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Analytic Semigroups and Degenerate Elliptic Operators with Unbounded Coefficients: A Probabilistic Approach

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By using techniques derived from the theory of stochastic differential equations, we prove that a class of second order degenerate elliptic operators having unbounded coefficients generates analytic semigroups in $C_b(\mathbb{R}^d)$, the space of uniformly continuous and bounded functions from $\mathbb{R}^d$ into $\mathbb{R}$. © 2000 Academic Press

1. INTRODUCTION

In the present paper we are dealing with generation of analytic semigroups in the space of continuous and bounded functions by suitable second order differential operators, which have unbounded coefficients and may be degenerate.

This means that we are away from the classical framework in the study of the generation of analytic semigroups by elliptic operators (see Lunardi [25] for a comprehensive overview on the more recent results in this field) under two respects: firstly, we are able to overcome the usual assumption of boundedness of coefficients; secondly our results adapt to a wide class of degenerate operators. Moreover, we do not use the more classical deterministic techniques developed beginning from the works of Stewart [27] and [28]. Actually, we regard our operators as the diffusion operators corresponding to suitable stochastic differential equations and hence we proceed by giving an explicit probabilistic representation of the semigroups.

The study of partial differential equations by probabilistic methods is classical by now. Starting with the books by Strook and Varadhan [29], Ethier and Kurtz [16] and others, many results have been proved about existence, uniqueness and regularity. These results have been extended in various aspects, including less growth restrictions and less regularity for the coefficients, as well as more degeneracy. To this purpose it is worthwhile to cite the interesting books by Friedlin [20] and Krylov [23]. All these

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books are not interested in the generation of analytic semigroups. As far as the problem of analyticity for Markovian diffusion semigroups is concerned, it has been widely developed starting with the book by Stein [26] and later by the works of Devinatz [13], Liskevich and Perelmuter [24] and others, but only in L^p spaces and in the symmetric case. In fact the situation we are dealing with here of non symmetric operators in the space of uniformly continuous and bounded functions seems to be completely new in the probabilistic literature.

The case of unbounded coefficients in the whole space, after the works by Aronson–Besala [1] and Besala [3], has been studied in some other papers. Among them we recall the papers by Cannarsa–Vespri [8] and [9], where, by assuming the uniform ellipticity of the operators and by giving suitable assumptions on the growth of the coefficients, the generation of analytic semigroups in $L^p(\mathbb{R}^d)$ and in the space of continuous functions is proved. Concerning the generation for degenerate operators, after the two papers by Feller (see [18] and [19]) and the paper by Brezis–Rosenkrantz–Singer [5], several authors as Clément and Timmermans, Campiti, Metafune and Pallara, Favini, J. Goldstein and Romanelli (see [11], [6], [7] and [17]) widely developed the case of ordinary differential operators with Ventcel's boundary conditions, both in L^p spaces and in spaces of continuous functions on a real interval. In other papers by Baouendi–Gaulaouic and Vespri (see [2] and [30]) the case of operators defined in bounded domains of $\mathbb{R}^d$, which are strongly elliptic everywhere but on the boundary, is considered. The d-dimensional case is also considered in the recent paper by Gozzi–Monte–Vespri [22].

In all these papers deterministic techniques are used and rather restrictive conditions are given on the way the coefficients vanish. Actually, they cannot vanish but in a negligible set and suitable assumptions are given on their behaviour near the zeros. Nevertheless, a differential operator having coefficients identically zero does generate an analytic semigroup in any functions space, so that the restriction on the set where the coefficients vanish seem to be mainly related to the methods used in the proofs. The aim of this paper is to show how, by using completely different techniques, derived from the theory of stochastic differential equations, we can avoid such restrictions.

We consider the following class of second order differential operators

$$\mathscr{A}_0 u(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x)\, D_{ij} u(x) + \sum_{i=1}^{d} b_i(x)\, D_i u(x), \qquad x \in \mathbb{R}^d, \tag{1.1}$$

where the symmetric matrix $a(x)$ is positive semi-definite and has quadratic growth and the vector field $b(x)$ has linear growth and is of class C^2. We assume that $a(x)$ is of class C^2 with bounded second derivatives. Whence, as proved by Freidlin in [20] (see also [29]), it can be written as

$$a(x) = \psi^2(x), \qquad x \in \mathbb{R}^d,$$

where the matrix valued function $\psi = \sqrt{a}\colon \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d)$ is Lipschitz continuous. Here we assume that $\sqrt{a}$ is twice differentiable with bounded derivatives. Moreover a compatibility condition is given among $\sqrt{a}$ and b. Namely we assume that there exists a bounded vector field $\beta\colon \mathbb{R}^d \to \mathbb{R}^d$ of class C^2 such that

$$b(x) = \sqrt{a(x)}\, \beta(x), \qquad x \in \mathbb{R}^d.$$

Notice that an analogous condition is also considered in almost all the papers quoted above and it seems to be quite natural. For instance, in the case of the Ornstein-Uhlenbeck operator (where $a(x) = A > 0$ does not depend on $x \in \mathbb{R}^d$ and $b(x) = Bx$, for a non zero matrix B) the boundedness of $\beta(x) = A^{-1}Bx$ fails to be true and, as proved in [12], the semigroup is not analytic. Finally, we give some technical conditions for the derivatives of $\sqrt{a}$ which will be specified later on.

As known from the theory of Markov processes, the operator $\mathscr{A}_0$ is the diffusion operator corresponding to the stochastic differential problem

$$d\xi(t) = b(\xi(t))\, dt + \sqrt{a(\xi(t))}\, dw(t), \qquad \xi(0) = x, \tag{1.2}$$

where $w(t) = (w_1(t), ..., w_d(t))$ is a standard d-dimensional Brownian motion. Due to the Itô's formula this means that the semigroup P_t corresponding to the operator $\mathscr{A}_0$ is the transition Markov semigroup relative to the equation (1.2). More precisely, if $\xi(t; x)$ denotes the solution of the problem (1.2) and if $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ is Borelian and bounded, then it holds

$$P_t\varphi(x) = \mathbb{E}\varphi(\xi(t; x)), \qquad t \geqslant 0, \quad x \in \mathbb{R}^d. \tag{1.3}$$

The crucial point here is that, by using the representation formula for the semigroup P_t given by (1.3), if $\varphi \in C_b^2(\mathbb{R}^d)$ we can express explicitly $\mathscr{A}_0(P_t\varphi)(x)$ in terms of φ and not of its derivatives, for any $t > 0$ and $x \in \mathbb{R}^d$. This allows us to get the fundamental estimate

$$\sup_{x \in \mathbb{R}^d} |\mathscr{A}_0(P_t\varphi)(x)| \leqslant c(t \wedge 1)^{-1} \sup_{x \in \mathbb{R}^d} |\varphi(x)|, \tag{1.4}$$

from which the analyticity of the semigroup follows, by functional analysis arguments.

We would like to stress that if $\mathscr{A}_0$ is strongly elliptic, then there exists $\varepsilon > 0$ such that $a(x) \geqslant \varepsilon I$, for any $x \in \mathbb{R}^d$, so that there exists $\sqrt{a}$ with the same regularity as a and $(\sqrt{a})^{-1}\colon \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d)$ is bounded. Hence, this implies that the semigroup P_t has a smoothing effect, that is P_t maps Borel and bounded functions into twice differentiable functions (we are assuming a and b of class C^2). Moreover, the Elworthy-Bismut formula holds for any continuous and bounded function $\varphi\colon \mathbb{R}^d \to \mathbb{R}$

$$\langle D(P_t\varphi)(x), h\rangle = \frac{1}{t}\,\mathbb{E}\varphi(\xi(t;x)) \int_0^t \langle (\sqrt{a(\xi(s;x))})^{-1}\, D\xi(s;x)\, h, dw(s)\rangle,$$

where $D\xi(s;x)\,h$ is the first mean-square derivative of $\xi(s;x)$ with respect to $x \in \mathbb{R}^d$ along the direction $h \in \mathbb{R}^d$. In our case, since $\mathscr{A}_0$ is possibly degenerate, we cannot prove that the semigroup P_t is regularizing. Nevertheless we can prove that for any twice differentiable function φ the following generalization of the Bismut-Elworthy formula holds

$$\langle D(P_t\varphi)(x), \sqrt{a(x)}\, h\rangle = \frac{1}{t}\,\mathbb{E}\varphi(\xi(t;x)) \int_0^t \langle v^h(s;x), dw(s)\rangle, \tag{1.5}$$

for a suitable integrable process $v^h(s;x)$. A similar formula is proved for the second derivative $\langle D^2(P_t\varphi)(x)\sqrt{a(x)}\, h, \sqrt{a(x)}\, h\rangle$, so that we give an expression for the derivatives of $P_t\varphi$ involving φ and not its derivatives, even if only along the directions $\sqrt{a(x)}\, h$, for $x, h \in \mathbb{R}^d$.

