0}$ is dissipative in $C(S_d)$. Hence, in order to prove the quasi-dissipativity of $\varepsilon\Delta + L_{A,0}$ in $C^2(S_d)$ we need to prove some a-priori estimates for the second order derivatives, which are uniform with respect to $\varepsilon > 0$.

Let $i, j \in I$ be fixed and let $x_0 \in S_d$ such that

$$D_{ij} u(x_0) = \min_{x \in S_d} D_{ij} u(x) = -\|D_{ij} u\|_{C(S_d)} \tag{4.16}$$

(if this is not the case, instead of u we can take $-u$). If $x_0 \in \partial S_d \setminus \{0\}$ in view of Remark 4.2 (see (3.11)) without loss of generality we can assume that x_0 is of the form

$$x_0 = (y_0, 0, \ldots, 0), \quad y_0 \in S_l^\circ,$$

for some $1 \leqslant l \leqslant d-1$. Now, if we set $f := \lambda u - (\varepsilon\Delta + L_{A,0})u$ and

$$\hat{u}(y) := u(y, 0, \ldots, 0), \quad y \in S_l,$$

as $u \in C^2_\sharp(S_d)$ and $A \in \mathcal{H}_A(S_d)$, it is easy to see that $\hat{u}$ solves the equation

$$\lambda \hat{u} = \varepsilon \Delta_l \hat{u} + L_{\hat{A}_l,0} \hat{u} + \hat{f}, \quad \text{on } S_l,$$

where Δ_l is the Laplacian in $\mathbb{R}^l$, $\hat{A}_l : S_l \to \mathcal{L}^+(\mathbb{R}^l)$ is defined as in (3.15) and

$$\hat{f}(y) := f(y, 0, \ldots, 0), \quad y \in S_l.$$

Note that the restricted matrix $\hat{A}_l$ belongs to $\mathcal{H}_A(S_l) \cap \mathcal{K}_A(S_l) \cap C^2(S_l; \mathcal{L}^+(\mathbb{R}^l))$. Thus, since $y_0 \in S_l^\circ$, the treatment of this case reduces to the treatment of the case $x_0 \in S_d^\circ$. This means that in what follows we can assume that either $x_0 \in S_d^\circ$ or $x_0 = 0$.

We start considering the case $i \neq j$ in (4.16). If $u \in D(\varepsilon\Delta + L_{A,0})$ we define $w := D_{ij}u$ and fix a sequence $\{u_n\} \subset C^4(S_d)$ which converges to u in $C^3(S_d)$. Moreover we define

$$f_n := \lambda u_n - (\varepsilon\Delta + L_{A,0})u_n, \quad n \geqslant 1.$$

Thus, setting $w_n := D_{ij}u_n$, according to (4.8) we have

$$\lambda w_n = (\varepsilon\Delta + L_{A,0})w_n + \langle b_A^i + b_A^j, Dw_n\rangle + c_A^{i,j} w_n + D_{ij} f_n. \tag{4.17}$$

Now, if $x_0 \in S_d^\circ$ we multiply both sides in (4.17) by $\varphi \in W^{1,\infty}(S_d) \cap C_0(S_d)$ and, integrating by parts, thanks to (3.1) and to Remark 3.2 we have

$$\int_{S_d} \left[(\lambda - c_A^{i,j}) w_n \varphi + \left\langle \left(\varepsilon + \frac{1}{2}A\right) Dw_n, D\varphi \right\rangle + \left\langle \frac{1}{2} DA - (b_A^i + b_A^j), Dw_n \right\rangle \varphi \right] dx$$
$$= -\int_{S_d} D_i f_n D_j \varphi \, dx.$$

Now, as u_n converges to u in $C^3(S_d)$ we have that w_n and f_n converge respectively to w and f in $C^1(S_d)$ and then we can take the limit above as n goes to infinity and obtain

$$\int_{S_d} \left[(\lambda - c_A^{i,j}) w \varphi + \left\langle \left(\varepsilon + \frac{1}{2}A\right) Dw, D\varphi \right\rangle + \left\langle \frac{1}{2} DA - (b_A^i + b_A^j), Dw \right\rangle \varphi \right] dx$$
$$= -\int_{S_d} D_i f D_j \varphi \, dx = \int_{S_d} D_{ij} f \varphi \, dx, \tag{4.18}$$

