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Schauder estimates for a degenerate second order elliptic operator on a cube ☆

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On the occasion of the 70th birthday of Giuseppe Da Prato

Abstract

In the present article we are concerned with a class of degenerate second order differential operators $L_{A,b}$ defined on the cube $[0,1]^d$, with $d \geqslant 1$. Under suitable assumptions on the coefficients A and b (among them the assumption of their Hölder regularity) we show that the operator $L_{A,b}$ defined on $C^2([0,1]^d)$ is closable and its closure is m-dissipative. In particular, its closure $\overline{L_{A,b}}$ is the generator of a C_0-semigroup of contractions on $C([0,1]^d)$ and $C^2([0,1]^d)$ is a core for it. The proof of such result is obtained by studying the solvability in Hölder spaces of functions of the elliptic problem $\lambda u(x) - L_{A,b}u(x) = f(x)$, $x \in [0,1]^d$, for a sufficiently large class of functions f.

MSC: 35B45; 60J70; 35B65; 35J25

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1. Introduction

In this paper we continue our study of a class of degenerate elliptic problems

$$\lambda u(x) - L_{A,b}u(x) = f(x), \quad x \in [0,1]^d, \tag{1.1}$$

where $d \geqslant 1$, λ is a positive constant and $L_{A,b}$ is the operator

$$L_{A,b} := \frac{1}{2}\operatorname{Tr}\big[AD^2\big] + \langle b, D\rangle, \tag{1.2}$$

with $\langle\cdot,\cdot\rangle$ the usual scalar product in $\mathbb{R}^d$. Here we assume that $A:[0,1]^d \to \mathcal{L}^+(\mathbb{R}^d)$ (where $\mathcal{L}^+(\mathbb{R}^d)$ is the space of symmetric and non-negative definite $(d \times d)$-matrices) and $b:[0,1]^d \to \mathbb{R}^d$ are continuous mappings such that

$$A\nu = 0, \quad \langle b, \nu\rangle \geqslant 0, \quad \text{on } \partial[0,1]^d, \tag{1.3}$$

where ν is the unit inward normal at $\partial[0,1]^d$. We recall that condition (1.3) is necessary and sufficient in order to have *stochastic invariance* (see [8]).

It is known (to this purpose we refer for example to [4]) that the operator $L_{A,b}$ defined on $C^2([0,1]^d)$ with values in $C([0,1]^d)$ is a pregenerator of a Markov semigroup on $C([0,1]^d)$, equipped with the supremum norm, in the sense of Liggett [13, Definitions 2.1]. A natural question to be addressed is whether the closure of $L_{A,b}$ is a generator, which is equivalent to the solvability of problem (1.1) for a dense subset of data f in $C([0,1]^d)$. In [6, Theorem 1.1] we showed that this is the case when the mappings A and b are of class $C^2([0,1]^d)$ and the components a_{ij} of the matrix A satisfy the further condition

$$a_{ij}(x) = a_{ij}(x_i, x_j), \quad x \in [0,1]^d.$$

Note that no non-degeneracy condition is assumed on A. It would be of interest to be able to exhibit an explicit class of functions f for which problem (1.1) has a solution in $C^2([0,1]^d)$, but the method we have used in [6] does not provide an answer to this question. However a first step in this direction is already contained in [6, Theorem 5.2] where we have proved that there exists some $\lambda_0 \geqslant 0$ such that for any $f \in C^1([0,1]^d)$ and $\lambda > \lambda_0$ there exists a unique *weak solution* u in $C^1([0,1]^d)$ to problem (1.1). That is there exists a unique $u \in C^1([0,1]^d)$ such that

$$\int_{[0,1]^d} \left[\lambda u\varphi + \frac{1}{2}\langle ADu, D\varphi\rangle + \left\langle \frac{1}{2}DA - b, Du\right\rangle\varphi\right] dx = \int_{[0,1]^d} f\varphi\, dx, \tag{1.4}$$

for any $\varphi \in W^{1,\infty}([0,1]^d)$. Moreover, we have shown that

$$\|u\|_{C^1([0,1]^d)} \leqslant \frac{1}{\lambda - \lambda_0}\|f\|_{C^1([0,1]^d)}. \tag{1.5}$$

We emphasize that in order to give a meaning to problem (1.1) in its weak formulation (1.4) we needed some differentiability property of A. Actually in [6, Lemma 3.4] we established a *maximum principle* when A is in $C^1([0,1]^d)$ and b is in $C([0,1]^d)$, which guarantees uniqueness of weak solutions in $C^1([0,1]^d)$.

In view of natural applications of these results to the theory of dynamics of populations and of interacting particles the assumption of C^2 regularity for A and b seems to be rather strong. In this paper, motivated by the work of Bass, Perkins and others (see [2] and [3]), we consider a matrix $A = (a_{ij})$ of the form

$$a_{ij}(x) = m_i(x) x_i(1-x_i)\delta_{ij}, \quad x \in [0,1]^d,\ i, j = 1, \dots, d,$$

where m_i are strictly positive functions in $C^\delta([0,1]^d)$, for some $\delta \in (0,1)$. Our assumptions on b include the special form

$$b(x) = c(\theta - x), \quad x \in [0,1]^d,$$

for some positive constant c and some vector θ in the interior of $[0,1]^d$. We would like to stress that by using the method developed in [2] and [3], the well-posedness of the associated martingale problem can be proved. This means that if for any $x \in [0,1]^d$ we denote by $\{\eta(t)\}_{t\geqslant 0}$ the canonical process on $C_x([0,+\infty);[0,1]^d)$ and by $\mathcal{E}$ and $\{\mathcal{E}_t\}_{t\geqslant 0}$ we denote the canonical σ-algebra with the canonical filtration on $C_x([0,+\infty);[0,1]^d)$, then there exists a unique probability measure $\mathbb{P}$ on $(C_x([0,+\infty);[0,1]^d), \mathcal{E})$ such that the process

$$t \in [0,+\infty) \mapsto \varphi\big(\eta(t)\big) - \int_0^t \mathcal{L}_{A,b}\varphi\big(\eta(s)\big)\,ds,$$

is an $\mathcal{E}_t$-martingale on $(C_x([0,+\infty);[0,1]^d), \mathcal{E}, \mathbb{P})$, for any $\varphi \in C^2([0,1]^d)$.

However, in the present paper our goal is to establish the much stronger result stated in Theorem 2.1. Namely, we show that the operator $L_{A,b}$ defined on $C^2([0,1]^d)$ is closable and its closure is m-dissipative. In particular $\overline{L_{A,b}}$ is the generator of a C_0-semigroup of contractions on $C([0,1]^d)$ and $C^2([0,1]^d)$ is a core for it. We emphasize that the regularity we impose on the coefficients m_k, b_k and c_k in Theorem 2.1 is Hölder regularity. It is by no means clear that the conclusions of the theorem hold when m_k, b_k and c_k are merely continuous and $d > 1$.

In order to prove Theorem 2.1, we use a change of variables, introduced in the one-dimensional case by Metafune in [16]. Actually, we write the operator $L_{A,b}$ as the sum of the operators

$$L_k u(x) = m_k(x) x_k(1-x_k) D_k^2 u(x) + c_k(x) b_k(x_k) D_k u(x), \quad x \in [0,1]^d,$$

for $k = 1, \dots, d$. For each k we take

$$D(L_k) := \Big\{ u \in C\big([0,1]^d\big):\ D_k u \in C\big([0,1]^d\big),\ D_k^2 u \in C\big([0,1]^d \cap \{0 < x_k < 1\}\big),$$
$$\lim_{x_k \to \{0,1\}} \sup_{\substack{x_i \in [0,1] \\ i \neq k}} x_k(1-x_k) D_k^2 u(x) = 0 \Big\}.$$

Clearly $C^2([0,1]^d)$ is contained in the intersection of all $D(L_k)$, so that $L_{A,b} \subset L_1 + \cdots + L_d$. Next, we introduce the following change of variables, by setting for any $v \in C([0,\pi]^d)$

$$[T_\Phi v](x) := v\big(\varphi^{-1}(x_1), \dots, \varphi^{-1}(x_d)\big), \quad x \in [0,1]^d,$$

where $\varphi(t) := (1 - \cos t)/2$, for $t \in [0,\pi]$. In these new coordinates, the operator L_k can be written as

$$N_k = T_\Phi^{-1} \circ L_k \circ T_\Phi,$$

so that

$$N_k v(y) = \mu_k(y)\left[D_k^2 v(y) + \frac{1}{\sin y_k}\left(\frac{2\gamma_k(y)\beta_k(y_k)}{\mu_k(y)} - \cos y_k\right) D_k v(y)\right], \quad y \in [0,\pi]^d,$$

for suitable mappings μ_k, γ_k and β_k. Finally, we define

$$\hat{N} := \sum_{k=1}^d N_k, \qquad D(\hat{N}) := \bigcap_{k=1}^d D(N_k) = T_\Phi^{-1}\left(\bigcap_{k=1}^d D(L_k)\right).$$

Although the transformed operator $\hat{N}$ has singular coefficients, it is possible to obtain for it Schauder type estimates in the usual Hölder spaces. This is the content of Theorem 2.2 (see also its Corollary 2.3) and the proof of such a theorem, which is realized in several steps through Sections 3–5, is the main task of the paper. Clearly, Theorem 2.2 could be reformulated for the operator $L_{A,b}$ in terms of inhomogeneous Hölder spaces, but we refrain to do it since it is obvious from Theorem 2.2 and since we do not have a direct application of it in this paper.

The proof of Theorem 2.2 is based on the method of *freezing coefficients*. Indeed, having optimal regularity estimates for the operator N (defined as the closure of the operator $\hat{N}$) when the coefficients μ_k, β_k and γ_k are constant, we are able to construct an approximate resolvent operator. Using the fact that, when the coefficients μ_k, β_k and γ_k are constant, the operator N is a *commutative sum* of partial differential operators acting on one variable only, we can apply the *method of sums* introduced by Grisvard in [11], see also [9] in the so-called parabolic case. In the present paper we provide a proof of it due to Da Prato [7], which applies when all operators are generators of analytic semigroups. Notice that this proof does not rely on complex methods.

Once we have Theorem 2.2, we introduce the closed operator

$$M := T_\Phi \circ N \circ T_\Phi^{-1}, \qquad D(M) := T_\Phi\big(D(N)\big),$$

and, thanks to the results proved for N, we show that $\lambda - M$ is a bijection from $D(M)$ into $C([0,1]^d)$ and $C^2([0,1]^d)$ is a core for M. We conclude the proof of Theorem 2.1, by showing that $M = \overline{L_{A,b}}$.

The paper is organized as follows. In Section 2 we state our main results and show how Theorem 2.1 can be deduced from Corollary 2.3. In Section 3 the required estimates for the partial differential operators acting on one variable are established. They rely on previous results of Angenent [1] and Metafune [16] and on an abstract result stated in Appendix A. In Section 4 we establish the basic a priori estimates (4.4), by using the method of sums given in Appendix B. In Section 5 we introduce a suitable partition of unity and construct an approximate resolvent.

We mention that by extending this approximate resolvent for complex values of λ we could prove the analyticity of the semigroup generated by N in $C([0,1]^d)$, but we refrain to do it in this paper.

2. Statement of main results

Fix $a \leqslant b$ and $d \in \mathbb{N}^\star$. We denote by $C([a,b]^d)$ the Banach space of continuous functions $u:[a,b]^d \to \mathbb{R}$, endowed with the sup-norm

$$|u|_0 := \max_{x\in[a,b]^d} \left|u(x)\right|.$$

For any $\delta \in (0,1)$, $C^\delta([a,b]^d)$ is the subspace of Hölder continuous functions, endowed with the norm

$$|u|_\delta := |u|_0 + [u]_\delta =: |u|_0 + \sup_{\substack{x,y\in[a,b]^d \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\delta}.$$

If $u, v \in C^\delta([a,b]^d)$ we have

$$[uv]_\delta \leqslant |u|_0[v]_\delta + [u]_\delta|v|_0. \tag{2.1}$$

For any $k \in \mathbb{N}$, we denote by $C^k([a,b]^d)$ the subspace of k-times continuously differentiable functions u, endowed with the norm

$$|u|_k := |u|_0 + \sum_{h=1}^k [u]_h =: |u|_0 + \sum_{h=1}^k \sup_{x\in[a,b]^d} \left|D^h u(x)\right|$$

and by $C^{k+\delta}([a,b]^d)$ the subspace of functions $u \in C^k([a,b]^d)$ having Hölder continuous kth derivative, endowed with the norm

$$|u|_{k+\delta} := |u|_k + \left[D^h u\right]_\delta.$$

Finally, for any $\delta \in (0,1)$ we denote by $h^\delta([a,b]^d)$ the space of little-Hölder continuous functions, consisting of all functions $u \in C^\delta([a,b]^d)$ such that

$$\lim_{\epsilon\to 0} \sup_{\substack{x,y\in[a,b]^d \\ |x-y|<\epsilon}} \frac{|u(x)-u(y)|}{|x-y|^\delta} = 0.$$

For any $k \in \mathbb{N}$, we denote by $h^{k+\delta}([a,b]^d)$ the subspace of functions in $C^k([a,b]^d)$ such that $D^k u \in h^\delta([a,b]^d)$.