2. ASSUMPTIONS AND NOTATIONS

We denote by $B_b(\mathbb{R}^d)$ the Banach space of bounded Borel functions $\varphi\colon \mathbb{R}^d \to \mathbb{R}$, endowed with the *sup-norm*

$$\|\varphi\|_0 = \sup_{x \in \mathbb{R}^d} |\varphi(x)|.$$

$C_b(\mathbb{R}^d)$ is the subspace of all uniformly continuous functions and, for any $k \in \mathbb{N}$, $C_b^k(\mathbb{R}^d)$ is the subspace of k-times differentiable functions having bounded derivatives, up to the kth order.

If E, F and G are three finite dimensional vector spaces, we denote by $\mathscr{L}(E;G)$ the vector space of all linear operators from E into G and by $\mathscr{L}(E \times F;G)$ the vector space of all bilinear operators from $E \times F$ into G. If $E = G$ we set $\mathscr{L}(E;E) = \mathscr{L}(E)$. Moreover we denote by $\mathscr{S}_+(E)$ the subspace of $\mathscr{L}(E)$ consisting of positive semi-definite and symmetric linear operators.

In the present paper we are assuming that the matrix $a(x)$ is symmetric and positive semi-definite for any $x \in \mathbb{R}^d$ and the mapping $a\colon \mathbb{R}^d \to \mathscr{S}_+(\mathbb{R}^d)$

is of class C^2 with bounded second derivatives. Thus, as proved in Freidlin [20]-Theorem 3.2.1. we have the following result

THEOREM 2.1 (Freidlin). *There exists* $\psi\colon \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d)$ *Lipschitz continuous such that*

$$a(x) = \psi(x)\, \psi^\star(x), \qquad x \in \mathbb{R}^d. \tag{2.1}$$

Moreover ψ *can be taken in* $\mathscr{S}_+(\mathbb{R}^d)$.

The matrix ψ coincides clearly with the square root $\sqrt{a}$ of a. Here $\sqrt{a}$ has to fulfil stronger regularity assumptions than Lipschitz continuity. But at present it is not clear what sort of stronger regularity assumptions has to be satisfied by a. In fact, if we consider

$$a(x) = |x|^2\, I, \qquad x \in \mathbb{R}^d,$$

we have that $a\colon \mathbb{R}^d \to \mathscr{S}_+(\mathbb{R}^d)$ is of class C^∞ with bounded derivatives beginning from order 2, but its square root $|x|\, I$ is not even once differentiable.

Hypothesis 2.2. 1. The functions $\sqrt{a}\colon \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d)$ and $b\colon \mathbb{R}^d \to \mathbb{R}^d$ are of class C^2 and have bounded first and second derivatives.

2. There exists a bounded vector field $\beta\colon \mathbb{R}^d \to \mathbb{R}^d$ of class C^2 having bounded derivatives such that

$$b(x) = \sqrt{a(x)}\, \beta(x), \qquad x \in \mathbb{R}^d. \tag{2.2}$$

3. There exist two bounded maps $\gamma_1\colon \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d; \mathscr{L}(\mathbb{R}^d))$ and $\gamma_2\colon \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d \times \mathbb{R}^d; \mathscr{L}(\mathbb{R}^d))$ such that for any $x, h, k \in \mathbb{R}^d$ it holds

$$\begin{aligned} D\sqrt{a(x)}\,\sqrt{a(x)}\, h &= \sqrt{a(x)}\, \gamma_1(x)\, h,\\ D^2\sqrt{a(x)}\,(\sqrt{a(x)}\, h, \sqrt{a(x)}\, k) &= \sqrt{a(x)}\, \gamma_2(x)(h, k). \end{aligned} \tag{2.3}$$

Notice that from (2.2) and (2.3) it immediately follows that there exist two bounded maps $\rho_1\colon \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d)$ and $\rho_2\colon \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$ such that for any $x, h, k \in \mathbb{R}^d$ it holds

$$\begin{aligned} Db(x)\sqrt{a(x)}\, h &= \sqrt{a(x)}\; \rho_1(x)\, h,\\ D^2 b(x)(\sqrt{a(x)}\, h, \sqrt{a(x)}\, k) &= \sqrt{a(x)}\; \rho_2(x)(h, k). \end{aligned} \tag{2.4}$$

Actually, instead of asking β to be twice differentiable with bounded derivatives, we could assume directly (2.4).

Remark 2.3. 1. If the operator $\mathscr{A}_0$ is assumed to be *strongly* elliptic, that is there exists $\nu > 0$ such that

$$\inf_{x \in \mathbb{R}^d} \sum_{i,\, j=1}^{d} a_{ij}(x)\, h_i h_j \geqslant \nu\, |h|^2, \qquad h \in \mathbb{R}^d,$$

and if $a\colon \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d)$ is of class C^2, then it is possible to show that there exists $\sqrt{a}\colon \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d)$ of class C^2. The matrix $\sqrt{a(x)}$ is clearly invertible for any $x \in \mathbb{R}^d$ and then, if b is any vector field of class C^2 having bounded derivatives, the hypotheses 2.2 are all satisfied.

2. The boundedness assumption for the function γ_2 implies that the second derivatives of $\sqrt{a}$ vanish at infinity as $1/|x|$. In particular, by some calculations we easily have that the hypothesis of boundedness for the second derivatives of a is necessary in order to have (2.3).

3. The authors which use deterministic techniques often assume that the coefficients a and b do not vanish outside a negligible set (see e.g. [22], [17], [7] and [6]). Here no conditions are given on the set where a degenerates. Actually, a can also be taken identically zero; in this case b is zero and the semigroup generated by $\mathscr{A}_0$ is the semigroup identically equal to identity, which is analytic.

4. Let $\sigma\colon \mathbb{R}^d \to \mathscr{S}_+(\mathbb{R}^d)$ be a mapping of class C^2 with bounded first and second derivatives and assume that for any $x \in \mathbb{R}^d$ the matrix $\sigma(x)$ is invertible and the mapping $\sigma^{-1}\colon \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d)$ is bounded. In particular σ has a square root of class C^2. Then, if a is any matrix valued function which satisfies the hypothesis 2.2, the matrix $\sqrt{\sigma}\, a \sqrt{\sigma}$ can be factorized as in (2.1) and $\sqrt{\sigma}\sqrt{a}$ satisfies the hypothesis 2.2, as well.

5. The conditions described in the hypothesis 2.2-3 are satisfied if we take

$$a(x) = g(x)\, a, \qquad x \in \mathbb{R}^d,$$

for any function $g\colon \mathbb{R}^d \to \mathbb{R}$ of class C^2 and any matrix $a \in \mathscr{L}(\mathbb{R}^d)$ which does not depend on $x \in \mathbb{R}^d$.