where last equality has followed integrating by parts once more. As proved in Lemma 4.6, the vector field $b_A^i + b_A^j$ belongs to $\mathcal{H}_b(S_d)$, then as $x_0 \in S_d^\circ$ and (4.18) holds for any $\varphi \in W^{1,\infty}(S_d) \cap C_0(S_d)$, according to Lemma 3.4 (applied to $B = \varepsilon I + 1/2A$) and to the proof of Corollary 3.7 we get

$$\|D_{ij}u\|_{C(S_d)} = \|w\|_{C(S_d)} \leqslant \frac{1}{\lambda - \lambda_0^{i,j}} \|D_{ij} f\|_{C(S_d)}, \quad \lambda > \lambda_0^{i,j}, \tag{4.19}$$

where

$$\lambda_0^{i,j} := \max_{x \in S_d} c_A^{i,j}(x).$$

Now, assume that $x_0 = 0$. In view of Remark 3.5 it is sufficient to prove (4.19) for $i = 1$ and $j = 2$. As shown above, if we set

$$\hat{u}(y) = u(y, 0, \ldots, 0), \quad y \in S_2,$$

we have that $\hat{u} \in C^3(S_2) \cap C_\sharp^2(S_2)$ and solves the problem

$$\lambda\hat{u} = \varepsilon\Delta_2\hat{u} + L_{\hat{A},0}\hat{u} + \hat{f}, \quad \text{on } S_2,$$

with Δ_2, $\hat{A} = \hat{A}_2$ and $\hat{f}$ defined as above. Moreover, if we define $w := D_{12}u$ we have

$$\hat{w}(y) = w(y, 0, \dots, 0) = D_{12}\hat{u}(y), \quad y \in S_2,$$

and

$$\hat{w}(0) = \min_{y \in S_2} \hat{w}(y).$$

If as above $\{\hat{u}_n\}$ is a sequence in $C^4(S_2)$ converging to $\hat{u}$ in $C^3(S_2)$ and

$$\hat{w}_n := D_{12}\hat{u}_n, \quad \hat{f}_n := \lambda \hat{u}_n - (\varepsilon\Delta_2 + L_{\hat{A},0})\hat{u}_n, \quad n \geqslant 1,$$

by using (4.8) we have

$$\lambda \hat{w}_n = (\varepsilon\Delta_2 + L_{\hat{A},0})\hat{w}_n + \langle b^1_{\hat{A}} + b^2_{\hat{A}}, D\hat{w}_n\rangle + c^{1,2}_{\hat{A}}\hat{w}_n + D_{12}\hat{f}_n.$$

Thus, multiplying each side by a test function $\varphi \in W^{1,\infty}(S_2)$ and integrating by parts, thanks to Lemma 3.1 we have

$$\int_{S_2} \left[(\lambda - c^{1,2}_{\hat{A}})\hat{w}_n\varphi + \left\langle\left(\varepsilon + \frac{1}{2}\hat{A}\right)D\hat{w}_n, D\varphi\right\rangle + \left\langle\frac{1}{2}D\hat{A} - (b^1_{\hat{A}} + b^2_{\hat{A}}), D\hat{w}_n\right\rangle\varphi\right] dy$$
$$= -\int_{S_2} D_2\hat{f}_n D_1\varphi\, dy + \int_{\partial S_2^{1,0}} D_2\hat{f}_n\varphi\, d\sigma - \int_{\partial S_2^{1,1}} D_2\hat{f}_n\varphi\, d\sigma - \varepsilon\int_{\partial S_2} \langle D\hat{w}_n, \nu\rangle\varphi\, d\sigma,$$

where ν is the unit inward normal at ∂S_2. Repeating the same arguments used above in the case $x_0 \in S_d^\circ$, by taking the limit as n goes to infinity this yields

$$\int_{S_2} \left[(\lambda - c^{1,2}_{\hat{A}})\hat{w}\varphi + \left\langle\left(\varepsilon + \frac{1}{2}\hat{A}\right)D\hat{w}, D\varphi\right\rangle + \left\langle\frac{1}{2}D\hat{A} - (b^1_{\hat{A}} + b^2_{\hat{A}}), D\hat{w}\right\rangle\varphi\right] dy$$
$$= \int_{S_2} D_{12}\hat{f}\varphi\, dy - \varepsilon\int_{\partial S_2} \langle D\hat{w}, \nu\rangle\varphi\, d\sigma.$$