It is immediate to check that $C^{\delta_1}([a,b]^d) \subset h^{\delta_2}([a,b]^d)$, for any $\delta_1 > \delta_2$. Moreover it is possible to prove that $h^{\delta_2}([a,b]^d)$ is the closure of $C^{\delta_1}([a,b]^d)$ in $C^{\delta_2}([a,b]^d)$ (for a proof see e.g. [14, Proposition 0.2.1]).

We are here concerned with the following second order elliptic operator

$$\begin{cases} Lu(x) = \sum_{k=1}^{d} \big[m_k(x)x_k(1-x_k)D_k^2u(x) + c_k(x)b_k(x_k)D_ku(x)\big], & x \in [0,1]^d, \\ u \in D(L) = C^2\big([0,1]^d\big). \end{cases} \tag{2.2}$$

We assume that the coefficients m_k, c_k and b_k satisfy the following conditions.

Hypothesis 1. There exists $\delta \in (0,1)$ such that for any $k = 1, \dots, d$

(1) the functions m_k and c_k belong to $C^\delta([0,1]^d)$ and are strictly positive;
(2) the function b_k belongs to $C^\delta([0,1])$, with $b_k(0) > 0$ and $b_k(1) < 0$.

Notice that under the conditions above, if we set

$$A_{hk}(x) := \delta_{hk} m_k(x) x_k(1-x_k), \quad \tilde{b}_k(x) := c_k(x) b_k(x_k),$$

we have

$$A(x)\nu(x) = 0, \quad \big\langle \tilde{b}(x), \nu(x)\big\rangle \geqslant 0, \quad x \in \partial[0,1]^d,$$

where ν is the unit inward normal at $\partial[0,1]^d$. As proved for example in [5, Lemma 3.3], the two conditions above imply that a *minimum principle* holds for the operator L. Namely, at any point $\bar{x} \in [0,1]^d$ where a function $u \in C^2([0,1]^d)$ achieves its minimum, we have $Lu(\bar{x}) \geqslant 0$. As is well known, since $L1 = 0$, this implies that the operator L is dissipative, that is for any $\lambda > 0$

$$|u|_0 \leqslant \frac{1}{\lambda}|\lambda u - Lu|_0, \quad u \in D(L). \tag{2.3}$$

Therefore, as $D(L) = C^2([0,1]^d)$ is dense in $C([0,1]^d)$, the operator $(L, D(L))$ is closable in $C([0,1]^d)$ and its closure $(\bar{L}, D(\bar{L}))$ is a dissipative operator on $C([0,1]^d)$.

Thanks to the Lumer–Phillips theorem, if we show that $\mathrm{Range}(\lambda - \bar{L}) = C([0,1]^d)$, for any $\lambda > 0$, the main result of this paper follows.

Theorem 2.1. *Under Hypothesis* 1*, the operator* $(L, D(L))$ *defined in* (2.2) *is closable and its closure is* m*-dissipative. Hence* $\bar{L}$ *generates a* C_0*-semigroup of contractions on* $C([0,1]^d)$ *and* $C^2([0,1]^d)$ *is a core for* $\bar{L}$.

In order to prove that $\mathrm{Range}(\lambda - \bar{L}) = C([0,1]^d)$, we study the solvability of the elliptic equation

$$\lambda u - Lu = f, \tag{2.4}$$

for all f in some dense subset of $C([0,1]^d)$. To this purpose, in what follows we shall introduce an auxiliary operator, obtained from L by a suitable change of variables.

For any $k = 1, \dots, d$ we define

$$L_k u(x) = m_k(x) x_k (1 - x_k) D_k^2 u(x) + c_k(x) b_k(x_k) D_k u(x), \quad x \in [0,1]^d,$$

for any $u \in D(L_k)$, where

$$D(L_k) := \Big\{ u \in C\big([0,1]^d\big)\colon\ D_k u \in C\big([0,1]^d\big),\ D_k^2 u \in C\big([0,1]^d \cap \{0 < x_k < 1\}\big), \\ \lim_{x_k \to \{0,1\}} \sup_{\substack{x_i \in [0,1] \\ i \neq k}} x_k (1 - x_k) D_k^2 u(x) = 0 \Big\}$$

and m_k, c_k and b_k are the coefficients of the operator L introduced in (2.2) and fulfilling Hypothesis 1. Clearly we have

$$Lu = (L_1 + \cdots + L_d) u, \quad u \in C^2\big([0,1]^d\big).$$

In order to study the operators L_k, we perform a change of variables as in [16]. We define

$$\Phi : [0,\pi]^d \to [0,1]^d, \quad y \mapsto \Phi(y) = \big(\varphi(y_1), \dots, \varphi(y_d)\big),$$

where $\varphi : [0,\pi] \to [0,1]$ is the homeomorphism

$$\varphi(t) := \frac{1}{2}(1 - \cos t), \quad t \in [0,\pi].$$

If for any $v : [0,\pi]^d \to \mathbb{R}$ we define

$$[T_\Phi v](x) = v\big(\Phi^{-1}(x)\big), \quad x \in [0,1]^d, \tag{2.5}$$

and set

$$N_k := T_\Phi^{-1} \circ L_k \circ T_\Phi,$$

it is immediate to check that

$$N_k v(y) = \mu_k(y) \bigg[D_k^2 v(y) + \frac{1}{\sin y_k} \bigg(\frac{2\gamma_k(y)\beta_k(y_k)}{\mu_k(y)} - \cos y_k \bigg) D_k v(y) \bigg], \tag{2.6}$$

where

$$\mu_k := m_k \circ \Phi, \qquad \gamma_k := c_k \circ \Phi, \qquad \beta_k := b_k \circ \Phi. \tag{2.7}$$

By proceeding as in [16, pp. 265–266], it is possible to show that if we set $D(N_k) = T_\Phi^{-1}(D(L_k))$, then

$$D(N_k) := \Big\{ v \in C\big([0,\pi]^d\big)\colon\ J_k v \in C^2\big([0,\pi]; C\big([0,\pi]^{d-1}\big)\big), \\ \lim_{t \to 0^+} (J_k v)'(t) = \lim_{t \to \pi^-} (J_k v)'(t) = 0 \Big\},$$

where J_k are the mappings from $C([0,\pi]^d)$ into $C([0,\pi]; C([0,\pi]^{d-1}))$ which are defined for any $k = 1,\dots,d$, $f \in C([0,\pi]^d)$, $t \in [0,\pi]$ and $(x_1,\dots,x_{k-1},x_{k+1},\dots,x_d) \in [0,\pi]^{d-1}$ by

$$\big[J_k f(t)\big](x_1,\dots,x_{k-1},x_{k+1},\dots,x_d) := f(x_1,\dots,x_{k-1},t,x_{k+1},\dots,x_d)$$

(for all details see Appendix A).

Now, we define

$$\hat{N} := \sum_{k=1}^{d} N_k, \qquad D(\hat{N}) := \bigcap_{k=1}^{d} D(N_k) = T_\Phi^{-1}\left(\bigcap_{k=1}^{d} D(L_k)\right). \tag{2.8}$$

The operator $\hat{N}$ is clearly densely defined and is dissipative in $C([0,\pi]^d)$, hence it is closable with dissipative closure $(N, D(N))$.

In Section 5 we shall prove the following result.

Theorem 2.2. *Under Hypothesis* 1, *for any* $\lambda > 0$ *the operator* $\lambda - N$ *is an isomorphism from* $C^{2+\delta}_{\mathcal{N}}([0,\pi]^d)$ *into* $C^\delta([0,\pi]^d)$, *where*

$$C^{2+\delta}_{\mathcal{N}}\big([0,\pi]^d\big) := \bigcap_{k=1}^{d}\big\{v \in C^{2+\delta}\big([0,\pi]^d\big)\colon\ D_k v(y) = 0,\ y \in [0,\pi]^d \cap \{y_k = 0,\pi\}\big\}. \tag{2.9}$$

As a consequence of the previous theorem, we have

Corollary 2.3. *Under Hypothesis* 1, *for any* $\delta' \in (0,\delta)$ *and* $\lambda > 0$ *the operator* $\lambda - N$ *is an isomorphism from* $h^{2+\delta'}_{\mathcal{N}}([0,\pi]^d)$ *into* $h^{\delta'}([0,\pi]^d)$, *where*

$$h^{2+\delta'}_{\mathcal{N}}\big([0,\pi]^d\big) := \bigcap_{k=1}^{d}\big\{u \in h^{2+\delta'}\big([0,\pi]^d\big)\colon\ D_k v(y) = 0,\ y \in [0,\pi]^d \cap \{y_k = 0,\pi\}\big\}.$$

Proof. This is a direct consequence of the fact that $\lambda - N$ is an isomorphism between $C^{2+\delta'}_{\mathcal{N}}([0,\pi]^d)$ and $C^{\delta'}([0,\pi]^d)$, for any $\delta' \leqslant \delta$, the spaces $h^{2+\delta'}([0,\pi]^d)$ and $h^{\delta'}([0,\pi]^d)$ are closed respectively in $C^{2+\delta'}([0,\pi]^d)$ and $C^{\delta'}([0,\pi]^d)$ and the spaces $C^{2+\delta}([0,\pi]^d)$ and $C^\delta([0,\pi]^d)$ are dense respectively in $h^{2+\delta'}([0,\pi]^d)$ and $h^{\delta'}([0,\pi]^d)$.

Actually, if $f \in h^{2+\delta'}([0,\pi]^d)$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C^{2+\delta}([0,\pi]^d)$ which converges to f in $C^{2+\delta'}([0,\pi]^d)$, so that the sequence $\{(\lambda - N)f_n\}_{n\in\mathbb{N}}$ converges to the function $(\lambda - N)f$ in $C^{\delta'}([0,\pi]^d)$. Now, as $C^\delta([0,\pi]^d) \subset h^{\delta'}([0,\pi]^d)$, for each $n \in \mathbb{N}$ we have $(\lambda - N)f_n \in h^{\delta'}([0,\pi]^d)$ and, as $h^{\delta'}([0,\pi]^d)$ is closed in $C^{\delta'}([0,\pi]^d)$, we conclude that $(\lambda - N)f \in h^{\delta'}([0,\pi]^d)$. $\square$

In particular, from Theorem 2.2 and its Corollary 2.3, we have that $(N, D(N))$ is a densely defined m-dissipative operator which generates a C_0-semigroup of contractions on $C([0,\pi]^d)$ and $h^{2+\delta'}_{\mathcal{N}}([0,\pi]^d)$ is a core for N, for any $\delta' \in (0,\delta)$.

Now, we introduce the space

$$Z := \big\{v \in C^4\big([0,\pi]^d\big)\colon\ (J_k v)'(t) = (J_k v)^{(3)}(t) = 0,\ t \in \{0,\pi\},\ k = 1,\dots,d\big\}.$$

For any fixed $\delta \in (0,1)$ and $u \in h_{\mathcal{N}}^{2+\delta}([0,\pi]^d)$, we can find a sequence $\{u_n\}_{n\mathbb{N}} \subset C^4([0,\pi]^d)$ which converges to u in the $C^{2+\delta}([0,\pi]^d)$ norm. As the function u satisfies a Neumann boundary condition, we can construct functions u_n which satisfy a Neumann boundary condition as well. Moreover, as the sequence $\{u_n\}_{n\in\mathbb{N}}$ converge to u in the $C^{2+\delta}([0,\pi]^d)$ norm, it is possible to construct the functions u_n in such a way that their third derivatives vanish at the boundary as well. This means that Z is dense in $h_{\mathcal{N}}^{2+\delta}([0,\pi]^d)$, with respect to the $C^{2+\delta}([0,\pi]^d)$ norm, and, in particular, for any $v \in h_{\mathcal{N}}^{2+\delta}([0,\pi]^d)$ there exists a sequence $\{v_n\}_n \subset Z$ such that

$$\lim_{n\to\infty} v_n = v, \qquad \lim_{n\to\infty} \hat{N} v_n = \hat{N} v, \quad \text{in } C\big([0,\pi]^d\big).$$

As $h_{\mathcal{N}}^{2+\delta'}([0,\pi]^d)$ is a core for N, for any $\delta' \in (0,\delta)$, this implies that Z is a core for N.

Now, we introduce the closed operator

$$M := T_\Phi \circ N \circ T_\Phi^{-1}, \qquad D(M) = T_\Phi\big(D(N)\big).$$

As T_Φ is an isomorphism from $C([0,\pi]^d)$ into $C([0,1]^d)$ and N is m-dissipative and densely defined, it follows that M is m-dissipative and then $\lambda - M$ is a bijection from $D(M)$ into $C([0,1]^d)$, for any $\lambda > 0$. Moreover, as Z is a core for N, it follows that $T_\Phi(Z)$ is a core for M.

In the next lemma we show that $T_\Phi(Z) \subseteq C^2([0,1]^d)$. Since $T_\phi(Z)$ is a core for M, this implies that $C^2([0,1]^d)$ is a core for M.

