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ON THE SMOLUCHOWSKI-KRAMERS APPROXIMATION FOR SPDES AND ITS INTERPLAY WITH LARGE DEVIATIONS AND LONG TIME BEHAVIOR

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Abstract. We discuss here the validity of the small mass limit (the so-called Smoluchowski-Kramers approximation) on a fixed time interval for a class of semi-linear stochastic wave equations, both in the case of the presence of a constant friction term and in the case of the presence of a constant magnetic field. We also consider the small mass limit in an infinite time interval and we see how the approximation is stable in terms of the invariant measure and of the large deviation estimates and the exit problem from a bounded domain of the space of square integrable functions.

1. **Introduction.** The motion of a particle of a mass μ in the field $b(q) + \sigma(q)\dot{W}$, with a constant damping proportional to the speed, is described, according to the Newton law, by the Langevin equation

$$\mu \ddot{q}^\mu_t = b(q^\mu_t) + \sigma(q^\mu_t)\dot{W}_t - \dot{q}_t, \qquad q^\mu_0 = q \in \mathbb{R}^n, \quad \dot{q}^\mu_0 = p \in \mathbb{R}^n, \tag{1.1}$$

(for the sake of simplicity the friction coefficient is taken equal to 1).

Here $b(q)$ is the deterministic component of the force and $\sigma(q)\dot{W}_t$, where $\dot{W}_t$ is the standard Gaussian white noise in $\mathbb{R}^n$ and $\sigma(q)$ is an $n \times n$-matrix, is the stochastic part. It is known that, for $0 < \mu \ll 1$, the random position q^μ_t of the particle can be approximated by the solution of the first order equation

$$\dot{q}_t = b(q_t) + \sigma(q_t)\dot{W}_t, \qquad q_0 = q \in \mathbb{R}^n, \tag{1.2}$$

in the sense that

$$\lim_{\mu \downarrow 0} P\left(\max_{0 \le t \le T} |q^\mu_t - q_t| > \delta\right) = 0, \tag{1.3}$$

for any $0 \le T < \infty$ and $\delta > 0$ fixed. Statement (1.3) is called then *Smoluchowski-Kramers approximation* of q^μ_t by q_t (see to this purpose [27, 38, 19]). This statement

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justifies the description of the motion of a small particle by the first order equation (1.2) instead of the second order equation (1.1). Several authors have considered generalizations of this phenomenon in the presence of a magnetic field (see [6, 28]) and for a non-constant friction, both in the case it is strictly positive, as in [20, 25, 24], and in the case it is possibly vanishing, as in [21]. Large deviations and exit problems in the small noise regime have also been studied (see [11]).

In the present paper, we will review some results about the Smoluchowski-Kramers approximation for systems with an infinite number of degrees of freedom, obtained by the three authors in a series of papers written in last several years (cfr. [3, 4, 8, 9, 10]).

Let $\mathcal{O}$ be a bounded smooth domain of $\mathbb{R}^d$, with $d \geq 0$. We are dealing here with the following stochastic semi-linear damped wave equation on $\mathcal{O}$

$$\left\{\begin{array}{l} \mu \dfrac{\partial^2 u}{\partial t^2}(t,x) + \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + b(x,u(t,x)) + g(x,u(t,x))\dfrac{\partial w^Q}{\partial t}(t,x), \\ \\ u(0,x) = u_0(x), \qquad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), \qquad u(t,x) = 0, \quad x \in \partial\mathcal{O}. \end{array}\right. \tag{1.4}$$

Equation (1.4) models the displacement of an elastic material with mass density $\mu > 0$ in the region $\mathcal{O}$, exposed to deterministic and random forces. The term $-\partial u/\partial t$ models the damping, the Laplacian Δ models the forces that neighboring particles exert on each other and the non-linearity b models some deterministic forcing. State-dependent stochastic perturbations are modeled by the term $g\,\partial w^Q/\partial t$, for some Wiener process w^Q, that is white in time and Q-correlated in space (see Section 3 for all definitions), and for some nonlinear coefficient g.

Many authors have studied stochastic wave equations under various assumptions on the non-linear coefficients b and g, on the correlation Q of the noise and on the domain $\mathcal{O}$ and the boundary conditions (see for example [1, 12, 13, 26, 29, 30, 31, 32, 36, 37]). We will specify our hypotheses in the following sections.

In this paper we explore the asymptotic behavior of the solution to (1.4) as the mass density of the material μ vanishes. This is the infinite dimensional analogue to the Smoluchowski-Kramers approximation described in (1.3). In Sections 3 and 4, where the case of additive and multiplicative noise are considered, respectively, we review the results of [3, 4] to show that as, $\mu \to 0$, the solution of equation (1.4) converges to the solution of the stochastic heat equation

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + b(x,u(t,x)) + g(x,u(t,x))\dfrac{\partial w^Q}{\partial t}(t,x), \;\; x \in \mathcal{O}, \;\; t \geq 0, \\ \\ u(0,x) = u_0(x), \quad u(t,x) = 0, \quad x \in \partial\mathcal{O}. \end{array}\right. \tag{1.5}$$

Specifically, if we denote by u^μ and u the solutions to (1.4) and (1.5), respectively, we show that for any $T > 0$ and $\delta > 0$ fixed,

$$\lim_{\mu \to 0} \mathbb{P}\left(\sup_{0 \leq t \leq T} \int_{\mathcal{O}} |u^\mu(t,x) - u(t,x)|^2 dx > \delta \right) = 0. \tag{1.6}$$

This means that when the mass density of a material is small, the stochastic heat equation approximates the stochastic wave equation well, on any finite time interval.

The proof of this infinite dimensional Smoluchowski-Kramers approximation requires uniform estimates on the Sobolev regularity in space and the Hölder regularity in time for the solutions to the stochastic wave equation (1.4). As known, such uniform bounds are used to establish tightness in an appropriate functional space. Once we obtain a weakly convergent subsequence, by the Prokhorov theorem, the identification of the unique limit, and hence the convergence of the whole sequence, is obtained by using a non-trivial integration-by-parts formula for SPDEs.

In Section 5 we review the results of [8], which concern the Smoluchowski-Kramers approximation for an electrically charged material in the presence of a uniform magnetic field. In this case, the displacement of the material is modeled by the equation

$$\left\{\begin{array}{l} \mu\dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + b(u(t,x),x,t) + \vec{m}\times\dfrac{\partial u}{\partial t} + g(u(t,x),x,t)\dfrac{\partial w^Q}{\partial t}(t,x), \\[10pt] u(0,x) = u_0(x), \quad \dfrac{\partial u}{\partial t}(0,x) + v_0(x),\ x\in\mathcal{O}, \quad u(t,x)=0,\ x\in\partial\mathcal{O}, \end{array}\right. \tag{1.7}$$

where $\vec{m} = (0,0,m)$ is a constant vector field that is perpendicular to the plane of motion of the material and that models a magnetic field (notice that here the symbol $\times$ denotes the usual vector product in $\mathbb{R}^3$). The material is assumed to be electrically charged and therefore the above equation describes the movement of an elastic material with constant mass density $\mu > 0$, that is exposed to the electric field, some deterministic forcing b, and a state dependent stochastic forcing $g\,\partial w^Q/\partial t$.

One might hope, based on the results from Sections 3 and 4, that, for any $T > 0$ and $\delta > 0$ fixed, a limit analogous to (1.6) holds, where u_μ is the solution to equation (1.7) and u is the solution to the equation

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t}(t,x) = J_0^{-1}\left[\Delta u(t,x) + b(u(t,x),x,t) + g(u(t,x),x,t)\,\dfrac{\partial w^Q}{\partial t}(t,x)\right], \\[10pt] u(t,x) = 0, \quad x\in\partial\mathcal{O}, \qquad\quad u(0,x) = u_0(x), \quad x\in\mathcal{O}, \end{array}\right. \tag{1.8}$$

where J_0^{-1} is the inverse of the matrix

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Unfortunately, because of the presence of the stochastic term, the small-mass limit (1.6) is not true anymore. A similar situation was explored in [6], where it was shown that, in the case of a system with a finite number of degrees of freedom, the difference $u_\mu - u$ does not converge to zero, as μ tends to zero. Thus, in Section 5 we study the problem of the small mass limit by adding a small friction $-\epsilon\,\partial u/\partial t$ to equation (1.7) and we show that, for any fixed $\epsilon > 0$, the solution to (1.7), with fixed positive friction, converges to the solution of the system of semi-linear stochastic heat-equations formally obtained by setting $\mu = 0$.

Here, we use a slightly different line of argument than in Sections 3 and 4, which allows us to prove the validity of the convergence in $L^p(\Omega; C([0,T]; L^2(\mathcal{O})))$. This means that, if u_μ^ϵ is the solution to the wave equation exposed to a magnetic field and an ϵ friction, and if u_ϵ is the solution to the equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = (\epsilon\, I + J_0)^{-1}\left[\Delta u(t,x) + b(u(t,x),x,t) + g(u(t,x),x,t)\,\dfrac{\partial w^Q}{\partial t}(t,x)\right], \\ \\ u(t,x) = 0, \quad x \in \partial\mathcal{O}, \qquad\qquad u(0,x) = u_0(x), \quad x \in \mathcal{O}, \end{cases}$$

then for any $p \geq 1$ and $T \geq 0$ fixed,

$$\lim_{\mu\to 0} \mathbb{E}\left(\sup_{t\leq T} \int_{\mathcal{O}} \left|u^\epsilon_\mu(t,x) - u_\epsilon(t,x)\right|^2 dx\right)^{\frac{p}{2}} = 0. \tag{1.9}$$

Notice that the L^p convergence can also be proven in the case there is no magnetic field, as long as b and g are suitably regular.

In Sections 6, 7, and 8, we investigate the multi-scale interactions between the small mass limit and long-time behaviors of the stochastic wave equation. In Sections 3 to 5, we demonstrate that the solutions to (1.4) converge to the solution to (1.5) in the topology of $C([0,T];L^2(\mathcal{O}))$. In fact (1.6) is only true for fixed finite $T > 0$ and, on longer time scales, the solutions will deviate arbitrarily far apart in a pathwise sense. But, despite the fact that the Smoluchowski-Kramers approximation is not valid in a pathwise sense on long-time scales, there are still many important long-time characteristics of the small-mass wave equation that are approximated by the heat equation.

In Section 6 we review some results from [3] concerning the relationship between the Smoluchowski-Kramers approximation and invariant measures for equations (1.4) and (1.5), in the case of gradient systems. We recall here that by gradient system we mean a system where the noise is additive (that is $g \equiv I$) and, if $B : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ is defined as

$$B(h)(x) = b(h(x)), \qquad x \in \mathcal{O},$$

then there exists a potential functional $F : L^2(\mathcal{O}) \to \mathbb{R}$, such that for any $h \in L^2(\mathcal{O})$

$$B(h) = -Q^2 DF(h),$$

where DF is the Fréchet derivative of F in $L^2(\mathcal{O})$ and Q is the correlation of the noise.

Due to the fact that (1.4) is a gradient system, by using suitable finite dimensional approximations, we show that the invariant measure for the pair $(u^\mu, \partial u^\mu/\partial t)$ solving (1.4) in the space $L^2(\mathcal{O}) \times H^{-1}(\mathcal{O})$ is given by the probability measure

$$\nu_\mu(du,dv) := \frac{1}{Z}\, e^{-2F(u)}\, \mathcal{N}\left(0,(-\Delta)^{-1}Q^2/2\right)(du) \times \mathcal{N}\left(0,(-\Delta)^{-1}Q^2/2\mu\right)(dv),$$

where Z is a normalizing constant, independent of $\mu > 0$. This means that we have an explicit expression of the density of the invariant measure with respect to a suitable Gaussian measure on the product space $L^2(\mathcal{O}) \times H^{-1}(\mathcal{O})$. In particular, due to the special form of ν_μ, its first marginal $\Pi_1\nu_\mu$ does not depend on $\mu > 0$ and coincides with the probability measure

$$\nu(du) := \frac{1}{Z}\, e^{-2F(u)}\, \mathcal{N}\left(0,(-\Delta)^{-1}Q^2/2\right)(du),$$

which is the invariant measure of system (1.5). In particular, we have that the Smoluchowski-Kramers approximation is valid in the sense that the stochastic heat equation and the damped stochastic wave equation have the same long-time behavior.

The case of non-gradient systems is still open, and of course we cannot expect to have any explicit expression for the invariant measures of systems (1.4) and (1.5). Nevertheless, we expect that if ν_μ and ν denote the invariant measures of (1.4) and (1.5), respectively, then some sort of convergence for $\Pi_1\nu_\mu$ to ν holds, in the small mass μ limit.

In Sections 7 and 8 we review results from [10] and [9] about the relationship between the small mass and the small noise asymptotics. More precisely, we consider for any $\epsilon > 0$ the stochastic damped wave equation

$$\mu\frac{\partial^2 u}{\partial t^2}(t,x) + \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + b(x,u(t,x)) + \sqrt{\epsilon}\,\frac{\partial w^Q}{\partial t}(t,x), \tag{1.10}$$

with appropriate initial and boundary conditions and the corresponding heat equation

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + b(x,u(t,x)) + \sqrt{\epsilon}\,\frac{\partial w^Q}{\partial t}(t,x), \tag{1.11}$$

with the same initial and boundary conditions. We are interested in the multiscale behavior of system (1.10), as both μ and ϵ go to zero.

From the earlier results, it is clear that if we first take the limit as $\mu \to 0$ and then as $\epsilon \to 0$, the large deviation principle for the heat equation should describe the behavior of the system. Our purpose here is to show that when the order in the two limits is reversed and we first take the limit as $\epsilon \to 0$ and then $\mu \to 0$, the large deviations principle for the heat equation is still appropriate for studying the long-time behaviors of the wave equation. In particular, this result provides a rigorous mathematical justification of what is done in applications, when, in order to study rare events and transitions between metastable states for the more complicated system (1.10), as well as exit times from basins of attraction and the corresponding exit places, the relevant quantities associated with the large deviations for system (1.11) are considered.

Because of the dissipation introduced by the friction term, under suitable conditions on b, the solution to the unperturbed version of (1.10) (that is with $\epsilon = 0$) will converge to 0 as $t \to +\infty$. When the system is exposed to small random perturbations (that is when $0 < \epsilon \ll 1$), the solution will deviate from this equilibrium point on long time scales. It is thus of interest to study exit times of the form

$$\tau^{\mu,\epsilon} = \inf\{t > 0 : u^\mu_\epsilon(t,\cdot) \notin D\},$$

where $D \subset L^2(\mathcal{O})$ contains the equilibrium solution 0.

By extending to this infinite dimensional setting well known results in the theory of large deviations for finite dimensional systems (see [17, 22]), in Section 8 we show that the logarithmic exit time asymptotics, as well as the logarithmic expected value of the exit time and the exit position $u^\mu_\epsilon(\tau^{\mu,\epsilon},\cdot)$, can be characterized by the quasi-potential $\bar{V}_\mu = \inf_{v\in H^{-1}} V^\mu(u,v)$. More precisely, we show that, for small enough fixed $\mu > 0$ and appropriate initial conditions

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\tau^{\mu,\epsilon} = \inf_{u\in\,\partial D} \bar{V}_\mu(u), \tag{1.12}$$

and

$$\lim_{\epsilon\to 0} \epsilon \log\left(\tau^{\mu,\epsilon}\right) = \inf_{u\in\,\partial D} \bar{V}_\mu(u), \qquad \text{in probability.} \tag{1.13}$$

We also prove that if $N \subset \partial D$ has the property that $\inf_{u \in N} \bar{V}_\mu(u) > \inf_{u \in \partial D} \bar{V}_\mu(u)$, then

$$\lim_{\epsilon \to 0} \mathbb{P}\left(u^\mu_\epsilon(\tau^{\mu,\epsilon}) \in N\right) = 0. \tag{1.14}$$

Thus, due to the role played by the quasi-potential in the description of these important asymptotic features of the system, our purpose here is to compare the quasi-potential $V^\mu(u,v)$ associated with (1.10), with the quasi-potential $V(u)$ associated with (1.11), and to show that for any closed set $N \subset L^2(\mathcal{O})$ it holds

$$\lim_{\mu \to 0} \inf_{u \in N} \bar{V}_\mu(u) := \lim_{\mu \to 0} \inf_{u \in N} \inf_{v \in H^{-1}(\mathcal{O})} V^\mu(u,v) = \inf_{u \in N} V(u). \tag{1.15}$$

This means that, in the description of the large deviation principle, taking first the limit as $\epsilon \downarrow 0$ (large deviation) and then taking the limit as $\mu \downarrow 0$ (Smoluchowski-Kramers approximation) is the same as first taking the limit as $\mu \downarrow 0$ and then as $\epsilon \downarrow 0$.

In Section 7, we address this problem in the particular case system (1.10) is of gradient type. As for the invariant measures studied in Section 6, in the case of gradient systems all relevant quantities associated with the large deviation can be explicitly computed. In particular, we show that for any $\mu > 0$

$$V^\mu(u,v) = \left|(-\Delta)^{1/2} Q^{-1} u\right|^2_{L^2(\mathcal{O})} + 2F(u) + \mu \left|Q^{-1} v\right|^2_{L^2(\mathcal{O})}, \tag{1.16}$$

for any $(u,v) \in \mathrm{Dom}((-\Delta)^{1/2} Q^{-1}) \times \mathrm{Dom}(Q^{-1})$. Therefore, as

$$V(u) = \left|(-\Delta)^{1/2} Q^{-1} u\right|^2_{L^2(\mathcal{O})} + 2F(u), \qquad u \in \mathrm{Dom}((-\Delta)^{1/2} Q^{-1}),$$

from (1.16) we have that for any $\mu > 0$,

$$\bar{V}_\mu(u) := \inf_{v \in H^{-1}(\mathcal{O})} V^\mu(u,v) = V^\mu(u,0) = V(u), \qquad u \in \mathrm{Dom}((-\Delta)^{1/2} Q^{-1}). \tag{1.17}$$

In particular, this means that $\bar{V}_\mu(u)$ does not just coincide with $V(u)$ at the limit, but for any fixed $\mu > 0$.

In the general non-gradient case considered in Section 8, the situation is considerably more delicate and we cannot expect anything explicit as in (1.16). The lack of an explicit expression for $V^\mu(u,v)$ and $V(u)$ makes the proof of (1.15) much more difficult and requires the introduction of new arguments and techniques.

The first key idea in order to prove (1.15) is to characterize $V^\mu(u,v)$ as the minimum value for a suitable functional. We recall that the quasi-potential $V^\mu(u,v)$ is defined as the minimum energy required to the system to go from the asymptotically stable equilibrium 0 to the point $(u,v) \in L^2(\mathcal{O}) \times H^{-1}(\mathcal{O})$, in any time interval. Namely

$$V^\mu(u,v) = \inf \left\{ I^\mu_{0,T}(z) \; ; \; z(0) = 0, \; z(T) = (u,v), \; T > 0 \right\},$$

where

$$I^\mu_{0,T}(z) = \frac{1}{2} \inf \left\{ |\psi|^2_{L^2((0,T);L^2(\mathcal{O}))} \, : \, z = z^\mu_\psi \right\},$$

is the large deviation action functional and $z^\mu_\psi = (u^\mu_\psi, \partial u^\mu_\psi / \partial t)$ is a mild solution of the skeleton equation associated with equation (1.10), with control $\psi \in L^2((0,T); L^2(\mathcal{O}))$,

$$\mu \frac{\partial^2 u^\mu_\psi}{\partial t^2}(t) = \Delta u^\mu_\psi(t) - \frac{\partial u^\mu_\psi}{\partial t}(t) + B(u^\mu_\psi(t)) + Q\psi(t), \qquad t \in [0,T]. \tag{1.18}$$

By working thoroughly with the skeleton equation (1.18), we show that, for small enough $\mu > 0$,

$$V^{\mu}(u,v) = \min\left\{ I^{\mu}_{-\infty,0}(z) : \lim_{t\to-\infty} |z(t)|_{L^2(\mathcal{O})\times H^{-1}(\mathcal{O})} = 0,\ z(0) = (u,v) \right\} \tag{1.19}$$

where the minimum is taken over all $z \in C((-\infty,0]; L^2(\mathcal{O}) \times H^{-1}(\mathcal{O}))$. In particular, we get that the level sets of V^{μ} and $\bar{V}_{\mu}$ are compact in $L^2(\mathcal{O}) \times H^{-1}(\mathcal{O})$ and $L^2(\mathcal{O})$, respectively. Moreover, we show that both V^{μ} and $\bar{V}_{\mu}$ are well defined and continuous in suitable Sobolev spaces of functions.