6. If we assume that for any $i, j = 1, ..., d$

$$a_{ij}(x) = a_i(x)\, \delta_{ij}, \qquad x \in \mathbb{R}^d,$$

then the conditions of the hypothesis 2.2-3. can be written as

$$\frac{\partial \sqrt{a_h}}{\partial x_i} \sqrt{a_i} = \sqrt{a_h} (\gamma_1 e_i)_h,$$

$$\frac{\partial^2 \sqrt{a_h}}{\partial x_i\, \partial x_j} \sqrt{a_i} \sqrt{a_j} = \sqrt{a_h} (\gamma_2(e_i, e_j))_h,$$

for any i, j, $h = 1, ..., d$. A simple case in which they are satisfied is when for any $h = 1, ..., d$

$$a_h(x) = f_h^2(x_h), \qquad x \in \mathbb{R}^d,$$

for any function $f_h \colon \mathbb{R} \to \mathbb{R}$ of class C^2 with bounded derivatives. Another situation in which the conditions above are both satisfied is when for any $h = 1, ..., d$

$$a_h(x) = f_h^6(x), \qquad x \in \mathbb{R}^d,$$

for some function $f_h \colon \mathbb{R}^d \to \mathbb{R}$ of class C^2 such that for any $x \in \mathbb{R}^d$

$$\left| f_i^3(x) \frac{\partial f_h}{\partial x_i}(x) \right| \leqslant c\, |f_h(x)|$$

and

$$\left| f_i^3(x)\, f_j^3(x) \left(2 \frac{\partial f_h}{\partial x_i}(x) \frac{\partial f_h}{\partial x_j}(x) + f_h(x) \frac{\partial^2 f_h}{\partial x_i\, \partial x_j}(x) \right) \right| \leqslant c\, |f_h^2(x)|.$$

3. SOME PROPERTIES OF THE SOLUTION OF THE CORRESPONDING SDE

The operator $\mathscr{A}_0$ can be rewritten in the following form

$$\mathscr{A}_0 u(x) = \tfrac{1}{2} \operatorname{Tr}[D^2 u(x)\, \psi^2(x)] + \langle b(x), Du(x) \rangle, \qquad x \in \mathbb{R}^d, \tag{3.1}$$

where $\psi(x) = \sqrt{a(x)}$. As well known from the theory of Markov processes (see [21]), $\mathscr{A}_0$ is the diffusion operator associated with the stochastic initial value problem

$$d\xi(t) = b(\xi(t))\, dt + \psi(\xi(t))\, dw(t), \qquad t \geqslant 0, \qquad \xi(0) = x, \tag{3.2}$$

where $w(t) = (w_1(t), ..., w_d(t))$ is a standard d-dimensional Brownian motion.

Since b and ψ are Lipschitz continuous, then for any $x \in \mathbb{R}^d$ the problem (3.2) admits a unique solution, that is there exists a $\mathbb{R}^d$-valued process $\xi(t; x)$, continuous with probability 1, such that

$$\xi(t; x) = x + \int_0^t b(\xi(s; x))\, ds + \int_0^t \psi(\xi(s; x))\, dw(s)$$

(for a proof see for example [21]). Moreover, it is possible to check that for any $x \in \mathbb{R}^d$

$$\mathbb{E}\, |\xi(t; x)|^2 \leqslant c\, |x|^2\, e^{ct}, \qquad t \geqslant 0, \tag{3.3}$$

for a suitable constant $c \in \mathbb{R}$.

We recall that a $\mathbb{R}^{d_1}$-valued stochastic process $\eta(z)$ of the parameter $z \in \mathbb{R}^{d_2}$ is said to be *mean-square differentiable* at the point z_0, with the random variable $\zeta(z_0)$ as its derivative, if

$$\lim_{|h| \to 0} \frac{1}{|h|^2}\, \mathbb{E}\, |\eta(z_0 + h) - \eta(z_0) - \zeta(z_0)\, h|^2 = 0.$$

Since we are assuming b and ψ to be twice differentiable with bounded derivatives, it can be proved that $\xi(t; x)$ is twice mean-square differentiable with respect to $x \in \mathbb{R}^d$ and the derivatives are solutions themselves of suitable stochastic differential equations that one obtains from (3.2) by differentiating the coefficients. Namely, if $\eta^h(t; x) = D\xi(t; x)\, h$ denotes the first mean-square derivative of $\xi(t; x)$ with respect to $x \in \mathbb{R}^d$ along the direction $h \in \mathbb{R}^d$, then we have

$$d\eta^h(t) = Db(\xi(t; x))\, \eta^h(t)\, dt + [D\psi(\xi(t; x))\, \eta^h(t)]\, dw(t), \qquad \eta^h(0) = h. \tag{3.4}$$

Similarly, if $\zeta^{hk}(t; x) = D^2\xi(t; x)(h, k)$ denotes the second mean-square derivative of $\xi(t; x)$ with respect to $x \in \mathbb{R}^d$ along the directions $h, k \in \mathbb{R}^d$, we have

$$\begin{cases} d\zeta^{hk}(t) = Db(\xi(t; x))\, \zeta^{hk}(t)\, dt + [D\psi(\xi(t; x))\, \zeta^{hk}(t)]\, dw(t) + d\theta^{hk}(t), \\ \zeta^{hk}(0) = 0, \end{cases} \tag{3.5}$$

where

$$\begin{aligned} d\theta^{hk}(t) = {} & D^2 b(\xi(t; x))(\eta^h(t; x), \eta^k(t; x))\, dt \\ & + D^2\psi(\xi(t; x))(\eta^h(t; x), \eta^k(t; x))\, dw(t). \end{aligned} \tag{3.6}$$

In the sequel we shall denote by $(g_i)_{i=1}^{2d}$ and $(e_i)_{i=1}^{d}$ the standard orthonormal bases of $\mathbb{R}^{2d}$ and $\mathbb{R}^d$, respectively.

LEMMA 3.1. *Assume that the hypothesis 2.2 holds and let $y(t)$ be an Itô process with drift $y_1\colon [0,+\infty)\to\mathbb{R}^d$ and covariance $y_2\colon [0,+\infty)\to\mathscr{L}(\mathbb{R}^d)$, that is*

$$y(t)=y(0)+\int_0^t y_1(s)\,ds+\int_0^t y_2(s)\,dw(s).$$

Then, if the function $\varphi\colon \mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ is defined by $\varphi(x,y)=\psi(x)\,y$ and if $z(t)=(\xi(t),y(t))^t$, where $\xi(t)=\xi(t;x)$ is the solution of the problem (3.2), we have

$$\begin{aligned}d\varphi(z(t))&=\psi(\xi(t))([\gamma_1(\xi(t))\,\beta(t)]\,y(t)+y_1(t))\,dt\\&\quad+\psi(\xi(t))\sum_{i=1}^d([\gamma_1(\xi(t))\,y_2^\star(t)\,e_i]\,e_i+\tfrac{1}{2}[\gamma_2(\xi(t))(e_i,e_i)]\,y(t))\,dt\\&\quad+\psi(\xi(t))([\gamma_1(\xi(t))\,dw(t)]\,y(t)+y_2(t)\,dw(t)).\end{aligned}\tag{3.7}$$

Proof. The process $z(t)=(\xi(t;x),y(t))^t$ is the solution of the problem

$$dz(t)=B(z(t))\,dt+\Sigma(z(t))\,dw(t),\qquad z(0)=(x,y(0))^t,$$

where, for any $t\geqslant 0$

$$B(z(t))=\begin{pmatrix}b(\xi(t))\\y_1(t)\end{pmatrix},\qquad \Sigma(z(t))=\begin{pmatrix}\psi(\xi(t))\\y_2(t)\end{pmatrix}.$$

Now, since ψ is twice continuously differentiable, then the function φ is twice differentiable with continuous derivatives and it holds

$$D\varphi(x,y)(h,k)=[D\psi(x)\,h]\,y+\psi(x)\,k$$

$$\begin{aligned}D^2\varphi(x,y)((h_1,k_1),(h_2,k_2))&=[D^2\psi(x)(h_1,h_2)]\,y\\&\quad+[D\psi(x)\,h_1]\,k_2+[D\psi(x)\,h_2]\,k_1.\end{aligned}$$

Then we can apply the Itô's formula and we get

$$\begin{aligned}d\varphi(z(t))&=D\varphi(z(t))\,dz(t)+\tfrac{1}{2}\operatorname{Tr}[D^2\varphi(z(t))(\Sigma\Sigma^\star)(z(t))]\,dt\\&=D\varphi(z(t))\,B(z(t))\,dt+D\varphi(z(t))\,\Sigma(z(t))\,dw(t)\\&\quad+\tfrac{1}{2}\operatorname{Tr}[D^2\varphi(z(t))(\Sigma\Sigma^\star)(z(t))]\,dt.\end{aligned}$$