Lemma 2.4. *For any $v \in Z$, we have that $T_\Phi v \in C^2([0,1]^d)$.*

Proof. Clearly $T_\Phi v \in C([0,1]^d)$. As far as the first derivatives are concerned we have

$$\big(\varphi^{-1}\big)'(s) = \frac{2}{\sin(\varphi^{-1}(s))}, \quad s \in (0,1),$$

so that, for any $k = 1,\dots,d$ and $x \in [0,1]^d \cap \{x_k \in (0,1)\}$

$$D_k(T_\Phi v)(x) = 2\frac{D_k v(y)}{\sin y_k},$$

where $y = \Phi^{-1}(x)$. As $D_k v(y) = 0$, for $y_k = 0,\pi$, this implies that

$$\lim_{x_k\to 0,1} D_k(T_\Phi v)(x) = 2 \lim_{y_k\to 0,\pi} \frac{D_k v(y)}{\sin y_k} = 2(-1)^{x_k} D_k^2 v\big(\Phi^{-1}(x)\big)_{|x_k=0,1},$$

so that $D_k(T_\Phi v) \in C([0,1]^d)$, for any $k = 1,\dots,d$, and hence $T_\Phi v \in C^1([0,1]^d)$.

As far as the second derivatives are concerned, for any $k = 1,\dots,d$ and $x \in [0,1]^d \cap \{x_k \in (0,1)\}$, we have

$$D_k^2(T_\Phi v)(x) = 4\frac{\sin y_k D_k^2 v(y) - \cos y_k D_k v(y)}{\sin^3 y_k},$$

where $y = \Phi^{-1}(x)$. Hence, as $D_k v(y) = D_k^3 v(y) = 0$, for $y_k = 0,\pi$, by the theorem of De L'Hopital, we have

$$\begin{aligned}\lim_{x_k\to 0,1} D_k^2(T_\Phi v)(x) &= 4 \lim_{y_k\to 0,\pi} \frac{\sin y_k D_k^3 v(y) + \sin y_k D_k v(y)}{3\sin^2 y_k \cos y_k} \\ &= \frac{4}{3}\lim_{y_k\to 0,\pi} \frac{D_k^3 v(y)}{\sin y_k \cos y_k} + \frac{D_k v(y)}{\sin y_k \cos y_k} \\ &= \frac{4}{3}\big[D_k^4 v\big(\Phi^{-1}(x)\big) + D_k^2 v\big(\Phi^{-1}(x)\big)\big]_{|x_k=0,1}.\end{aligned}$$

This means that $D_k^2(T_\Phi v) \in C([0,1]^d)$. Concerning mixed derivatives, for any $k \neq h = 1,\dots,d$ and $x \in [0,1]^d \cap \{x_k, x_h \in (0,1)\}$

$$D_{hk}^2(T_\Phi v)(x) = 2\frac{D_{hk}^2 v(y)}{\sin y_k \sin y_h},$$

where $y = \Phi^{-1}(x)$. As $D_{hk}^2 v(y) = 0$, both for $y_k = 0,\pi$ and for $y_h = 0,\pi$, we get

$$\lim_{x_k\to 0,1} D_{hk}^2(T_\Phi v)(x) = 2\lim_{y_k\to 0,\pi} 2\frac{D_{hk}^2 v(y)}{\sin y_k \sin y_h} = 2(-1)^{x_k}\frac{D_{hk}^3 v(\Phi^{-1}(x))}{\sin(\varphi^{-1}(x_h))}|_{x_k=0,1}$$

and analogously for the limit as x_h goes to 0 and 1. This allows to conclude that $D_{hk}^2(T_\Phi v) \in C([0,1]^d)$, for any $h,k = 1,\dots,d$, so that $T_\Phi v \in C^2([0,1]^d)$. □

Now, if we show that $\bar{L} = M$, since $\lambda - M$ is a bijection from $D(M)$ into $C([0,1]^d)$, for any $\lambda > 0$, we have that Range$(\lambda - \bar{L}) = C([0,1]^d)$, for any $\lambda > 0$ and Theorem 2.1 is proved. In order to prove that $\bar{L} = M$, we notice that

$$D(L) = C^2\big([0,1]^d\big) \subseteq \bigcap_{k=1}^{d} D(L_k).$$

Then, due to (2.8) we have $T_\Phi^{-1}(D(L)) \subseteq D(\hat{N})$, so that $L \subseteq M$ and hence

$$\bar{L} \subseteq \bar{M} = M.$$

On the other hand, since $C^2([0,\pi]^d)$ is a core for M, for any $u \in D(M)$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^2([0,\pi]^d)$ such that

$$\lim_{n\to\infty} u_n = u, \qquad \lim_{n\to\infty} Mu_n = Mu.$$

But $Mu_n = Lu_n$, so that $u \in D(\bar{L})$ and $\bar{L}u = Mu$.

3. The one-dimensional case

In the present section we denote by E the space $C([0,\infty];\mathbb{C})$, equipped with the sup-norm $|\cdot|_0$, and by F the space $\{u \in C^2([0,\infty];\mathbb{C})\colon u'(0) = 0\}$, equipped with the C^2-norm $|\cdot|_2$.

We start with the following result by Angenent (see [1, Theorem 4.2]).

Proposition 3.1. *For any* $b \in \mathbb{C}$, *set*

$$A_b u(t) := D^2 u(t) + \frac{b-1}{t} Du(t), \quad t > 0, \qquad A_b u(0) := bu''(0).$$

Then, $A_b \in \mathcal{L}(F; E)$. *Moreover, if* $\operatorname{Re} b > 0$, *then* $\lambda - A_b \in \operatorname{Isom}(F; E)$, *for any* $\lambda \in \mathbb{C} \setminus (-\infty, 0]$.

If we set

$$U := \{(\lambda, b) \in \mathbb{C}^2: \lambda \notin (-\infty, 0], \operatorname{Re} b > 0\},$$

then U is an open subset of $\mathbb{C}^2$ and the map

$$(\lambda, b) \in U \mapsto \lambda - A_b \in \operatorname{Isom}(F; E)$$

is analytic. Since the mapping $B \mapsto B^{-1}$ from $\operatorname{Isom}(F; E)$ into $\operatorname{Isom}(E; F)$ is analytic, the map

$$(\lambda, b) \in U \mapsto (\lambda - A_b)^{-1} \in \operatorname{Isom}(E; F)$$

is analytic. Then, given $b \in \mathbb{C}$, with $\operatorname{Re} b > 0$, $\mu \in I := [\mu_1, \mu_2]$, with $0 < \mu_1 < \mu_2$, and $\theta \in [0, \pi)$, we have that there exists $M_0 = M_0(\theta, I) > 0$ such that

$$\left\| (\lambda - A_b)^{-1} \right\|_{\mathcal{L}(E;F)} \leqslant M_0, \quad |\lambda| = 1, \quad |\arg \lambda| \leqslant \theta, \quad \mu \in I. \tag{3.1}$$

By using a rescaling argument, as in [1] and [16, Proposition 2.7], we obtain

Proposition 3.2. *Let* $b \in \mathbb{C}$, *with* $\operatorname{Re} b > 0$, *and* $\mu > 0$. *Then*

$$\mathbb{C} \setminus (-\infty, 0] \subset \rho(A_{\mu b}).$$

Moreover, for every $\theta \in [0, \pi)$ *and* $I := [\mu_1, \mu_2]$, *with* $0 < \mu_1 < \mu_2$, *there exists* $M_1 = M_1(\theta, I) > 0$ *such that*

$$\left\| (\lambda - A_{\mu b})^{-1} \right\|_{\mathcal{L}(E)} \leqslant \frac{M_1}{|\lambda|}, \quad \lambda \in \rho(A_{\mu b}), \quad |\arg \lambda| \leqslant \theta, \quad \mu \in I. \tag{3.2}$$

By using the same argument as in [1] or [16], we see that if the constant b is replaced by a function $b \in C([0, \infty); \mathbb{C})$, with $\operatorname{Re} b(0) > 0$, then $A_b \in \mathcal{L}(F; E)$ and (3.2) holds.

Next, for any $\mu > 0$ and $b \in C([0, \pi])$, we consider the operator

$$B_\mu v(t) := \begin{cases} D^2 v(t) + (\sin t)^{-1}(\mu b(t) - \cos t) Dv(t), & t \in (0, \pi), \\ \lim_{s \to 0, \pi} B_\mu v(s), & t = 0, \pi, \end{cases}$$

with

$$D(B_\mu) := \{v \in C^2([0, \pi]): v'(0) = v'(\pi) = 0\}.$$

By following the same arguments used in [16, Theorem 2.11] we obtain

Proposition 3.3. *Let $b \in C([0,\pi]^d)$ satisfy $b(0) > 0$ and $b(\pi) < 0$ and let $I := [\mu_1, \mu_2]$, with $0 < \mu_1 < \mu_2$. Then, there exist $\theta \in [0,\pi]$ and $\rho > 0$ such that for every $\lambda \in \mathbb{C}$, with $|\lambda| \geqslant \rho$ and $|\arg \lambda| \leqslant \theta$, and every $\mu \in I$ it holds $0 \in \rho(\lambda - B_\mu)$ and*

$$\left\|(\lambda - B_\mu)^{-1}\right\|_{\mathcal{L}(E)} \leqslant \frac{M_2}{|\lambda|}, \tag{3.3}$$

for some $M_2 = M_2(\theta, \rho, I) > 0$.

The previous proposition implies that if we denote by e^{tB_μ}, $t \geqslant 0$, the analytic semigroup on E generated by B_μ, then for any $b \in C([0,\pi])$, with $b(0) > 0$ and $b(\pi) < 0$, and for any $I := [\mu_1, \mu_2]$, with $0 < \mu_1 < \mu_2$, there exists a constant $M_3 = M_3(I) > 0$ such that

$$\left\|tB_\mu e^{tB_\mu}\right\|_{\mathcal{L}(E)} \leqslant M_3, \quad t > 0.$$

Replacing B_μ by mB_μ, with $m \in J := [m_1, m_2]$ and $0 < m_1 < m_2$, we obtain

$$\left\|tmB_\mu e^{tmB_\mu}\right\|_{\mathcal{L}(E)} \leqslant M_3, \quad t > 0, \tag{3.4}$$

for some $M_3 = M_3(I, J) > 0$.

We conclude the present section by recalling some well-known estimates in Hölder spaces for the operators $(\sin t)^{-1}D$ and mB_μ.

Lemma 3.4. *Let $u \in C^2([0,\pi])$ such that $u'(0) = u'(\pi) = 0$ and fix $\delta \in (0,1)$.*

1. *We have*

$$\left|(\sin t)^{-1}Du\right|_0 \leqslant c|u''|_0, \qquad \left[(\sin t)^{-1}Du\right]_\delta \leqslant c_\delta\left[u''\right]_\delta.$$

2. *If $mB_\mu u \in C^\delta([0,\pi])$, with $m \in J$ and $\mu \in I$, we have that $u \in C^{2+\delta}([0,\pi])$ and there exists $M_4 = M_4(I, J) > 0$ such that*

$$[u'']_\delta \leqslant M_4[mB_\mu u]_\delta.$$

4. The d-dimensional commutative case

In the present section we are concerned with the operator $(\hat{N}, D(\hat{N}))$ introduced in (2.8), in the case its coefficients μ_k and γ_k, introduced in (2.7), are constant.

In what follows we shall denote

$$\mu := \inf_{\substack{y\in[0,\pi]^d \\ k=1,\dots,d}} \mu_k(y), \qquad M := \sup_{\substack{y\in[0,1\pi]^d \\ k=1,\dots,d}} \mu_k(y),$$

and

$$\gamma := \inf_{\substack{y\in[0,\pi]^d \\ k=1,\dots,d}} \gamma_k(y), \qquad \Gamma := \sup_{\substack{x\in[0,\pi]^d \\ k=1,\dots,d}} \gamma_k(y).$$

Lemma 4.1. *Assume that μ_k and γ_k are constant, for any $k = 1, \dots, d$. Then, under Hypothesis* 1 *the operator $\hat{N}$ is closable in $C([0,\pi]^d)$ and, if we denote by N its closure, for any $\lambda > 0$ we have that $\lambda - N$ is an isomorphism between $C^{2+\delta}_{\mathcal{N}}([0,\pi]^d)$ and $C^\delta([0,\pi]^d)$ and for any $v \in X$*

$$\left|(\lambda - N)^{-1} v\right|_0 \leqslant \frac{1}{\lambda} |v|_0.$$

Moreover, there exists $K = K(\mu, M, \gamma, \Gamma) > 0$ such that for any $\lambda > 0$ and $v \in X$

$$\left[D_k D_h (\lambda - N)^{-1} v\right]_\delta \leqslant K\left(\lambda^{-\delta} |v|_0 + [v]_\delta\right). \tag{4.1}$$

Proof. As shown in Proposition 3.3, for any $k = 1, \dots, d$ the operator $\mu_k B_{2\gamma_k/\mu_k}$ is densely defined and m-dissipative and generates an analytic semigroup $e^{t\mu_k B_{2\gamma_k/\mu_k}}$ on $C([0,\pi]^d)$. Then, according to Theorem A.2, we denote by $E_k(\mu_k B_{2\gamma_k/\mu_k})$ the unique densely defined and m-dissipative operator, which generates an analytic semigroup on $C([0,\pi]^d)$, such that

$$e^{tE_k(\mu_k B_{2\gamma_k/\mu_k})} = E_k\left(e^{t\mu_k B_{2\gamma_k/\mu_k}}\right), \quad t \geqslant 0.$$

Due to (A.10)

$$E_k(\mu_k B_{2\gamma_k/\mu_k}) e^{tE_k(\mu_k B_{2\gamma_k/\mu_k})} = E_k\left(\mu_k B_{2\gamma_k/\mu_k} e^{t\mu_k B_{2\gamma_k/\mu_k}}\right),$$

and then, thanks to (3.4) and to (A.5)

$$\begin{aligned}
&\left\| t E_k(\mu_k B_{2\gamma_k/\mu_k}) e^{tE_k(\mu_k B_{2\gamma_k/\mu_k})}\right\|_{\mathcal{L}(C([0,\pi]^d))} \\
&\quad = \left\| t\mu_k B_{2\gamma_k/\mu_k} e^{t\mu_k B_{2\gamma_k/\mu_k}}\right\|_{\mathcal{L}(C([0,\pi]))} \leqslant M_3,
\end{aligned} \tag{4.2}$$

for some $M_3 = M_3(\mu, M, \gamma, \Gamma) > 0$. Next, if for any $u \in C([0,1])$ and $v \in C([0,1]^{d-1})$ we denote, as in (A.3),

$$[u \otimes_k v](x) := u(x_k) v(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_d), \quad x \in [0,1]^d,$$

it is immediate to check that, when μ_k and γ_k are constant,

$$N_k[u \otimes_k v] = (\mu_k B_{2\gamma_k/\mu_k} u) \otimes_k v = E_k(\mu_k B_{2\gamma_k/\mu_k})[u \otimes_k v],$$

last equality following from (A.6). Then, by Theorem A.2 we obtain that, for any $k = 1, \dots, d$, $N_k = E_k(\mu_k B_{2\gamma_k/\mu_k})$.