The second key idea is based on the fact that, as in [11] where the finite dimensional case is studied, for all functions $z \in C((-\infty,0]; L^2(\mathcal{O}) \times H^{-1}(\mathcal{O}))$ that are regular enough, if we denote $\varphi(t) = \Pi_1 z(t)$, we have

$$\begin{aligned} I^{\mu}_{-\infty}(z) &= I_{-\infty}(\varphi) + \frac{\mu^2}{2}\int_{-\infty}^{0} \left| Q^{-1}\frac{\partial^2\varphi}{\partial t^2}(t)\right|^2_{L^2(\mathcal{O})} dt \\ &+\mu \int_{-\infty}^{0} \left\langle Q^{-1}\frac{\partial^2\varphi}{\partial t^2}(t), Q^{-1}\left(\frac{\partial\varphi}{\partial t}(t) - A\varphi(t) - B(\varphi(t))\right)\right\rangle_{L^2(\mathcal{O})} dt \\ &=: I_{-\infty}(\varphi) + J^{\mu}_{-\infty}(z). \end{aligned} \tag{1.20}$$

Thus, if $\bar{z}^{\mu}$ is the minimizer of $\bar{V}_{\mu}(u)$, whose existence is guaranteed by (1.19), and if $\bar{z}^{\mu}$ has enough regularity to guarantee that all terms in (1.20) are meaningful, we obtain

$$\bar{V}_{\mu}(u) = I_{-\infty}(\bar{\varphi}_{\mu}) + J^{\mu}_{-\infty}(\bar{z}^{\mu}) \geq V(u) + J^{\mu}_{-\infty}(\bar{z}^{\mu}). \tag{1.21}$$

In the same way, if $\bar{\varphi}$ is a minimizer for $V(u)$ and is regular enough, then

$$\bar{V}_{\mu}(u) \leq I^{\mu}_{-\infty}(\bar{\varphi}, \partial\bar{\varphi}/\partial t) = V(u) + J^{\mu}_{-\infty}((\bar{\varphi}, \partial\bar{\varphi}/\partial t)). \tag{1.22}$$

If we could prove that

$$\liminf_{\mu\to 0} J^{\mu}_{-\infty}(\bar{z}^{\mu}) = \limsup_{\mu\to 0} J^{\mu}_{-\infty}((\bar{\varphi}, \partial\bar{\varphi}/\partial t)) = 0, \tag{1.23}$$

from (1.21) and (1.22) we could conclude that (1.15) holds true. But unfortunately, neither $\bar{z}^{\mu}$ nor $\bar{\varphi}$ have the required regularity to justify (1.23). Thus, we have to proceed with suitable approximations, which, among other things, require us to prove the continuity of the mappings $\bar{V}_{\mu} : \mathrm{Dom}((-\Delta)^{1/2}Q^{-1}) \to \mathbb{R}$, uniformly with respect to $\mu \in (0,1]$.

In the second part of Section 8, we apply (1.15) to the study of the exit time and of the exit place of u^{μ}_{ϵ} from a given domain in $L^2(D)$. If

$$\tau^{\epsilon} = \inf\{t > 0 : u_{\epsilon}(t) \notin D\}$$

is the exit time from D for the solution of (1.11), and $V(u)$ is the quasi-potential associated with this system, the exit time and exit place results for the first-order system are analogous to (1.12), (1.13), and (1.14).

As a consequence of (1.17), in the gradient case, (1.12), and (1.13) imply that, for any fixed $\mu > 0$, the exit time and exit place asymptotics of (1.11) match those of (1.10). In particular, for any $\mu > 0$

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\tau^{\mu,\epsilon} = \inf_{u\in\partial D} V(u) = \lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\tau^{\epsilon}, \tag{1.24}$$

and

$$\lim_{\epsilon\to 0} \epsilon \log \tau^{\mu,\epsilon} = \inf_{u\in\partial D} V(u) = \lim_{\epsilon\to 0} \epsilon \log \tau^{\epsilon}, \qquad \text{in probability.} \tag{1.25}$$

In the general non-gradient case, we cannot have (1.24) and (1.25). Nevertheless, in view of (1.15), the exit time and exit place asymptotics of (1.10) can be approximated by V. Namely

$$\lim_{\mu\to 0}\lim_{\epsilon\to 0}\epsilon\log\mathbb{E}\tau^{\mu,\epsilon}=\inf_{u\in\partial D}V(u)=\lim_{\epsilon\to 0}\epsilon\log\mathbb{E}\tau^{\epsilon},$$

and

$$\lim_{\mu\to 0}\lim_{\epsilon\to 0}\epsilon\log\tau^{\mu,\epsilon}=\inf_{u\in\partial D}V(u)=\lim_{\epsilon\to 0}\epsilon\log\tau^{\epsilon},\qquad\text{in probability.}$$

2. **Notations, assumptions and a few preliminary results.** We denote by $\mathcal{O}$ a bounded open subset of $\mathbb{R}^d$, with $d\geq 1$, and we assume that $\mathcal{O}$ is of class C^3. In what follows we shall denote by $\{e_k\}_{k\in\mathbb{N}}$ the complete orthonormal basis which diagonalizes the Laplace operator Δ, endowed with Dirichlet boundary conditions on $\partial\mathcal{O}$. Moreover we shall denote by $\{-\alpha_k\}_{k\in\mathbb{N}}$ the corresponding sequence of eigenvalues.

For any $\delta\in\mathbb{R}$, we denote by $H^\delta(\mathcal{O})$ the completion of $C_0^\infty(\mathcal{O})$ with respect to the norm

$$|h|^2_{H^\delta(\mathcal{O})}=\sum_{i=1}^\infty\alpha_i^\delta\,\langle h,e_i\rangle^2_{L^2(\mathcal{O})}\,.$$

$H^\delta(\mathcal{O})$ is a Hilbert space, endowed with the scalar product

$$\langle h,k\rangle_{H^\delta(\mathcal{O})}=\sum_{i=1}^\infty\alpha_i^\delta\langle h,e_i\rangle\langle k,e_i\rangle,\qquad h,k\in H^\delta(\mathcal{O}).$$

Next, for any $\delta\in\mathbb{R}$ we denote by $\mathcal{H}_\delta$ the Hilbert space $H^\delta(\mathcal{O})\times H^{\delta-1}(\mathcal{O})$, endowed with the natural scalar product and norm inherited from each component. In what follows, we shall denote $L^2(\mathcal{O})=H^0(\mathcal{O})=:H$ and $\mathcal{H}_0=:\mathcal{H}$.

For any $\mu>0$ and $\delta\in\mathbb{R}$, we define the unbounded operator A_μ by setting

$$A_\mu(h,k)=\frac{1}{\mu}\left(\mu k,\Delta h-k\right),\qquad(h,k)\in D(A_\mu):=\mathcal{H}_{\delta+1}.$$

The operators A_μ defined on different $\mathcal{H}_\delta$ are all consistent. It is known that A_μ is the generator of a semigroup of bounded linear transformations $\{S_\mu(t)\}_{t\in\mathbb{R}}$ on $\mathcal{H}_\delta$ which is strongly continuous (for a proof see e.g. [35, section 7.4]). This means that for any $(u_0,v_0)\in\mathcal{H}_\delta$ and for any $\mu>0$, $S_\mu(t)(u_0,v_0)$ is the solution of the deterministic linear system

$$\left\{\begin{array}{l}\dfrac{\partial u}{\partial t}(t,x)=v(t,x),\qquad\mu\dfrac{\partial v}{\partial t}(t,x)=\Delta u(t,x)-v(t,x),\qquad t>0,\quad x\in\mathcal{O},\\[10pt] u(0)=u_0,\quad v(0)=v_0,\qquad u(t,x)=0,\qquad t\geq 0,\quad x\in\partial\mathcal{O},\end{array}\right.$$

which can be written as the following abstract evolution problem in $\mathcal{H}_\delta$

$$\frac{dz}{dt}(t)=A_\mu z(t),\qquad z(0)=(u_0,v_0),$$

where $z(t):=(u(t),v(t))$.

If we set $\Pi_1(u,v):=u$ and $\Pi_2(u,v):=v$, we have that

$$\Pi_1\,S_\mu(t)(u,v)=\sum_{k=1}^\infty f_k^\mu(t;u,v)e_k,\qquad\Pi_2\,S_\mu(t)(u,v)=\sum_{k=1}^\infty g_k^\mu(t;u,v)e_k,$$

where, for each $k \in \mathbb{N}$ and $\mu > 0$, the pair $(f_k^\mu(t;u,v), g_k^\mu(t;u,v))$ solves the system

$$\left\{\begin{array}{l} f'(t) = g(t), \qquad f(0) = \langle u, e_k\rangle_{L^2(\mathcal{O})} \\ \\ \mu\, g'(t) = -\alpha_k\, f(t) - g(t), \qquad g(0) = \langle v, e_k\rangle_{L^2(\mathcal{O})}. \end{array}\right. \tag{2.1}$$

In fact, both f_k^μ and g_k^μ can be explicitly computed (to this purpose see [3, Prop 2.2]). Moreover, in view of the explicit formulas, it is possible to prove suitable bounds for $f_k^\mu(t;u,v)$, which imply that for any fixed $\mu > 0$ and for any $\delta \in \mathbb{R}$ the semigroup $\{S_\mu(t)\}_{t\geq 0}$ is of negative type in $\mathcal{H}_\delta$. This means that there exist some $\omega_\mu > 0$ and $M_\mu > 0$ such that

$$\|S_\mu(t)\|_{\mathcal{L}(\mathcal{H}_\delta)} \leq M_\mu\, e^{-\omega_\mu t}, \qquad t \geq 0. \tag{2.2}$$

Notice that, in the case $\mathcal{O} = [0, L]$ and $\delta < 1/2$, $S_\mu(t)$ has an integral representation, in terms of a suitable kernel $K_\mu(t,x,y)$. That is, for any $v \in H^{-\delta}(0,L)$ it holds

$$\Pi_1 S_\mu(t)(0,v)(x) = \int_0^L K_\mu(t,x,y)v(y)\,dy, \qquad (t,x) \in [0,\infty) \times [0,L],$$

where $K_\mu : [0,\infty) \times [0,L]^2 \to \mathbb{R}$ is defined by

$$K_\mu(t,x,y) := \sum_{k=1}^\infty f_k^\mu(t;0,e_k)e_k(x)e_k(y).$$

The kernel $K_\mu(t,x,y)$ satisfies some regularity properties both in the time and in the space variables.

3. **The approximation in the case of additive noise.** We are here concerned with the following stochastic damped semilinear equation

$$\left\{\begin{array}{l} \mu\dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) - \dfrac{\partial u}{\partial t}(t,x) + b(x,u(t,x)) + \dfrac{\partial w^Q}{\partial t}(t,x), \qquad t > 0, \quad x \in \mathcal{O}, \\ \\ u(0,x) = u_0(x), \qquad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), \qquad u(t,x) = 0, \qquad t \geq 0, \quad x \in \partial\mathcal{O}. \end{array}\right. \tag{3.1}$$

Our aim is proving that the solution $u^\mu(t)$ converges to the solution of the stochastic semi-linear heat equation

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + b(x,u(t,x)) + \dfrac{\partial w^Q}{\partial t}(t,x), \qquad t > 0, \quad x \in \mathcal{O}, \\ \\ u(0,x) = u_0(x), \qquad u(t,x) = 0, \qquad t \geq 0, \quad x \in \partial\mathcal{O}, \end{array}\right. \tag{3.2}$$

as the parameter μ converges to zero.

Here and in what follows, $w^Q(t,x)$ is a cylindrical Wiener process. We shall assume that for any $h, k \in H$ and $t, s \geq 0$

$$\mathbb{E}\left\langle w^Q(t), h\right\rangle_H \left\langle w^Q(s), k\right\rangle_H = (t \wedge s)\left\langle Qh, k\right\rangle_H, \tag{3.3}$$

for some operator $Q \in \mathcal{L}(H)$. We will assume that Q satisfies the following condition.

Hypothesis 1. *The bounded linear operator $Q : H \to H$ is diagonal with respect to the basis $\{e_k\}_{k\in\mathbb{N}}$. If $\{\lambda_k\}_{k\in\mathbb{N}}$ denotes the corresponding sequence of eigenvalues, there exists a constant $\theta \in (0,1)$ such that*

$$\sum_{k=1}^{\infty} \frac{\lambda_k^2}{\alpha_k^{1-\theta}}\,|e_k|_\infty < \infty. \tag{3.4}$$

Remark 3.1. 1. *In several cases, as for example in the case of space dimension $d = 1$ or in the case $\mathcal{O}$ is a hypercube and the Laplace operator Δ is endowed with Dirichlet boundary conditions, the eigenfunctions e_k are equi-bounded in the sup-norm and then condition (3.4) becomes*

$$\sum_{k=1}^{\infty} \frac{\lambda_k^2}{\alpha_k^{1-\theta}} < \infty.$$

In general it holds

$$|e_k|_\infty \leq c\,k^\alpha, \qquad k \in \mathbb{N},$$

for some $\alpha \geq 0$. Thus, condition (3.4) is fulfilled if

$$\sum_{k=1}^{\infty} \frac{\lambda_k^2\,k^\alpha}{\alpha_k^{1-\theta}} < \infty.$$

2. *In case*

$$\alpha_k \sim k^{2/d}, \qquad k \in \mathbb{N},$$

(and this is true for several reasonable domains) (3.4) becomes

$$\sum_{k=1}^{\infty} \frac{\lambda_k^2}{k^{2(1-\theta)/d}}|e_k|_\infty < +\infty.$$

Thus, in dimension $d = 1$ condition (3.4) is fulfilled by a white noise in space and time. As soon as one goes to higher dimension, this of course is no longer possible. It is important to stress, however, that if the sup-norms of the eigenfunctions e_k are equi-bounded, condition (3.4) does not require a noise with trace-class covariance, no matter how large the space dimension is.

Concerning the nonlinearity b, we shall assume the following condition

Hypothesis 2. *The mapping $b : \bar{\mathcal{O}} \times \mathbb{R} \to \mathbb{R}$ is measurable and*

$$\sup_{x\in\bar{\mathcal{O}}} |b(x,\sigma) - b(x,\rho)| \leq L\,|\sigma - \rho|, \qquad \sigma,\, \rho \in \mathbb{R},$$

for some positive constant L. Moreover

$$\sup_{x\in\bar{\mathcal{O}}} |b(x,0)| =: b_0 < \infty.$$

3.1. **Estimates for the stochastic convolution.** For each $\mu > 0$, we consider the linear problem

$$\begin{cases} \mu\dfrac{\partial^2\eta}{\partial t^2}(t,x) = \Delta\eta(t,x) - \dfrac{\partial\eta}{\partial t}(t,x) + \dfrac{\partial w^Q}{\partial t}(t,x), & t > 0, \quad x \in \mathcal{O}, \\[2ex] \eta(0) = 0, \qquad \dfrac{\partial\eta}{\partial t}(0) = 0, \qquad \eta(t,x) = 0, & t \geq 0, \quad x \in \partial\mathcal{O}. \end{cases} \tag{3.5}$$

It is well known that if for some $\theta \in \mathbb{R}$ condition (3.4) holds, then for any $\mu > 0$ there exists a unique solution η^μ to problem (3.5) such that for any $T > 0$ and $p \geq 1$

$$\eta^\mu \in L^p(\Omega; C([0,T]; H^\theta(\mathcal{O}))), \qquad \frac{\partial \eta^\mu}{\partial t} \in L^p(\Omega; C([0,T]; H^{\theta-1}(\mathcal{O})))$$

(for a proof we refer for example to [14] and [13]).

If the constant θ above is strictly positive (as in Hypothesis 1), then for any $\mu > 0$ and $\delta < \theta/2$ the process η^μ has a version (which we still denote by η^μ) which is δ-Hölder continuous with respect to $(t,x) \in [0,T] \times \bar{\mathcal{O}}$, for any $T > 0$. Moreover, for any $p \geq 1$

$$\sup_{\mu>0} \mathbb{E}\, |\eta^\mu|^p_{C^\delta([0,T]\times\bar{\mathcal{O}})} =: c_{T,p} < \infty. \tag{3.6}$$

Due to (3.3) and Hypothesis 1, the cylindrical Wiener process $w^Q(t,x)$ can be written as

$$w^Q(t,x) = \sum_{k=1}^\infty Q e_k\, \beta_k(t) = \sum_{k=1}^\infty \lambda_k e_k(x)\, \beta_k(t), \qquad t \geq 0, \quad x \in \mathcal{O},$$

where $\{e_k\}_{k\in\mathbb{N}}$ is the complete orthonormal basis in H which diagonalizes Δ and $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is a sequence of mutually independent standard Brownian motions all defined on some stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. This means that for all $(t,x) \in [0,\infty) \times \bar{\mathcal{O}}$ we have

$$\eta^\mu(t,x) = \sum_{k=1}^\infty \eta^\mu_k(t) e_k(x),$$

where, for each $k \in \mathbb{N}$, $\eta^\mu_k(t)$ is the solution of the one dimensional problem

$$\left\{ \begin{array}{l} \mu \dfrac{d^2\eta^\mu_k}{dt^2}(t) = -\alpha_k \eta^\mu_k(t) - \dfrac{d\eta^\mu_k}{dt}(t) + \lambda_k \dfrac{d\beta_k}{dt}(t), \\[10pt] \eta^\mu_k(0) = 0, \qquad \dfrac{d\eta^\mu_k}{dt}(0) = 0. \end{array} \right. \tag{3.7}$$

Notice that the second order equation (3.7) can be rewritten more rigorously as the following system

$$\left\{ \begin{array}{l} d\eta^\mu_k(t) = \theta^\mu_k(t)\, dt \\[10pt] \mu\, d\theta^\mu_k(t) = -\left(\alpha_k \eta^\mu_k(t) + \theta^\mu_k(t)\right)\, dt + \lambda_k\, d\beta_k(t), \\[10pt] \eta^\mu_k(0) = 0, \qquad \theta^\mu_k(0) = 0. \end{array} \right.$$

Then, by the variation of constants formula, it is immediate to check that

$$\eta^\mu_k(t) = \frac{\lambda_k}{\mu} \int_0^t f^\mu_k(t-s; 0, e_k)\, d\beta_k(s),$$

and

$$\frac{d\eta^\mu_k}{dt}(t) = \theta^\mu_k(t) = \frac{\lambda_k}{\mu} \int_0^t g^\mu_k(t-s; 0, e_k)\, d\beta_k(s),$$

with f^μ_k and g^μ_k defined as the solutions of system (2.1).

In fact, in [3, Lemma 3.2] it has been shown that some estimates for the second moments of the increments of η^μ, both in time and in space, which are independent of $\mu > 0$, hold. As for any $t, s \geq 0$ and $x, y \in \bar{\mathcal{O}}$ the random variable $\eta^\mu(t,x) -$

$\eta^\mu(s,y)$ is Gaussian, such estimates imply that for any $p \geq 1$ there exists a constant c_p such that

$$\mathbb{E}\,|\eta^\mu(t,x)-\eta^\mu(s,y)|^p \leq c_p\left(|t-s|^2+|x-y|^2\right)^{\frac{\theta p}{4}}.$$

Due to the arbitrariness of $p \geq 1$, the inequality above implies (3.6).

3.2. **The convergence result.** First of all, we show that if the initial conditions $u_0 \in H^1$ and $v_0 \in H$, then the families of functions

$$\{\Pi_1 S_\mu(\cdot)(u_0,v_0)\}_{\mu\in(0,1)} \subset L^\infty(0,\infty;H^1), \quad \{\Pi_2 S_\mu(\cdot)(u_0,v_0)\}_{\mu\in(0,1)} \subset L^2(0,\infty;H)$$

are equi-bounded. Hence, by the Ascoli-Arzelà theorem, we have

$$\{\Pi_1 S_\mu(\cdot)(u_0,v_0)\}_{\mu\in(0,1)} \subset C([0,T];H), \qquad \text{compactly},$$

for any $T > 0$.