We have

$$(\Sigma\Sigma^\star)(z(t))=\begin{pmatrix}\psi^2(\xi(t)) & \psi(\xi(t))\,y_2^\star(t)\\ y_2(t)\,\psi^\star(\xi(t)) & (y_2y_2^\star)(t)\end{pmatrix}$$

and then, from easy calculations it follows that if $i=1, ..., d$

$$D^2\varphi(z(t))((\Sigma\Sigma^\star)(z(t))\, g_i, g_i)$$
$$= [D^2\psi(\xi(t))(\psi^2(\xi(t))\, e_i, e_i)]\, y(t) + [D\psi(\xi(t))\, e_i]\, y_2(t)\, \psi^\star(\xi(t))\, e_i \quad (3.8)$$

and if $i=d+1, ..., 2d$

$$D^2\varphi(z(t))((\Sigma\Sigma^\star)(z(t))\, g_i, g_i) = [D\psi(\xi(t))\, \psi(\xi(t))\, y_2^\star(t)\, e_i]\, e_i. \quad (3.9)$$

It is easy to check that, if $A \in \mathscr{L}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, then it holds

$$\sum_{i=1}^d [D\psi(x)\, e_i]\, Ae_i = \sum_{i=1}^d [D\psi(x)\, A^\star e_i]\, e_i.$$

In the same way, if $B \in \mathscr{L}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, it is possible to prove that

$$\sum_{i=1}^d D^2\psi(x)(BB^\star e_i, e_i) = \sum_{i=1}^d D^2\psi(x)(Be_i, Be_i).$$

Due to (3.8) and (3.9) this implies that

$$\frac{1}{2}\,\mathrm{Tr}[D^2\varphi(z(t))(\Sigma\Sigma^\star)(z(t))]$$
$$= \sum_{i=1}^d \big(\tfrac{1}{2}[D^2\psi(\xi(t))(\psi(\xi(t))\, e_i, \psi(\xi(t))\, e_i)]\, y(t)$$
$$+ [D\psi(\xi(t))\, \psi(\xi(t))\, y_2^\star(t)\, e_i]\, e_i\big).$$

Hence we can conclude that

$$d\varphi(z(t)) = ([D\psi(\xi(t))\, b(\xi(t))]\, y(t) + \psi(\xi(t))\, y_1(t))\, dt$$
$$+ \sum_{i=1}^d \big(\tfrac{1}{2}[D^2\psi(\xi(t))(\psi(\xi(t))\, e_i, \psi(\xi(t))\, e_i)]\, y(t)\, dt$$
$$+ [D\psi(\xi(t))\, \psi(\xi(t))\, y_2^\star(t)\, e_i]\, e_i\big)\, dt$$
$$+ [D\psi(\xi(t))\, \psi(\xi(t))\, dw(t)]\, y(t) + \psi(\xi(t))\, y_2(t)\, dw(t).$$

From the hypothesis 2.2-2. we have

$$[D\psi(\xi(t))\, b(\xi(t))]\, y(t) = [D\psi(\xi(t))(\psi\beta)(\xi(t))]\, y(t)$$

and then from the hypothesis 2.2-3. it follows

$$[D\psi(\xi(t))\, b(\xi(t))]\, y(t) = \psi(\xi(t))[\gamma_1(\xi(t))\, \beta(\xi(t))]\, y(t). \quad (3.10)$$

Again, due to the hypothesis 2.2-3. we have

$$[D^2\psi(\xi(t))(\psi(\xi(t))\, e_i, \psi(\xi(t))\, e_i)]\, y(t) = \psi(\xi(t))[\gamma_2(\xi(t))(e_i, e_i)]\, y(t) \tag{3.11}$$

and

$$[D\psi(\xi(t))\, \psi(\xi(t))\, dw(t)]\, y(t) = \psi(\xi(t))[\gamma_1(\xi(t))\, dw(t)]\, y(t). \tag{3.12}$$

Finally, from the hypothesis 2.2-3. it follows

$$[D\psi(\xi(t))\, \psi(\xi(t))\, y_2^\star(t)\, e_i]\, e_i = \psi(\xi(t))[\gamma_1(\xi(t))\, y_2^\star(\xi(t))\, e_i]\, e_i. \tag{3.13}$$

Therefore, by using (3.10), (3.11), (3.12) and (3.13), we get (3.7). ∎

PROPOSITION 3.2. *Under the hypothesis 2.2, for any $x \in \mathbb{R}^d$ and $t \geqslant 0$ there exists a d-dimensional process $v^h(t; x)$ such that*

$$D\xi(t; x)\, \psi(x)\, h = \psi(\xi(t; x))\, v^h(t; x), \; \mathbb{P}\text{-a.s.} \tag{3.14}$$

and

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}\, |v^h(t; x)|^2 \leqslant c(t)\, |h|^2, \tag{3.15}$$

for a continuous increasing function $c(t) \geqslant 0$.

Proof. Since $D\xi(t; x)\, \psi(x)\, h$ is the unique solution of the problem (3.4) with initial value equal to $\psi(x)\, h$, in order to prove (3.14) we have to show that there exist suitable mappings $v_1^h(\cdot\,; x)\colon [0, +\infty) \to \mathbb{R}^d$ and $v_2^h(\cdot\,; x)\colon [0, +\infty) \to \mathscr{L}(\mathbb{R}^d)$ such that

$$v^h(t; x) = h + \int_0^t v_1^h(s; x)\, ds + \int_0^t v_2^h(s; x)\, dw(s)$$

and

$$d\varphi(z(t)) = Db(\xi(t))\, \psi(\xi(t))\, v^h(t)\, dt + [D\psi(\xi(t))\, \psi(\xi(t))\, v^h(t)]\, dw(t),$$

where $z(t) = (\xi(t; x), v^h(t; x))^t$ and $\varphi(x, y) = \psi(x)\, y$ is the function introduced in the previous lemma.

By using (2.4) we have

$$Db(\xi(t))\, \psi(\xi(t))\, v^h(t) = \psi(\xi(t))\, \rho_1(\xi(t))\, v^h(t)$$

and from (2.3) we have

$$[D\psi(\xi(t))\, \psi(\xi(t))\, v^h(t)]\, dw(t) = \psi(\xi(t))[\gamma_1(\xi(t))\, v^h(t)]\, dw(t).$$

Therefore, if we assume that

$$\psi(\xi(t))\, v_1^h(t) = \psi(\xi(t))(\rho_1(\xi(t))\, v^h(t) - [\gamma_1(\xi(t))\, \beta(\xi(t))]\, v^h(t)) - \psi(\xi(t)) \sum_{i=1}^{d} (\tfrac{1}{2}[\gamma_2(\xi(t))(e_i, e_i)]\, v^h(t) + [\gamma_1(\xi(t))(v_2^h(t))^{\star}\, (t)\, e_i]\, e_i)$$

and for any $k \in \mathbb{R}^d$

$$\psi(\xi(t))\, v_2^h(t)\, k = \psi(\xi(t))([\gamma_1(\xi(t))\, v^h(t)]\, k - [\gamma_1(\xi(t))\, k]\, v^h(t)),$$

due to (3.7), we obtain (3.14). In particular, a possible choice for $v_2^h(t; x)\, k$ is

$$v_2^h(t; x)\, k = A_2^h(t; x)\, v^h(t; x) = [\gamma_1(\xi(t))\, v^h(t)]\, k - [\gamma_1(\xi(t))\, k]\, v^h(t), \tag{3.16}$$

so that for $v_1^h(t; x)$ we can take

$$v_1^h(t; x) = A_1^h(t; x)\, v^h(t; x) = \rho_1(\xi(t))\, v^h(t) - [\gamma_1(\xi(t))\, \beta(\xi(t))]\, v^h(t) - \sum_{i=1}^{d} r(\tfrac{1}{2}[\gamma_2(\xi(t))(e_i, e_i)]\, v^h(t) + [\gamma_1(\xi(t))(A_2^h(t; x)\, v^h(t; x))^{\star}\, e_i]\, e_i). \tag{3.17}$$