Now, as $u \in C^2_\sharp(S_d)$, it is immediate to check that for any $(y_1, y_2) \in S_2$

$$D_1\hat{w}(0, y_2) = D_1 w(0, y_2, 0, \dots, 0) = 0,$$
$$D_2\hat{w}(y_1, 0) = D_2 w(y_1, 0, \dots, 0) = 0,$$

so that

$$\big\langle D\hat{w}(y), \nu(y)\big\rangle = 0, \quad y \in \partial S_2.$$

This means that $\hat{w}$ verifies

$$\int_{S_2} \left[(\lambda - c^{1,2}_{\hat{A}})\hat{w}\varphi + \left\langle\left(\varepsilon + \frac{1}{2}\hat{A}\right)D\hat{w}, D\varphi\right\rangle + \left\langle\frac{1}{2}D\hat{A} - (b^1_{\hat{A}} + b^2_{\hat{A}}), D\hat{w}\right\rangle\varphi\right] dy$$
$$= \int_{S_2} D_{12}\hat{f}\varphi\, dy,$$

for any $\varphi \in W^{1,\infty}(S_2)$. Then, as for $\varepsilon > 0$ fixed the matrix $\varepsilon I + 1/2A$ is strictly non-degenerate, by classical results on the maximum principle for weak solutions of the Neumann problem (see for a proof in the case of Dirichlet boundary conditions [12]) we can say that

$$\|D_{12}\hat{u}\|_{C(S_2)} = \|\hat{w}\|_{C(S_2)} \leqslant \frac{1}{\lambda - \lambda_0^{1,2}} \|D_{12}\hat{f}\|_{C(S_2)}. \tag{4.20}$$

This allows us to conclude that (4.19) holds, as

$$\begin{aligned}\|D_{12}u\|_{C(S_d)} &= -D_{12}u(0) = -D_{12}\hat{u}(0) = \|D_{12}\hat{u}\|_{C(S_2)} \\ &\leqslant \frac{1}{\lambda - \lambda_0^{1,2}} \|D_{12}\hat{f}\|_{C(S_2)} \leqslant \frac{1}{\lambda - \lambda_0^{1,2}} \|D_{12} f\|_{C(S_d)}.\end{aligned}$$

Next, we consider the case $i = j$. In this case we can assume that there exists $x_0 \in S_d^\circ$ such that

$$D_i^2 u(x_0) = -\|D_i^2 u\|_{C(S_d)} = \min_{x \in S_d} D_i^2 u(x).$$

Actually, if

$$-\|D_i^2 u\|_{C(S_d)} = D_i^2 u(0) = \min_{x \in S_d} D_i^2 u(x),$$

as $u \in C_\sharp^2(S_d)$ we have $D_i^2 u(0) = 0$ and then

$$\|D_i^2 u\|_{C(S_d)} = 0 \leqslant \frac{1}{\lambda - \lambda_0} \|D_i^2 f\|_{C(S_d)},$$

for any $0 \leqslant \lambda_0 < \lambda$.

Now, if as in the case $i \neq j$ we fix a sequence $\{u_n\} \subset C^4(S_d)$ converging to u in $C^3(S_d)$ and define $f_n := \lambda u_n - (\varepsilon\Delta + L_{A,0})u_n$, due to (4.7) we have that $w_n := D_i^2 u_n$ satisfies

$$\begin{aligned}\lambda w_n &= (\varepsilon\Delta + L_{A,0})w_n + 2\langle b_A^i, Dw_n\rangle + \langle \gamma_A^i, D(D_i u_n)\rangle + D_i^2 f \\ &= (\varepsilon\Delta + L_{A,0})w_n + 2\langle b_A^i, Dw_n\rangle + (\gamma_A^i)_i w_n + \sum_{\substack{i,j=1 \\ j \neq i}}^{d} (\gamma_A^i)_j D_{ij} u_n + D_i^2 f.\end{aligned}$$

Multiplying as above each side by $\varphi \in W^{1,\infty}(S_d) \cap C_0(S_d)$ and integrating by parts, we obtain

$$\begin{aligned}&\int_{S_d} \left[\left(\lambda - (\gamma_A^i)_i\right) w_n \varphi + \left\langle \left(\varepsilon + \frac{1}{2}A\right) Dw_n, D\varphi \right\rangle + \left\langle \frac{1}{2} DA - 2b_A^i, Dw_n \right\rangle \varphi \right] dx \\ &\quad = \int_{S_d} \left[\sum_{\substack{j=1 \\ j \neq i}}^{d} (\gamma_A^i)_j D_{ij} u_n \varphi + D_i f_n D_i \varphi \right] dx.\end{aligned}$$