Now, for any $\mu > 0$ and $\delta \in [0,1]$, we define the operators

$$B_\mu(h,k)(x) := \frac{1}{\mu}(0, b(x,h(x))), \quad (h,k) \in \mathcal{H}_\delta, \quad x \in \mathcal{O}, \tag{3.8}$$

and

$$\mathcal{Q}_\mu h = \frac{1}{\mu}(0, Qh), \qquad h \in H. \tag{3.9}$$

Note that, since $\delta \in [0,1]$, for any $z_1 = (u_1,v_1)$ and $z_2 = (u_2,v_2) \in \mathcal{H}_\delta$

$$|B_\mu(z_1)-B_\mu(z_2)|_{\mathcal{H}_\delta} = \frac{1}{\mu}\,|b(\cdot,u_1)-b(\cdot,u_2)|_{H^{\delta-1}(\mathcal{O})} \leq \frac{c}{\mu}\,|b(\cdot,u_1)-b(\cdot,u_2)|_H,$$

and then, thanks to Hypothesis 2

$$|B_\mu(z_1)-B_\mu(z_2)|_{\mathcal{H}_\delta} \leq \frac{c\,L}{\mu}|u_1-u_2|_H \leq \frac{c\,L}{\mu}|z_1-z_2|_{\mathcal{H}_\delta}.$$

Definition 3.2. Let $\delta \in [0,1]$. A process $z^\mu(t) = (u^\mu(t), v^\mu(t))$, $t \geq 0$, is a *mild solution* of problem (3.1) in $L^2(\Omega; C([0,T];\mathcal{H}_\delta)$, if

$$u^\mu \in L^2(\Omega; C([0,T]; H^\delta(\mathcal{O}))), \qquad v^\mu = \frac{\partial u^\mu}{\partial t} \in L^2(\Omega; C([0,T]; H^{\delta-1}(\mathcal{O}))),$$

for any $T > 0$, and

$$z^\mu(t) = S_\mu(t)(u_0,v_0) + \int_0^t S_\mu(t-s)B_\mu(z^\mu(s))\,ds + \int_0^t S_\mu(t-s)\,dw^{\mathcal{Q}_\mu}(s).$$

Note that with these notations, the weak solution η^μ of problem (3.5) studied in Subsection 3.1 can be written as

$$\eta^\mu(t) = \Pi_1 \int_0^t S_\mu(t-s)\,dw^{\mathcal{Q}_\mu}(s), \qquad t \geq 0,$$

and hence, if $z^\mu = (u^\mu, v^\mu)$ is a mild solution of problem (3.1), we have

$$u^\mu(t) = \Pi_1 S_\mu(t)(u_0,v_0) + \Pi_1 \int_0^t S_\mu(t-s)B_\mu(u^\mu(s), v^\mu(s))\,ds + \eta^\mu(t), \qquad t \geq 0.$$

Due to the Lipschitz continuity of b, and hence of B_μ, it is possible to prove that problem (3.1) is well posed in $L^2(\Omega; C([0,T];\mathcal{H}_\delta)$, for every $\mu > 0$ fixed (for a proof see e.g. [15, Theorem 5.3.1]).

Actually, for any $\mu > 0$ and for any initial data $u_0 \in H^\theta(\mathcal{O})$ and $v_0 \in H^{\theta-1}(\mathcal{O})$, there exists a unique mild solution $z^\mu(t)$ to problem (3.1). Moreover, for any $T > 0$ and $p \geq 1$ there exists $c_{p,\mu}(T) > 0$ such that

$$\mathbb{E} \sup_{t\in[0,T]} |z^\mu(t)|^p_{\mathcal{H}_\theta} \leq c_{p,\mu}(T) \left(1 + |(u_0, v_0)|^p_{\mathcal{H}_\theta}\right). \tag{3.10}$$

Once one has the well-posedness of equation (3.1), for every $\mu > 0$, in order to prove the validity of the Smoluchowski-Kramers approximation we have to show that the family of probability measures $\{\mathcal{L}(u^\mu)\}_{\mu\in[0,1]}$ is tight on $C([0,T]; L^2(\mathcal{O}))$, for any $T > 0$. In [3, Proposition 4.3] we show that this is the case if $u_0 \in H^1(\mathcal{O})$ and $v_0 \in L^2(\mathcal{O})$.

Next, once one has the tightness of the family $\{\mathcal{L}(u^\mu)\}_{\mu\in[0,1]}$ on $C([0,T]; H)$, by the Prokhorov theorem for every sequence $\{\mathcal{L}(u^{\mu_n})\}_{n\in\mathbb{N}}$ there exists a subsequence $\{\mathcal{L}(u^{\mu_{n_k}})\}_{k\in\mathbb{N}}$ and a probability $\mathbb{Q}$ on $C([0,T]; H)$ such that $\mathcal{L}(u^{\mu_{n_k}})$ weakly converges to $\mathbb{Q}$.

The final step of our proof consists in identifying the probability $\mathbb{Q}$ with $\mathcal{L}(u)$ and showing that in fact the family u^μ converges to u in $C([0,T]; H)$ in the sense of convergence in probability.

The following integration by parts formula can be proven for any $u_0 \in H^\theta(\mathcal{O})$ and $v_0 \in H^{\theta-1}(\mathcal{O})$, for any $\mu > 0$ and for any $\varphi \in C^2([0,T] \times \bar{\mathcal{O}})$, such that $\varphi \equiv 0$ on $\partial\mathcal{O}$

$$\begin{aligned}
&\int_{\mathcal{O}} u^\mu(t,x)\varphi(t,x)\,dx \\
&= \int_{\mathcal{O}} u_0(x)\varphi(0,x)\,dx + \int_0^t \int_{\mathcal{O}} u^\mu(s,x)\left[\frac{\partial\varphi}{\partial t}(s,x) + \Delta\varphi(s,x)\right] ds\,dx \\
&+ \int_0^t \int_{\mathcal{O}} b(x, u^\mu(s,x))\varphi(s,x)\,ds\,dx + \int_0^t \int_{\mathcal{O}} \varphi(s,x)\,w^Q(ds,dx) + R_\mu(t).
\end{aligned} \tag{3.11}$$

Here $R_\mu(t)$ is a suitable remainder term such that

$$\lim_{\mu\to 0} \mathbb{E}\,|R_\mu(t)|^2 = 0, \qquad t \geq 0. \tag{3.12}$$

Due to the tightness in $C([0,1]; H)$ of the sequence $\{\mathcal{L}(u^\mu)\}_{\mu\in[0,1]}$, the Skorokhod theorem assures that for any two sequences $\{\mu_n\}_n$ and $\{\mu_m\}_m$ converging to zero there exist subsequences $\{\mu_{n(k)}\}_{k\in\mathbb{N}}$ and $\{\mu_{m(k)}\}_{k\in\mathbb{N}}$ and a sequence of random elements

$$\{\rho_k\}_{k\in\mathbb{N}} := \left\{(u_1^k, u_2^k, \hat{w_k}^Q)\right\}_{k\in\mathbb{N}},$$

in $\mathcal{C} := C([0,T]; H)^2 \times C([0,T]; \mathcal{D}'(\mathcal{O}))$, defined on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, such that the law of ρ_k coincides with the law of $(u^{\mu_{n(k)}}, u^{\mu_{m(k)}}, w^Q)$, for each $k \in \mathbb{N}$, and ρ_k converges $\hat{\mathbb{P}}$-a.s. to some random element $\rho := (u_1, u_2, \hat{w}^Q) \in \mathcal{C}$.

Now, if we show that $u_1 = u_2$, we have that there exists some $u \in C([0,T]; H)$ such that u^μ converges to u in probability. Actually, as observed by Gyöngy and Krylov in [23], if E is any Polish space equipped with the Borel σ-algebra, a sequence $\{\rho_n\}$ of E-valued random variables converges in probability if and only if for every pair of subsequences $\{\rho_m\}$ and $\{\rho_l\}$ there exists an E^2-valued subsequence $w_k := (\rho_{m(k)}, \rho_{l(k)})$ converging weakly to a random variable w supported on the diagonal $\{(h,k) \in E^2\ :\ h = k\}$.

Note that both u_1^k and u_2^k solve equation (3.1) with w^Q replaced by $\hat{w}_k{}^Q$. Then they both verify formula (3.11), with R_1^k and R_2^k obtained replacing u^μ respectively with u_1^k and u_2^k and w^Q with $\hat{w}_k^Q$. According to (3.12), we have that both R_1^k and R_2^k converge to zero in $L^2(\hat{\Omega})$, as $m_{n(k)}$ and $\mu_{m(k)}$ go to zero, and then, possibly for a subsequence, they converge $\hat{\mathbb{P}}$-a.s. to zero. Due to formula (3.11) this implies

$$\begin{aligned}
&\int_{\mathcal{O}} u_i(t,x)\varphi(t,x)\,dx\\
&=\int_{\mathcal{O}} u_0(x)\varphi(0,x)\,dx+\int_0^t\int_{\mathcal{O}} u_i(t,x)\left[\frac{\partial\varphi}{\partial t}(s,x)+\Delta\varphi(s,x)\right]ds\,dx\\
&+\int_0^t\int_{\mathcal{O}} b(x,u_i(s,x))\varphi(s,x)\,ds\,dx+\int_0^t\int_{\mathcal{O}}\varphi(s,x)\,\hat{w}^Q(ds,dx),\qquad i=1,2,
\end{aligned}$$

and then they coincide with the solution of the semi-linear heat equation perturbed by the noise $\hat{w}^Q$, which is unique.

As we have recalled above, thanks to the Gyöngy-Krylov remark in [23] this implies that u^μ converges in probability to some random variable $u\in C([0,T];H)$. But, by using again formula (3.11) and Lemma 3.12 we have that u solves the heat equation (3.2).

We have just proved the main result of this section.

Theorem 3.3. *Assume Hypotheses 1 and 2 and fix $u_0\in H^1(\mathcal{O})$ and $v_0\in L^2(\mathcal{O})$. Then, if u^μ is the solution of the stochastic semi-linear damped wave equation (3.1) and u is the solution of the stochastic semi-linear heat equation (3.2), for any $T>0$ and for any $\epsilon>0$ we have*

$$\lim_{\mu\to 0}\mathbb{P}\left(|u^\mu-u|_{C([0,T];H)}>\epsilon\right)=0.$$

4. **The approximation in the case of multiplicative noise.** We are dealing here with the following damped semi-linear wave equation perturbed by multiplicative noise in the interval $[0,L]$

$$\left\{\begin{array}{l}
\displaystyle\mu\frac{\partial^2 u}{\partial t^2}(t,x)+\frac{\partial u}{\partial t}(t,x)=\frac{\partial^2 u}{\partial x^2}(t,x)+b(x,u(t,x))+g(x,u(t,x))\frac{\partial w}{\partial t}(t,x),\\[10pt]
\displaystyle u(0,x)=u_0(x),\qquad \frac{\partial u}{\partial t}(0,x)=v_0(x),\qquad u(t,0)=u(t,L)=0.
\end{array}\right. \tag{4.1}$$

In the present section, we assume that $w(t,x)$ is a cylindrical Wiener process, white in space and time. The coefficients b and g are measurable from $[0,L]\times\mathbb{R}$ with values in $\mathbb{R}$ and g is Lipschitz-continuous in the second variable, uniformly with respect the first one. The coefficient b is either assumed to be Lipschitz-continuous in the second variable, uniformly with respect the first one, or satisfying some polynomial growth and dissipation conditions in the spirit of the Klein-Gordon model (and in this second case g is assumed bounded).

As in the case of Lipschitz continuous b and additive noise in any space dimension, considered in Section 3, we want to show that for any $\epsilon>0$ and $T>0$

$$\lim_{\mu\to 0}\mathbb{P}\left(\sup_{t\in[0,T]}|u^\mu(t)-u(t)|_{L^2(0,L)}>\epsilon\right)=0,$$

where $u(t)$ is the solution of the parabolic problem

$$\left\{\begin{array}{l}\dfrac{\partial u}{\partial t}(t,x)=\Delta u(t,x)+b(x,u(t,x))+g(x,u(t,x))\dfrac{\partial w}{\partial t}(t,x),\qquad x\in\,[0,L],\\[10pt] u(0,x)=u_0(x),\qquad u(t,0)=u(t,L)=0.\end{array}\right. \tag{4.2}$$

4.1. **The coefficients b and g.** Concerning the coefficients b and g, as we already mentioned above,we shall consider two different cases. Here is described the first one.

Hypothesis 3. 1. *The mapping $b:[0,L]\times\mathbb{R}\to\mathbb{R}$ is measurable and there exists $M>0$ such that*

$$\sup_{x\in\,[0,L]}|b(x,\sigma)-b(x,\rho)|\leq M\,|\sigma-\rho|,$$

for any $\sigma,\rho\in\,\mathbb{R}$. Moreover, $\sup_{x\in\,[0,L]}|b(x,0)|=:b_0<\infty$.

2. *The mapping $g:[0,L]\times\mathbb{R}\to\mathbb{R}$ is measurable and there exists $M>0$ such that*

$$\sup_{x\in\,[0,L]}|g(x,\sigma)-g(x,\rho)|\leq M\,|\sigma-\rho|,$$

for any $\sigma,\rho\in\,\mathbb{R}$. Moreover, $\sup_{x\in\,[0,L]}|g(x,0)|=:g_0<\infty$.

In particular from the assumptions above we have that both b and g have linear growth in the second variable, uniformly with respect to the first.

As we are assuming that $b(x,\cdot)$ is Lipschitz continuous, uniformly with respect to $x\in\,[0,L]$, we have seen that the mapping B_μ defined in (3.8) is Lipschitz continuous from $\mathcal{H}_\delta$ into itself.

Now, for any $\mu>0$ and $\delta\in\,[0,1]$, we define

$$[G_\mu(u,v)h](x):=\frac{1}{\mu}\,(0,g(x,u(x)))h(x),\quad x\in\,[0,L],\quad (u,v)\in\,\mathcal{H}_\delta,\quad h\in\,L^\infty(0,L).$$

Due to Hypothesis 3, the mapping $G_\mu(\cdot)h:\mathcal{H}_\delta\to\mathcal{H}_\delta$ is Lipschitz continuous, for any fixed $h\in\,L^\infty(0,L)$.

The second case that we shall consider is described below.

Hypothesis 4. 1. *The mapping $b:[0,L]\times\mathbb{R}\to\mathbb{R}$ is measurable and $b(x,\cdot):\mathbb{R}\to\mathbb{R}$ is of class C^1, for almost all $x\in\,[0,L]$. Moreover,*

(a) *there exist $\lambda\in\,(1,3]$ and $c_1>0$ such that for any $\sigma\in\,[0,L]$*

$$\sup_{x\in\,[0,L]}|b(x,\sigma)|\leq c_1\,\left(1+|\sigma|^\lambda\right),\qquad \sup_{x\in\,[0,L]}|\partial_\sigma\,b(x,\sigma)|\leq c_1\,\left(1+|\sigma|^{\lambda-1}\right);\tag{4.3}$$

(b) *there exists $c_2>0$ such that for any $\sigma\in\mathbb{R}$*

$$\sup_{x\in\,[0,L]}\int_0^\sigma b(x,\rho)\,d\rho\leq\,c_2\,\left(1-|\sigma|^{\lambda+1}\right);\tag{4.4}$$

(c) *for any $(x,\sigma)\in\,[0,L]\times\mathbb{R}$*

$$\partial_\sigma\,b(x,\sigma)\leq 0.\tag{4.5}$$

2. *The mapping $g:[0,L]\times\mathbb{R}\to\mathbb{R}$ is measurable and there exists $M>0$ such that*

$$\sup_{x\in\,[0,L]}|g(x,\sigma)-g(x,\rho)|\leq M\,|\sigma-\rho|,$$

for any $\sigma, \rho \in \mathbb{R}$. *Moreover,*

$$\sup_{(x,\sigma)\in[0,L]\times\mathbb{R}} |g(x,\sigma)| =: g_0 < \infty. \tag{4.6}$$

Remark 4.1. 1. *A typical example of a function* b *fulfilling conditions* (4.3), (4.4) *and* (4.5) *is*

$$b(x,\sigma) = -\alpha\,|\sigma|^{\lambda-1}\sigma,$$

for any strictly positive constant α *(the Klein-Gordon equation).*

2. *Here we are assuming the condition* $\partial_\sigma b(x,\sigma) \leq 0$, *for any* $(x,\sigma) \in [0,L]\times\mathbb{R}$, *just for simplicity of notations. Actually we could also treat the case*

$$\sup_{(x,\sigma)\in[0,L]\times\mathbb{R}} \partial_\sigma b(x,\sigma) \leq c,$$

for some constant c, *by setting* $b(x,\sigma) = b_1(x,\sigma) + b_2(x,\sigma)$, *where* $b_1(x,\sigma) = b(x,\sigma) - c\,\sigma$ *fulfills conditions* (4.3), (4.4) *and* (4.5), *and* $b_2(x,\sigma) = c\,\sigma$ *is a Lipschitz perturbation.*

As far as the diffusion coefficient G_μ is concerned, due to (4.6) in this second case the mapping $G_\mu(\cdot)h : \mathcal{H}_\delta \to \mathcal{H}_\delta$ is bounded, for any fixed $h \in L^\infty(0,L)$.

4.2. **Uniform bounds for the stochastic convolution.** For any $\mu > 0$ and $T > 0$ and for any $z \in L^p(\Omega; C([0,T];\mathcal{H}_\delta))$ we define

$$\Gamma_\mu(z)(t) := \int_0^t S_\mu(t-s)G_\mu(z(s))\,dw(s), \qquad t \in [0,T].$$

Our aim in this section is providing some a-priori estimates for $\Gamma_\mu(z)$, which are uniform with respect to $\mu \in (0,1]$. These estimates represent the key point in the proof of the tightness of the family of probability measures $\{\mathcal{L}(u^\mu)\}_{\mu\in(0,1]}$ in $C([0,T];H)$ and, as a consequence, of the Smoluchowski-Kramers approximation (see Section 3).

As shown in [4, Proposition 3.1], uniform bounds for $\Pi_1\Gamma_\mu(z)$, as a function of $t \in [0,T]$ with values in $H^\delta(0,L)$, hold.

Actually, for any $p > 4$, $\delta < 1/2 - 2/p$ and $z \in L^p(\Omega; L^p(0,T;\mathcal{H}))$ we have

$$\sup_{\mu>0} E\,|\Pi_1\Gamma_\mu(z)|^p_{C([0,T];H^\delta(0,L))} \leq c_p(T)\left(1 + \mathbb{E}\int_0^T |\Pi_1 z(t)|_H^p\,dt\right). \tag{4.7}$$

The estimates above for the $H^\delta(0,L)$-norm of $\Pi_1\Gamma_\mu(z)$, are obtained only for $\delta < 1/2$. This means that they do not provide any bound in the space of continuous functions. In fact, we need some pointwise uniform bounds, both in the space and in the time variables, which will lead us to uniform bounds in the space of Hölder continuous functions.

Proposition 4.2. [4, Proposition 3.2] *Assume that* $g : [0,L]\times\mathbb{R} \to \mathbb{R}$ *is a measurable mapping such that*

$$\sup_{x\in[0,L]} |g(x,\sigma)| \leq c\,(1 + |\sigma|^\kappa), \qquad \sigma \in \mathbb{R},$$

for some $\kappa \in [0,1]$. *Then, for any* $\mu > 0$, $\epsilon \in (0, 1/2\kappa)$, $\rho \in (0, 1/4 - \kappa(1/2 - \kappa\epsilon)/2)$, $p \geq 1$ *and* $z \in L^p(\Omega; C([0,T];\mathcal{H}_{\kappa\epsilon}))$, *the process* $\Pi_1\,\Gamma_\mu(z)$ *has a version which is* ρ*-Hölder continuous with respect to* $(t,x) \in [0,T]\times[0,L]$. *Moreover,*

$$\sup_{\mu>0} E\,|\Pi_1\,\Gamma_\mu(z)|^p_{C^\rho([0,T]\times[0,L])} \leq c\,\left(1 + \mathbb{E}\,|\Pi_1 z|^{p\kappa}_{C([0,T];H^{\kappa\epsilon})}\right),$$

for some constat $c = c(\epsilon, \rho, p)$.