This means that the process $v^h(t; x)$ is the solution of the following linear stochastic equation with random coefficients

$$dv^h(t) = A_1^h(t; x)\, v^h(t)\, dt + A_2^h(t; x)\, v^h(t)\, dw(t), \qquad v^h(0) = h. \tag{3.18}$$

Thanks to the boundedness of the functions γ_i and ρ_i, $i = 1, 2$, we have that there exists $c > 0$ such that

$$\sup_{t \geqslant 0,\, x \in \mathbb{R}^d} |A_1^h(t; x)|_{\mathscr{L}(\mathbb{R}^r)} + |A_2^h(t; x)|_{\mathscr{L}(\mathbb{R}^d;\, \mathscr{L}(\mathbb{R}^d))} \leqslant c, \qquad \mathbb{P}\text{-}a.s.,$$

and then the equation (3.18) admits a unique solution, which is $v^h(t; x)$. Moreover, by the Itô's formula we have

$$\tfrac{1}{2} d\, |v^h(t)|^2 = (\langle A_1^h(t; x)\, v^h(t), v^h(t) \rangle \tfrac{1}{2} \operatorname{Tr}[A_2^h(t; x)(A_2^h(t; x))^{\star}])\, dt + \langle A_2^h(t; x)\, v^h(t), v^h(t) \rangle\, dw(t),$$

so that, by using the boundedness of $A_1^h(t;x)$ and $A_2^h(t;x)$, from standard calculations we get (3.15). ∎

Now we prove a similar result concerning the second derivative of $\xi(t;x)$.

PROPOSITION 3.3. *Under the hypothesis* 2.2, *for any* $x \in \mathbb{R}^d$ *and* $t \geqslant 0$ *there exists a d-dimensional process* $u^h(t;x)$ *such that*

$$D^2\xi(t;x)(\psi(x)\,h,\psi(x)\,h) = \psi(\xi(t;x))\,u^h(t;x), \qquad \mathbb{P}\text{-a.s.} \tag{3.19}$$

and

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\,|u^h(t;x)|^2 \leqslant c(t)\,|h|^4, \tag{3.20}$$

for a suitable increasing continuous function $c(t) \geqslant 0$.

Proof. We proceed as in the proof of the previous proposition. Actually, the process $D^2\xi(t;x)(\psi(x)\,h,\psi(x)\,h)$ is the unique solution of the problem (3.5), with

$$\begin{aligned} d\theta(t) &= D^2b(\xi(t;x))(D\xi(t;x)\,\psi(x)\,h, D\xi(t;x)\,\psi(x)\,h)\,dt \\ &\quad + D^2\psi(\xi(t;x))(D\xi(t;x)\,\psi(x)\,h, D\xi(t;x)\,\psi(x)\,h)\,dw(t). \end{aligned}$$

Then, if $z(t) = (\xi(t;x), u^h(t;x))^t$ and $u^h(t;x)$ is an Itô process having drift term $u_1^h(t;x)$ and covariance term $u_2^h(t;x)$ and such that $u^h(0;x) = 0$, we have to impose the condition

$$\begin{aligned} d\varphi(z(t)) &= Db(\xi(t))\,\psi(\xi(t;x))\,u^h(t)\,dt \\ &\quad + [D\psi(\xi(t))\,\psi(\xi(t;x))\,u^h(t)]\,dw(t) + d\theta(t). \end{aligned}$$

By using (2.3) and (2.4) as in the proof of the previous proposition, we get an explicit expression for $u_1^h(t;x)$ and $u_2^h(t;x)$ and we prove the estimate (3.20), due to the boundedness of the coefficients. ∎

4. THE TRANSITION SEMIGROUP

For any $\varphi \in B_b(\mathbb{R}^d)$ we set

$$P_t\varphi(x) = \mathbb{E}\varphi(\xi(t;x)) = \int_{\mathbb{R}^d} \varphi(y)\,P_t(x,dy), \tag{4.1}$$

where $P_t(x, dy)$, $t \geqslant 0$ and $x \in \mathbb{R}^d$, is the transition probabilities family corresponding to the equation (3.2). Since $P_t(x, dy)$ fulfils the Chapman–Kolmogorov equation, P_t defines a semigroup of contractions from $B_b(\mathbb{R}^d)$ into itself. Moreover it is possible to prove that for any $t \geqslant 0$ and $x, y \in \mathbb{R}^d$

$$\mathbb{E}\,|\xi(t; x) - \xi(t; y)|^2 \leqslant c e^{ct}\,|x - y|^2.$$

Then, we easily have that P_t maps $C_b(\mathbb{R}^d)$ into itself.

As we recalled in the previous section, since we are assuming b and ψ to be twice differentiable with bounded derivatives, the solution $\xi(t; x)$ of the problem (3.2) is twice mean-square differentiable with respect to $x \in \mathbb{R}^d$. Then, if $\varphi \in C_b^2(\mathbb{R}^d)$, by deriving under the sign of integral in (4.1), we get that $P_t\varphi \in C_b^2(\mathbb{R}^d)$ and for any $x, h, k \in \mathbb{R}^d$ it holds

$$\begin{aligned}\langle D(P_t\varphi)(x), h\rangle &= \mathbb{E}\langle D\varphi(\xi(t; x)), D\xi(t; x)\, h\rangle,\\ \langle D^2(P_t\varphi)(x)\, h, k\rangle &= \mathbb{E}\langle D^2\varphi(\xi(t; x))\, D\xi(t; x)\, h, D\xi(t; x)\, k\rangle \qquad (4.2)\\ &\quad + \mathbb{E}\langle D\varphi(\xi(t; x)), D^2\xi(t; x)(h, k)\rangle.\end{aligned}$$

In the strictly non degenerate case there exists $\psi^{-1}(x)$, for any $x \in \mathbb{R}^d$, and $\sup_{x \in \mathbb{R}^d} |\psi^{-1}(x)| < \infty$. Thus it is possible to prove the following Bismut–Elworthy formula for the first derivative of $P_t\varphi$

$$\langle D(P_t\varphi)(x), h\rangle = \frac{1}{t}\,\mathbb{E}\varphi(\xi(t; x)) \int_0^t \langle \psi^{-1}(\xi(s; x))\, D\xi(s; x)\, h, dw(s)\rangle$$

(see [4] and [15] for a proof). Such a formula is first proved for functions $\varphi \in C_b^2(\mathbb{R}^d)$ and hence it is extended to all functions $\varphi \in C_b(\mathbb{R}^d)$. By using the semigroup law, the differentiability of $P_t\varphi$ follows, for any $\varphi \in B_b(\mathbb{R}^d)$. In our case, which is possibly degenerate, the semigroup P_t has not this regularizing effect. Nevertheless, by adapting the proof of the Bismut–Elworthy formula to the present situation, we give an explicit expression for $\langle D(P_t\varphi)(x), \psi(x)\, h\rangle$ and $\langle D^2(P_t\varphi)(x)\, \psi(x)\, h, \psi(x)\, h\rangle$, which does not involve the derivatives of φ.