This means that the operator N_k is densely defined and m-dissipative and generates an analytic semigroup e^{tN_k}, $t \geqslant 0$, in $C([0,\pi]^d)$ and all the semigroups e^{tN_k} are commuting. Moreover, thanks to (4.2) there exists some $\bar{M} = \bar{M}(m, M, \gamma, \Gamma) > 0$ such that

$$\sup_{t>0} \left\| tN_k e^{tN_k}\right\|_{\mathcal{L}(C([0,\pi]^d))} \leqslant \bar{M}, \quad k = 1, \dots, d. \tag{4.3}$$

Now, we are in the position to apply Theorem B.2, in the case $E = C([0,\pi]^d)$. To this purpose we need to identify the spaces X and Y introduced respectively in (B.3) and (B.4), for $\theta = \delta/2$. Noting that for any $k = 1, \dots, d$

$$D\big(J_k N_k J_k^{-1}\big) = \big\{v \in C^2\big([0,\pi]; C\big([0,\pi]^{d-1}\big)\big):\ v'(0) = v'(\pi) = 0\big\},$$

as proved for example in Lunardi [14], we have

$$D_{J_k N_k J_k^{-1}}(\delta/2, \infty) = C^\delta\big([0,1]; C\big([0,\pi]^{d-1}\big)\big).$$

Moreover, we have

$$D_{N_k}(\delta/2,\infty) = \big\{u \in C\big([0,\pi]^d\big):\ J_k u \in D_{J_k N_k J_k^{-1}}(\delta/2,\infty)\big\}$$

and then, as

$$X = \bigcap_{k=1}^{d} D_{N_k}(\delta/2,\infty),$$

according to Lemma C.1 we conclude that $X = C^\delta([0,\pi]^d)$, and the norm introduced in (B.2) is equivalent to the usual norm in $C^\delta([0,\pi]^d)$.

Concerning the space Y, we have

$$Y = \bigcap_{k=1}^{d} \big\{u \in D(N_k):\ N_k u \in D_{N_k}(\delta/2,\infty)\big\}.$$

Proceeding as before for X and using Lemma 3.4 in his Banach space-valued version, we find that for any $k = 1, \dots, d$

$$\begin{aligned}
&\big\{u \in D(N_k):\ N_k u \in D_{N_k}(\delta/2,\infty)\big\}\\
&\quad = \big\{u \in C\big([0,\pi]^d\big):\ J_k u \in C^2\big([0,\pi]; C\big([0,\pi]^{d-1}\big)\big),\\
&\qquad (J_k u)'(0) = (J_k u)'(\pi) = 0,\ (J_k N_k u) \in C^\delta\big([0,\pi]; C\big([0,\pi]^{d-1}\big)\big)\big\}\\
&\quad = \big\{u \in C\big([0,\pi]^d\big):\ J_k u \in C^{2+\delta}\big([0,\pi]; C\big([0,\pi]^{d-1}\big)\big),\ (J_k u)'(0) = (J_k u)'(\pi) = 0\big\}.
\end{aligned}$$

In view of Lemma C.2 we conclude that

$$Y = \big\{u \in C^{2+\delta}\big([0,\pi]^d\big):\ D_k u_{|_{y_k=0}} = D_k u_{|_{y_k=\pi}} = 0\big\} = C^{2+\delta}_{\mathcal{N}}\big([0,\pi]^d\big).$$

Therefore, thanks to Theorem B.2 we have that the operator $\hat{N}$ defined in (2.8) is closable in $C([0,\pi]^d)$ and, if we denote by N its closure, $\lambda - N$ is an isomorphism from Y onto X, for any $\lambda > 0$, and estimate (4.1) holds, with $D_k D_h$ replaced by N_k. Now, by applying again Lemma 3.4 in its Banach space valued version, we obtain estimate (4.1) for D_k^2 and then we obtain the general case $D_k D_h$ from Lemma C.2. □

By using interpolation (see [12, Theorem 3.2.1]), as an immediate consequence of the previous lemma we have

Corollary 4.2. *Under the same conditions of Lemma* 4.1*, for any* $\theta \in (0, 2+\delta)$ *there exists* $K = K(\theta, m, M, \gamma, \Gamma) > 0$ *such that for any* $\lambda \geqslant 1$ *and* $v \in X$

$$\left[(\lambda - N)^{-1} v\right]_\theta \leqslant K \lambda^{\frac{\theta}{2+\delta} - 1} |v|_\delta. \tag{4.4}$$

5. The variable coefficient case

In order to prove our result in the case of non-constant coefficients, we use the method of *approximate resolvents* (cf. for example [1] and [12]). This procedure is rather heavy and we will proceed in several steps.

First we introduce a partition of unity for the hypercube $[0,\pi]^d$. Namely, for any $n \in \mathbb{N}$ fixed, we construct a family of functions $\{\Phi^n_\alpha\}_{\alpha \in I_{n,d}} \subset C^\infty([0,\pi]^d)$, where $I_{n,d}$ is a suitable set of indices, such that for any $y \in [0,\pi]^d$

$$0 \leqslant \Phi^n_\alpha(y) \leqslant 1, \qquad \sum_{\alpha \in I_{n,d}} \left(\Phi^n_\alpha(y)\right)^2 = 1.$$

What is important is that functions Φ^n_α satisfy the following bounds for the sup-norm and for the Hölder seminorm

$$\left|\Phi^n_\alpha\right|_0 \leqslant 1, \quad \left[\Phi^n_\alpha\right]_\delta \leqslant k_3(d) n^\delta, \quad \alpha \in I_{n,d},$$

for some constant $k_3(d)$ independent of n and α. In this way, for each $n \in \mathbb{N}$ we can introduce an equivalent norm on $C^\delta([0,\pi]^d)$ taking into account of the estimate above,

$$|v|_{\delta,n} := |v|_0 + \frac{n^{-\delta}}{k_3(d)} [v]_\delta.$$

Once we have such a partition of unity, for each $n \in \mathbb{N}$ and $\alpha \in I_{n,d}$ we introduce the operator $N_{\alpha,n}$, where we have substituted the coefficients μ_k and γ_k appearing in each operator N_k with the constants $\mu_k(y_\alpha)$ and $\gamma_k(y_\alpha)$, for some points $y_\alpha \in \operatorname{supp}(\Phi^n_\alpha)$. Now, operators $N_{\alpha,n}$ have constant coefficients and all results proved in Section 4 for the commutative case can be applied to such operators.

Next, for any $\lambda > 0$ and $n \in \mathbb{N}$ we introduce the operator

$$S_n(\lambda) := \sum_{\alpha \in I_{n,d}} \Phi^n_\alpha \circ (\lambda - N_{\alpha,n})^{-1} \circ \Phi^n_\alpha.$$

We show that $S_n(\lambda)$ belongs to $\mathcal{L}(C^\delta([0,\pi]^d), C^{2+\delta}_{\mathcal{N}}([0,\pi]^d))$ and

$$(\lambda - N) S_n(\lambda) = I + C_n(\lambda),$$

for a suitable operator $C_n(\lambda)$. In Lemma 5.4 we show that there exists some $n_0 \in \mathbb{N}$ and $\lambda_0 > 0$ such that

$$\left|C_{n_0}(\lambda)v\right|_{\delta,n_0} \leqslant \frac{1}{2}|v|_{\delta,n_0}, \quad \lambda \geqslant \lambda_0,$$

so that $I + C_{n_0}(\lambda) \in \mathrm{Iso}(C^\delta([0,\pi]^d))$. We will show that this implies that $\lambda - N$ is an isomorphism from $C^{2+\delta}_{\mathcal{N}}([0,\pi]^d)$ into $C^\delta([0,\pi]^d)$, for any $\lambda \geqslant \lambda_0$ and hence for all $\lambda > 0$.

5.1. *A partition of unity*

Let $\psi \in C^\infty([0,\pi])$ be a function satisfying

$$\begin{cases} 0 \leqslant \psi(t) \leqslant 1, \quad t \in [0,\pi], \\ \psi(0) = 0, \quad \psi(\pi) = 1, \quad \psi(\pi - t) = 1 - \psi(t), \ t \in [0,\pi], \\ \psi'(t) > 0, \ t \in (0,\pi), \quad \psi^{(k)}(0) = \psi^{(k)}(\pi) = 0, \ k \geqslant 1. \end{cases} \tag{5.1}$$

For any $n \geqslant 3$, we set $h_n := \pi/n$ and define

$$\psi_1^n(t) := \begin{cases} 1, & 0 \leqslant t \leqslant h_n, \\ \psi(\pi - n(t - h_n)), & h_n \leqslant t \leqslant 2h_n, \\ 0, & t \geqslant 2h_n, \end{cases}$$
$$\psi_{n-1}^n(t) := \psi_1^n(\pi - t),$$
$$\psi_2^n(t) := \begin{cases} \psi_{n-1}^n(t + \pi - 3h_n), & 0 \leqslant t \leqslant 2h_n, \\ \psi_1^n(t - h_n), & 2h_n \leqslant t \leqslant 3h_n, \\ 0, & t \geqslant 3h_n, \end{cases}$$
$$\psi_k^n(t) := \psi_2^n(t + 2h_n - kh_n), \quad 2 < k \leqslant n - 1.$$

It is immediate to check that

$$\sum_{k=1}^{n-1} \psi_k^n(t) = 1, \quad t \in [0,\pi]. \tag{5.2}$$

Now, for any $d, n \in \mathbb{N}$ we set

$$I_{n,d} := \left\{\alpha = (\alpha_1, \ldots, \alpha_d), \ \alpha_i = 1, \ldots, n-1\right\}$$

and for any $\alpha \in I_{n,d}$ we define

$$\varphi_\alpha^n(y_1, \ldots, y_d) := \prod_{l=1}^{d} \psi_{\alpha_l}^n(y_l), \quad y = (y_1, \ldots, y_d) \in [0,\pi]^d.$$

Clearly,

$$\operatorname{supp} \varphi_\alpha^n = \operatorname{supp} \psi_{\alpha_1}^n \times \cdots \times \operatorname{supp} \psi_{\alpha_d}^n.$$

Moreover the following lower bound is satisfied.