Hence, as w_n, f_n and $D_{ik} u_n$ converge respectively to $w := D_i^2 u$, f and $D_{ik} u$ in $C^1(S_d)$, by taking the limit above, as n goes to infinity, we have

$$\int_{S_d}\left[\left(\lambda-(\gamma_A^i)_i\right)w\varphi+\left\langle\left(\varepsilon+\frac{1}{2}A\right)Dw,D\varphi\right\rangle+\left\langle\frac{1}{2}DA-2b_A^i,Dw\right\rangle\varphi\right]dx$$
$$=\int_{S_d}\left[\sum_{\substack{j=1\\j\neq i}}^{d}(\gamma_A^i)_jD_{ij}u+D_i^2f\right]\varphi\,dx.$$

Now, since $x_0\in S_d^\circ$ we can apply Lemma 3.4 (for $B=\varepsilon I+1/2A$) and Corollary 4.6 and we get

$$\|D_i^2u\|_{C(S_d)}\leqslant\frac{1}{\lambda-\lambda_0^{i,i}}\|D_i^2f\|_{C(S_d)}+\frac{\lambda_0^{i,i}}{\lambda-\lambda_0^{i,i}}\sum_{\substack{i,j=1\\j\neq i}}^{d}\|D_{ij}u\|_{C(S_d)},\quad\lambda>\lambda_0^{i,i},$$

where

$$\lambda_0^{i,i}:=\sup_{k=1,\dots,d}\left\|(\gamma_A^i)_k\right\|_{C(S_d)}.$$

This implies that

$$\lambda\|D_i^2u\|_{C(S_d)}\leqslant\|D_i^2f\|_{C(S_d)}+\lambda_0^{i,i}\sum_{j=1}^{d}\|D_{ij}u\|_{C(S_d)},\quad\lambda>\lambda_0^{i,i},$$

and then, if we set $\lambda_0:=2\max_{i,j=1,\dots,d}\lambda_0^{i,j}$, summing on $i=1,\dots,d$ we obtain

$$\lambda\sum_{i=1}^{d}\|D_i^2u\|_{C(S_d)}\leqslant\sum_{i=1}^{d}\|D_i^2f\|_{C(S_d)}+\frac{\lambda_0}{2}\sum_{i,j=1}^{d}\|D_{ij}u\|_{C(S_d)},\quad\lambda>\lambda_0.\tag{4.21}$$

In the same way, summing on $i\neq j$ in (4.19) we easily have

$$\lambda\sum_{\substack{i,j=1\\j\neq i}}^{d}\|D_{ij}u\|_{C(S_d)}\leqslant\sum_{\substack{i,j=1\\j\neq i}}^{d}\|D_{ij}f\|_{C(S_d)}+\frac{\lambda_0}{2}\sum_{\substack{i,j=1\\j\neq i}}^{d}\|D_{ij}u\|_{C(S_d)},\tag{4.22}$$

and then combining together (4.21) and (4.22) we obtain

$$\lambda\sum_{i,j=1}^{d}\|D_{ij}u\|_{C(S_d)}\leqslant\sum_{i,j=1}^{d}\|D_{ij}f\|_{C(S_d)}+\lambda_0\sum_{i,j=1}^{d}\|D_{ij}u\|_{C(S_d)}.$$

Therefore, as $(\varepsilon\Delta+L_{A,0})$ is dissipative in $C(S_d)$, for any $\lambda>\lambda_0$ this yields

$$\|u\|_{C^2(S_d)}=\|u\|_{C(S_d)}+\sum_{i,j=1}^{d}\|D_{ij}u\|_{C(S_d)}$$
$$\leqslant\frac{1}{\lambda}\|f\|_{C(S_d)}+\frac{1}{\lambda-\lambda_0}\sum_{i,j=1}^{d}\|D_{ij}f\|_{C(S_d)}\leqslant\frac{1}{\lambda-\lambda_0}\|f\|_{C^2(S_d)},$$

which is exactly what we wanted to prove. □

5. Proof of main results

As a consequence of the quasi-dissipativity in $C^1(S_d)$ and $C^2(S_d)$ of the approximating operator $\varepsilon\Delta + L_{A,b}$ proved in Theorem 4.3 we can finally give a proof of Theorem 1.1.