PROPOSITION 4.1. *Under the hypothesis 2.2, for any $\varphi \in C_b^2(\mathbb{R}^d)$, $t > 0$ and $x, h \in \mathbb{R}^d$, we have*

$$\langle D(P_t\varphi)(x), \psi(x)\, h\rangle = \frac{1}{t}\,\mathbb{E}\varphi(\xi(t; x)) \int_0^t \langle v^h(s; x), dw(s)\rangle, \qquad (4.3)$$

where $v^h(t;x)$ *is the random variable introduced in the Proposition* 3.2. *Moreover it holds*

$$\sup_{x\in\mathbb{R}^d} |\langle D(P_t\varphi)(x), \psi(x)\, h\rangle| \leqslant c(t\wedge 1)^{-1/2}\, \|\varphi\|_0\, |h|. \tag{4.4}$$

Proof. According to the Itô's formula, it is possible to prove that for any $\varphi \in C_b^2(\mathbb{R}^d)$ it holds

$$\varphi(\xi(t;x)) = P_t\varphi(x) + \int_0^t \langle D(P_{t-s}\varphi)(\xi(s;x)), \psi(\xi(s;x))\, dw(s)\rangle.$$

Thanks to (3.15) we can multiply each side by $\int_0^t \langle v^h(s;x), dw(s)\rangle$ and by taking the expectation we get

$$\mathbb{E}\varphi(\xi(t;x)) \int_0^t \langle v^h(s;x), dw(s)\rangle$$

$$= \mathbb{E}\int_0^t \langle D(P_{t-s}\varphi)(\xi(s;x)), \psi(\xi(s;x))\, v^h(s;x)\rangle\, ds.$$

Due to (3.14), it follows that

$$\mathbb{E}\varphi(\xi(t;x)) \int_0^t \langle v^h(s;x), dw(s)\rangle$$

$$= \mathbb{E}\int_0^t \langle D(P_{t-s}\varphi)(\xi(s;x)), D\xi(s;x)\, \psi(x)\, h\rangle\, ds$$

$$= \left\langle D\left(\int_0^t \mathbb{E}P_{t-s}\varphi(\xi(s;\cdot))\, ds\right)(x), \psi(x)\, h\right\rangle.$$

By using the Markov property of $\xi(t;x)$ we have

$$\mathbb{E}P_{t-s}\varphi(\xi(s;x)) = P_t\varphi(x), \tag{4.5}$$

and then

$$\mathbb{E}\varphi(\xi(t;x)) \int_0^t \langle v^h(s;x), dw(s)\rangle = t\langle D(P_t\varphi)(x), \psi(x)\, h\rangle,$$

which gives (4.3). The estimate (4.4) easily follows from (3.15) and the semigroup law. Indeed, if $0<t\leqslant 1$ we have

$$\frac{1}{t}\left|\,\mathbb{E}\varphi(\xi(t;x))\int_0^t\langle v^h(s;x),dw(s)\rangle\,\right|$$

$$\leqslant\frac{1}{t}\,\|\varphi\|_0\left(\mathbb{E}\left|\int_0^t\langle v^h(s;x),dw(s)\rangle\right|^2\right)^{1/2}$$

$$=\frac{1}{t}\,\|\varphi\|_0\left(\mathbb{E}\int_0^t|v^h(s;x)|^2\,ds\right)^{1/2}\leqslant ct^{-1/2}\,\|\varphi\|_0\,|h|.$$

If $t>1$ we have

$$\langle D(P_t\varphi)(x),h\rangle=\langle D(P_1(P_{t-1}\varphi))(x),h\rangle$$

and then

$$\sup_{x\in\mathbb{R}^d}|\langle D(P_t\varphi)(x),h\rangle|\leqslant c\,\|P_{t-1}\varphi\|_0\,|h|\leqslant c\,\|\varphi\|_0\,|h|,$$

so that (4.4) follows. ∎

Concerning the second derivative of $P_t\varphi$ we have the following result.

PROPOSITION 4.2. *Assume that the hypothesis* 2.2 *holds. Then, if* $\varphi\in C_b^2(\mathbb{R}^d)$, *for any* $x,h\in\mathbb{R}^d$ *and* $t>0$ *we have*

$$\begin{aligned}&\langle D^2(P_t\varphi)(x)\,\psi(x)\,h,\psi(x)\,h\rangle\\&=\frac{2}{t}\,\mathbb{E}\langle D(P_{t/2}\varphi)(\xi(t/2;x)),D\xi(t/2;x)\,\psi(x)\,h\rangle\int_0^{t/2}\langle v^h(s;x),dw(s)\rangle\\&\quad-\frac{2}{t}\,\mathbb{E}\int_0^{t/2}\langle D(P_{t-s}\varphi)(\xi(s;x)),[D\psi(\xi(s;x))\,D\xi(s;x)\,\psi(x)\,h]\,v^h(s;x)\rangle\,ds\\&\quad+\frac{2}{t}\,\mathbb{E}\int_0^{t/2}\langle D(P_{t-s}\varphi)(\xi(s;x)),D^2\xi(s;x)(\psi(x)\,h,\psi(x)\,h)\rangle\,ds\end{aligned}\tag{4.6}$$

and the following estimate holds

$$\sup_{x\in\mathbb{R}^d}|\langle D^2(P_t\varphi)(x)\,\psi(x)\,h,\psi(x)\,h\rangle|\leqslant c\,(t\wedge 1)^{-1}\,\|\varphi\|_0\,|h|^2.\tag{4.7}$$

Proof. Let us fix $T>0$ and $0\leqslant t\leqslant T$. Since $\varphi\in C_b^2(\mathbb{R}^d)$, due to the Itô's formula we have

$$P_{T-t}\varphi(\xi(t;x))=P_T\varphi(x)+\int_0^t\langle D(P_{t-s}\varphi)(\xi(s;x)),\,\psi(\xi(s;x))\,dw(s)\rangle.$$

Then, by taking for each side the derivative with respect to $x\in\mathbb{R}^d$ along the direction $\psi(x)\,h$, we get

$$\begin{aligned}\langle D(P_{T-t}\varphi)&(\xi(t;x)),\,D\xi(t;x)\,\psi(x)\,h\rangle\\ &=\langle D(P_T\varphi)(x),\,\psi(x)\,h\rangle\\ &\quad+\int_0^t\langle D^2(P_{t-s}\varphi)(\xi(s;x))\,D\xi(s;x)\,\psi(x)\,h,\,\psi(\xi(s;x))\,dw(s)\rangle\\ &\quad+\int_0^t\langle D(P_{t-s}\varphi)(\xi(s;x)),\,[D\psi(\xi(s;x))\,D\xi(s;x)\,\psi(x)\,h]\,dw(s)\rangle.\end{aligned}$$

Thus, if we set $t=T$ we have

$$\begin{aligned}\langle D\varphi(\xi(T;x)),&\,D\xi(T;x)\,\psi(x)\,h\rangle\\ &=\langle D(P_T\varphi)(x),\,\psi(x)\,h\rangle\\ &\quad+\int_0^T\langle D^2(P_{T-s}\varphi)(\xi(s;x))\,D\xi(s;x)\,\psi(x)\,h,\,\psi(\xi(s;x))\,dw(s)\rangle\\ &\quad+\int_0^T\langle D(P_{T-s}\varphi)(\xi(s;x)),\,[D\psi(\xi(s;x))\,D\xi(s;x)\,\psi(x)\,h]\,dw(s)\rangle.\end{aligned}$$

Now, we multiply each side by $\int_0^T\langle v^h(s;x),dw(s)\rangle$ and by taking the expectation and recalling (3.14) it follows

$$\begin{aligned}&\mathbb{E}\langle D\varphi(\xi(T;x)),\,D\xi(T;x)\,\psi(x)\,h\rangle\int_0^T\langle v^h(s;x),\,dw(s)\rangle\\ &=\mathbb{E}\int_0^T\langle D^2(P_{T-s}\varphi)(\xi(s;x))\,D\xi(s;x)\,\psi(x)\,h,\,D\xi(s;x)\,\psi(x)\,h\rangle\,ds\\ &\quad+\mathbb{E}\int_0^T\langle D(P_{T-s}\varphi)(\xi(s;x)),\,[D\psi(\xi(s;x))\,D\xi(s;x)\,\psi(x)\,h]\,v^h(s;x)\rangle\,ds.\end{aligned}$$

If we derive twice with respect to $x \in \mathbb{R}^d$ each side of (4.5) (with $t = T$) along the directions $\psi(x)\,h$, we get

$$\begin{aligned}&\langle D^2(P_T\varphi)(x)\,\psi(x)\,h, \psi(x)\,h\rangle\\&\quad= \mathbb{E}\langle D^2(P_{T-s}\varphi)(\xi(s;x))\,D\xi(s;x)\,\psi(x)\,h, D\xi(s;x)\,\psi(x)\,h\rangle\\&\qquad+ \mathbb{E}\langle D(P_{T-s}\varphi)(\xi(s;x)), D^2\xi(s;x)(\psi(x)\,h, \psi(x)\,h)\rangle.\end{aligned}$$

Therefore we can conclude that

$$\begin{aligned}&T\langle D^2(P_T\varphi)(x)\,\psi(x)\,h, \psi(x)\,h\rangle\\&= \mathbb{E}\langle D\varphi(\xi(T;x)), D\xi(T;x)\,\psi(x)\,h\rangle \int_0^T \langle v^h(s;x), dw(s)\rangle\\&\quad+ \mathbb{E}\int_0^T \langle D(P_{T-s}\varphi)(\xi(s;x)), D^2\xi(s;x)(\psi(x)\,h, \psi(x)\,h)\rangle\,ds\\&\quad- \mathbb{E}\int_0^T \langle D(P_{T-s}\varphi)(\xi(s;x)), [D\psi(\xi(s;x))\,D\xi(s;x)\,\psi(x)\,h]\,v^h(s;x)\rangle\,ds.\end{aligned}$$

If we replace T with $t/2$ and φ with $P_{t/2}\varphi$, then (4.6) follows.