Remark 4.3. *In particular, if Hypothesis 3 is satisfied, then* g *has linear growth* ($\kappa = 1$)*, so that as a consequence of Proposition 4.2 for any* $\epsilon \in (0, 1/2)$, $\rho < \epsilon/2$ *and* $p \geq 1$ *there exists some constant* $c = c(\epsilon, \rho, p, T)$ *such that*

$$\sup_{\mu>0} E\,|\Pi_1\,\Gamma_\mu(z)|^p_{C^\rho([0,T]\times[0,L])} \leq c\,\left(1 + \mathbb{E}\,|\Pi_1 z|^p_{C([0,T];H^\epsilon)}\right),$$

for any $z \in L^p(\Omega; C([0,T]; \mathcal{H}_\epsilon))$.

Analogously, if Hypothesis 4 is verified, then g *is bounded* ($\kappa = 0$)*, so that for any* $\rho < 1/4$ *and* $p \geq 1$ *there exists some constant* $c = c(\rho, p, T)$ *such that*

$$\sup_{\mu>0} E\,|\Pi_1\,\Gamma_\mu(z)|^p_{C^\rho([0,T]\times[0,L])} \leq c, \tag{4.8}$$

for any $z \in L^p(\Omega; C([0,T]; \mathcal{H}_0))$.

4.3. **The convergence result.** If we set $z := (u, \partial u/\partial t)$ equation (4.1) can be written in the following abstract form

$$dz(t) = [A_\mu z(t) + B_\mu(z(t))]\,dt + G_\mu(z(t))\,dw(t), \qquad z(0) = z_0 = (u_0, v_0). \tag{4.9}$$

As in Definition 3.2, a process $z_\mu = (u_\mu, v_\mu) \in L^p(\Omega; C([0,T]; \mathcal{H}_\delta))$ is a mild solution of (4.9), if

$$u^\mu \in L^p(\Omega; C([0,T]; H^\delta(0,L))), \qquad v^\mu \in L^p(\Omega; C([0,T]; H^{\delta-1}(0,L))),$$

and

$$z^\mu(t) = S_\mu(t)(u_0, v_0) + \int_0^t S_\mu(t-s) B_\mu(z^\mu(s))\,ds + \int_0^t S_\mu(t-s) G_\mu(z^\mu(s))\,dw(s).$$

The existence and uniqueness of a mild solution of equation (4.9) for any fixed $\mu > 0$ is a well known fact in the literature, both under Hypothesis 3 (see [1]) and under Hypothesis 4 (see [30] in the more delicate case of space dimension $d = 2$, under more restrictive conditions on the noise and on the initial data u_0 and v_0, due to the higher dimension). Namely, we have

Theorem 4.4. [4, Theorem 4.2] *Fix* $\mu > 0$ *and* $\delta \in [0, 1/2)$ *and assume Hypothesis 1. Then, both under Hypothesis 3 and under Hypothesis 4, for any* $T > 0$ *and* $p \geq 1$ *and for any initial datum* $z_0 = (u_0, v_0) \in \mathcal{H}_1$ *there exists a unique mild solution* z^μ *to problem* (4.9) *in* $L^p(\Omega; C([0,T]; \mathcal{H}_\delta))$.

The proof of the theorem above is obtained by considering an analog of equation (4.1), obtained by replacing the coefficient b with the truncated coefficients b_n. Uniform estimates are obtained for the solutions of the approximating problem and a global solution is obtained by introducing stopping times.The uniform bounds are obtained by using a splitting method and the uniform bounds for the stochastic convolution given in Proposition 4.2.

Remark 4.5.

1. *In the case of Lipschitz continuous* b*, in order to have solutions in* $L^p(\Omega; C([0,T]; \mathcal{H}_\delta))$ *it is not necessary to take the initial data* $z_0 = (u_0, v_0)$ *in* $\mathcal{H}_1$*, but it is sufficient to take them in* $\mathcal{H}_\delta$.
2. *The solution* z^μ *is also unique in* $L^p(\Omega; L^p(0,T; \mathcal{H}_\delta))$.

Now, the key point in the proof of our convergence result is showing that the family of probabilities $\{\mathcal{L}(u^\mu)\}_{\mu \in (0,1]}$ is tight in $C([0,T]; H)$.

Remark 4.6. *In the proof of tightness it is shown that if b is Lipschitz continuous, then*

$$\sup_{\mu\in(0,1]}\mathbb{E}\sup_{s\in[0,T]}|u^\mu(s)|^p_{H^\delta}\leq c_p(T)\left(1+|v_0|^p_{H^{\delta-1}}+|u_0|^p_{H^\delta}\right).$$

This implies that the family $\{u^\mu\}_{\mu\in(0,1]}$ is bounded in $L^p(\Omega;C([0,T];H^\delta))$, for any $p\geq 1$ and $\delta<1/2$. In the case that b is not Lipschitz continuous we cannot prove that. Nevertheless, we have that for any $\theta<1/4$ the family $\{u^\mu\}_{\mu\in(0,1]}$ is uniformly integrable in $C([0,T];C^\theta([0,L]))$, that is

$$\lim_{R\to\infty}\sup_{\mu\in(0,1]}\mathbb{P}\left(\sup_{t\in[0,T]}|u^\mu(t)|_{C^\theta([0,L])}>R\right)=0. \tag{4.10}$$

Now, as in the case of additive noise, it is possible to prove the following integration by parts formula. For any $\varphi\in C^2([0,T]\times[0,L])$, such that $\varphi(t,0)=\varphi(t,L)=0$,

$$\begin{aligned}\int_0^L u^\mu(t,x)\varphi(t,x)\,dx&=\int_0^L u_0(x)\varphi(0,x)\,dx\\&+\int_0^t\int_0^L u^\mu(s,x)\left[\frac{\partial\varphi}{\partial t}(s,x)+\Delta\varphi(s,x)\right]ds\,dx+\int_0^t\int_0^L b(x,u^\mu(s,x))\varphi(s,x)\,ds\,dx\\&+\int_0^t\int_0^L\varphi(s,x)g(x,u^\mu(s,x))\,w(ds,dx)+R_\mu(t),\end{aligned} \tag{4.11}$$

where R_μ is a suitable remainder such that

$$\lim_{\mu\to 0}\mathbb{P}\left(|R_\mu(t)|>\epsilon\right)=0. \tag{4.12}$$

As in the case of additive noise, the tightness of the family $\{\mathcal{L}(u^\mu)\}_{\mu\in(0,1]}$ of measures in $C([0,T];H)$, together with the integration by parts formula (4.11) and limit (4.12), imply the following result.

Theorem 4.7. [4, Theorem 4.9] *For any $\mu>0$, let $u^\mu=\Pi_1 z^\mu$, where z^μ is the mild solution of equation (4.9). Then, under either Hypothesis 3 or Hypothesis 4, for any $z_0=(u_0,v_0)\in\mathcal{H}_1$, $T>0$ and $\epsilon>0$ we have*

$$\lim_{\mu\to 0}\mathbb{P}\left(|u^\mu-u|_{C([0,T];H)}>\epsilon\right)=0,$$

where u is the solution of the semi-linear stochastic heat equation (4.2).

5. **The Smoluchowski-Kramers approximation in presence of a magnetic field.** We consider here the following two dimensional system of stochastic PDEs

$$\left\{\begin{array}{l}\mu\dfrac{\partial^2 u_\mu}{\partial t^2}(t,x)=\Delta u_\mu(t,x)+B(u_\mu(\cdot,t),t)+\vec{m}\times\dfrac{\partial u_\mu}{\partial t}(t,x)+G(u_\mu(\cdot,t),t)\,\dfrac{\partial w^Q}{\partial t},\\[10pt] u_\mu(0,x)=u_0(x),\quad\dfrac{\partial u_\mu}{\partial t}(0,x)=v_0(x),\quad x\in\mathcal{O},\qquad u_\mu(t,x)=0,\quad x\in\partial\mathcal{O},\end{array}\right. \tag{5.1}$$

where $\mathcal{O}$ is a bounded regular domain in $\mathbb{R}^d$, with $d\geq 1$, B and G are suitable nonlinearities, $\vec{m}=(0,0,m)$ is a constant vector and $w^Q(t,x)$ is a cylindrical Wiener process, white in time and colored in space, in the case of space dimension $d>1$.

By Newton's law, the vector field $u_\mu : [0,+\infty)\times\mathcal{O} \to \mathbb{R}^2$ models the displacement of a continuum of electrically charged particles with constant mass density $\mu > 0$ in the region $\mathcal{O} \subset \mathbb{R}^d$, in the presence of a noisy perturbation and a constant magnetic field $\vec{m} = (0, 0, m)$, which is orthogonal to the plane where the motion occurs (in what follows we shall assume just for simplicity of notations $m = 1$). For example, if $d = 1$ and $\mathcal{O} = [0, 1]$, this could model the displacement of a charged one-dimensional string, with fixed endpoints, that can move through two other spacial dimensions, where the Laplacian Δ models the forces neighboring particles exert on each other, the uniform magnetic field points in the direction of $\mathcal{O}$, B is some nonlinear forcing, and $\partial w^Q/\partial t$ is a Gaussian random forcing field, whose intensity G may depend on the state u_μ.

In Section 3 and Section 4, we have studied the validity of the Smoluchowski-Kramers approximation, in the case the magnetic field is replaced by a constant friction. Namely, it has been shown that, as μ tends to 0, the solutions of the second order system converge to the solution of the first order system which is obtained simply by taking $\mu = 0$.

One might hope that a similar result would be true in the case treated in the present section. Namely, one would expect that for any $T > 0$, $\delta > 0$,

$$\lim_{\mu\to 0} \mathbb{P}\left(\sup_{t\in[0,T]} |u_\mu(t) - u(t)|_{L^2(\mathcal{O};\mathbb{R}^2)} > \delta\right) = 0,$$

where $u(t)$ is the solution of the following system of stochastic PDEs

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t}(t,x) = J_0^{-1}\Big[\Delta u(t,x) + B(u(\cdot,t),t) + G(u(\cdot,t),t)\dfrac{\partial w^Q}{\partial t}(t,x)\Big] \\[10pt] u(t,x) = 0, \quad x \in \partial\mathcal{O}, \qquad u(0,x) = u_0(x), \quad x \in \mathcal{O}, \end{array}\right. \tag{5.2}$$

where

$$J_0^{-1} = -\,J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Unfortunately, as shown in [6] such a limit is not valid, even for finite dimensional analogues of this problem. Actually, one can prove that if the stochastic term in (5.1) is replaced by a continuous function, then u_μ would converge uniformly in $[0, T]$ to the solution of (5.2). But if we have the white noise term, this is not true anymore. An explanation of this lies in the fact that, while for any continuous function $\varphi(s)$ it holds

$$\lim_{\mu\to 0}\int_0^t \sin(s/\mu)\,\varphi(s)\,ds = 0,$$

if we consider a stochastic integral and replace $\varphi(s)ds$ with $dB(s)$, we have

$$\lim_{\mu\to 0}\int_0^t \sin(s/\mu)\,dB(s) \neq 0,$$

since

$$\mathrm{Var}\left(\int_0^t \sin(s/\mu)\,dB(s)\right) = \int_0^t \sin^2(s/\mu)\,ds \to \frac{t}{2}, \qquad \text{as} \quad \mu \downarrow 0.$$

Nevertheless, the problem under consideration can be regularized in such a way that a counterpart of the Smoluchowski-Kramers approximation is still valid. To this purpose, there are various ways to regularize the problem. One possible way consists in regularizing the noise (to this purpose, see [6] and [28] for the analysis of finite dimensional systems, both in the case of constant and in the case of state dependent magnetic field). Another possible way, which is the one we are using here, consists in introducing a small friction proportional to the velocity in equation (5.1) and considering the regularized problem

$$\left\{\begin{array}{l}\mu\dfrac{\partial^2 u^\epsilon_\mu}{\partial t^2}(t)=\Delta u^\epsilon_\mu(t)+B(u^\epsilon_\mu(t),t)+\vec{m}\times\dfrac{\partial u^\epsilon_\mu}{\partial t}(t)-\epsilon\,\dfrac{\partial u^\epsilon_\mu}{\partial t}(t)+G(u^\epsilon_\mu(t),t)\dfrac{\partial w^Q}{\partial t},\\[10pt] u^\epsilon_\mu(0)=u_0,\quad \dfrac{\partial u^\epsilon_\mu}{\partial t}(0)=v_0,\qquad u^\epsilon_\mu(t,x)=0,\quad x\in\partial\mathcal{O},\end{array}\right. \tag{5.3}$$

which now depends on two small positive parameters ϵ and μ. Our purpose here is showing that, for any fixed $\epsilon>0$, we can take the limit as μ goes to 0. Namely, we want to prove that for any $T>0$ and $p\geq 1$

$$\lim_{\mu\to 0}\mathbb{E}\sup_{t\in[0,T]}|u^\epsilon_\mu(t)-u_\epsilon(t)|^p_{L^2(\mathcal{O};\mathbb{R}^2)}=0,$$

where $u_\epsilon(t)$ is the unique mild solution of the problem

$$\left\{\begin{array}{l}\dfrac{\partial u_\epsilon}{\partial t}(t,x)=(J_0+\epsilon I)^{-1}\left[\Delta u_\epsilon(t,x)+B(u_\epsilon(\cdot,t),t)+G(u_\epsilon(\cdot,t),t)\dfrac{\partial w^Q}{\partial t}(t,x)\right],\\[10pt] u_\epsilon(t,x)=0,\quad x\in\partial\mathcal{O},\qquad u_\epsilon(0,x)=u_0(x),\qquad x\in\mathcal{O},\end{array}\right.$$

which is precisely what we get from (5.3) when we formally set $\mu=0$.

5.1. **Assumptions and notations.** In the present section, unlike in the rest of the paper, we shall denote by H the Hilbert space $L^2(\mathcal{O},\mathbb{R}^2)$, endowed with the scalar product

$$\langle(h_1,k_1),(h_2,k_2)\rangle_H=\int_{\mathcal{O}}h_1(x)h_2(x)\,dx+\int_{\mathcal{O}}k_1(x)k_2(x)\,dx,$$

and the corresponding norm $|\cdot|_H$.

Now, if Δ denotes the realization of the Laplace operator in $L^2(\mathcal{O})$, endowed with Dirichlet boundary conditions, there exists an orthonormal basis $\{\hat{e}_k\}$ for $L^2(\mathcal{O})$ and a positive sequence $\{\hat{\alpha}_k\}$ such that $\Delta\hat{e}_k=-\hat{\alpha}_k\hat{e}_k$, with $0<\hat{\alpha}_1\leq\hat{\alpha}_k\leq\hat{\alpha}_{k+1}$. Thus, if we define for any $k\in\mathbb{N}$,

$$e_{2k-1}=(\hat{e}_k,0),\ \ \alpha_{2k}=\hat{\alpha_k},$$
$$e_{2k}=(0,\hat{e}_k),\ \ \alpha_{2k+1}=\hat{\alpha_k},$$

we have that $\{e_k\}_{k=1}^\infty$ is a complete orthonormal basis of H. Moreover, if we define

$$D(A)=D(\Delta)\times D(\Delta),\quad A(h,k)=(\Delta h,\Delta k),\quad (h,k)\in D(A),$$

we have that

$$Ae_k=-\alpha_k e_k,\qquad k\in\mathbb{N}.$$

Next, in the same way as in section 3 and 4, for any $\delta\in\mathbb{R}$, we define H^δ to be the completion of $C_0^\infty(\mathcal{O};\mathbb{R}^2)$ with respect to the norm

$$|u|^2_{H^\delta}=\sum_{k=1}^\infty\alpha_k^\delta\,\langle u,e_k\rangle^2_H\,.$$

Moreover, we define $\mathcal{H}_\delta:=H^\delta\times H^{\delta-1}$, and in the case $\delta=0$ we simply set $\mathcal{H}:=\mathcal{H}_0$. Finally, for any $(h,k)\in\mathcal{H}_\delta$, we denote

$$\Pi_1(h,k)=h,\qquad \Pi_2(h,k)=k.$$

The cylindrical Wiener process $w^Q(t,x)$ is defined as the formal sum

$$w^Q(t,x)=\sum_{k=1}^{\infty} Qe_k(x)\beta_k(t),$$

where $Q=(Q_1,Q_2)\in \mathcal{L}(H)$, $\{\beta_k\}_{k\in\mathbb{N}}$ is a sequence of identical, independently distributed one-dimensional, Brownian motions defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ and $\{e_k\}_{k\in\mathbb{N}}$ is the orthonormal basis of $\mathcal{H}$ introduced above.

Concerning the non-linearity B we assume the following conditions

Hypothesis 5. *The mapping $B: H\times[0,+\infty)\to H$ is measurable. Moreover, for any $T>0$ there exists $\kappa_B(T)>0$ such that*

$$|B(u_1,t)-B(u_2,t)|_H\leq \kappa_B(T)|u_1-u_2|_H,\qquad u_1,u_2\in H,\quad t\in[0,T],$$

and

$$\sup_{t\in[0,T]}|B(0,t)|_H\leq\kappa_B(T).$$

In the case there exists some measurable $b:\mathbb{R}\times\mathcal{O}\times[0,+\infty)\to\mathbb{R}$ such that for any $h\in H$ and $t\geq 0$

$$B(h,t)(x)=b(h(x),x,t),\quad x\in\mathcal{O},$$

then Hypothesis 5 is satisfied if $b(\cdot,x,t):\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous and has linear growth, uniformly with respect to $x\in\mathcal{O}$ and $t\in[0,T]$, for any $T>0$.

Concerning the diffusion coefficient G, we assume the following

Hypothesis 6. *The mapping $G: H\times[0,+\infty)\to\mathcal{L}(L^\infty(\mathcal{O});H)$ is measurable and for any $T>0$ there exists $\kappa_G(T)>0$ such that*

$$|[G(h_1,t)-G(h_2,t)]\,z|_H\leq\kappa_G(T)|h_1-h_2|_H|z|_\infty,\ h_1,h_2\in H, z\in L^\infty(\mathcal{O}), t\in[0,T],$$

and

$$\sup_{t\in[0,T]}|G(0,t)z|_H\leq\kappa_G(T)|z|_\infty,\qquad z\in L^\infty(\mathcal{O}),\quad t\in[0,T].$$

If for any $h\in L^2(\mathcal{O})$ and $z\in L^\infty(\mathcal{O})$ we define

$$[G(h,t)z](x)=g(h(x),x,t)z(x),\qquad x\in\mathcal{O},$$

for some measurable $g:\mathbb{R}^2\times\mathcal{O}\times[0,+\infty]\to\mathcal{L}(\mathbb{R}^2)$, then Hypothesis 6 is satisfied if

$$\sup_{x\in\mathcal{O}}\sup_{t\in[0,T]}|g(h_1,x,t)-g(h_2,x,t)|_{\mathcal{L}(\mathbb{R}^2)}\leq\kappa_T|h_1-h_2|_{\mathbb{R}^2}$$

and it has linear growth

$$\sup_{x\in\mathcal{O}}\sup_{t\in[0,T]}|g(h,x,t)|_{\mathcal{L}(\mathbb{R}^2)}\leq\kappa_T(1+|h|_{\mathbb{R}^2}).$$

Now, for any $\mu>0$ and $\delta\in\mathbb{R}$, as in Section 2 we define on $\mathcal{H}_\delta$ the unbounded linear operator

$$A_\mu(u,v)=\frac{1}{\mu}(\mu v, Au-J_0v),\qquad (u,v)\in D(A_\mu)=\mathcal{H}_{\delta+1},$$

where J_0 is the skew symmetric 2×2 matrix

$$J_0=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

It can be proven that A_μ is the generator of a strongly continuous group of bounded linear operators $\{S_\mu(t)\}_{t\geq 0}$ on each $\mathcal{H}_\delta$ (for a proof see [35, Section 7.4]).