Lemma 5.1. *For any $n, d \in \mathbb{N}$ it holds*

$$\inf_{y\in[0,\pi]^d} \sum_{\alpha\in I_{n,d}} \left[\varphi_\alpha^n(y)\right]^2 \geqslant \left(\frac{1}{4}\right)^d. \tag{5.3}$$

Proof. We proceed by induction on the dimension d. If $d = 1$ we have

$$\sum_{\alpha\in I_{n,1}} \left[\varphi_\alpha^n(y)\right]^2 = \sum_{k=1}^{n-1}\left[\psi_k^n(y)\right]^2.$$

Note that

$$\psi_1^n(y) = 1, \quad y \in [0, h_n], \qquad \psi_{n-1}^n(y) = 1, \quad y \in [\pi - h_n, \pi],$$

so that

$$\sum_{\alpha\in I_{n,1}} \left[\varphi_\alpha^n(y)\right]^2 = 1 \geqslant \frac{1}{4}, \quad y \in [0, h_n] \cup [\pi - h_n, \pi].$$

Moreover

$$\psi_k^n(y) + \psi_{k+1}^n(y) = 1, \quad y \in \left[kh_h, (k+1)h_n\right],\ k = 1, \dots, n-2,$$

so that

$$\max\left\{\psi_k^n(y), \psi_{k+1}^n(y)\right\} \geqslant \frac{1}{2}, \quad y \in \left[kh_h, (k+1)h_n\right],$$

and

$$\sum_{\alpha\in I_{n,1}} \left[\varphi_\alpha^n(y)\right]^2 \geqslant \left(\frac{1}{2}\right)^2 = \frac{1}{4}, \quad y \in \left[h_n, (n-1)h_n\right].$$

Now, for any $d \geqslant 2$ we have

$$\begin{aligned}
\sum_{\alpha\in I_{n,d}} \left[\varphi_\alpha^n(y)\right]^2 &= \sum_{k=1}^{n-1} \sum_{\beta\in I_{n,d-1}} \left[\psi_k^n(y_1)\right]^2 \prod_{l=2}^{d}\left[\psi_{\beta_l}^n(y_l)\right]^2 \\
&= \sum_{k=1}^{n-1}\left[\psi_k^n(y_1)\right]^2 \sum_{\beta\in I_{n,d-1}} \left[\varphi_\beta^n(y_2, \dots, y_d)\right]^2 \\
&\geqslant \frac{1}{4} \sum_{\beta\in I_{n,d-1}} \left[\varphi_\beta^n(y_2, \dots, y_d)\right]^2.
\end{aligned}$$

Then, if we assume that

$$\sum_{\beta\in I_{n,d-1}} \left[\varphi^n_\beta(y_2,\dots,y_d)\right]^2 \geqslant \left(\frac{1}{4}\right)^{d-1},$$

we have

$$\sum_{\alpha\in I_{n,d}} \left[\varphi^n_\alpha(y)\right]^2 \geqslant \left(\frac{1}{4}\right)^{d-1}\frac{1}{4} = \left(\frac{1}{4}\right)^{d}, \quad y\in[0,\pi]^d,$$

and we can conclude that (5.3) holds. □

Next, for any $n\in\mathbb{N}$ and $\alpha\in I_{n,d}$ we define

$$\Phi^n_\alpha(y) := \left(\sum_{\alpha\in I_{n,d}} \left[\varphi^n_\alpha(y)\right]^2\right)^{-\frac{1}{2}} \varphi^n_\alpha(y), \quad y\in[0,\pi]^d.$$

As $\psi\in C^\infty[0,\pi]$ and (5.3) holds, we have that $\Phi^n_\alpha\in C^\infty([0,\pi]^d)$. Moreover $0\leqslant\Phi^n_\alpha(y)\leqslant 1$, for any $y\in[0,\pi]^d$,

$$\sum_{\alpha\in I_{n,d}} \left(\Phi^n_\alpha(y)\right)^2 = 1, \quad y\in[0,\pi]^d,$$

and for any $n\in\mathbb{N}$ and $\alpha\in I_{n,d}$ it holds

$$\operatorname{supp}\Phi^n_\alpha = \operatorname{supp}\varphi^n_\alpha = \left[(\alpha_1-1)h_n,(\alpha_1+1)h_n\right]\times\cdots\times\left[(\alpha_d-1)h_n,(\alpha_d+1)h_n\right].$$

Our aim here is giving an estimate for the C^δ semi-norm of functions Φ^n_α. For each $n\in\mathbb{N}$ and $k=1,\dots,n-1$, we have

$$\left|\psi^n_k\right|_0\leqslant 1, \qquad \left[\psi^n_k\right]_\delta = k_1 n^\delta,$$

for some $k_1>0$. Hence, by using (2.1) we have

$$\left[\varphi^n_\alpha\right]_\delta = \left[\prod_{l=1}^d \psi^n_{\alpha_l}\right]_\delta \leqslant \sum_{l=1}^d \left[\psi^n_{\alpha_l}\right]_\delta \leqslant d k_1 n^\delta. \tag{5.4}$$

Lemma 5.2. *There exists some $k_2=k_2(d)>0$ such that for any $\alpha\in I_{n,d}$ and $v_\alpha\in C([0,\pi]^d)$*

$$\left|\sum_{\alpha\in I_{n,d}} \varphi^n_\alpha v_\alpha\right|_0 \leqslant k_2(d)\max_{\alpha\in I_{n,d}}\left|\varphi^n_\alpha v_\alpha\right|_0, \qquad \left|\sum_{\alpha\in I_{n,d}} \Phi^n_\alpha v_\alpha\right|_0 \leqslant k_2(d)\max_{\alpha\in I_{n,d}}\left|\Phi^n_\alpha v_\alpha\right|_0.$$

Moreover, if $v_\alpha \in C^\delta([0,\pi]^d)$*, we have*

$$\Big[\sum_{\alpha\in I_{n,d}} \varphi^n_\alpha v_\alpha\Big]_\delta \leqslant k_2(d) \max_{\alpha\in I_{n,d}} \big[\varphi^n_\alpha v_\alpha\big]_\delta, \qquad \Big[\sum_{\alpha\in I_{n,d}} \Phi^n_\alpha v_\alpha\Big]_\delta \leqslant k_2(d) \max_{\alpha\in I_{n,d}} \big[\Phi^n_\alpha v_\alpha\big]_\delta.$$

Proof. It follows from the *smallness* of the supports of the functions φ^n_α that for each cell of the form

$$\prod_{i=1}^d \Big[\frac{1}{n}\beta_i, \frac{1}{n}(\beta_i+1)\Big], \quad \beta_i = 0, 1, \dots, n-1,$$

the sums inside the (semi)norms can be reduced to at most 3^d terms. Therefore $k_2(d)$ can be taken equal 3^d. □

As a consequence of the previous lemma, we have an estimate both for the sup-norm and for the Hölder seminorm of Φ^n_α.

Corollary 5.3. *There exists* $k_3 = k_3(d) > 0$ *such that for any* $n \in \mathbb{N}$ *and* $\alpha \in I_{n,d}$

$$\big|\Phi^n_\alpha\big|_0 \leqslant 1, \qquad \big[\Phi^n_\alpha\big]_\delta \leqslant k_3(d) n^\delta. \tag{5.5}$$

Proof. The bound on the sup-norm has been already seen. In general, if $v \in C^\delta([0,\pi]^d)$, with $v(y) \geqslant v_0 > 0$, for any $y \in [0,\pi]^d$ we have

$$[1/v]_\delta \leqslant \big|1/v^2\big|_0 [v]_\delta, \qquad [\sqrt{v}\,]_\delta \leqslant |1/v|_0 [v]_\delta.$$

Hence, thanks to (5.3), (2.1) and (5.4) we have

$$\begin{aligned}\big[\Phi^n_\alpha\big]_\delta &= \bigg[\varphi^n_\alpha\Big(\sum_{\alpha\in I_{n,d}} \big(\varphi^n_\alpha\big)^2\Big)^{-1/2}\bigg]_\delta \leqslant 2^d\big[\varphi^n_\alpha\big]_\delta + \bigg[\Big(\sum_{\alpha\in I_{n,d}} \big(\varphi^n_\alpha\big)^2\Big)^{-1/2}\bigg]_\delta \\ &\leqslant 2^d\big[\varphi^n_\alpha\big]_\delta + \bigg|1\Big/\sum_{\alpha\in I_{n,d}} \big(\varphi^n_\alpha\big)^2\bigg|_0^2 \Big[\sum_{\alpha\in I_{n,d}} \big(\varphi^n_\alpha\big)^2\Big]_\delta \leqslant 2^d\big[\varphi^n_\alpha\big]_\delta + 16^d \Big[\sum_{\alpha\in I_{n,d}} \big(\varphi^n_\alpha\big)^2\Big]_\delta.\end{aligned}$$

According to (2.1) and to Lemma 5.2 this implies

$$\big[\Phi^n_\alpha\big]_\delta \leqslant \big[\varphi^n_\alpha\big]_\delta + k_2(d)32^d \max_{\alpha\in I_{n,d}} \big[\big(\varphi^n_\alpha\big)^2\big]_\delta \leqslant \big[\varphi^n_\alpha\big]_\delta + 2k_2(d)32^d \max_{\alpha\in I_{n,d}} \big|\varphi^n_\alpha\big|_0 \big[\varphi^n_\alpha\big]_\delta,$$

so that, thanks to (5.4)

$$\big[\Phi^n_\alpha\big]_\delta \leqslant d k_3'(d) k_1 n^\delta.$$

If we set $k_3(d) := d k_1 k_3'(d)$, we obtain the thesis. □

In view of (5.5) we introduce an equivalent norm on $C^\delta([0,\pi]^d)$. For any $v \in C^\delta([0,\pi]^d)$ we set

$$|v|_{\delta,n} := |v|_0 + \frac{n^{-\delta}}{k_3(d)}[v]_\delta. \tag{5.6}$$

Clearly we have

$$|vw|_{\delta,n} \leqslant |v|_{\delta,n}|w|_{\delta,n},$$

so that, thanks to (5.5), for any $n \in \mathbb{N}$ and $\alpha \in I_{n,d}$ and for any $v \in C^\delta([0,\pi]^d)$

$$\left|\Phi^n_\alpha v\right|_{\delta,n} \leqslant \left|\Phi^n_\alpha\right|_{\delta,n}\big(|v|_0 + [v]_{\delta,n,\operatorname{supp}\Phi^n_\alpha}\big) \leqslant 2\big(|v|_0 + [v]_{\delta,n,\operatorname{supp}\Phi^n_\alpha}\big), \tag{5.7}$$

where

$$[v]_{\delta,n,\operatorname{supp}\Phi^n_\alpha} := \frac{n^{-\delta}}{k_3(d)} \sup_{\substack{y,z\in \operatorname{supp}\Phi^n_\alpha \\ y\neq z}} \frac{|v(y)-v(z)|}{|y-z|^\delta}.$$

5.2. *Proof of Theorem 2.2*

Our aim here is proving that the operator $\lambda - N$ is an isomorphism from $C^{2+\delta}_{\mathcal{N}}([0,\pi]^d)$ into $C^\delta([0,\pi]^d)$, for any $\lambda > 0$.

To this purpose, for any $n \geqslant 3$ and $\alpha \in I_{n,d}$ we introduce the operator

$$N_{\alpha,n} := \sum_{k=1}^d N_k,$$

where in each operator N_k the functions μ_k and γ_k are replaced by the constants $\mu_k(y_\alpha)$ and $\gamma_k(y_\alpha)$, for some points $y_\alpha \in \operatorname{supp}\Phi^n_\alpha$. For these operators Lemma 4.1 holds.

Now, for any $\lambda > 0$ and $n \in \mathbb{N}$ we introduce the operator

$$S_n(\lambda) := \sum_{\alpha\in I_{n,d}} \Phi^n_\alpha \circ (\lambda - N_{\alpha,n})^{-1} \circ \Phi^n_\alpha,$$

where the Φ^n_α's have to be interpreted as the multiplication operators by Φ^n_α. Observe that for any $\alpha \in I_{n,\alpha}$, the operator Φ^n_α maps $C^\delta([0,\pi]^d)$ into itself and $C^{2+\delta}_{\mathcal{N}}([0,\pi]^d)$ into itself as a bounded operator, so that

$$S_n(\lambda) \in \mathcal{L}\big(C^\delta\big([0,\pi]^d\big), C^{2+\delta}_{\mathcal{N}}\big([0,\pi]^d\big)\big), \quad \lambda > 0.$$

Moreover,

$$(\lambda - N)S_n(\lambda) = I + C_n(\lambda), \tag{5.8}$$

where

$$C_n(\lambda) := \sum_{\alpha\in I_{n,d}} \big[\Phi_\alpha^n, N\big](\lambda - N_{\alpha,n})^{-1} \circ \Phi_\alpha^n + \sum_{\alpha\in I_{n,d}} \Phi_\alpha^n \circ (N_{\alpha,n} - N)(\lambda - N_{\alpha,n})^{-1} \circ \Phi_\alpha^n$$

and for any $v \in C^{2+\delta}_{\mathcal{N}}([0,\pi]^d)$

$$\big[\Phi_\alpha^n, N\big]v = \Phi_\alpha^n(Nv) - N\big(\Phi_\alpha^n v\big).$$

If we show that there exist some $n_0 \in \mathbb{N}$ and $\lambda_0 \geqslant 1$ such that for any $v \in C^\delta([0,\pi]^d)$

$$\big|C_{n_0}(\lambda)v\big|_{\delta,n_0} \leqslant \frac{1}{2}|v|_{\delta,n_0}, \quad \lambda \geqslant \lambda_0, \tag{5.9}$$

where $|\cdot|_{\delta,n_0}$ is the equivalent norm introduced in (5.6), then it follows that $I + C_{n_0}(\lambda) \in \operatorname{Isom}(C^\delta([0,\pi]^d))$ and from (5.8)

$$(\lambda - N)S_{n_0}(\lambda)\big(I + C_{n_0}(\lambda)\big)^{-1} = I, \quad \text{in } C^\delta\big([0,\pi]^d\big).$$

By the minimum principle, the operator $(\lambda - N)$ is injective, and then $S_{n_0}(\lambda)(I + C_{n_0}(\lambda))^{-1}$ is the inverse of $(\lambda - N)$, so that $(\lambda - N) \in \operatorname{Isom}(C^{2+\delta}_{\mathcal{N}}([0,\pi]^d), C^\delta([0,\pi]^d))$, for any $\lambda \geqslant \lambda_0$.

On the other hand, if $0 < \lambda < \lambda_0$ we have that the equation $\lambda v = Nv + f$ is equivalent to

$$v = (\lambda_0 - N)^{-1}(\lambda_0 - \lambda)v + (\lambda_0 - N)^{-1} f.$$

Then, as the operator $K := (\lambda_0 - N)^{-1}(\lambda_0 - \lambda)$ is compact in X and, by the minimum principle, $\operatorname{Ker}(I - K) = \{0\}$, we obtain that $\lambda - N \in \operatorname{Isom}(C^{2+\delta}_{\mathcal{N}}([0,\pi]^d), C^\delta([0,\pi]^d))$, for all $\lambda > 0$.