Finally, we prove the estimate (4.7). We can assume $0 < t \leqslant 1$. Actually, the general case $t > 0$ follows from the semigroup law as in the proof of the previous proposition. Due to (3.14) we have

$$\begin{aligned}&\frac{2}{t}\,\mathbb{E}\langle D(P_{t/2}\varphi)(\xi(t/2;x)), D\xi(t/2;x)\,\psi(x)\,h\rangle \int_0^{t/2} \langle v^h(s;x), dw(s)\rangle\\&\quad= \frac{2}{t}\,\mathbb{E}\langle D(P_{t/2}\varphi)(\xi(t/2;x)), \psi(\xi(t/2;x))\,v^h(t/2;x)\rangle\\&\qquad\times \int_0^{t/2} \langle v^h(s;x), dw(s)\rangle,\end{aligned}$$

then, since (4.4) holds, we have

$$\begin{aligned}&\frac{2}{t}\left|\,\mathbb{E}\langle D(P_{t/2}\varphi)(\xi(t/2;x)), D\xi(t/2;x)\,\psi(x)\,h\rangle \int_0^{t/2} \langle v^h(s;x), dw(s)\rangle\right|\\&\qquad\leqslant c t^{-3/2}\,\|\varphi\|_0\,(\mathbb{E}\,|v^h(t/2;x)|^2)^{1/2} \left(\mathbb{E}\int_0^{t/2} |v^h(s;x)|^2\,ds\right)^{1/2}.\end{aligned}$$

Thanks to (3.15), it follows that

$$\frac{2}{t}\left|\mathbb{E}\langle D(P_{t/2}\varphi)(\xi(t/2;x)), D\xi(t/2;x)\,\psi(x)\,h\rangle \int_0^{t/2}\langle v^h(s;x), dw(s)\rangle\right|$$

$$\leqslant ct^{-1}\,\|\varphi\|_0\,|h|^2. \tag{4.8}$$

By using (3.19) we have

$$\frac{2}{t}\,\mathbb{E}\int_0^{t/2}\langle D(P_{t-s}\varphi)(\xi(s;x)), D^2\xi(s;x)(\psi(x)\,h, \psi(x)\,h)\rangle\,ds$$

$$=\frac{2}{t}\,\mathbb{E}\int_0^{t/2}\langle D(P_{t-s}\varphi)(\xi(s;x)), \psi(\xi(s;x))\,u^h(s;x)\rangle\,ds,$$

and then, due to (3.20) and (4.4), we get

$$\frac{2}{t}\left|\mathbb{E}\int_0^{t/2}\langle D(P_{t-s}\varphi)(\xi(s;x)), D^2\xi(s;x)(\psi(x)\,h, \psi(x)\,h)\rangle\,ds\right|$$

$$\leqslant ct^{-1}\,\|\varphi\|_0\int_0^{t/2}(t-s)^{-1/2}\,\mathbb{E}\,|u^h(s;x)|\,ds \leqslant ct^{-1}\,\|\varphi\|_0\,|h|^2. \tag{4.9}$$

Due to the condition given by the hypothesis 2.2-3. and due to (3.14), we have that

$$\begin{aligned}&[D\psi(\xi(s;x))\,D\xi(s;x)\,\psi(x)\,h]\,v^h(s;x)\\&\quad=[D\psi(\xi(s;x))\,\psi(\xi(s;x))\,v^h(s;x)]\,v^h(s;x)\\&\quad=\psi(\xi(s;x))[\gamma_1(\xi(s;x))\,v^h(s;x)]\,v^h(s;x).\end{aligned}$$

Therefore, by using (4.4), we have

$$\frac{2}{t}\left|\mathbb{E}\int_0^{t/2}\langle D(P_{t-s}\varphi)(\xi(s;x)), [D\psi(\xi(s;x))\,D\xi(s;x)\,\psi(x)\,h]\,v^h(s;x)\rangle\,ds\right|$$

$$\leqslant ct^{-1}\,\|\varphi\|_0\,|h|^2\int_0^{t/2}(t-s)^{-1/2}\,\mathbb{E}\,|v^h(s;x)|^2\,ds \leqslant ct^{-1}\,\|\varphi\|_0\,|h|^2. \tag{4.10}$$

Hence, from (4.8), (4.9) and (4.10), the estimate (4.7) follows. ∎

As an immediate consequence of the Itô's formula and of the previous two propositions we have

COROLLARY 4.3. *Assume the hypothesis 2.2. Then for any* $\varphi \in C_b^2(\mathbb{R}^d)$ *and* $x \in \mathbb{R}^d$ *the mapping* $(0, +\infty) \to \mathbb{R}$, $t \mapsto P_t\varphi(x)$, *is differentiable and it holds*

$$\frac{d}{dt} P_t\varphi(x) = \mathscr{A}_0(P_t\varphi)(x).$$

Moreover we have

$$\sup_{x \in \mathbb{R}^d} \left| \frac{d}{dt} P_t\varphi(x) \right| = \|\mathscr{A}_0(P_t\varphi)\|_0 \leqslant c(t \wedge 1)^{-1} \|\varphi\|_0. \tag{4.11}$$

Proof. Since φ is assumed to be twice differentiable with bounded derivative, we can apply the Itô's formula and we have that

$$\frac{d}{dt} P_t\varphi(x) = \frac{1}{2} \sum_{i=1}^{d} \langle D^2(P_t\varphi)(x)\, \psi(x)\, e_i, \psi(x)\, e_i \rangle + \langle D(P_t\varphi)(x), b(x) \rangle$$
$$= \mathscr{A}_0(P_t\varphi)(x).$$

Now, since $b(x) = \psi(x)\, \beta(x)$, we have that

$$\langle D(P_t\varphi)(x), b(x) \rangle = \langle D(P_t\varphi)(x), \psi(x)\, \beta(x) \rangle$$

and recalling that β is assumed to be bounded, by applying the Propositions 4.1 and 4.2, we get (4.11). ∎

5. THE GENERATION RESULT

For any $\varphi \in C_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the mapping $[0, +\infty) \to \mathbb{R}$, $t \mapsto P_t\varphi(x)$, is continuous. Indeed, if $\varphi \in C_b^1(\mathbb{R}^d)$ we have

$$|P_t\varphi(x) - \varphi(x)| \leqslant \mathbb{E}\, |\varphi(\xi(t; x)) - \varphi(x)| \leqslant \|D\varphi\|_0\, (\mathbb{E}\, |\xi(t; x) - x|^2)^{1/2} \to 0,$$

as t goes to zero, since the process $\xi(t; x)$ is continuous with respect to t, with probability 1, and (3.3) holds. The general case of $\varphi \in C_b(\mathbb{R}^d)$ follows by approximating φ by means of a sequence $(\varphi_n) \subset C_b^1(\mathbb{R}^d)$.