Moreover, for any $\mu > 0$ we define

$$B_\mu : \mathcal{H} \times [0, +\infty) \to \mathcal{H}, \qquad (z,t) \in \mathcal{H} \times [0,+\infty) \mapsto \frac{1}{\mu}(0, B(\Pi_1 z, t)),$$

and

$$G_\mu : \mathcal{H} \times [0, +\infty) \to \mathcal{L}(L^\infty(\mathcal{O}), \mathcal{H}), \qquad (z,t) \in \mathcal{H} \times [0,+\infty) \mapsto \frac{1}{\mu}(0, G(\Pi_1 z, t)).$$

With these notations, if we set

$$z_\mu(t) = \left(u_\mu(t), \frac{\partial u_\mu}{\partial t}(t) \right),$$

system (5.1) can be rewritten as the following stochastic equation in the Hilbert space $\mathcal{H}$

$$dz_\mu(t) = [A_\mu z_\mu(t) + B_\mu(z_\mu(t), t)]\, dt + G_\mu(z_\mu(t), t) dw^Q(t), \qquad z_\mu(0) = (u_0, v_0).$$

5.2. **The approximating semigroup.** For any $\mu, \epsilon > 0$ and $\delta \in \mathbb{R}$, we define

$$A_\mu^\epsilon(u,v) = \frac{1}{\mu}(\mu\, v, Au - J_\epsilon v), \qquad (u,v) \in D(A_\mu^\epsilon) = \mathcal{H}_{\delta+1},$$

where

$$J_\epsilon = J_0 + \epsilon I = \begin{pmatrix} \epsilon & 1 \\ -1 & \epsilon \end{pmatrix}, \qquad \epsilon > 0.$$

As we have seen for A_μ, it is possible to prove that for any $\mu, \epsilon > 0$ the operator A_μ^ϵ generates a strongly continuous group of bounded linear operators $S_\mu^\epsilon(t)$, $t \geq 0$, on $\mathcal{H}_\delta$. Moreover, it is possible to prove that there exist $M_{\mu,\epsilon}$, and $\omega_{\mu,\epsilon} > 0$ such that

$$\|S_\mu^\epsilon(t)\|_{\mathcal{L}(\mathcal{H}_\theta)} \leq M_{\mu,\epsilon}\, e^{-\omega_{\mu,\epsilon} t}, \qquad t \geq 0,$$

and for any $\mu, \epsilon > 0$, and for any $\theta \in \mathbb{R}$ and $\gamma \in [0,1]$ it holds

$$\left|\Pi_1 S_\mu^\epsilon(t)(0,v)\right|_{H^\theta} \leq 2^\gamma \mu^{\frac{1+\gamma}{2}} |v|_{H^{\theta+\gamma-1}}, \quad t \geq 0, \quad v \in H^{\theta+\gamma-1}. \tag{5.4}$$

Now, analogously as in Sections 3 and 4 for any $\mu > 0$ we define the bounded linear operator

$$Q_\mu : H \to \mathcal{H}, \qquad u \in H \mapsto \frac{1}{\mu}(0, Qu) \in \mathcal{H}.$$

Next, for any $\epsilon > 0$ we define

$$A_\epsilon := J_\epsilon^{-1} A = \frac{1}{1+\epsilon^2} \begin{pmatrix} \epsilon & -1 \\ 1 & \epsilon \end{pmatrix} \Delta,$$

and we denote by $T_\epsilon(t)$, $t \geq 0$, the strongly continuous semigroup generated by A_ϵ in H^θ, for any $\theta \in \mathbb{R}$. We can prove that

$$\|T_\epsilon(t)\|_{\mathcal{L}(H^\theta)} \leq e^{-\frac{\epsilon \alpha_1}{1+\epsilon^2} t}, \qquad t \geq 0.$$

Moreover, we denote

$$Q_\epsilon = J_\epsilon^{-1} Q.$$

In view of some fundamental estimates for $S_\mu^\epsilon(t)$ and $T_\epsilon(t)$, we have that for any $\epsilon > 0$ and $T > 0$ and for any $(u,v) \in \mathcal{H}$,

$$\lim_{\mu \to 0} \sup_{t \leq T} |\Pi_1 S_\mu^\epsilon(t)(u,v) - T_\epsilon(t) u|_H = 0, \tag{5.5}$$

and, for any $v \in H$ and $0 < t_0 \leq T$,

$$\lim_{\mu \to 0} \sup_{t_0 \leq t \leq T} \left| \frac{1}{\mu} \Pi_1 S^\epsilon_\mu(t)\,(0,v) - T_\epsilon(t) J_\epsilon^{-1} v \right|_H = 0. \tag{5.6}$$

Moreover, for any $\epsilon > 0$, $T > 0$ and $p \geq 1$ and for any $\psi \in L^p(\Omega; L^p([0,T]; H))$,

$$\lim_{\mu \to 0} \mathbb{E} \sup_{t \in [0,T]} \left| \frac{1}{\mu} \int_0^t \Pi_1 S^\epsilon_\mu(t-s)(0,\psi(s)) ds - \int_0^t T_\epsilon(t-s) J_\epsilon^{-1} \psi(s) ds \right|_H^p = 0. \tag{5.7}$$

5.3. **Approximation by small friction for additive noise.** We assume here that the noisy perturbation in system (5.1) is of additive type, that is $G(u,t) = I$, for any $u \in H$ and $t \geq 0$. Moreover, we assume that the covariance operator Q satisfies the following condition.

Hypothesis 7. *There exists a non-negative sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ such that $Qe_k = \lambda_k e_k$, for any $k \in \mathbb{N}$. Moreover, there exists $\delta > 0$ such that*

$$\sum_{k=1}^\infty \frac{\lambda_k^2}{\alpha_k^{1-\delta}} < \infty.$$

With the notations we have introduced above and in Section 3, if we denote

$$z^\epsilon_\mu(t) = (u^\epsilon_\mu(t), \frac{\partial u^\epsilon_\mu}{\partial t}(t)), \quad t \geq 0,$$

the regularized system (5.3) can be rewritten as the abstract evolution equation

$$dz^\epsilon_\mu(t) = \left[A^\epsilon_\mu z^\epsilon_\mu(t) + B_\mu(z^\epsilon_\mu(t), t) \right] dt + Q_\mu dw(t), \qquad z^\epsilon_\mu(0) = (u,v) \tag{5.8}$$

in the Hilbert space $\mathcal{H}$.

Our purpose here is to show that for any fixed $\epsilon > 0$ the process $u^\epsilon_\mu(t)$ converges to the solution $u_\epsilon(t)$ of the following system of stochastic PDEs

$$\left\{ \begin{array}{l} \displaystyle \frac{\partial u_\epsilon}{\partial t}(t) = J_\epsilon^{-1} \Delta u_\epsilon(t) + B_\epsilon(u_\epsilon(t), t) + \frac{\partial w^{Q_\epsilon}}{\partial t}(t) \\[10pt] u_\epsilon(0) = u_0, \quad u_\epsilon(t,x) = 0, \quad x \in \partial \mathcal{O}, \end{array} \right. \tag{5.9}$$

where for any $\epsilon > 0$ we have defined $Q_\epsilon = J_\epsilon^{-1} Q$ and

$$B_\epsilon(u,t) = J_\epsilon^{-1} B(u,t), \quad u \in H, \quad t \geq 0.$$

Notice that with these notations, system (5.9) can be rewritten as the abstract evolution equation

$$du_\epsilon(t) = \left[A_\epsilon u_\epsilon(t) + B_\epsilon(u_\epsilon(t), t) \right] dt + Q_\epsilon dw(t), \qquad u_\epsilon(0) = u_0, \tag{5.10}$$

in the Hilbert space H.

Due to our assumptions, the stochastic convolution

$$\Gamma^\epsilon_\mu(t) := \int_0^t S^\epsilon_\mu(s) Q_\mu\, dw(s), \qquad t \geq 0,$$

takes values in $L^p(\Omega; C([0,T]; \mathcal{H}))$, for any $T > 0$ and $p \geq 1$ (for a proof see [14]). Therefore, as the mapping $B_\mu(\cdot, t) : \mathcal{H} \to \mathcal{H}$ is Lipschitz-continuous, uniformly with respect to $t \in [0,T]$, we have that there exists a unique process $z^\epsilon_\mu \in L^p(\Omega; C([0,T]; \mathcal{H}))$ which solves equation (5.8) in the mild sense, that is

$$z^\epsilon_\mu(t) = S^\epsilon_\mu(t)(u_0, v_0) + \int_0^t S^\epsilon_\mu(t-s) B_\mu(z^\epsilon_\mu(s), s)\, ds + \Gamma^\epsilon_\mu(t).$$

In the same way, the stochastic convolution

$$\Gamma_\epsilon(t) := \int_0^t T_\epsilon(s) Q_\epsilon\, dw(s), \qquad t \geq 0,$$

takes values in $L^p(\Omega; C([0,T]; H))$, for any $T > 0$ and $p \geq 1$, so that, as the mapping $B_\epsilon(\cdot, t) : H \to H$ is Lipschitz-continuous, uniformly with respect to $t \in [0,T]$, we can conclude that there exists a unique process $u_\epsilon \in L^p(\Omega; C([0,T]; H))$ solving equation (5.10) in mild sense, that is

$$u_\epsilon(t) = T_\epsilon(t) u_0 + \int_0^t T_\epsilon(t-s) B_\epsilon(u_\epsilon(s), s) ds + \Gamma_\epsilon(t).$$

Theorem 5.1. [8, Theorem 4.1] *Under Hypotheses 5 and 7, for any $\epsilon > 0$, $T > 0$ and $p \geq 1$ and for any initial conditions $z_0 = (u_0, v_0) \in \mathcal{H}$, we have*

$$\lim_{\mu \to 0} \mathbb{E} \sup_{t \leq T} \left| u^\epsilon_\mu(t) - u_\epsilon(t) \right|^p_H = 0.$$

5.4. **Approximation by small friction for multiplicative noise.** In this section we assume that the space dimension $d = 1$ and $\mathcal{O}$ is a bounded interval, the diffusion coefficient G satisfies Hypothesis 6 and the covariance operator Q satisfies the following condition.

Hypothesis 8. *There exists a bounded non-negative sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ such that*

$$Q e_k = \lambda_k e_k, \qquad k \in \mathbb{N}.$$

We begin by studying the stochastic convolutions

$$\Gamma^\epsilon_\mu(z)(t) := \int_0^t S^\epsilon_\mu(t-s) G_\mu(z(s), s) dw^Q(s), \qquad z \in L^p(\Omega, C([0,T]; \mathcal{H})),$$

and

$$\Gamma_\epsilon(u)(t) = \int_0^t T_\epsilon(t-s) G_\epsilon(u(s), s) dw^Q(s), u \in L^p(\Omega, C([0,T]; H)).$$

By using a stochastic factorization argument, and limits (5.5), (5.6) and (5.7), we can prove that for any fixed $\epsilon > 0$, $T > 0$ and $p \geq 1$,

$$\lim_{\mu \to 0} \mathbb{E} \left| \Pi_1 \Gamma^\epsilon_\mu((u, 0)) - \Gamma_\epsilon(u) \right|^p_{C([0,T];H)} = 0. \tag{5.11}$$

With the notations introduced above, the regularized system (5.3) can be rewritten as

$$dz^\epsilon_\mu(t) = \left[A^\epsilon_\mu z^\epsilon_\mu(t) + B_\mu(z^\epsilon_\mu(t), t) \right] dt + G_\mu(z^\epsilon_\mu(t), t)\, dw^Q(t), \qquad z^\epsilon_\mu(0) = (u_0, v_0), \tag{5.12}$$

and the limiting problem (5.9) can be rewritten as

$$du_\epsilon(t) = \left[A_\epsilon u_\epsilon(t) + B_\epsilon(u_\epsilon(t), t) \right] dt + G_\epsilon(u_\epsilon(t), t)\, dw^Q(t), \qquad u_\epsilon(0) = u_0, \tag{5.13}$$

where

$$G_\epsilon(u, t) = J_\epsilon^{-1} G(u, t).$$

Notice that since the mapping $B_\epsilon(\cdot, t) : H \to H$ is Lipschitz continuous, uniformly for $t \in [0,T]$, we have that for any initial condition $u_0 \in \mathcal{H}$, system (5.12) admits a unique adapted mild solution $u_\epsilon \in L^p(\Omega; C([0,T]; H))$.

Theorem 5.2. *Let $z^\epsilon_\mu = (u^\epsilon_\mu, v^\epsilon_\mu)$ and u_ϵ be the mild solutions of problems (5.12) and (5.13), with initial conditions $z_0 \in \mathcal{H}$ and $u_0 = \Pi_1 z_0 \in H$, respectively. Then, under Hypotheses 2, 3 and 11, for any $T > 0$, $\epsilon > 0$ and $p \geq 1$ we have*

$$\lim_{\mu \to 0} \mathbb{E}|u^\epsilon_\mu - u_\epsilon|^p_{C([0,T];H)} = 0. \tag{5.14}$$

We have

$$u^\epsilon_\mu(t) = \Pi_1 S^\epsilon_\mu(t)(u_0, v_0) + \Pi_1 \int_0^t S^\epsilon_\mu(t-s) B_\mu(z^\epsilon_\mu(s), s)ds + \Pi_1 \Gamma^\epsilon_\mu(z^\epsilon_\mu)(t),$$

and

$$u_\epsilon(t) = T_\epsilon(t)u_0 + \int_0^t T_\epsilon(t-s)B_\epsilon(u_\epsilon(s), s) + \Gamma_\epsilon(u_\epsilon)(t).$$

Then

$$\begin{aligned}
&\left|u^\epsilon_\mu(t) - u_\epsilon(t)\right|_H \leq \left|\Pi_1 S^\epsilon_\mu(t)(u_0, v_0) - T_\epsilon(t)u_0\right|_H \\
&+ \left|\int_0^t \Pi_1 S^\epsilon_\mu(t-s)[B_\mu(z^\epsilon_\mu(s), s) - B_\mu((u_\epsilon(s), 0), s)]ds\right|_H \\
&+ \left|\frac{1}{\mu}\int_0^t \Pi_1 S^\epsilon_\mu(t-s)(0, B(u_\epsilon(s), s))ds - \int_0^t T_\epsilon(t-s)J_\epsilon^{-1} B(u_\epsilon(s), s)ds\right|_H \\
&+ \left|\Pi_1\left[\Gamma^\epsilon_\mu(z^\epsilon_\mu)(t) - \Gamma^\epsilon_\mu((u_\epsilon(t), 0))\right]\right|_H + \left|\Pi_1 \Gamma^\epsilon_\mu(u_\epsilon(t), 0) - \Gamma_\epsilon(u_\epsilon)(t)\right|_H.
\end{aligned}$$

This allows to conclude that (5.14) holds, as a consequence of (5.4), (5.5), (5.7), and (5.11).

5.5. **The convergence for $\epsilon \downarrow 0$.** Limit (5.14) is not uniform in $\epsilon > 0$, and it is not true for $\epsilon = 0$. In this section we want to show that

$$\lim_{\epsilon \to 0} \mathbb{E}\,|u_\epsilon - u|^p_{C([0,T];H)} = 0, \tag{5.15}$$

where u is the mild solution of the problem

$$du(t) = [A_0 u(t) + B_0(u(t), t)]\, dt + G_0(u(t), t)\, dw^Q(t), \qquad u(0) = u_0, \tag{5.16}$$

with

$$A_0 := J_0^{-1} A, \quad B_0 = J_0^{-1} B, \quad G_0 = J_0^{-1} G.$$

This statement is not true unless we strengthen Hypothesis 8.

Hypothesis 9. *There exists a non-negative sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ such that $Q e_k = \lambda_k e_k$, for any $k \in \mathbb{N}$, and*

$$\sum_{k=1}^\infty \lambda_k^2 < +\infty.$$

In what follows, we shall denote by $T_0(t)$, $t \geq 0$, the semigroup generated by the differential operator A_0 in H, with $D(A_0) = D(A)$. The semigroup $T_0(t)$ is strongly continuous in H. Moreover, if we define $u(t) = T_0(t)x$, for $x \in D(A_0)$, we have

$$\left\{\begin{array}{l}
\dfrac{\partial u_1}{\partial t}(t) = -\Delta u_2(t), \qquad u_1(0) = x_1 \\[10pt]
\dfrac{\partial u_2}{\partial t}(t) = \Delta u_1(t), \qquad u_2(0) = x_2
\end{array}\right.$$

This means that if we take the scalar product in H^θ of the first equation by u_1 and of the second equation by u_2, we get

$$\frac{d}{dt}|u(t)|^2_{H^\theta} = 0,$$

so that

$$|T_0(t)x|_{H^\theta} = |x|_{H^\theta}, \qquad t \geq 0, \tag{5.17}$$

for any $\theta \in \mathbb{R}$ and $x \in H$.

Now, let us consider the stochastic convolution associated with problem (5.16), in the simple case $G = I$

$$\Gamma(t) = \int_0^t T_0(t-s)Q dw(s), \qquad t \geq 0.$$

As a consequence of (5.17), we have

$$\mathbb{E}\,|\Gamma(t)|^2_H = \int_0^t \sum_{k=1}^\infty |T_0(s)Qe_k|^2_H\, ds = \int_0^t \sum_{k=1}^\infty |Qe_k|^2_H\, ds = t\sum_{k=1}^\infty \lambda_k^2,$$

and this implies that Hypothesis 9 is necessary in order to have a solution in H for the limiting equation (5.16).

Now, u_ϵ is the unique mild solution in $L^p(\Omega; C([0,T];H)$ of problem (5.10) (in the case of additive noise) or problem (5.13) (in the case of multiplicative noise), so that

$$u_\epsilon(t) = T_\epsilon(t)u_0 + \int_0^t T_\epsilon(t-s)B_\epsilon(u_\epsilon(s),s)\, ds + \Gamma_\epsilon(u_\epsilon)(t).$$

Moreover, $u(t)$ is the unique mild solution in $L^p(\Omega; C([0,T];H))$ of the problem

$$du(t) = [A_0u(t) + B_0(u(t),t)]\ dt + G_0(u(t),t)\, dw^Q(t), \qquad u(0) = u_0,$$

with $G_0 = J_0^{-1}I$ or $G_0 = J_0^{-1}G$, so that

$$u(t) = T_0(t)u_0 + \int_0^t T_0(t-s)B_0(u_\epsilon(s),s)\, ds + \Gamma_0(u_\epsilon)(t).$$

Then, we have the following result.

Theorem 5.3. [8, Theorem 6.4] *Assume either G satisfies Hypothesis 3 or $G(x,t) = I$. Then, under Hypotheses 2 and 10, we have that for any $T > 0$ and $p \geq 1$*

$$\lim_{\epsilon\to 0} \mathbb{E}\,|u_\epsilon - u|^p_{C([0,T];H)} = 0.$$

In fact, it is possible to show that the convergence result proved above for $\epsilon \downarrow 0$ is also valid for the second order system, that is for every $\mu > 0$ fixed.

Theorem 5.4. [8, Theorem 6.5] *Assume either G satisfies Hypothesis 6 or $G(x,t) = I$. Then, under Hypotheses 5 and 9, we have that for any initial conditions (u_0, v_0) and $\mu > 0$*

$$\lim_{\epsilon\to 0} \mathbb{E}\,|z_\mu^\epsilon - z_\epsilon|^p_{C([0,T];\mathcal{H})},$$

for any $T > 0$ and $p \geq 1$.

6. **The long time behavior.** In this section we study the relation between the stationary distributions of the processes $u^\mu(t)$ and $u(t)$, defined respectively as the solution of the semi-linear stochastic damped wave equation (3.1) and as the solution of the semi-linear stochastic heat equation (3.2).

If we set

$$z^\mu(t) := (u^\mu(t), v^\mu(t)), \qquad t \geq 0, \quad \mu > 0,$$

with the notations introduced in Section 3 we can write equation (3.1) as the abstract evolution equation on the Hilbert space $\mathcal{H} = H \times H^{-1} = L^2(\mathcal{O}) \times H^{-1}(\mathcal{O})$

$$dz^\mu(t) = [A_\mu z^\mu(t) + B_\mu(z^\mu(t))]\, dt + dw^{\mathcal{Q}_\mu}(t), \qquad z^\mu(0) = (u_0, v_0), \tag{6.1}$$

where B_μ and $\mathcal{Q}_\mu$ are the operators defined in (3.8) and (3.9), respectively.