Hence, in order to conclude the proof of Theorem 2.2 we have to prove that (5.9) holds.

Lemma 5.4. *For any $n \geqslant 3$ let us define*

$$C_n(\lambda) := \sum_{\alpha\in I_{n,d}} \big[\Phi_\alpha^n, N\big](\lambda - N_{\alpha,n})^{-1} \circ \Phi_\alpha^n + \sum_{\alpha\in I_{n,d}} \Phi_\alpha^n \circ (N_{\alpha,n} - N)(\lambda - N_{\alpha,n})^{-1} \circ \Phi_\alpha^n.$$

Then, there exists $n_0 \in \mathbb{N}$ and $\lambda_0 \geqslant 1$ such that for any $\lambda \geqslant \lambda_0$

$$\big|C_{n_0}(\lambda)v\big|_{\delta,n_0} \leqslant \frac{1}{2}|v|_{\delta,n_0}. \tag{5.10}$$

Proof. If we apply Lemma 3.4 in its Banach space-valued version, to the functions $D_k^2 v$, with $v \in C^{2+\delta}_{\mathcal{N}}([0,\pi]^d), C^\delta([0,\pi]^d)$ and $k = 1,\dots,d$, we obtain

$$\big|(\sin y_k)^{-1} D_k v\big|_0 \leqslant k_4\big|D_k^2 v\big|_0, \qquad \big[(\sin y_k)^{-1} D_k v\big]_\delta \leqslant k_4\big[D_k^2 v\big]_\delta. \tag{5.11}$$

We recall here that any function $v \in C^{2+\delta}_{\mathcal{N}}([0,\pi]^d), C^\delta([0,\pi]^d)$ is the restriction to $[0,\pi]^d$ of a function $\bar{v} \in C^{2+\delta}(\mathbb{R}^d)$. Actually, if $v \in C^{2+\delta}_{\mathcal{N}}([0,\pi]^d), C^\delta([0,\pi]^d)$, due to the boundary conditions we can extend it by evenness and then by 2π periodicity to the whole space $\mathbb{R}^d$. Hence for any $0 \leqslant s \leqslant r < 3$ and $w \in C^r([0,\pi]^d)$ the following interpolation inequality holds

$$[w]_s \leqslant N[w]_r^{s/r}|w|_0^{1-s/r}, \tag{5.12}$$

for some constant $N = N(d,r)$ (for a proof see [12, Theorem 3.2.1]).

Now, let us estimate the norm of

$$\sum_{\alpha\in I_{n,d}} \Phi_\alpha^n \circ (N_{\alpha,n} - N)(\lambda - N_{\alpha,n})^{-1} \circ \Phi_\alpha^n.$$

For any $v \in X$, if we set $w := (\lambda - N_{\alpha,n})^{-1}\Phi_\alpha^n v$, thanks to Lemma 5.2 and to (5.7)

$$\begin{aligned}
&\Big|\sum_{\alpha\in I_{n,d}} \Phi_\alpha^n (N_{\alpha,n} - N)(\lambda - N_{\alpha,n})^{-1}\Phi_\alpha^n v\Big|_{\delta,n} \\
&\quad\leqslant k_2(d) \max_{\alpha\in I_{n,d}} \big|\Phi_\alpha^n (N_{\alpha,n} - N)w\big|_{\delta,n} \\
&\quad\leqslant 2k_2(d) \max_{\alpha\in I_{n,d}} \big(\big|(N_{\alpha,n} - N)w\big|_0 + \big[(N_{\alpha,n} - N)w\big]_{\delta,n,\operatorname{supp}\Phi_\alpha^n}\big).
\end{aligned} \tag{5.13}$$

Now, for any $n \in \mathbb{N}$ and $\alpha \in I_{n,d}$,

$$\begin{aligned}
&\big[(N_{\alpha,n} - N)w\big]_{\delta,n,\operatorname{supp}\Phi_\alpha^n} \\
&\quad\leqslant \sum_{k=1}^d \big[\big(\mu_k - \mu_k(y_\alpha)\big)D_k^2 w\big]_{\delta,n,\operatorname{supp}\Phi_\alpha^n} \\
&\qquad + \sum_{k=1}^d \big[2b_k\big(\gamma_k - \gamma_k(y_\alpha)\big)(\sin y_k)^{-1} D_k w\big]_{\delta,n,\operatorname{supp}\Phi_\alpha^n} \\
&\quad\leqslant \sum_{k=1}^d \Big(\big[D_k^2 w\big]_{\delta,n} \sup_{y\in\operatorname{supp}\Phi_\alpha^n} \big|\mu_k(y) - \mu_k(y_\alpha)\big| + \big|D_k^2 w\big|_0 [\mu_k]_{\delta,n}\Big) \\
&\qquad + 2\sum_{k=1}^d \Big(\big[(\sin y_k)^{-1} D_k w\big]_{\delta,n} \\
&\qquad \times \sup_{y\in\operatorname{supp}\Phi_\alpha^n} \big|\gamma_k(y) - \gamma_k(y_\alpha)\big||b_k|_0 + \big|(\sin y_k)^{-1} D_k w\big|_0 [b_k\gamma_k]_{\delta,n}\Big).
\end{aligned}$$

According to (4.4) and (5.5), if $\lambda \geqslant 1$

$$\begin{aligned}
\big[D_k^2 w\big]_{\delta,n} &= \big[D_k^2 (\lambda - N_{\alpha,n})^{-1}\Phi_\alpha^n v\big]_{\delta,n} \leqslant K\frac{n^{-\delta}}{k_3(d)}\big(\big|\Phi_\alpha^n v\big|_0 + \big[\Phi_\alpha^n v\big]_\delta\big) \\
&\leqslant K\frac{n^{-\delta}}{k_3(d)}\big(\big(1 + k_3(d)\big)|v|_0 + [v]_\delta\big) \leqslant K(d)\big(|v|_0 + [v]_{\delta,n}\big)
\end{aligned}$$

and according to (5.11)

$$\big[(\sin y_k)^{-1} D_k w\big]_{\delta,n} = \big[(\sin y_k)^{-1} D_k (\lambda - N_{\alpha,n})^{-1} \Phi_\alpha^n v\big]_{\delta,n}$$
$$\leqslant k_4 \frac{n^{-\delta}}{k_3(d)} \big[D_k^2 (\lambda - N_{\alpha,n})^{-1} \Phi_\alpha^n v\big]_\delta \leqslant k_4(d)\big(|v|_0 + [v]_{\delta,n}\big).$$

Now, if we choose $n_0 \in \mathbb{N}$ large enough such that

$$K(d) \sum_{k=1}^d \sup_{y \in \operatorname{supp} \Phi_\alpha^n} \big|\mu_k(y) - \mu_k(y_\alpha)\big| + 2k_4(d) \sum_{k=1}^d \sup_{y \in \operatorname{supp} \Phi_\alpha^n} \big|\gamma_k(y) - \gamma_k(y_\alpha)\big| |b_k|_0$$
$$\leqslant \frac{1}{32 k_2(d)},$$

we get for any $n \geqslant n_0$

$$\big[(N_{\alpha,n} - N) w\big]_{\delta,n,\operatorname{supp} \Phi_\alpha^n}$$
$$\leqslant \frac{1}{32 k_2(d)} |v|_{\delta,n} + \sum_{k=1}^d \Big(\big|D_k^2 w\big|_0 [\mu_k]_{\delta,n} + \big|(\sin y_k)^{-1} D_k w\big|_0 [b_k \gamma_k]_{\delta,n}\Big).$$

Next, due to (5.7), (4.4) and (5.11)

$$\big|D_k^2 w\big|_0 [\mu_k]_{\delta,n} + \big|(\sin y_k)^{-1} D_k w\big|_0 [b_k \gamma_k]_{\delta,n}$$
$$\leqslant \big([\mu_k]_{\delta,n} + [b_k \gamma_k]_{\delta,n}\big)(1 + k_4) \big|D_k^2 w\big|_0 \leqslant k_n (1 + k_4) \big|D_k^2 \big[(\lambda - N_{n,\alpha})^{-1} \Phi_\alpha^n v\big]\big|_0$$
$$\leqslant K k_n (1 + k_4) \lambda^{-\frac{\delta}{2+\delta}} \big|\Phi_\alpha^n v\big|_\delta \leqslant \big(1 + k_1 k_3(d) n^\delta\big) K k_n (1 + k_4) \lambda^{-\frac{\delta}{2+\delta}} \big|\Phi_\alpha^n v\big|_{\delta,n}$$
$$\leqslant 2\big(1 + k_1 k_3(d) n^\delta\big) K k_n (1 + k_4) \lambda^{-\frac{\delta}{2+\delta}} |v|_{\delta,n}.$$

Now, if we choose $n = n_0$ and $\lambda_1 \geqslant 1$ such that

$$2\big(1 + k_1 k_3(d) n_0^\delta\big) K k_{n_0} (1 + k_4) \lambda^{-\frac{\delta}{2+\delta}} \leqslant \frac{1}{32 k_2(d)},$$

for any $\lambda \geqslant \lambda_1$, we have

$$\big[(N_{\alpha,n_0} - N) w\big]_{\delta,n_0,\operatorname{supp} \Phi_\alpha^{n_0}} \leqslant \frac{1}{16 k_2(d)} |v|_{\delta,n_0}. \tag{5.14}$$

Next, we have to estimate $|(N_{\alpha,n_0} - N) w|_0$. Thanks to (5.11), we have

$$\big|(N_{\alpha,n_0} - N) w\big|_0 \leqslant |N_{\alpha,n_0} w|_0 + |N w|_0 \leqslant M \big[(\lambda - N_{n,\alpha})^{-1} \Phi_\alpha^n v\big]_2$$

and, as above, from (4.4) for some constant k_{n_0}

$$\big|(N_{\alpha,n_0} - N) w\big|_0 \leqslant M K k_{n_0} \lambda^{-\frac{\delta}{2+\delta}} |v|_{\delta,n_0}.$$

Thus, due to (5.13) and (5.14) we can choose $\lambda_2 \geqslant \lambda_1$ such that for any $\lambda \geqslant \lambda_2$

$$\left| \sum_{\alpha \in I_{n_0,d}} \Phi_\alpha^{n_0} (N_{\alpha,n_0} - N)(\lambda - N_{\alpha,n_0})^{-1} \Phi_\alpha^{n_0} v \right|_{\delta,n_0} \leqslant \frac{1}{4} |v|_{\delta,n_0}. \tag{5.15}$$

Now, in order to conclude the proof of the lemma, we have to estimate

$$\sum_{\alpha \in I_{n_0,d}} \big[\Phi_\alpha^{n_0}, N\big](\lambda - N_{\alpha,n_0})^{-1} \Phi_\alpha^{n_0}.$$

Notice that, as $\Phi_\alpha^n \in C_{\mathcal{N}}^{2+\delta}([0,\pi]^d)$ and the coefficients μ_k and γ_k are in $C^\delta([0,\pi]^d)$, by using (5.11) we easily obtain

$$\left| \sum_{\alpha \in I_{n_0,d}} \big[\Phi_\alpha^{n_0}, N\big](\lambda - N_{\alpha,n_0})^{-1} \Phi_\alpha^{n_0} v \right|_\delta \leqslant k_{n_0} \big|(\lambda - N_{\alpha,n_0})^{-1} \Phi_\alpha^{n_0} v\big|_{1+\delta},$$

and by proceeding as before, we can find $\lambda_0 \geqslant \lambda_2$ such that for any $\lambda \geqslant \lambda_0$

$$\left| \sum_{\alpha \in I_{n_0,d}} \big[\Phi_\alpha^{n_0}, N\big](\lambda - N_{\alpha,n_0})^{-1} \Phi_\alpha^{n_0} v \right|_\delta \leqslant \frac{1}{4} |v|_{\delta,n_0}.$$

Together with (5.15), this implies (5.10). □

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Appendix A

Let B be a Banach space and fix $a \leqslant b$. For any $u \in C([a,b])$ and $h \in B$, we denote by $u \otimes h$ the mapping in $C([a,b]; B)$ defined by

$$[u \otimes h](t) = u(t)h, \quad t \in [a,b].$$

Moreover, we denote by $C([a,b]) \otimes B$ the subspace of $C([a,b]; B)$ given by finite linear combinations of element $u \otimes h$. By using a Banach-valued version of Bernstein polynomials, it is immediate to check that $C([a,b]) \otimes B$ is dense in $C([a,b]; B)$. The proof of the following lemma can be found in [10, Example 6, pp. 224–225]. We thank B. de Pagter for showing us this reference.