Proof of Theorem 1.1. Our aim is proving that the operator

$$L_{A,b}u = \frac{1}{2}\operatorname{Tr}[AD^2u] + \langle b, Du\rangle, \quad u \in C^2(S_d),$$

is essentially m-dissipative in $C(S_d)$, so that it is closable in $C(S_d)$ and its closure is the generator of a C_0-semigroup of contractions in $C(S_d)$ (and in particular the martingale problem associated with $L_{A,b}$ is well posed in $C(S_d)$).

Since $A \in \mathcal{H}_A(S_d)$, with the same arguments used in the proof of Lemma 4.2 (see [4]), we have that the operator $L_{A,b}$ is dissipative in $C(S_d)$ (see also Remark 3.8). Moreover, it is densely defined in $C(S_d)$ and hence, according to Proposition 2.2, is closable with dissipative closure. Thus, in order to prove the essential m-dissipativity of $L_{A,b}$ (which thanks to the Lumer–Phillips Theorem 2.4 implies that the closure of $L_{A,b}$ generates a strongly continuous semigroup of contractions on $C(S_d)$), according to Proposition 2.3 it remains to prove that for some $\lambda > 0$ large enough the range of $\lambda - L_{A,b}$ is dense in $C(S_d)$.

As $C^2(S_d)$ is dense in $C(S_d)$, if we fix $f \in C^2(S_d)$ and show that there exists a sequence $\{f_n\} \subset R(\lambda - L_{A,b})$ converging to f in $C(S_d)$, as n goes to infinity, we are clearly done. In view of Theorem 4.1, for any $n \in \mathbb{N}$ there exists a unique $u_n \in C^3(S_d) \cap C^2_\sharp(S_d)$ such that

$$\lambda u_n - \frac{1}{n}\Delta u_n - L_{A,b}u_n = f.$$

Thus, if we define

$$f_n := f + \frac{1}{n}\Delta u_n,$$

we have that f_n converges to f. Indeed, thanks to Theorem 4.3 there exists $\mu_2 \geqslant 0$ such that for any $\lambda > \mu_2$ we have

$$\|\Delta u_n\|_{C(S_d)} \leqslant \|u_n\|_{C^2(S_d)} \leqslant \frac{1}{\lambda - \mu_2}\|f\|_{C^2(S_d)}, \quad n \in \mathbb{N},$$

and hence $1/n\Delta u_n$ converges to zero in $C(S_d)$. □

Remark 5.1. Theorem 1.1 can be extended to the following class of operators

$$Lu := \frac{1}{2}\operatorname{Tr}[\gamma AD^2u] + \langle b, Du\rangle, \quad u \in D(L) = C^2(S_d),$$

where A and b are as in Theorem 1.1 and $\gamma \in C(S_d)$ is a strictly positive function such that

$$\frac{b}{\gamma} \in C^2(S_d; \mathbb{R}^d).$$

Actually, if we set $\hat{b} := b/\gamma$ and denote by G the multiplication operator by the function γ, we can write

$$Lu = \frac{1}{2}\,\mathrm{Tr}[\gamma A D^2 u] + \langle b, Du\rangle = \gamma\left[\frac{1}{2}\,\mathrm{Tr}[A D^2 u] + \langle \hat{b}, Du\rangle\right] = G L_{A,\hat{b}} u.$$

Now, since γ is continuous and strictly positive, we have that $\gamma A \in \mathcal{H}_A(S_d)$ and then L is dissipative in $C(S_d)$. As $D(L)$ is dense in $C(S_d)$, this means that L is closable in $C(S_d)$ and its closure is dissipative. Thus, in view of Lemma 2.6, in order to prove the essential m-dissipativity of L it is sufficient to prove the essential m-dissipativity of $L_{A,\hat{b}}$.

But, since γ is positive we clearly have that $\hat{b} \in \mathcal{H}_b(S_d)$. Hence, as by our assumptions on γ we have that $\hat{b} \in C^2(S_d)$, we can apply Theorem 1.1 to $L_{A,\hat{b}}$ and the general result for L follows.

Another consequence of the quasi-dissipativity in $C^1(S_d)$ and $C^2(S_d)$ of the approximating operator $\varepsilon\Delta + L_{A,b}$ is the existence of weak solutions in $C^1(S_d)$.