Note that the adjoint of the operator $\mathcal{Q}_\mu : H \to \mathcal{H}$ is the operator $\mathcal{Q}^\star_\mu : \mathcal{H} \to H$ defined by

$$\mathcal{Q}^\star_\mu(u, v) = \frac{1}{\mu}(-\Delta)^{-1} Q v.$$

In particular we have that $\mathcal{Q}_\mu \mathcal{Q}^\star_\mu : \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{Q}_\mu \mathcal{Q}^\star_\mu\,(u, v) = \frac{1}{\mu^2}\,(0, (-\Delta)^{-1} Q^2 v), \qquad (u, v) \in \mathcal{H}.$$

Next, for any $\mu > 0$, we introduce the operator $C_\mu \in \mathcal{L}^+(\mathcal{H})$ by setting

$$C_\mu := \int_0^\infty S_\mu(s)\, \mathcal{Q}_\mu \mathcal{Q}^\star_\mu\, S^\star_\mu(s)\, ds,$$

where $\{S^\star_\mu(t)\}_{t \geq 0}$ is the semigroup generated by $A^\star_\mu$, the adjoint to the operator A_μ.

In fact, we have an explicit expression for the operator C_μ.

Proposition 6.1. [3, Proposition 5.1] *For every $\mu > 0$, we have*

$$C_\mu(u, v) = \frac{1}{2}\left((-\Delta)^{-1} Q^2 u, \frac{1}{\mu}(-\Delta)^{-1} Q^2 v\right), \qquad (u, v) \in \mathcal{H}. \tag{6.2}$$

In particular, if we assume that $Qe_k = \lambda_k e_k$, for every $k \in \mathbb{N}$, where $\{\lambda_k\}_{k \in \mathcal{N}}$ is a non-negative sequence such that

$$\sum_{k=1}^\infty \frac{\lambda_k^2}{\alpha_k} < \infty,$$

then C_μ is a trace-class operator with

$$\operatorname{Tr} C_\mu = \frac{1}{2}\left(1 + \frac{1}{\mu}\right) \sum_{k=1}^\infty \frac{\lambda_k^2}{\alpha_k}.$$

This means that C_μ is a non-negative, symmetric trace-class operator in $\mathcal{H}$, and we can conclude that the centered Gaussian measure

$$\nu_\mu := \mathcal{N}(0, C_\mu),$$

of covariance C_μ is well defined in $\mathcal{H}$. Moreover, ν_μ can be written as a product of two centered gaussian measures on H and H^{-1}, respectively, that is

$$\nu_\mu = \mathcal{N}(0, (-\Delta)^{-1} Q^2/2) \times \mathcal{N}(0, (-\Delta)^{-1} Q^2/2\mu).$$

6.1. **The linear case.** Our aim here is studying the invariant measure of the linear system

$$dz(t) = A_\mu z(t)\,dt + d\,w^{Q_\mu}(t), \qquad z(0) = (u_0, v_0) \in \mathcal{H}, \tag{6.3}$$

and showing that the stationary distribution for the solution of the linear damped wave equation

$$\mu\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) - \frac{\partial u}{\partial t}(t,x) + \frac{\partial w^Q}{\partial t}(t,x), \qquad u(t,x) = 0, \quad x \in \partial\mathcal{O}, \tag{6.4}$$

coincides for all $\mu > 0$ with the unique invariant measure of the stochastic heat equation

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \frac{\partial w^Q}{\partial t}(t,x), \qquad u(0,x) = 0, \quad x \in \partial\mathcal{O}. \tag{6.5}$$

In what follows, we shall assume the following conditions on Q.

Hypothesis 10. *The linear operator Q is bounded in H and diagonal with respect to the basis $\{e_k\}_{k\in\mathbb{N}}$ which diagonalizes A. Moreover, if $\{\lambda_k\}_{k\in\mathbb{N}}$ is the corresponding sequence of eigenvalues, we have*

$$\sum_{k=1}^{\infty}\frac{\lambda_k^2}{\alpha_k} < +\infty.$$

In particular, if $d = 1$ we can take $Q = I$, but if $d > 1$ the noise has to colored in space.

Theorem 6.2. [3, Theorem 5.2] *Under Hypothesis 10, the Gaussian measure $\nu_\mu = \mathcal{N}(0, C_\mu)$ is the unique invariant measure of system (6.3), for each $\mu > 0$, and for any $\varphi \in C_b(\mathcal{H})$ and $z_0 \in \mathcal{H}$*

$$\lim_{t\to\infty}\mathbb{E}^{z_0}\,\varphi(z^\mu(t)) = \int_{\mathcal{H}}\varphi(z)\,\mathcal{N}(0, C_\mu)(dz), \tag{6.6}$$

so that $\mathcal{N}(0, C_\mu)$ is ergodic and strongly mixing.

Moreover the Gaussian measure $\Pi_1\nu_\mu = \mathcal{N}(0, (-\Delta)^{-1}Q^2/2)$ is the stationary distribution of (6.4). In particular, $\Pi_1\nu_\mu$ does not depend on $\mu > 0$ and coincides with the unique invariant measure ν of the stochastic heat equation (6.5).

The operator C_μ is non-negative, symmetric and of trace-class on $\mathcal{H}$. Thus problem (6.3) admits an invariant measure of the form

$$\rho_\mu \star \mathcal{N}(0, C_\mu),$$

where ρ_μ is an invariant measure for the semigroup $S_\mu(t)$ and $\mathcal{N}(0, C_\mu)$ is the Gaussian measure, with zero mean and covariance operator C_μ (for a proof see e.g. [14, Theorem 11.7]). Moreover, as the semigroup $\{S_\mu(t)\}_{t\geq 0}$ is of negative type (2.2), due to [14, Theorem 11.11] $\mathcal{N}(0, C_\mu)$ is the unique invariant measure for (6.1) and (6.6) holds. As well known this implies that $\mathcal{N}(0, C_\mu)$ is ergodic and strongly mixing.

Next, due to (6.2) the measure $\mathcal{N}(0, C_\mu)$ defined on $\mathcal{H}$ is the product of two Gaussian measures, defined respectively on $L^2(\mathcal{O})$ and on $H^{-1}(\mathcal{O})$. Namely

$$\mathcal{N}(0, C_\mu) = \mathcal{N}\left(0, (-\Delta)^{-1}Q^2/2\right) \times \mathcal{N}\left(0, (-\Delta)^{-1}Q^2/2\mu\right).$$

In particular the marginal measure $\Pi_1\mathcal{N}(0, C_\mu)$ equals $\mathcal{N}(0, (-\Delta)^{-1}Q^2/2)$, so that it does not depend on $\mu > 0$ and coincides with the unique invariant measure ν of the Ornstein-Uhlenbeck process solving problem (6.5).

This allows us to conclude, as the process $\bar{u}^\mu(t) = \Pi_1 \bar{z}^\mu(t)$ with

$$\bar{z}^\mu(t) = (\bar{u}^\mu(t), \bar{v}^\mu(t)) := \int_{-\infty}^t S_\mu(t-s)\, d\bar{w}^{\mathcal{Q}_\mu}(s)$$

is the stationary solution to problem (6.4) and its distribution does coincides with the measure $\Pi_1 \mathcal{N}(0, C_\mu)$.

6.2. **The semi-linear case.** We show that an analogous result holds also in the non-linear case, when (3.1) is a gradient system. To this purpose, we need to assume that the non-linearity B has some special structure.

Hypothesis 11. *There exists $F : H \to \mathbb{R}$ of class C^1, with $F(0) = 0$, $F(h) \geq 0$ and $\langle DF(h), h\rangle \geq 0$ for all $h \in H$, such that*

$$B(h) = -Q^2 DF(h), \qquad h \in H.$$

Moreover, there exists some $\kappa > 0$ such that

$$|DF(h) - DF(k)|_H \leq \kappa\, |h-k|_H\,, \qquad h, k \in H.$$

Example 6.3. 1. *Assume $d = 1$ and take $Q = I$. Let $b : \mathbb{R} \to \mathbb{R}$ be a decreasing Lipschitz continuous function with $b(0) = 0$. Then the composition operator $B(h)(x) = b(h(x))$, $x \in \mathcal{O}$, is of gradient type. Actually, if we set*

$$F(h) = -\int_{\mathcal{O}} \int_0^{h(x)} b(\eta)\, d\eta\, dx, \qquad h \in H,$$

we have

$$B(h) = -DF(h), \qquad h \in H.$$

Moreover, it is clear that $F(0) = 0$, $F(h) \geq 0$ for all $h \in H$, and

$$\langle DF(h), h\rangle = -\int_{\mathcal{O}} b(h(x))h(x)\, dx \geq 0, \qquad h \in H.$$

2. *Assume now $d \geq 1$, so that Q is a general bounded operator in H, satisfying Hypothesis 1. Let $b : \mathbb{R} \to \mathbb{R}$ be a function of class C^1, with Lipschitz-continuous first derivative, such that $b(0) = 0$ and $b(x) \geq 0$, for all $x \in \mathbb{R}$. Moreover, the only local minimum of b occurs at 0. Let*

$$F(h) = \int_{\mathcal{O}} b(h(x))\, dx, \qquad h \in H.$$

It is immediate to check that $F(0) = 0$ and $F(h) \geq 0$, for all $h \in H$. Furthermore, for any $h \in H$

$$DF(h)(x) = b'(h(x)), \qquad x \in \mathcal{O}.$$

Therefore, the nonlinearity

$$B(h) = -Q^2 b'(h(\cdot)), \qquad h \in H,$$

satisfies Hypothesis 11.

By using suitable finite dimensional approximations, we show that, under the above conditions, system (6.1) is of gradient type and the measure

$$\nu_\mu(dz) := \frac{1}{Z}\, e^{-2F(u)}\, \mathcal{N}(0, C_\mu)(dz), \qquad \mu > 0,$$

is invariant for system (6.1). Moreover the distribution

$$\Pi_1 \nu_\mu(du) = \nu(du) := \frac{1}{Z}\, e^{-2F(u)}\, \mathcal{N}(0, Q^2(-\Delta)^{-1}/2)(du)$$

is stationary for equation (3.1) and coincides with the unique invariant measure for the stochastic semi-linear heat equation

$$du(t) = \left[\Delta u(t) - Q^2 DF(u(t))\right] dt + dw^Q(t), \qquad u(0) = u_0. \tag{6.7}$$

7. **Large deviations in the gradient case.** We are here interested in the small noise behavior of the solution of the evolution equation (6.1) in $\mathcal{H}$ and in comparing it with the small mass behavior of the solution of the evolution equation (6.7).

Namely, we introduce the equations

$$dz^\mu_\epsilon(t) = [A_\mu z^\mu_\epsilon(t) + B_\mu(z^\mu_\epsilon(t))]\, dt + \sqrt{\epsilon}\, dw^{\mathcal{Q}_\mu}(t), \qquad z^\mu_\epsilon(0) = (u_0, v_0) \in \mathcal{H}, \tag{7.1}$$

and

$$du^\epsilon(t) = [Au^\epsilon(t) + B(u^\epsilon(t))]dt + \sqrt{\epsilon} dw^Q(t), \qquad u^\epsilon(0) = u_0 \in H, \tag{7.2}$$

for some parameters $0 < \epsilon, \mu \ll 1$. We assume that Hypotheses 10 and 11 are verified so that the non-linearity B has a gradient structure.

We keep $\mu > 0$ fixed and let ϵ tend to zero, to study some relevant quantities associated with the large deviation principle for this system, as the quasi-potential that describes also the asymptotic behavior of the expected exit time from a domain and the corresponding exit places. Due to the gradient structure of (7.1), as in the finite dimensional case studied in [19], we are here able to calculate explicitly the quasi-potentials $V^\mu(u, v)$ for system (7.1) as

$$V^\mu(u, v) = \left|(-\Delta)^{\frac{1}{2}} Q^{-1} u\right|^2_H + 2F(u) + \mu \left|Q^{-1} v\right|^2_H. \tag{7.3}$$

7.1. **A characterization of the quasi-potential.** For any $\mu > 0$ and $t_1 < t_2$, and for any $z \in C([t_1, t_2]; \mathcal{H})$ and $z_0 \in \mathcal{H}$, we define

$$I^\mu_{t_1,t_2}(z) = \frac{1}{2} \inf \left\{ |\psi|^2_{L^2((t_1,t_2);H)} : z = z^\mu_{z_0,\psi} \right\}$$

where z^μ_ψ solves the skeleton equation associated with (7.1)

$$\frac{dz^\mu_{z_0,\psi}}{dt}(t) = A_\mu z^\mu_{z_0,\psi}(t) + B_\mu(z^\mu_{z_0,\psi}(t)) + \sqrt{\epsilon} \mathcal{Q}_\mu \psi(t), \qquad z^\mu_{z_0,\psi}(0) = z_0 \in \mathcal{H}.$$

Analogously, for $t_1 < t_2$, and for any $\varphi \in C([t_1, t_2]; H)$ and $u_0 \in H$, we define

$$I_{t_1,t_2}(\varphi) = \frac{1}{2} \inf \left\{ |\psi|^2_{L^2((t_1,t_2);H)} : \varphi = \varphi_{\psi,u_0} \right\},$$

where $\varphi_{u_0,\psi}$ solves the problem

$$\frac{d\varphi_{u_0,\psi}}{dy}(t) = \Delta \varphi_{u_0,\psi} - Q^2 DF(\varphi_{u_0,\psi}(t)) + Q\, \psi(t), \qquad \varphi_{u_0,\psi}(0) = u_0.$$

In what follows, we shall also denote

$$I^\mu_{-\infty}(z) = \sup_{t<0} I^\mu_{t,0}(z), \qquad I_{-\infty}(\varphi) = \sup_{t<0} I_{t,0}(\varphi).$$

Since (7.1) and (7.2) have additive noise, as a consequence of the contraction lemma we have that the family $\{z^\mu_\epsilon\}_{\epsilon>0}$ satisfies the large deviations principle in $C([0, T]; \mathcal{H})$, with respect to the rate function $I^\mu_{0,T}$ and the family $\{u^\epsilon\}_{\epsilon>0}$ satisfies the large deviations principle in $C([0, T]; H)$, with respect to the rate function $I_{0,T}$.

In what follows, for any fixed $\mu > 0$ we shall denote by V^μ the quasi-potential associated with system (7.1), namely

$$V^\mu(u, v) = \inf \left\{ I^\mu_{0,T}(z) \,:\, z(0) = 0,\ z(T) = (u, v),\ T > 0 \right\}.$$

Analogously, we shall denote by V the quasi-potential associated with equation (7.2), that is

$$V(u) = \inf\{I_{0,T}(\varphi)\,:\,\varphi(0)=0,\ \varphi(T)=u,\ T>0\}.$$

Moreover, for any $\mu>0$ we shall define

$$\bar{V}_\mu(u) = \inf_{v\in H^{-1}(\mathcal{O})} V^\mu(u,v) = \inf\left\{I^\mu_{0,T}(z)\,:\,z(0)=0,\ \Pi_1 z(T)=u,\ T>0\right\}.$$

In [7, Proposition 5.4], it has been proven that $V(u)$ can be represented as

$$V(u) = \inf\left\{I_{-\infty}(\varphi)\,:\,\varphi(0)=u,\ \lim_{t\to-\infty}|\varphi(t)|_H = 0\right\}.$$

In fact, we have proved that a similar representations for $V^\mu(u,v)$ holds, for any fixed $\mu>0$.

Theorem 7.1. [10, Theorem 3.1] *For any $\mu>0$ and $(u,v)\in\mathcal{H}$ we have*

$$V^\mu(u,v) = \inf\left\{I^\mu_{-\infty}(z)\,:\,z(0)=(u,v),\ \lim_{t\to-\infty}\left|C_\mu^{-1/2}z(t)\right|_H = 0\right\}.$$

7.2. **The main result.** If $z\in C((-\infty,0];\mathcal{H})$ is such that $I^\mu_{-\infty,0}(z)<+\infty$, then we have for $\varphi=\Pi_1 z$,

$$I^\mu_{-\infty}(z) = \frac{1}{2}\int_{-\infty}^0 \left|Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t)+\frac{\partial\varphi}{\partial t}(t)-\Delta\varphi(t)+Q^2DF(\varphi(t))\right)\right|_H^2 dt. \tag{7.4}$$

If $I^\mu_{-\infty,0}(z)<+\infty$, then there exists $\psi\in L^2((-\infty,0);H)$ such that $\varphi=\Pi_1 z$ is a weak solution to

$$\mu\frac{\partial^2\varphi}{\partial t^2}(t) = \Delta\varphi(t)-\frac{\partial\varphi}{\partial t}(t)-Q^2DF(\varphi(t))+Q\psi.$$

This means that

$$\psi(t) = Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t)+\frac{\partial\varphi}{\partial t}(t)-\Delta\varphi(t)+Q^2DF(\varphi(t))\right)$$

and (7.4) follows.

By the same argument, if $I_{-\infty,0}(\varphi)<+\infty$, then it follows that

$$I_{-\infty}(\varphi) = \frac{1}{2}\int_{-\infty}^0 \left|Q^{-1}\left(\frac{\partial\varphi}{\partial t}(t)-\Delta\varphi(t)+Q^2DF(\varphi(t))\right)\right|_H^2 dt.$$

Theorem 7.2. [10, Theorem 4.1] *For any fixed $\mu>0$ and $(u,v)\in D((-\Delta)^{1/2}Q^{-1})\times D(Q^{-1})$ it holds*

$$V^\mu(u,v) = \left|(-\Delta)^{\frac{1}{2}}Q^{-1}u\right|_H^2 + 2F(u) + \mu\left|Q^{-1}v\right|_H^2.$$

Moreover

$$V(u) = \left|(-\Delta)^{\frac{1}{2}}Q^{-1}u\right|_H^2 + 2F(u).$$

In particular, for any $\mu>0$,

$$\bar{V}_\mu(u) = \inf_{v\in H^{-1}} V^\mu(u,v) = V^\mu(u,0) = V(u).$$

First, we observe that if $\varphi(t)=\Pi_1 z(t)$, then

$$I^\mu_{-\infty}(z)=\frac{1}{2}\int_{-\infty}^0\left|Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t)-\frac{\partial\varphi}{\partial t}(t)-\Delta\varphi(t)+Q^2DF(\varphi(t))\right)\right|_H^2 dt$$
$$+2\int_{-\infty}^0\left\langle Q^{-1}\frac{\partial\varphi}{\partial t}(t),Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t)-\Delta\varphi(t)\right)+QDF(\varphi(t))\right\rangle_H dt. \tag{7.5}$$

Now, if

$$\lim_{t\to-\infty}|C_\mu^{-1/2}z(t)|_{\mathcal{H}}=0,$$

then

$$\lim_{t\to-\infty}\left|(-\Delta)^{\frac{1}{2}}Q^{-1}\varphi(t)\right|_H+\left|Q^{-1}\frac{\partial\varphi}{\partial t}(t)\right|_H=0,$$

so that

$$2\int_{-\infty}^0\left\langle Q^{-1}\frac{\partial\varphi}{\partial t}(t),Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t)-\Delta\varphi(t)\right)+QDF(\varphi(t))\right\rangle_H dt$$
$$=\left|(-\Delta)^{\frac{1}{2}}Q^{-1}\varphi(0)\right|_H^2+2F(\varphi(0))+\mu\left|Q^{-1}\frac{\partial\varphi}{\partial t}(0)\right|_H^2.$$

This yields

$$V^\mu(u,v)\geq\left|(-\Delta)^{\frac{1}{2}}Q^{-1}u\right|_H^2+2F(u)+\mu\left|Q^{-1}v\right|_H^2.$$

Due to (7.5), we conclude by finding a suitable $\hat\varphi$ satisfying

$$\mu\frac{\partial^2\hat\varphi}{\partial t^2}(t)=\Delta\hat\varphi(t)+\frac{\partial\hat\varphi}{\partial t}(t)-Q^2DF(\hat\varphi(t)),\quad \hat\varphi(0)=u,\quad \frac{\partial\hat\varphi}{\partial t}(0)=v.$$

8. **Large deviations in the non-gradient case.** Here, we assume that the smooth bounded domain $\mathcal{O}\subset\mathbb{R}^d$ is regular enough so that

$$\alpha_k\sim k^{2/d},\qquad k\in\mathbb{N}, \tag{8.1}$$

where $\{\alpha_k\}_{k\in\mathbb{N}}$ are the eigenvalues of the Laplacian Δ in H. This happens for example in the case of a *strongly regular* open set $\mathcal{O}$, both with Dirichlet and with Neumann boundary conditions, see [16, Theorem 1.9.6].