Lemma A.1. *Let B be a Banach space. For any $N \in \mathcal{L}(C([a,b]))$ there exists a unique $N_B \in \mathcal{L}(C([a,b]; B))$ such that*

$$N_B(u \otimes h) = Nu \otimes h, \quad u \in C\big([a,b]\big),\ h \in B. \tag{A.1}$$

Moreover, it holds

$$\|N_B\|_{\mathcal{L}(C([a,b];B))} = \|N\|_{\mathcal{L}(C([a,b]))}. \tag{A.2}$$

Now, for any $u \in C([a,b])$ and $v \in C([a,b]^{d-1})$ and for any $k = 1, \dots, d$ we define

$$[u \otimes_k v](x) := u(x_k)v(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_d), \quad x \in [a,b]^d, \tag{A.3}$$

and we denote by $C([a,b]) \otimes_k C([a,b]^{d-1})$ the subspace of $C([a,b]^d)$ given by finite linear combinations of elements $u \otimes_k v$. Clearly, $C([a,b]) \otimes_k C([a,b]^{d-1})$ is a dense subspace of $C([a,b]^d)$.

Next, for any $k = 1, \dots, d$, $f \in C([a,b]^d)$, $t \in [a,b]$ and $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_d) \in [a,b]^{d-1}$ we define

$$\big[J_k f(t)\big](x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_d) := f(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_d). \tag{A.4}$$

With this definition,

$$J_k : C\big([a,b]^d\big) \to C\big([a,b]; C\big([a,b]^{d-1}\big)\big)$$

is a surjective isometry. Then, for any $N \in \mathcal{L}(C([a,b]))$ and $k = 1, \dots, d$ we define

$$E_k(N) := J_k^{-1} N_{C([a,b]^{d-1})} J_k,$$

where the operator $N_{C([a,b]^{d-1})}$ is defined in Lemma A.1, by taking $B = C([a,b]^{d-1})$. According to Lemma A.1, it is immediate to check that $E_k(N) \in \mathcal{L}(C([a,b]^d)$ and

$$\big\|E_k(N)\big\|_{\mathcal{L}(C([a,b]^d))} = \|N\|_{\mathcal{L}(C([a,b]))}. \tag{A.5}$$

Moreover, for any $u \in C([a,b])$ and $v \in C([a,b]^{d-1})$

$$E_k(N)[u \otimes_k v] = Nu \otimes_k v.$$

Theorem A.2. *Let $A : D(A) \subset C([a,b]) \to C([a,b])$ be a densely defined and m-dissipative operator generating an analytic semigroup e^{tA}. Then, for any $k = 1, \dots, d$ there exists a unique densely defined and m-dissipative operator $E_k(A) : D(E_k(A)) \subset C([a,b]^d) \to C([a,b]^d)$ generating an analytic semigroup $e^{tE_k(A)}$, such that*

$$e^{tE_k(A)} = E_k\big(e^{tA}\big), \quad t \geqslant 0.$$

Moreover,

$$D\big(E_k(A)\big) \supseteq D(A) \otimes_k C\big([a,b]^{d-1}\big),$$

$D(A) \otimes_k C([a,b]^{d-1})$ is a core for $E_k(A)$, and

$$E_k(A)(u \otimes_k v) = Au \otimes_k v, \quad u \in D(A),\ v \in C\big([a,b]^{d-1}\big). \tag{A.6}$$

Proof. For any $\lambda > 0$ we set

$$J(\lambda) := R(\lambda, A) = (\lambda - A)^{-1}.$$

Due to the assumptions on A, we have that the range of $J(\lambda)$ is dense in $C([a, b])$ and

$$\|\lambda J(\lambda)\|_{\mathcal{L}(C([a,b]))} \leqslant 1, \quad \lambda > 0.$$

With the notations introduced above, for any $\lambda > 0$ we define the operator $E_k(J(\lambda)) \in \mathcal{L}(C([a, b]^d))$. Due to (A.5) we have

$$\|\lambda E_k(J(\lambda))\|_{\mathcal{L}(C([a,b]^d))} \leqslant 1, \quad \lambda > 0.$$

We first show that $\{E_k(J(\lambda))\}_{\lambda>0}$ is a pseudo-resolvent, that is

$$E_k(J(\lambda)) - E_k(J(\mu)) = (\mu - \lambda) E_k(J(\lambda)) E_k(J(\mu)), \quad \lambda, \mu > 0. \tag{A.7}$$

As $C([a, b]) \otimes_k C([a, b]^{d-1})$ is dense in $C([a, b]^d)$ and $E_k(J(\lambda))$ is a bounded operator on $C([a, b]^d)$ for any $\lambda > 0$, it is sufficient to prove (A.7) only on $C([a, b]) \otimes_k C([a, b]^{d-1})$. For any $u \in C([a, b])$ and $v \in C([a, b]^{d-1})$, we have

$$\begin{aligned} E_k(J(\lambda))(u \otimes_k v) - E_k(J(\mu))(u \otimes_k v) &= [J(\lambda)u] \otimes_k v - [J(\mu)u] \otimes_k v \\ &= [J(\lambda)u - J(\mu)u] \otimes_k v = [(\mu - \lambda) J(\lambda) J(\mu) u] \otimes_k v \\ &= (\mu - \lambda) E_k(J(\lambda)) E_k(J(\mu))(u \otimes_k v), \end{aligned}$$

and the same is true for finite linear combinations of elements of the type $u \otimes_k v$.

Next, we show that the range of $E_k(J(\lambda))$ is dense in $C([a, b]^d)$. To this purpose, as $C([a, b]) \otimes_k C([a, b]^{d-1})$ is dense in $C([a, b]^d)$, it is sufficient to prove that the range of $E_k(J(\lambda))$ is dense in $C([a, b]) \otimes_k C([a, b]^{d-1})$. Let $u \in C([a, b])$ and $v \in C([a, b]^{d-1})$. As the range of $J(\lambda)$ is dense in $C([a, b])$, there exists a sequence $\{u_n\}_n \subset C([a, b])$ such that $J(\lambda)u_n \to u$, so that $E_k(J(\lambda))(u_n \otimes_k v) = J(\lambda)u_n \otimes_k v \to u \otimes_k v$.

Hence, since the family $\{E_k(J(\lambda))\}_{\lambda>0}$ is a pseudo-resolvent and the range of the operator $E_k(J(\lambda))$ is dense in $C([a, b]^d)$, in view of [17, Theorem 9.4] we can conclude that $\{E_k(J(\lambda))\}_{\lambda>0}$ is the resolvent of a unique densely defined closed m-dissipative linear operator $E_k(A)$, which generates a C_0-contraction semigroup $e^{tE_k(A)}$.

We remark that

$$D(E_k(A)) = \text{Range}\, E_k(J(\lambda)) \supset [\text{Range}\, J(\lambda)] \otimes_k C([a, b]^{d-1}) = D(A) \otimes_k C([a, b]^{d-1}).$$

Moreover

$$\begin{aligned} E_k(J(\lambda))(\lambda I - E_k(A))(u \otimes_k v) = u \otimes_k v &= [J(\lambda)(\lambda I - A)u] \otimes_k v \\ &= E_k(J(\lambda))[(\lambda I - A)u \otimes_k v], \end{aligned}$$

so that, as $E_k(J(\lambda))$ is injective, we have

$$E_k(A)(u \otimes_k v) = \lambda u \otimes_k v - \big(\lambda I - E_k(A)\big)(u \otimes_k v)$$
$$= \lambda u \otimes_k v - \big[(\lambda I - A)u\big] \otimes_k v = Au \otimes_k v.$$

Next, we show that $D(A) \otimes_k C([a,b]^{d-1})$ is a core for $E_k(A)$. Let $g \in D(E_k(A))$ and let $f \in C([a,b]^d)$ such that $g = E_k(J(\lambda))f$. As $C([a,b]) \otimes_k C([a,b]^{d-1})$ is dense in $C([a,b]^d)$, there exists $\{f_n\} \in C([a,b]) \otimes_k C([a,b]^{d-1})$ such that

$$f_n \to f, \quad E_k\big(J(\lambda)\big) f_n \to E_k\big(J(\lambda)\big) f.$$

We have

$$f_n = \sum_{j=1}^{j_n} u_n^j \otimes_k v_n^j,$$

so that

$$g_n := E_k\big(J(\lambda)\big) f_n = \sum_{j=1}^{j_n} \big[J(\lambda) u_n^j\big] \otimes_k v_n^j \in D(A) \otimes_k C\big([a,b]^{d-1}\big),$$

and

$$g_n \to E_k\big(J(\lambda)\big) f = g, \quad \big(\lambda - E_k(A)\big) g_n = f_n \to f = (\lambda - A) g.$$

In order to conclude the proof we have to show that the semigroup $e^{tE_k(A)}$ is analytic. We show that there exists some constant $M > 0$ such that for any $f \in C([a,b]^d)$

$$\big|t E_k(A) e^{tE_k(A)} f\big|_{C([a,b]^d)} \leqslant M |f|_{C([a,b]^d)}, \quad t > 0. \tag{A.8}$$

By the exponential formula we have

$$e^{tE_k(A)}(u \otimes_k v) = \lim_{n\to\infty} \big(I - t/n E_k(A)\big)^{-n} (u \otimes_k v) = \lim_{n\to\infty} \big(t/n E_k\big(J(t/n)\big)\big)^n (u \otimes_k v)$$

and then, by iteration, we obtain

$$e^{tE_k(A)}(u \otimes_k v) = \lim_{n\to\infty} \big[\big(t/n J(t/n)\big)^n u\big] \otimes_k v = e^{tA} u \otimes_k v. \tag{A.9}$$

As e^{tA} is analytic, for any $t > 0$ the operator Ae^{tA} is bounded on $C([a,b])$ and there exists $M > 0$ such that

$$\big|Ae^{tA}\big|_{\mathcal{L}(C([a,b]))} \leqslant M/t, \quad t > 0.$$

Hence, if we show that

$$E_k(A) e^{tE_k(A)} = E_k\big(Ae^{tA}\big), \quad t > 0, \tag{A.10}$$

we can conclude that (A.8) holds. For any $u \in C([a,b])$ and $v \in C([a,b]^{d-1})$, due to (A.9) we have

$$E_k\big(Ae^{tA}\big)(u \otimes_k v) = \big[Ae^{tA}u\big] \otimes_k v = E_k(A)\big(e^{tA}u \otimes_k v\big) = E_k(A)e^{tE_k(A)}(u \otimes_k v)$$

and by linearity we have that (A.10) holds on $C([a,b]) \otimes_k C([a,b]^{d-1})$. By density we can conclude that (A.10) is true on $C([a,b]^d)$. □

Appendix B

Let E be a real Banach space endowed with the norm $|\cdot|_E$ and let $\{S_k(t)\}_{k=1}^d$ be a family of d commuting C_0-contraction analytic semigroups on E with corresponding generators $\{A_k\}_{k=1}^d$. We assume that there exists $M > 0$ such that for any $k = 1, \dots, d$

$$\sup_{t>0} t \big\| A_k S_k(t) \big\|_{\mathcal{L}(E)} \leqslant M.$$

In what follows we shall denote by $S(t)$, $t \geqslant 0$, the product of the semigroups S_k, that is

$$S(t) := \prod_{k=1}^d S_k(t), \quad t \geqslant 0.$$

It is well known that $S(t)$, $t \geqslant 0$, is also an analytic C_0-contraction semigroup and its generator, which will be denoted by A, is the closure in E of the operator $\sum_{k=1}^d A_k$ defined on

$$D\left(\sum_{k=1}^d A_k\right) := \bigcap_{k=1}^d D(A_k).$$

Proceeding as in [9], for any $\theta \in (0,1)$ and $p \in [1,\infty]$ we set

$$D_A(\theta, p) := \big(E, D(A)\big)_{\theta,p},$$

where $(E, D(A))_{\theta,p}$ is the real interpolation space between E and $D(A)$ of exponents θ and p. We recall that in the case $p = \infty$

$$|u|_{D_A(\theta,p)} := |u|_E + \sup_{t>0} t^{\theta-1} \big|AS(t)u\big|_E =: |u|_E + [u]_{D_A(\theta,p)}.$$

The proof of the following lemma is due to Alessandra Lunardi [15], whom we thank.

Lemma B.1. *Under the above assumptions, for any $\theta \in (0,1)$ and $p \in [1,\infty]$*

$$D_A(\theta, p) = \left(E, \bigcap_{k=1}^d D(A_k)\right)_{\theta,p}. \tag{B.1}$$

Proof. Recall that

$$\left(E, \bigcap_{k=1}^{d} D(A_k)\right)_{\theta,p} = \bigcap_{k=1}^{d} D_{A_k}(\theta, p).$$

Since we have

$$D(A) \supset \bigcap_{k=1}^{d} D(A_k),$$

due to Grisvard [11] we have

$$D_A(\theta, p) = \big(E, D(A)\big)_{\theta,p} \supset \bigcap_{k=1}^{d} \big(E, D(A_k)\big)_{\theta,p} = \bigcap_{k=1}^{d} D_{A_k}(\theta, p),$$

so that one inclusion in (B.1) is proved.