Thus, for any complex λ such that $\operatorname{Re} \lambda > 0$ and for any $\varphi \in C_b(\mathbb{R}^d)$ we can define

$$F(\lambda)\, \varphi(x) = \int_0^{+\infty} e^{-\lambda t} P_t\varphi(x)\, dt, \qquad x \in \mathbb{R}^d.$$

As shown in [10], $F(\lambda)$ is a bounded linear operator in $C_b(\mathbb{R}^d)$ which fulfils the resolvent law and such that $\operatorname{Ker} F(\lambda) = \{0\}$, for any $\lambda \in \{\operatorname{Re} \lambda > 0\}$.

This allows us to introduce the *weak generator* $\mathscr{A}$ of the semigroup P_t as the unique closed operator $\mathscr{A}: D(\mathscr{A}) \subseteq C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ such that

$$R(\lambda, \mathscr{A}) = F(\lambda), \qquad \lambda \in \{\operatorname{Re} \lambda > 0\}.$$

In [10] it is proved that $P_t D(\mathscr{A}) \subseteq D(\mathscr{A})$, for any $t \geqslant 0$, and if $\varphi \in D(\mathscr{A})$ then $P_t(\mathscr{A}\varphi) = \mathscr{A}(P_t\varphi)$. Moreover, if $\varphi \in D(\mathscr{A})$ the mapping $[0, +\infty) \to \mathbb{R}$, $t \mapsto P_t\varphi(x)$, is differentiable, for any fixed $x \in \mathbb{R}^d$, and

$$\frac{d}{dt} P_t\varphi(x) = P_t(\mathscr{A}\varphi)(x) = \mathscr{A}(P_t\varphi)(x).$$

Lemma 5.1. *If $\varphi \in C_b^2(\mathbb{R}^d)$, we have that $P_t\varphi \in D(\mathscr{A})$, for any $t \geqslant 0$, and*

$$\mathscr{A}(P_t\varphi) = \mathscr{A}_0(P_t\varphi). \tag{5.1}$$

Proof. We first remark that if $\varphi \in C_b^2(\mathbb{R}^d)$ then

$$\mathscr{A}_0(P_t\varphi) = P_t(\mathscr{A}_0\varphi). \tag{5.2}$$

Indeed, from the Itô's formula we have

$$\begin{aligned}\frac{P_{t+h}\varphi(x) - P_t\varphi(x)}{h} &= \frac{1}{h}\,\mathbb{E}(\varphi(\xi(t+h; x)) - \varphi(\xi(t; x)))\\ &= \frac{1}{h}\,\mathbb{E}\int_t^{t+h} (\tfrac{1}{2}\operatorname{Tr}[a(\xi(s; x))\, D^2\varphi(\xi(s; x))]\\ &\quad + \langle D\varphi(\xi(s; x)), b(\xi(s; x))\rangle)\, ds.\end{aligned}$$

Then, by using the dominated convergence theorem, it follows

$$\begin{aligned}\lim_{h\to 0} \frac{P_{t+h}\varphi(x) - P_t\varphi(x)}{h} &= \mathbb{E}(\tfrac{1}{2}\operatorname{Tr}[a(\xi(t; x))\, D^2\varphi(\xi(t; x))] + \langle D\varphi(\xi(t; x)), b(\xi(t; x))\rangle)\\ &= P_t(\mathscr{A}_0\varphi)(x).\end{aligned}$$

Recalling that

$$\frac{d}{dt} P_t\varphi(x) = \mathscr{A}_0(P_t\varphi)(x),$$

this yields (5.2).

Now, for any $t \geqslant 0$ we define

$$\psi_t = \lambda P_t \varphi - \mathscr{A}_0(P_t \varphi).$$

By using (5.2) and the definition of $\mathscr{A}$, we have

$$\begin{aligned} R(\lambda, \mathscr{A})\, \psi_t &= \int_0^{+\infty} e^{-\lambda s} P_s(\lambda P_t \varphi - \mathscr{A}_0(P_t \varphi))(x)\, ds \\ &= \int_0^{+\infty} e^{-\lambda s}(\lambda P_{t+s} \varphi(x) - \mathscr{A}_0(P_{t+s} \varphi)(x))\, ds, \end{aligned}$$

so that

$$R(\lambda, \mathscr{A})\, \psi_t(x) = -\int_0^{+\infty} \frac{d}{ds}\,(e^{-\lambda s} P_{t+s} \varphi(x))\, ds = P_t \varphi(x).$$

This means that $P_t \varphi \in D(\mathscr{A})$ and $\mathscr{A}(P_t \varphi) = \mathscr{A}_0(P_t \varphi)$. ∎

THEOREM 5.2. *Assume the hypothesis 2.2. Then, for any $\varphi \in C_b(\mathbb{R}^d)$ and $t > 0$ we have that $P_t \varphi \in D(\mathscr{A})$. In particular, for any fixed $x \in \mathbb{R}^d$, the mapping $(0, +\infty) \to \mathbb{R}$, $t \mapsto P_t \varphi(x)$ is differentiable and*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{d}{dt}\,(P_t \varphi)(x) \right| = \|\mathscr{A}(P_t \varphi)\|_0 \leqslant c\,(t \wedge 1)^{-1}\, \|\varphi\|_0. \tag{5.3}$$

Proof. We have seen in the lemma above that if $\varphi \in C_b^2(\mathbb{R}^d)$ then $P_t \varphi \in D(\mathscr{A})$, for any $t > 0$, and (5.3) holds. Now, let us fix $\varphi \in C_b(\mathbb{R}^d)$ and a sequence $\{\varphi_n\} \subset C_b^2(\mathbb{R}^d)$ converging to φ in $C_b(\mathbb{R})$. Then, by the dominated convergence theorem we have that

$$\lim_{n \to +\infty} P_t \varphi_n = P_t \varphi. \tag{5.4}$$

Moreover, by using (4.11) and the previous lemma, we have

$$\|\mathscr{A}(P_t \varphi_n) - \mathscr{A}(P_t \varphi_m)\|_0 = \|\mathscr{A}_0(P_t(\varphi_n - \varphi_m))\|_0 \leqslant c(t \wedge 1)^{-1}\, \|\varphi_n - \varphi_m\|_0,$$

so that $\{\mathscr{A}(P_t \varphi_n)\}$ is a Cauchy sequence in $C_b(\mathbb{R}^d)$. Then, since $\mathscr{A}$ is a closed operator, we have that $P_t \varphi \in D(\mathscr{A})$ and

$$\mathscr{A}(P_t \varphi) = \lim_{n \to +\infty} \mathscr{A}(P_t \varphi_n).$$

In particular, as (5.3) holds for φ_n, (5.3) follows for any $\varphi \in C_b(\mathbb{R}^d)$. ∎

From the Theorem 5.3, by proceeding as in the standard case of analytic semigroups, it easily follows that, for any $\varphi \in C_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the mapping $(0, +\infty) \to \mathbb{R}$, $t \mapsto P_t \varphi(x)$ is infinitely differentiable and

$$\frac{d^n}{dt^n} (P_t \varphi(x)) = (\mathscr{A}(P_{t/n}))^n \varphi(x), \qquad t > 0,$$

so that from (5.3) we have

$$\sup_{x \in \mathbb{R}^d} \left| \frac{d^n}{dt^n} (P_t \varphi(x)) \right| \leqslant c^n \left(\frac{t}{n} \wedge 1 \right)^{-n} \|\varphi\|_0. \tag{5.5}$$

As well known, this implies that the mapping $(0, +\infty) \to \mathbb{R}$, $t \mapsto P_t \varphi(x)$, admits an analytic extension to a sector around the positive real axis, which is independent of $x \in \mathbb{R}^d$. Therefore, as proved in [31]-theorem IX-10, we have that there exists a constant $c > 0$ such that for any $\varphi \in C_b(\mathbb{R}^d)$

$$\|\lambda R(\lambda, \mathscr{A}) \varphi\|_0 \leqslant c \|\varphi\|_0, \qquad \operatorname{Re} \lambda > 0. \tag{5.6}$$

Then $\mathscr{A}$ generates an analytic semigroup (see [25]-proposition 2.1.11) and such a semigroup coincides clearly with P_t.

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