So far, we have assumed that the noise $w^Q(t,x)$ is a cylindrical Wiener process that can be written as

$$w^Q(t,x)=\sum_{k=1}^\infty Qe_k(x)\beta_k(t),$$

where $\{e_k\}_{k\in\mathbb{N}}$ is the complete orthonormal basis in H that diagonalizes Δ and $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is a sequence of mutually independent Brownian motions, all defined on the same stochastic basis. Moreover, we have assumed that Q is diagonal with respect to the basis $\{e_k\}_{k\in\mathbb{N}}$, with $Qe_k=\lambda_k\,e_k$. In what follows we shall assume the following for the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$.

Hypothesis 12. *If $\{\lambda_k\}_{k\in\mathbb{N}}$ is the sequence of eigenvalues of Q, corresponding to the eigenbasis $\{e_k\}_{k\in\mathbb{N}}$, we have*

$$\frac{1}{c}\alpha_k^{-\beta}\leq\lambda_k\leq c\,\alpha_k^{-\beta},\qquad k\in\mathbb{N}, \tag{8.2}$$

for some $c>0$ and $\beta>(d-2)/4$.

Remark 8.1. 1. *If $d = 1$, according to Hypothesis 12 we can consider space-time white noise, thai is $Q = I$.*

2. *Thanks to (8.1), condition (8.2) implies that if $d \geq 2$, then there exists $\gamma < 2d/(d-2)$ such that*
$$\sum_{k=1}^{\infty} \lambda_k^{\gamma} < \infty.$$
Moreover
$$\sum_{k=1}^{\infty} \frac{\lambda_k^2}{\alpha_k} < \infty.$$

3. *As a consequence of (8.2), for any $\delta \in \mathbb{R}$*
$$\mathrm{Dom}((-A)^{\delta/2} Q^{-1}) = H^{\delta+2\beta}$$
and there exists $c_\delta > 0$ such that for any $x \in H^{\delta+2\beta}$
$$\frac{1}{c_\delta}\, |(-A)^{\delta/2} Q^{-1} x|_H \leq |x|_{\delta+2\beta} \leq c_\delta\, |(-A)^{\delta/2} Q^{-1} x|_H$$

Concerning the nonlinearity B, we shall assume the following conditions.

Hypothesis 13. *For any $\delta \in [0, 1+2\beta]$, the mapping $B : H^\delta \to H^\delta$ is Lipschitz continuous, with*
$$[B]_{\mathrm{Lip}(H^\delta)} =: \gamma_\delta < \alpha_1.$$
Moreover $B(0) = 0$. We also assume that B is differentiable in the space $H^{2\beta}$, with
$$\sup_{u \in H^{2\beta}(D)} \|DB(u)\|_{L(H^{2\beta})} = \gamma_{2\beta}.$$

Remark 8.2. 1. *The assumption that B is differentiable is made for convenience to simplify the proof of lower bounds in Theorem 8.5. We believe that by approximating the Lipschitz continuous B with a sequence of differentiable functions whose C^1 semi-norm is controlled by the Lipschitz semi-norm of B, the results proved in Theorem 8.5 should remain true.*

2. *If for any $h \in H$*
$$B(h)(x) = b(x, h(x)), \quad x \in \mathcal{O},$$
as in Section 3 and after, and if we assume that $b(x,\cdot) \in C^{2k}(\mathbb{R})$, for $k \in [\beta + \delta/2 - 5/4, \beta + \delta/2 - 1/4]$, and
$$\frac{\partial^j b}{\partial \sigma^j}(x,\sigma)_{|\sigma=0} = 0, \quad x \in \overline{\mathcal{O}},$$
then B maps H^δ into itself, for any $\delta \in [0, 1+2\beta]$. The Lipschitz continuity of B in H^δ and the bound on the Lipschitz norm, are satisfied if the derivatives of $b(x,\cdot)$ are small enough.

In Section 6 we have compared the small noise asymptotic behavior of the solution of equation (7.1) with the small noise asymptotic behavior of the solution of equation (7.2). We have seen that in the gradient case (that is, when B satisfies Hypothesis 11), the quasi-potential $V^\mu(u,v)$ associated with equation (7.1) is given by
$$V^\mu(u,v) = |(-\Delta)^{\frac{1}{2}} Q^{-1} u|_H^2 + 2F(u) + \mu\, |Q^{-1} v|_H^2, \tag{8.3}$$
so that
$$\bar{V}_\mu(u) := \inf_{v \in H^{-1}} V^\mu(u,v) = |(-\Delta)^{\frac{1}{2}} Q^{-1} u|_H^2 + 2F(u) =: V(u), \tag{8.4}$$

and $V(u)$ is, as is well known, the quasi-potential for equation (7.2).

In [9], we have studied the same problem in the more delicate situation the system under consideration is not of gradient type and hence there is no explicit expression for $V^\mu(u,v)$ and $V(u)$, as in (8.3) and (8.4).

8.1. **The skeleton equation.** For any $\psi \in L^2((-\infty,0);H^{2\alpha})$, with $\alpha \in [0,1/2]$, and for any $\mu>0$, we denote by $z^\mu_\psi \in C((-\infty,0);\mathcal{H})$ the solution of the skeleton equation

$$z^\mu_\psi(t) = \int_{-\infty}^t S_\mu(t-s)B_\mu(z^\mu_\psi(s))\,ds + \int_{-\infty}^t S_\mu(t-s)Q_\mu\psi(s)\,ds, \quad t\in\mathbb{R}. \tag{8.5}$$

In [9, Lemma 4.3] we have proved that if

$$\lim_{t\to-\infty}|z^\mu_\psi(t)|_{\mathcal{H}} = 0, \tag{8.6}$$

we have $z^\mu_\psi \in C((-\infty,0);\mathcal{H}_{1+2(\alpha+\beta)})$ and

$$\lim_{t\to-\infty}\left|z^\mu_\psi(t)\right|_{\mathcal{H}_{1+2(\alpha+\beta)}} = 0. \tag{8.7}$$

Remark 8.3. 1. *From the previous result, we have that if $z^\mu_\psi \in C((-\infty,0);\mathcal{H})$ solves equation (8.5) and limit (8.6) holds, then $z^\mu_\psi(t) \in \mathcal{H}_{1+2\beta}$, for any $t\le 0$. In particular $z^\mu_\psi(0)\in\mathcal{H}_{1+2\beta}$.*

2. *In [7, Lemma 3.5], it has been proven that the same holds for equation (7.2). Actually, if $\varphi_\psi \in C((-\infty,0);H)$ is the solution to*

$$\varphi_\psi(t) = \int_{-\infty}^t e^{(t-s)A}B(\varphi_\psi(s))ds + \int_{-\infty}^t e^{(t-s)A}Q\psi(s)ds,$$

for $\psi \in L^2((-\infty,0);H)$, and

$$\lim_{t\to-\infty}|\varphi_\psi(t)|_H = 0,$$

then $\varphi_\psi \in C((-\infty,0);H^{1+2\beta})$ and there exists a constant such that for all $t\le 0$,

$$|\varphi_\psi(t)|_{H^{1+2\beta}} \le c\,|\psi|_{L^2((-\infty,0;H)}.$$

Moreover,

$$\lim_{t\to-\infty}|\varphi_\psi(t)|_{H^{1+2\beta}} = 0.$$

8.2. **A characterization of the quasi-potential.** For any $t_1<t_2$, $\mu>0$ and $z\in C((t_1,t_2);\mathcal{H})$, we define

$$I^\mu_{t_1,t_2}(z) = \frac{1}{2}\inf\left\{|\psi|^2_{L^2((t_1,t_2);H)}\,:\, z = z^\mu_{\psi,z_0}\right\}, \tag{8.8}$$

where z^μ_{ψ,z_0} is a mild solution of the skeleton equation associated with equation (7.1), with deterministic control $\psi\in L^2((t_1,t_2);H)$ and initial conditions z_0, namely

$$\frac{dz^\mu_{\psi,z_0}}{dt}(t) = A_\mu z^\mu_{\psi,z_0}(t) + B_\mu(z^\mu_{\psi,z_0}(t)) + Q_\mu\psi(t), \quad t_1\le t\le t_2.$$

For $\epsilon,\mu>0$ and $z_0\in\mathcal{H}$ we denote by $z^\mu_{\epsilon,z_0}\in L^2(\Omega;C([0,T];\mathcal{H}))$ the mild solution of equation (7.1). Since the mapping $B_\mu:\mathcal{H}\to\mathcal{H}$ is Lipschitz-continuous and the noisy perturbation in (7.1) is of additive type, as an immediate consequence of the contraction lemma, for any fixed $\mu>0$ the family $\{\mathcal{L}(z^\mu_{\epsilon,z_0})\}_{\epsilon>0}$ satisfies a large deviation principle in $C([t_1,t_2];\mathcal{H})$, with action functional $I^\mu_{t_1,t_2}$. In particular, for any $\delta>0$ and $T>0$,

$$\liminf_{\epsilon\to 0}\epsilon\log\left(\inf_{z_0\in\mathcal{H}}\mathbb{P}\left(\left|z^\mu_{\epsilon,z_0}-z^\mu_{\psi,z_0}\right|_{C([0,T];\mathcal{H})}<\delta\right)\right)\geq -\frac{1}{2}|\psi|^2_{L^2((0,T);H)}$$

and, if $K^\mu_{0,T}(r)=\{z\in C([0,T];\mathcal{H}):I^\mu_{0,T}(z)\leq r\}$,

$$\limsup_{\epsilon\to 0}\epsilon\log\left(\sup_{z_0\in\mathcal{H}}\mathbb{P}\left(\operatorname{dist}_{\mathcal{H}}(z^\mu_{\epsilon,z_0},K^\mu_{0,T}(r))\geq\delta\right)\right)\leq -r.$$

Analogously, if for any $\epsilon>0$ u_ϵ denotes the mild solution of equation (7.2), the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon>0}$ satisfies a large deviation principle in $C([t_1,t_2];H)$ with action functional

$$I_{t_1,t_2}(\varphi)=\inf\left\{\frac{1}{2}\,|\psi|^2_{L^2([t_1,t_2];H)}\;:\;\varphi=\varphi_\psi\right\},$$

where φ_ψ is a mild solution of the skeleton equation associated with equation (7.2)

$$\frac{du}{dt}(t)=Au(t)+B(u(t))+Q\psi(t),\qquad t_1\leq t\leq t_2.$$

In particular, the functionals $I^\mu_{t_1,t_2}$ and I_{t_1,t_2} are lower semi-continuous and have compact level sets. Moreover, it is not difficult to show that for any compact sets $E\subset H$ and $\mathcal{E}\subset\mathcal{H}$, the level sets

$$K_{E,t_1,t_2}(r)=\{\varphi\in C([t_1,t_2];H)\;;\;I_{t_1,t_2}(\varphi)\leq r,\;\varphi(t_1)\in E\}$$

and

$$K^\mu_{\mathcal{E},t_1,t_2}(r)=\left\{z\in C([t_1,t_2];\mathcal{H})\;;\;I^\mu_{t_1,t_2}(z)\leq r,\;z(t_1)\in\mathcal{E}\right\}$$

are compact.

In what follows, for the sake of brevity, for any $\mu>0$ and $t\in(0,+\infty]$ we shall define $I^\mu_t:=I^\mu_{0,t}$ and $I^\mu_{-t}:=I^\mu_{-t,0}$ and, analogously, for any $t\in(0,+\infty]$ we shall define $I_t:=I_{0,t}$ and $I_{-t}:=I_{-t,0}$. In particular, we shall set

$$I^\mu_{-\infty}(z)=\sup_{t>0}I^\mu_{-t}(z),\qquad I_{-\infty}(\varphi)=\sup_{t>0}I_{-t}(\varphi).$$

Moreover, for any $r>0$ we shall set

$$K^\mu_{-\infty}(r)=\left\{z\in C((-\infty,0];\mathcal{H})\;;\;\lim_{t\to-\infty}|z(t)|_{\mathcal{H}}=0,\;I^\mu_{-\infty}(z)\leq r\right\}$$

and

$$K_{-\infty}(r)=\left\{\varphi\in C((-\infty,0];H)\;;\;\lim_{t\to-\infty}|\varphi(t)|_H=0,\;I_{-\infty}(\varphi)\leq r\right\}.$$

Once we have introduced the action functionals $I^\mu_{t_1,t_2}$ and I_{t_1,t_2}, as we have already seen in Section 6 we can introduce the corresponding *quasi-potentials*, by setting for any $\mu>0$ and $(u,v)\in\mathcal{H}$

$$V^\mu(u,v)=\inf\left\{I^\mu_{0,T}(z)\;;\;z(0)=0,\;z(T)=(u,v),T>0\right\},$$

and for any $u\in H$

$$V(u)=\inf\left\{I_{0,T}(\varphi)\;;\;\varphi(0)=0,\;\varphi(T)=u,\;T\geq 0\right\}.$$

Moreover, for any $\mu>0$ and $u\in H$, we shall define

$$\bar{V}_\mu(u)=\inf_{v\in H^{-1}}V^\mu(u,v).$$

In [7, Proposition 5.1] it has been proved that the level set $K_{-\infty}(r)$ is compact in the space $C((-\infty,0];H)$, endowed with the uniform convergence on bounded sets, and in [7, Proposition 5.4] it has been proven that the quasi-potential for (7.2) has the characterization

$$V(u)=\min\left\{ I_{-\infty}(\varphi)\ ;\ \varphi\in\ C((-\infty,0];H),\ \lim_{t\to-\infty}|\varphi(t)|_H=0,\ \varphi(0)=u\right\}.$$

In fact, as shown in [9, Theorem 5.1], for small enough $\mu>0$, the level sets $K^\mu_{-\infty}(r)$ are compact in the topology of uniform convergence on bounded intervals.

As a consequence of previous theorem, we have that there exists $\mu_0>0$ such that for any $\psi\in L^2((-\infty,0);H)$ and $\mu\leq\mu_0$ there exists $z^\mu_\psi\in C((-\infty,0];\mathcal{H})$ such that

$$z^\mu_\psi(t)=\int_{-\infty}^t S_\mu(t-s)B_\mu(z^\mu_\psi(s))ds+\int_{-\infty}^t S_\mu(t-s)Q_\mu\psi(s)ds,\qquad t\leq 0. \tag{8.9}$$

Moreover,

$$\lim_{t\to-\infty}|z^\mu_\psi(t)|_{\mathcal{H}}=0. \tag{8.10}$$

As $K_{-\infty}(r)$ is compact in $C((-\infty,0];H)$ with respect to the uniform convergence on bounded intervals, we have analogously that for any $\varphi\in\ L^2((-\infty,0)$ there exists $\varphi_\psi\in\ C((-\infty,0];H)$ such that

$$\varphi_\psi(t)=\int_{-\infty}^t e^{(t-s)A}B(\varphi(s))ds+\int_{-\infty}^t e^{(t-s)A}Q\psi(s)ds,$$

and

$$\lim_{t\to-\infty}|\varphi_\psi(t)|_H=0.$$

The next crucial result shows that an analogous result holds for $V^\mu(u,v)$ and $\bar V_\mu(u)$, at least for μ sufficiently small.

Theorem 8.4. [9, Theorem 5.3] *For small enough $\mu>0$, we have the following representation for the quasi-potentials $V^\mu(u,v)$*

$$V^\mu(u,v)=\min\left\{I^\mu_{-\infty}(z):\lim_{t\to-\infty}|z(t)|_{\mathcal{H}}=0,z(0)=(u,v)\right\},$$

and for $\bar V_\mu(u)$

$$\bar V_\mu(u)=\min\left\{I^\mu_{-\infty}(z):\lim_{t\to-\infty}|z(t)|_{\mathcal{H}}=0,\Pi_1z(0)=u\right\}, \tag{8.11}$$

whenever these quantities are finite. Moreover, both V^μ and $\bar V_\mu$ have compact level sets.

8.3. **Continuity of V^μ and $\bar V_\mu$.** As a consequence of the compactness of the level sets of V^μ and $\bar V_\mu$, the mappings $V^\mu:\mathcal{H}\to[0,+\infty]$ and $\bar V_\mu:H\to[0,+\infty]$ are lower semicontinuous. In fact, it is possible to show that this implies that the mappings

$$V^\mu:\mathcal{H}_{1+2\beta}\to[0,+\infty),\quad \bar V_\mu:H^{1+2\beta}\to[0,+\infty)$$

are well defined and continuous, uniformly in $0<\mu<1$.

Actually, there exists $c>0$ such that for any $\mu>0$ and $(u,v)\in\ \mathcal{H}_{1+2\beta}$, we have

$$V^\mu(u,v)\leq c(1+\mu+\mu^2)|(u,v)|^2_{\mathcal{H}_{1+2\beta}} \tag{8.12}$$

and

$$\bar V_\mu(u)\leq c(1+\mu)\,|u|^2_{H^{1+2\beta}}\,. \tag{8.13}$$

Moreover,

$$\lim_{n\to\infty}|(u,v)-(u_n,v_n)|_{\mathcal{H}_{1+2\beta}}=0\Longrightarrow \lim_{n\to\infty}\sup_{0<\mu<1}|V^\mu(u,v)-V^\mu(u_n,v_n)|=0.$$

and

$$\lim_{n\to\infty}|u-u_n|_{H^{1+2\beta}}=0\Longrightarrow \lim_{n\to\infty}\sup_{0<\mu<1}\left|\bar{V}_\mu(u)-\bar{V}_\mu(u_n)\right|=0.$$

8.4. **Upper bound.** In this section we show that for any closed set $N\subset H$

$$\limsup_{\mu\downarrow 0}\inf_{u\in N}\bar{V}_\mu(u)\leq \inf_{u\in N}V(u).$$

First of all, we notice that if $I_{-\infty}(\varphi)<\infty$, then

$$\varphi\in L^2((-\infty,0);H^{2(1+\beta)}),\quad \frac{\partial\varphi}{\partial t}\in L^2((-\infty,0);H^{2\beta}),\tag{8.14}$$

and

$$I_{-\infty}(\varphi)=\frac{1}{2}\int_{-\infty}^0\left|Q^{-1}\left(\frac{\partial\varphi}{\partial t}(t)-A\varphi(t)-B(\varphi(t))\right)\right|_H^2dt.\tag{8.15}$$

Actually, if φ solves

$$\varphi(t)=\int_{-\infty}^t e^{(t-s)A}B(\varphi(s))ds+\int_{-\infty}^t e^{(t-s)A}Q\psi(s)ds$$

then we can check that (8.14) holds and

$$\psi(t)=Q^{-1}\left(\frac{\partial\varphi}{\partial t}(t)-A\varphi(t)-B(\varphi(t))\right),$$

so that (8.15) follows. Moreover, if

$$\varphi\in L^2((-\infty,0);H^{2(1+\beta)}),\qquad \frac{\partial\varphi}{\partial t},\frac{\partial^2\varphi}{\partial t^2}\in L^2((-\infty,0);H^{2\beta}),$$

then

$$I^\mu_{-\infty}(z)=\frac{1}{2}\int_{-\infty}^0\left|Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t)+\frac{\partial\varphi}{\partial t}(t)-A\varphi(t)-B(\varphi(t))\right)\right|_H^2dt,$$

where

$$z(t)=(\varphi(t),\frac{\partial\varphi}{\partial t}(t)).$$

In particular, as in [11], where the finite dimensional case is studied, this means

$$\begin{aligned}I^\mu_{-\infty}(z)&=I_{-\infty}(\varphi)+\frac{\mu^2}{2}\int_{-\infty}^0\left|Q^{-1}\frac{\partial^2\varphi}{\partial t^2}(t)\right|_H^2dt\\&+\mu\int_{-\infty}^0\left\langle Q^{-1}\frac{\partial^2\varphi}{\partial t^2}(t),Q^{-1}\left(\frac{\partial\varphi}{\partial t}(t)-A\varphi(t)-B(\varphi(t))\right)\right\rangle_H dt,\end{aligned}\tag{8.16}$$

where $\varphi(t)=\Pi_1 z(t)$, as long as all of these terms are finite.