The proof of the opposite inclusion is more delicate. We give it in the case $d = 2$. The general case $d \geqslant 2$ follows by similar arguments. As in [9, p. 325] we set

$$D_{A_k}(1, p) := \big(E, D\big(A_k^2\big)\big)_{\frac{1}{2},p}.$$

From [9, Lemma 3.9] we have

$$D(A) \subset D_{A_1}(1, \infty) \cap D_{A_2}(1, \infty) \subset D_{A_1}(1, p) \cap D_{A_2}(1, p),$$

for any $\theta \in (0, 1)$ and $p \in [1, \infty]$, so that for $k = 1, 2$

$$D_A(\theta, \infty) \subset \big(E, D_{A_k}(1, \infty)\big)_{\theta,\infty}.$$

Hence, due to the Reiteration Theorem (see [14, Theorem 1.2.15]) for $k = 1, 2$ we obtain

$$D_A(\theta, \infty) \subset \big(E, D(A_k)\big)_{\theta,\infty}.$$

By using again [11] this yields

$$D_A(\theta, \infty) \subset \big(E, D(A_1)\big)_{\theta,\infty} \cap \big(E, D(A_2)\big)_{\theta,\infty} = \big(E, D(A_1) \cap D(A_2)\big)_{\theta,\infty}. \qquad \square$$

In what follows we shall denote by X the space $D_A(\theta, \infty)$, equipped with the corresponding norm

$$|u|_X := |u|_E + [u]_X := |u|_E + \sum_{k=1}^{d} [u]_{D_{A_k}(\theta,\infty)}. \tag{B.2}$$

Note that, in view of Lemma B.1

$$X = \bigcap_{k=1}^{d} D_{A_k}(\theta, \infty). \tag{B.3}$$

Furthermore we shall denote by Y the space

$$Y := \bigcap_{k=1}^{d} \bigl\{u \in D(A_k)\colon\ A_k u \in D_{A_k}(\theta, \infty)\bigr\}, \tag{B.4}$$

equipped with the norm

$$|u|_Y := |u|_X + \sum_{k=1}^{d} |A_k u|_X.$$

Note that due to Lemma B.1

$$Y \subset \bigcap_{k=1}^{d} D(A_k) \subset D(A) \subset X \subset E,$$

with continuous embeddings and, if we endow $D(A)$ with the graph norm, for any $\lambda > 0$ we have

$$\lambda - A : D(A) \to E,$$

is an isomorphism.

Now we can state the following abstract Schauder theorem.

Theorem B.2. *Let $\{S_k(t)\}_{k=1}^d$ be a family of d commuting C_0-contraction analytic semigroups on a Banach space E with corresponding generators $\{A_k\}_{k=1}^d$. Assume that there exists $M > 0$ such that for any $k = 1, \ldots, d$*

$$\sup_{t>0} t \bigl\| A_k S_k(t) \bigr\|_{\mathcal{L}(E)} \leqslant M. \tag{B.5}$$

Then, for any $\lambda > 0$ the operator $\lambda - A : Y \to X$ is an isomorphism, for X and Y defined respectively in (B.3) *and* (B.4)*. Moreover, for any $k = 1, \ldots, d$*

$$\bigl[A_k(\lambda - A)^{-1} f\bigr]_X \leqslant K[f]_X, \tag{B.6}$$

where

$$K := M^2 c(\theta)(3 + d) \int_0^\infty \frac{\sigma^{\theta-1}}{1+\sigma}\, d\sigma.$$

Remark B.3. The importance of the theorem above concerning the equation

$$\lambda u = \sum_{k=1}^{d} A_k u + f, \quad u \in \bigcap_{k=1}^{d} D(A_k), \tag{B.7}$$

is twofold. First, it provides a specific subspace $X \subset E$ of data f for which Eq. (B.7) is solvable in a strict sense.

Secondly, the theorem above provides the important a priori estimate (B.6) which will be used in an essential way in the proof of the main result. We emphasize that we have a precise control of the constant K in terms of the dimension d, of the Hölder exponent θ and of the constant M which we have introduced in (B.5).

Proof of Theorem B.2. A proof of this result, in a more general formulation, was given by Grisvard in [11], by using complex methods. Here we give a proof of this special case by using a real method which is due to Da Prato (unpublished lecture notes) and, for the sake of simplicity, we only consider the case $p = \infty$.

For any $f \in E$, we define

$$u := \int_0^\infty e^{-\lambda s} \prod_{k=1}^d S_k(s) f \, ds.$$

In order to prove our result, it is sufficient to show that the following conditions are satisfied:

(1) if $f \in D_{A_k}(\theta, \infty)$, for some $\theta \in (0,1)$ and $k = 1, \dots, d$, then $u \in D(A_k)$ and

$$A_k u \in \bigcap_{j=1}^d D_{A_j}(\theta, \infty); \tag{B.8}$$

(2) if $f \in X$, then

$$\lambda u = \sum_{j=1}^d A_k u + f \tag{B.9}$$

and $A_k u \in X$, for any $k = 1, \dots, d$;
(3) if $f = 0$, then $u = 0$ is the only solution to (B.9) in the space $\bigcap_{j=1}^d D(A_j)$;
(4) if $f \in X$, then (B.6) holds.

Let $f \in D_{A_k}(\theta, \infty)$, for some $\theta \in (0,1)$ and $k = 1, \dots, d$. For each $n \geqslant 1$ we define

$$u_n := \int_{1/n}^\infty e^{-\lambda s} \prod_{h=1}^d S_h(s) f \, ds.$$

We have $u_n \in D(A_k)$ and

$$A_k u_n = \int_{1/n}^\infty e^{-\lambda s} A_k S_k(s) \prod_{\substack{h=1 \\ h \neq k}}^d S_h(s) f \, ds.$$

If we show that

$$\int_0^\infty e^{-\lambda s} \left| A_k S_k(s) \prod_{\substack{h=1\\h\neq k}}^d S_h(s) f \right|_E ds < \infty, \tag{B.10}$$

we have that

$$\lim_{n\to\infty} A_k u_n = \int_0^\infty e^{-\lambda s} A_k S_k(s) \prod_{\substack{h=1\\h\neq k}}^d S_h(s) f\, ds,$$

and then, as u_n converges to u, by the closedness of A_k we conclude that $u \in D(A_k)$ and

$$A_k u = \int_0^\infty e^{-\lambda s} A_k S_k(s) \prod_{\substack{h=1\\h\neq k}}^d S_h(s) f\, ds.$$

The semigroups $S_h(t)$ are all commuting and are contractions, then, thanks to (B.5), we have

$$\int_0^\infty e^{-\lambda s} \left| A_k S_k(s) \prod_{\substack{h=1\\h\neq k}}^d S_h(s) f \right|_E ds \leqslant \int_0^\infty e^{-\lambda s} \left\| \prod_{\substack{h=1\\h\neq k}}^d S_h(s) \right\|_{\mathcal{L}(E)} \left| A_k S_k(s) f \right|_E ds$$

$$\leqslant M \int_0^\infty e^{-\lambda s} s^{\theta-1}\, ds |f|_{D_{A_k}(\theta,\infty)} = M\Gamma(\theta)\lambda^{-\theta} |f|_{D_{A_k}(\theta,\infty)}.$$

This implies (B.10). Moreover, we have

$$|A_k u|_E \leqslant M\Gamma(\theta)\lambda^{-\theta} |f|_{D_{A_k}(\theta,\infty)}. \tag{B.11}$$

Next, we show that (B.8) holds. In fact we prove something more. Namely, we show that there exists $M = M(\theta) > 0$ such that

$$\sup_{t\in(0,1]} t^{1-\theta} \left| A_j S_j(t) A_k u \right|_E \leqslant 4M^2 c(\theta), \quad j = 1, \dots, d, \tag{B.12}$$

where

$$c(\theta) = \int_0^\infty \frac{\sigma^{\theta-1}}{1+\sigma}\, d\sigma.$$

Assume first that $j \neq k$. By using again (B.5), for any $t \in (0, 1]$ we have

$$\begin{aligned}t^{1-\theta}\big|A_j S_j(t) A_k u\big|_E &\leqslant t^{1-\theta}\int_0^\infty e^{-\lambda s}\Bigg|A_j S_j(t+s)\prod_{\substack{h=1\\ h\neq k,j}}^d S_h(s) A_k S_k(s) f\Bigg|_E\,ds\\ &\leqslant M^2 t^{1-\theta}\int_0^\infty e^{-\lambda s}\frac{1}{t+s}s^{\theta-1}\,ds|f|_{D_{A_k}(\theta,\infty)}\\ &\leqslant M^2 t^{1-\theta}\int_0^\infty \frac{1}{t+s}s^{\theta-1}\,ds|f|_{D_{A_k}(\theta,\infty)}.\end{aligned}$$

Now, it is immediate to check that

$$t^{1-\theta}\int_0^\infty \frac{1}{t+s}s^{\theta-1}\,ds=\int_0^\infty \frac{\sigma^{\theta-1}}{1+\sigma}\,d\sigma<\infty,$$

so that (B.12) follows. Now, assume $j=k$. In this case we have

$$\begin{aligned}t^{1-\theta}\big|A_k S_k(t) A_k u\big|_E &\leqslant t^{1-\theta}\int_0^\infty e^{-\lambda s}\Bigg|A_k^2 S_k(t)\prod_{h=1}^d S_h(s) f\Bigg|_E\,ds\\ &\leqslant t^{1-\theta}\int_0^\infty e^{-\lambda s}\Bigg|S_k(t/2)\prod_{\substack{h=1\\ h\neq k}}^d S_h(s) A_k S_k\big((t+s)/2\big) A_k S_k(s/2) f\Bigg|_E\,ds\\ &\leqslant 4M^2 t^{1-\theta}\int_0^\infty e^{-\lambda s}\frac{1}{t+s}s^{\theta-1}\,ds|f|_{D_{A_k}(\theta,\infty)},\end{aligned}$$

and then we can conclude as above. This means that

$$\sum_{j=1}^d \sup_{t>0} t^{1-\theta}\big|A_j S_j(t) A_k u\big|_E \leqslant M^2(3+d)c(\theta)|f|_{D_{A_k}(\theta,\infty)}. \tag{B.13}$$

Therefore, collecting all terms, from (B.11) and (B.13) we have

$$|A_k u|_X \leqslant \big(M\Gamma(\theta)\lambda^{-\theta}+M^2(3+d)c(\theta)\big)|f|_{D_{A_k}(\theta,\infty)}. \tag{B.14}$$

Next, assume $f\in X$. We have

$$\sum_{k=1}^d A_k u=\int_0^\infty e^{-\lambda s}\sum_{k=1}^d A_k\prod_{h=1}^d S_h(s) f\,ds$$

$$= \int_0^\infty e^{-\lambda s} \frac{d}{ds}\left(\prod_{h=1}^d S_h(s) f\right) ds$$

$$= -f - \int_0^\infty \frac{d}{ds}\left[e^{-\lambda s} \prod_{h=1}^d S_h(s) f\right] ds = -f + \lambda u,$$

and this proves (B.9).

Next, let us prove uniqueness. The operators $A_1, \dots, A_d$ are dissipative and densely defined, hence strongly dissipative, in the sense of [18]. It follows that the operator $A_1 + \cdots + A_d$ defined on $\bigcup_{k=1}^d D(A_k)$ is also dissipative. This implies that if u satisfies (B.9), then

$$|u|_E \leqslant \frac{1}{\lambda} |f|_E,$$

which implies uniqueness.

Finally, (B.6) follows from (B.14) and the previous points. □

Appendix C

With the notations of Appendix A, for any $k = 1, \dots, d$ we denote by J_k the mapping from $C([a,b]^d)$ into $C([a,b]; C([a,b]^{d-1}))$ defined by

$$\big[J_k u(t)\big](x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_d).$$

We have the following characterization of the space $C^\delta([a,b]^d)$. The proof is straightforward.

Lemma C.1. *For any $a \leqslant b$ and $\delta \in (0,1)$*

$$C^\delta\big([a,b]^d\big) = \bigcap_{k=1}^d \big\{u \in C\big([a,b]^d\big):\ J_k u \in C^\delta\big([a,b]; C\big([a,b]^{d-1}\big)\big)\big\},$$

and

$$|u|_\delta \approx |u|_0 + \sum_{k=1}^d [J_k u]_\delta.$$

Next we give a characterization of $C^{2+\delta}_{\mathcal{N}}([a,b]^d)$, the space of $C^{2+\delta}([a,b]^d)$-functions, endowed with Neumann boundary conditions.

Lemma C.2. *For any $a \leqslant b$ and $\delta \in (0,1)$*

$$\big\{u \in C^{2+\delta}\big([a,b]^d\big):\ D_k u_{|_{x_k=a}} = D_k u_{|_{x_k=b}} = 0\big\}$$
$$= \bigcap_{k=1}^d \big\{u \in C\big([a,b]^d\big):\ J_k u \in C^{2+\delta}\big([a,b]; C\big([a,b]^{d-1}\big)\big),\ (J_k u)'(a) = (J_k u)'(b) = 0\big\},$$

and

$$|u|_{2+\delta} \approx |u|_0 + \sum_{k=1}^{d} \big[D_k^2 J_k u\big]_\delta.$$

Proof. If $u \in C([a,b]^d)$ is such that $J_k u \in C^{2+\delta}([a,b]; C([a,b]^{d-1}))$, for any $k = 1, \dots, d$, and $(J_k u)'(a) = (J_k u)'(b) = 0$, due to the conditions on the derivative of $J_k u$ we can extend it by evenness and then by $2(b-a)$ periodicity to the whole space $\mathbb{R}^d$. Then, as proved for example in [12, Theorem 3.4.1], for any $h, l = 1, \dots, d$

$$\big[D_{kl}^2 u\big]_\delta \leqslant c_\delta \sum_{k=1}^{d} \big|D_k^2 u\big|_\delta,$$

and this implies that $u \in C^{2+\delta}([a,b]^d)$. □

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