Theorem 5.2. *If $A \in \mathcal{H}_A(S_d) \cap \mathcal{K}_A(S_d) \cap C^2(S_d; \mathcal{L}^+(S_d))$ and $b \in \mathcal{H}_b(S_d) \cap C^2(S_d; \mathbb{R}^d)$, there exists $\lambda_0 \geqslant 0$ such that for any $f \in C^1(S_d)$ and $\lambda > \lambda_0$ there exists a unique weak solution $u \in C^1(S_d)$ of problem* (1.6). *That is there exists a unique $u \in C^1(S_d)$ such that*

$$\int_{S_d}\left[\lambda u\varphi + \frac{1}{2}\langle ADu, D\varphi\rangle + \left\langle \frac{1}{2}DA - b, Du\right\rangle\varphi\right]dx = \int_{S_d} f\varphi\,dx,$$

for any $\varphi \in W^{1,\infty}(S_d)$. Moreover,

$$\|u\|_{C^1(S_d)} \leqslant \frac{1}{\lambda - \lambda_0}\|f\|_{C^1(S_d)}. \tag{5.1}$$

Proof. According to Theorem 4.1, if $f \in C^2(S_d)$ for any $\varepsilon, \lambda > 0$ there exists a unique $u_\varepsilon \in C^2_\sharp(S_d) \cap C^3(S_d)$ such that

$$\lambda u_\varepsilon - (\varepsilon\Delta + L_{A,b})u_\varepsilon = f. \tag{5.2}$$

Moreover, thanks to Theorem 4.3 for each $i = 1, 2$ there exists $\mu_i \geqslant 0$ such that if $\lambda > \mu_i$ then

$$\|u_\varepsilon\|_{C^i(S_d)} \leqslant \frac{1}{\lambda - \mu_i}\|f\|_{C^i(S_d)}, \quad \varepsilon > 0.$$

Due to the Ascoli–Arzela theorem this implies that there exists a subsequence $\{u_{\varepsilon_n}\} \subset \{u_\varepsilon\}$ converging in $C^1(S_d)$ to some u such that

$$\|u\|_{C^1(S_d)} \leqslant \frac{1}{\lambda - \mu_1}\|f\|_{C^1(S_d)}. \tag{5.3}$$

Now, since u_{ε_n} is a classical solution to problem (5.2), thanks to Lemma 3.1 for any $\varphi \in W^{1,\infty}(S_d)$ and $n \in \mathbb{N}$ we have

$$\int_{S_d}\left[\lambda u_{\varepsilon_n}\varphi+\left\langle\left(\varepsilon_n+\frac{1}{2}A\right)Du_{\varepsilon_n}, D\varphi\right\rangle+\left\langle\frac{1}{2}DA-b, Du_{\varepsilon_n}\right\rangle\varphi\right]dx$$
$$=\int_{S_d} f\varphi\,dx-\varepsilon_n\int_{\partial S_d}\langle Du_{\varepsilon_n},\nu\rangle\,d\sigma,$$

where ν is the unit inward normal vector. Thus, by taking the limit for n going to infinity we obtain

$$\int_{S_d}\left[\lambda u\varphi+\left\langle\frac{1}{2}ADu, D\varphi\right\rangle+\left\langle\frac{1}{2}DA-b, Du\right\rangle\varphi\right]dx=\int_{S_d} f\varphi\,dx.$$

As shown in Corollary 3.7 this implies that

$$\|u\|_{C(S_d)}\leqslant\frac{1}{\lambda}\|f\|_{C(S_d)},$$

so that u is the unique weak solution of problem (1.6).

Next, assume that $f\in C^1(S_d)$. In this case we fix a sequence $\{f_n\}\subset C^2(S_d)$ converging to f in $C^1(S_d)$. For each n we denote by u_n the unique weak solution of problem (1.6) corresponding to the datum f_n. Thanks to (5.3) for each $n,m\in\mathbb{N}$ and $\lambda>\lambda_0:=\mu_1\wedge\mu_2$ we have

$$\|u_n-u_m\|_{C^1(S_d)}\leqslant\frac{1}{\lambda-\lambda_1}\|f_n-f_m\|_{C^1(S_d)},$$

and hence the sequence $\{u_n\}$ converges in $C^1(S_d)$ to some $u\in C^1(S_d)$ which is the unique weak solution of problem (1.6) corresponding to the datum f. □

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