Let φ be the minimizer of $V(u)$. This means $\varphi(0)=u$, (8.15) holds and $I_{-\infty}(\varphi)=V(u)$. For each $\mu>0$, we construct a function φ_μ such that

$$\varphi_\mu\in L^2((-\infty,0);H^{2(1+\beta)})\cap C((-\infty,0];H^{1+2\beta}),\quad \frac{\partial\varphi_\mu}{\partial t}\in L^2((-\infty,0);H^{2\beta}),$$

and satisfying suitable uniform bounds with respect to $\mu>0$.

Moreover,

$$\lim_{\mu\to 0}|x-\varphi_\mu(0)|_{H^{1+2\beta}}+\sup_{t\leq 0}|\varphi_\mu(t)-\varphi(t)|_{H^{1+2\beta}}=0, \tag{8.17}$$

and

$$\lim_{\mu\to 0}|\varphi_\mu-\varphi|_{L^2((-\infty,0);H^{2(1+\beta)})}+\left|\frac{\partial\varphi_\mu}{\partial t}-\frac{\partial\varphi}{\partial t}\right|_{L^2((-\infty,0);H^{2\beta})}=0. \tag{8.18}$$

It is clear that, if $u_\mu=\varphi_\mu(0)$, then

$$\bar{V}_\mu(u_\mu)\leq I^\mu_{-\infty}(z_\mu),$$

where

$$z_\mu(t)=(\varphi_\mu(t),\frac{\partial\varphi_\mu}{\partial t}(t)),\quad t\leq 0.$$

As φ_μ has the required regularity, we can apply (8.16) and we have

$$I^\mu_{-\infty}(z_\mu)\leq\frac{c\,\mu^2}{2}\int_{-\infty}^0|\frac{\partial^2\varphi_\mu}{\partial t^2}(t)|^2_{H^{2\beta}}dt+I_{-\infty}(\varphi_\mu)$$

$$+\mu\int_{-\infty}^0\left\langle Q^{-1}\frac{\partial^2\varphi_\mu}{\partial t^2}(t),Q^{-1}\left(\frac{\partial\varphi_\mu}{\partial t}(t)-A\varphi_\mu(t)-B(\varphi_\mu(t))\right)\right\rangle_H dt$$

$$\leq\frac{\mu^2}{2}\int_{-\infty}^0|\frac{\partial^2\varphi_\mu}{\partial t^2}(t)|^2_{H^{2\beta}}+I_{-\infty}(\varphi_\mu)+\mu\left(\int_{-\infty}^0|\frac{\partial^2\varphi_\mu}{\partial t^2}t)|^2_{H^{2\beta}}dt\right)^{1/2}(I_{-\infty}(\varphi_\mu))^{1/2}.$$

We can prove that there exists $\alpha>0$ such that

$$I^\mu_{-\infty}(z_\mu)\leq I_{-\infty}(\varphi_\mu)+c\mu^{2\alpha}\left|\frac{\partial\varphi}{\partial t}\right|^2_{L^2((-\infty,0);H^{2\beta})}+c\mu^\alpha\,(I_{-\infty}(\varphi_\mu))^{1/2}\left|\frac{\partial\varphi}{\partial t}\right|_{L^2((-\infty,0);H^{2\beta})}$$

and by (8.18)

$$\lim_{\mu\downarrow 0}I_{-\infty}(\varphi_\mu)=I_{-\infty}(\varphi)=V(u).$$

Therefore, we get

$$\limsup_{\mu\downarrow 0}\bar{V}_\mu(u_\mu)\leq\limsup_{\mu\downarrow 0}I^\mu_{-\infty}(z_\mu)\leq V(u).$$

In view of (8.17) and of continuity,

$$\limsup_{\mu\downarrow 0}\bar{V}_\mu(u_\mu)=\limsup_{\mu\downarrow 0}\bar{V}_\mu(u),$$

so that we can conclude that

$$\limsup_{\mu\downarrow 0}\bar{V}_\mu(u)\leq V(u). \tag{8.19}$$

8.5. **Lower bound.** Let $N\subset H$ be a closed set with $N\cap H^{1+2\beta}\neq\emptyset$. In particular, by (8.13) we have $\inf_{u\in N}\bar{V}_\mu(u)<+\infty$. Due to (8.11), there exists $z^\mu\in C((-\infty,0];\mathcal{H})$ such that

$$u^\mu:=\Pi_1 z^\mu(0)\in N,\qquad I^\mu_{-\infty}(z^\mu)=\bar{V}_\mu(u^\mu)=\inf_{u\in N}\bar{V}_\mu(u).$$

Now, let $\psi^\mu\in L^2((-\infty,0);H)$ be the minimal control such that

$$z^\mu(t)=\int_{-\infty}^t S_\mu(t-s)B_\mu(z^\mu(s))ds+\int_{-\infty}^t S_\mu(t-s)Q_\mu\psi^\mu(s)ds,$$

and

$$\inf_{u\in N} \bar{V}_\mu(u) = \bar{V}_\mu(u^\mu) = \frac{1}{2}\,|\psi^\mu|^2_{L^2((-\infty,0);H)}\,. \tag{8.20}$$

In what follows, we shall denote $v^\mu = \Pi_2 z^\mu(0)$. For any $\delta > 0$, we define the approximate control

$$\psi^{\mu,\delta}(t) = (I-\delta A)^{-\frac{1}{2}}\psi^\mu(t), \quad t \le 0,$$

and in view of (8.9) and (8.10) we can define $z^{\mu,\delta}$ to be the solution to the corresponding control problem

$$z^{\mu,\delta}(t) = \int_{-\infty}^t S_\mu(t-s)B_\mu(z^{\mu,\delta}(s))ds + \int_{-\infty}^t S_\mu(t-s)Q_\mu\psi^{\mu,\delta}(s)ds.$$

It is possible to prove that there exists $\mu_0 > 0$ such that,

$$\lim_{\delta\to 0}\sup_{\mu\le\mu_0}\left|u^\mu - u^{\mu,\delta}\right|^2_{H^{2\beta}} = 0, \tag{8.21}$$

where $(u^{\mu,\delta}, v^{\mu,\delta}) = z^{\mu,\delta}(0)$.

This implies the following result.

Theorem 8.5. [9, Theorem 8.2] *For any closed $N \subset H$, we have*

$$\inf_{u\in N} V(u) \le \liminf_{\mu\downarrow 0}\inf_{u\in N}\bar{V}_\mu(u). \tag{8.22}$$

If the right hand side in (8.22) is infinite, the theorem is trivially true. Therefore, in what follows we can assume that

$$\liminf_{\mu\to 0}\inf_{u\in N}\bar{V}_\mu(u) < +\infty. \tag{8.23}$$

We first observe that, if we define

$$\varphi^{\mu,\delta}(t) = \Pi_1 z^{\mu,\delta}(t), \quad t \le 0,$$

in view of (8.16)

$$V(x^{\mu,\delta}) \le I_{-\infty}(\varphi^{\mu,\delta}) = I^\mu_{-\infty}(z^{\mu,\delta}) - \frac{\mu^2}{2}\int_{-\infty}^0 \left|Q^{-1}\frac{\partial^2\varphi^{\mu,\delta}}{\partial t^2}(t)\right|^2_H dt$$

$$-\mu\int_{-\infty}^0 \left\langle Q^{-1}\frac{\partial^2\varphi^{\mu,\delta}}{\partial t^2}(t), Q^{-1}\frac{\partial\varphi^{\mu,\delta}}{\partial t}(t) - Q^{-1}A\varphi^{\mu,\delta}(t) - Q^{-1}B(\varphi^{\mu,\delta}(t))\right\rangle_H dt.$$

Since

$$|\psi^{\mu,\delta}(t)|_H = |(I-\delta A)^{-1/2}\psi^\mu(t)|_H \le |\psi^\mu(t)|_H, \quad t \le 0, \tag{8.24}$$

we have

$$I^\mu_{-\infty}(z^{\mu,\delta}) \le I^\mu_{-\infty}(z^\mu) = \inf_{x\in N}\bar{V}_\mu(u),$$

so that

$$V(u^{\mu,\delta}) \le \inf_{u\in N}\bar{V}_\mu(u)$$

$$-\mu\int_{-\infty}^0 \left\langle Q^{-1}\frac{\partial^2\varphi^{\mu,\delta}}{\partial t^2}(t), Q^{-1}\frac{\partial\varphi^{\mu,\delta}}{\partial t}(t) - Q^{-1}A\varphi^{\mu,\delta}(t) - Q^{-1}B(\varphi^{\mu,\delta}(t))\right\rangle_H dt.$$

Thanks to (8.7) and Hypothesis 13, by integrating by parts

$$V(u^{\mu,\delta}) \leq \inf_{u\in N} \bar{V}_\mu(u)$$

$$-\frac{\mu}{2}|Q^{-1}v^{\mu,\delta}|_H^2 - \mu\left\langle(-A)Q^{-1}u^{\mu,\delta}, Q^{-1}v^{\mu,\delta}\right\rangle_H + \mu\left\langle Q^{-1}B(u^{\mu,\delta}), Q^{-1}v^{\mu,\delta}\right\rangle_H$$

$$+c\,\mu\int_{-\infty}^0 \left|\frac{\partial\varphi^{\mu,\delta}}{\partial t}(t)\right|^2_{H^{1+2\beta}} dt + c\,\gamma_{2\beta}\mu\int_{-\infty}^0 \left|\frac{\partial\varphi^{\mu,\delta}}{\partial t}(t)\right|^2_{H^{2\beta}} dt = \inf_{u\in N}\bar{V}_\mu(u) + \mathcal{I}^{\mu,\delta}. \tag{8.25}$$

By analyzing all terms in $\mathcal{I}^{\mu,\delta}$, we obtain,

$$V(u^{\mu,\delta}) \leq \inf_{u\in N}\bar{V}_\mu(u) + c(\mu+\sqrt{\mu})(1+\delta^{-1/2})\inf_{u\in N}\bar{V}_\mu(u).$$

From this, choosing $\delta = \sqrt{\mu}$, and due to (8.23), we see that

$$\liminf_{\mu\to 0} V(u^{\mu,\sqrt{\mu}}) \leq \liminf_{\mu\to 0}\inf_{u\in N}\bar{V}_\mu(u).$$

Since we are assuming (8.23), and, by [7, Proposition 5.1], the level sets of V are compact, there is a sequence $\mu_n \to 0$ and $u^0 \in H$ such that

$$\lim_{n\to\infty}|u^{\mu_n,\sqrt{\mu_n}} - u^0|_H = 0, \quad V(u^0) \leq \liminf_{\mu\to 0} V(u^{\mu,\sqrt{\mu}}).$$

By (8.21), we have that u^{μ_n} converges to u^0 in H, so that $u^0 \in N$. This means that we can conclude, as

$$\inf_{u\in N} V(u) \leq V(u^0) \leq \liminf_{\mu\to 0} V(u^{\mu,\sqrt{\mu}}) \leq \liminf_{\mu\to 0}\inf_{u\in N}\bar{V}_\mu(u).$$

8.6. **Application to the exit problem.** We are here interested in the problem of the exit of the solution u^μ_ϵ of equation (7.1) from a domain $D \subset H$, for any $\mu > 0$ fixed. Then we apply the upper and lower bounds proved previously to show that, when μ is small, the relevant quantities in the exit problem from D for the solution u^μ_ϵ of equation (7.1) can be approximated by the corresponding ones arising for equation (7.2).

First, let us give some assumptions on the set D.

Hypothesis 14. *The domain $D \subset H$ is an open, bounded, connected set, such that $0 \in D$. Moreover, for any $h \in \partial D \cap H^{1+2\beta}$ there exists a sequence $\{h_n\}_{n\in\mathbb{N}} \subset \bar{D}^c \cap H^{1+2\beta}$ such that*

$$\lim_{n\to+\infty}|h_n - h|_{\mathcal{H}_{1+2\beta}} = 0. \tag{8.26}$$

Assume now that D is an open, bounded and connected set such that, for any $h \in \partial D \cap H^{1+2\beta}$, there exists a $k \in \bar{D}^c \cap H^{1+2\beta}$ such that

$$\{tk + (1-t)h : 0 < t \leq 1\} \subset \bar{D}^c. \tag{8.27}$$

Then it is immediate to check that (8.26) is satisfied. Condition (8.27) is true, for example, if D is convex, because of the Hahn-Banach separation theorem and the density of $H^{1+2\beta}$ in H.

We can prove that Hypothesis 14 implies that

$$\bar{V}_\mu(\partial D) := \inf_{u\in\partial D}\bar{V}_\mu(u) = \bar{V}_\mu(u_{D,\mu}) < \infty,$$

for some $u_{D,\mu} \in \partial D \cap \mathcal{H}_{1+2\beta}$.

Now, if we denote by $z^\mu_{\epsilon,z_0} = (u^\mu_{\epsilon,z_0}, v^\mu_{\epsilon,z_0})$ the mild solution of (7.1), with initial position and velocity $z_0 = (u_0, v_0) \in \mathcal{H}$, and by $u^\epsilon_{u_0}$ the mild solution of (7.2), with initial position $u_0 \in H$, we define the exit times

$$\tau^{\mu,\epsilon}_{z_0} = \inf\left\{t > 0 : u^\mu_{\epsilon,z_0}(t) \notin D\right\}, \text{ and } \tau^\epsilon_{u_0} = \inf\left\{t > 0 : u^\epsilon_{u_0}(t) \notin D\right\}.$$

Here is the main result of this section

Theorem 8.6. [9, Theorem 9.2] *There exists $\mu_0 > 0$ such that for $\mu < \mu_0$ the following conditions are verified. For any $z_0 = (u_0, v_0) \in \mathcal{H}$ such that $u_0 \in D$ and $u^\mu_{0,z_0}(t) \in D$, for $t \geq 0$,*

1. *The exit time has the following asymptotic growth*
$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\left(\tau^{\mu,\epsilon}_{z_0}\right) = \inf_{u\in\partial D} \bar{V}_\mu(u),$$
and for any $\eta > 0$,
$$\lim_{\epsilon\to 0} \mathbb{P}\left(\exp\left(\frac{1}{\epsilon}(\bar{V}_\mu(\partial D) - \eta)\right) \leq \tau^{\mu,\epsilon}_{z_0} \leq \exp\left(\frac{1}{\epsilon}(\bar{V}_\mu(\partial D) + \eta)\right)\right) = 1.$$
2. *For any closed $N \subset \partial D$ such that $\inf_{u\in N} \bar{V}_\mu(u) > \inf_{u\in\partial D} \bar{V}_\mu(u)$, it holds*
$$\lim_{\epsilon\to 0} \mathbb{P}\left(u^\mu_{\epsilon,z_0}(\tau^{\mu,\epsilon}_{z_0}) \in N\right) = 0.$$

In view of what we have proven in Sections 8.4 and 8.5 and of Theorem 8.6, this implies that the following Smoluchowski-Kramers approximations holds for the exit time.

Theorem 8.7. [9, Theorem 9.4]

1. *For any initial conditions $z_0 = (u_0, v_0)$,*
$$\lim_{\mu\to 0} \lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\left(\tau^{\mu,\epsilon}_{z_0}\right) = \lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\left(\tau^\epsilon_{u_0}\right) = \inf_{x\in\partial D} V(x).$$
2. *For any $\eta > 0$, there exists $\mu_0 > 0$ such that for $\mu < \mu_0$*
$$\lim_{\epsilon\to 0} \mathbb{P}\left(e^{\frac{1}{\epsilon}(\bar{V}-\eta)} \leq \tau^{\mu,\epsilon}_{z_0} \leq e^{\frac{1}{\epsilon}(\bar{V}+\eta)}\right) = 1 \tag{8.28}$$
3. *For any $N \subset \partial D$ such that $\inf_{x\in N} V(x) < \inf_{x\in\partial D} V(x)$, there exits $\mu_0 > 0$ such that for all $\mu < \mu_0$,*
$$\lim_{\epsilon\to 0} \mathbb{P}_{z_0}\left(u^\mu_\epsilon(\tau^{\mu,\epsilon}) \in N\right) = 0.$$

We recall that in Theorem 7.2 we have proved that, in the case of gradient systems, for any $\mu > 0$

$$\bar{V}_\mu(x) = V(x), \qquad x \in H.$$

This means that in this case for any $z_0 = (u_0, v_0) \in \mathcal{H}$ and $\mu > 0$ such that the unperturbed system $u^\mu_{0,z_0}(t) \in D$ for all $t > 0$

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\left(\tau^{\mu,\epsilon}_{z_0}\right) = \lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\left(\tau^\epsilon_{u_0}\right) = \inf_{x\in\partial D} V(x).$$

and (8.28) holds for any $\mu > 0$.

9. **Further developments.** There are still several open problems related to the Smoluchowski-Kramers approximation for systems with an infinite number of degrees of freedom, that are worth of future investigation.

9.1. **S-K approximation for state dependent friction.** It would be interesting to study the problem of the validity of the Smolochowski-Kramers approximation in the case the friction coefficient depends on the state $u(t,x)$, as described in the following equation

$$\left\{\begin{array}{l} \mu\dfrac{\partial^2 u}{\partial t^2}(t,x)+\alpha(x,u(t,x))\,\dfrac{\partial u}{\partial t}(t,x)=\Delta u(t,x)+b(x,u(t,x))+\dfrac{\partial w^Q}{\partial t}(t,x),\\[10pt] u(0,x)=u_0(x),\qquad \dfrac{\partial u}{\partial t}(0,x)=v_0(x),\qquad u(t,x)=0,\quad x\in\partial\mathcal{O}. \end{array}\right. \tag{9.1}$$

Here $\alpha:\mathcal{O}\times\mathbb{R}\to\mathbb{R}$ is some bounded function, such that $\alpha(x,\sigma)\geq\alpha_0>0$, for any $x\in\mathcal{O}$ and $\sigma\in\mathbb{R}$.

It turns out that in this case the solution $u^\mu(t)$ of the wave equation does not converge, in general, to the solution $u(t)$ of the parabolic problem ($\mu=0$). Actually, it can be proved that, even in finite dimension, the stochastic convolution for the second order equation does not converge to the stochastic convolution for the first order problem, unlike as it happens in the case of constant dumping coefficient. Nevertheless, we expect that, as in the finite dimensional case, we can get a modified limiting equation. Namely, we expect to obtain a quasi linear SPDE where a correction term appears.

9.2. **S-K approximation for vanishing friction.** It would be even more challenging to consider the same problem as above, in the case the function α vanishes in some regions. A similar problem has been studied by Freidlin, Hu and Wentcell in finite dimension in the paper [21], but the infinite dimensional case seems to be considerably more complicated.

9.3. **Long time behavior in the non-gradient case.** We have proved that in the gradient case, the first marginal of the stationary distribution of the stochastic wave equation does coincide with the invariant measure of the stochastic parabolic problem. In the non-gradient case, we do not have explicit formulas for the stationary distribution and the invariant measure. Nonetheless, we believe that one should be able to establish the convergence of statistically invariant states of the damped wave equation in the zero mass limit to the invariant measure of the parabolic problem. This should be true when the semigroup associated with the limiting equation is a strict contraction, for some time sufficiently large, and when the Smoluchowski-Kramers approximation holds for any fixed time.

9.4. **S-K approximation for multiplicative noise in any dimension.** We have proved the validity of the S.-K. approximation for systems perturbed by a multiplicative noise, both in the presence of a pure damping term and in the presence of a magnetic field, only in the case of space dimension $d=1$. This seems related to some technical problem that would be worth of further investigation. Actually, it would be interesting to understand if an analogous result can be proved also in the case of space dimension larger than 1.

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