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Schauder theorems for Ornstein-Uhlenbeck equations in infinite dimension

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Abstract

We prove Schauder type estimates for stationary and evolution equations driven by the classical Ornstein-Uhlenbeck operator in a separable Banach space, endowed with a centered Gaussian measure.

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1. Introduction

Let X be a separable Banach space, endowed with a centered Gaussian measure γ, and let $H \subset X$ be the corresponding Cameron-Martin space. In this context, an important differential operator that plays a central role in the Malliavin Calculus is the classical Ornstein-Uhlenbeck operator,

$$\mathcal{L}u = \mathrm{div}_\gamma \nabla_H u,$$

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where div_γ is the Gaussian divergence and ∇_H is the gradient along H. It plays the role played by the Laplacian with respect to the Lebesgue measure in $\mathbb{R}^d$, being the operator associated to the quadratic Dirichlet form

$$(u,v) \mapsto \int_X \langle \nabla_H u, \nabla_H v\rangle_H \, d\gamma, \quad u,\ v \in W^{1,2}(X,\gamma).$$

The corresponding Markov semigroup is explicitly represented by

$$T(t)f(x) = \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}\,y)\gamma(dy), \quad t>0,\ f \in C_b(X).$$

Its realization T_p in $L^p(X,\gamma)$ is a contraction, strongly continuous semigroup for every $p \in [1,+\infty)$, and it is analytic if $p>1$. In the latter case, the well known Meyer estimates imply that the domain of its infinitesimal generator L_p coincides with the Sobolev space $W^{2,p}(X,\gamma)$. In particular, for every $\lambda > 0$ and $f \in L^p(X,\gamma)$, the equation

$$\lambda u - \mathcal{L}u = f \tag{1.1}$$

has a unique solution $u \in W^{2,p}(X,\gamma)$, and $\|u\|_{W^{2,p}(X,\gamma)} \le C\|f\|_{L^p(X,\gamma)}$, with C independent of f. See e.g. [2, Ch. 5] for a survey on Sobolev spaces with respect to Gaussian measures, and on the operators L_p.

Here we consider a realization L of $\mathcal{L}$ in the space $C_b(X)$ of the continuous and bounded functions from X to $\mathbb{R}$, whose resolvent $R(\lambda, L)$ is given, for $\lambda>0$, by

$$R(\lambda,L)f(x) = \int_0^\infty e^{-\lambda t} T(t)f(x)\,dt, \quad f \in C_b(X).$$

The realizations of elliptic differential operators in spaces of continuous functions exhibit typical difficulties. Even in finite dimension, the solution of (1.1) does not belong to $C^2(\mathbb{R}^d)$ for general $f \in C_b(\mathbb{R}^d)$, while Schauder theorems are available both for non-degenerate ([7]) and for degenerate hypoelliptic ([8]) Ornstein-Uhlenbeck operators.

In our general setting we prove Schauder type regularity results for the solution to (1.1), that are the Hölder counterpart of the above mentioned maximal L^p regularity results. The appropriate Hölder spaces (as well as the Sobolev spaces $W^{k,p}(X,\gamma)$) have to be chosen according to the structure of $\mathcal{L}$: indeed, it is well known that $T(t)$ and $R(\lambda,L)$ are smoothing operators along the directions of the Cameron-Martin space H only. So, we use Hölder spaces along H, defined as

$$C^\alpha_H(X,Y) := \left\{ f \in C_b(X,Y) :\ [f]_{C^\alpha_H(X,Y)} := \sup_{x\in X,\, h\in H\setminus\{0\}} \frac{\|f(x+h)-f(x)\|_Y}{\|h\|_H^\alpha} < +\infty \right\},$$
$$\|f\|_{C^\alpha_H(X,Y)} := \sup_{x\in X} \|f(x)\|_Y + [f]_{C^\alpha_H(X,Y)},$$

for $\alpha \in (0,1)$ and for any Banach space Y. We prove that for every $f \in C^\alpha_H(X,\mathbb{R})$ and for every $\lambda>0$, the unique solution $u \in D(L)$ of (1.1) belongs to $C^{2+\alpha}_H(X,\mathbb{R})$. This means that u

is twice continuously differentiable along H, it has bounded and continuous H-gradient $\nabla_H u$ and H-Hessian operator $D^2_H u$, with values respectively in H and in the space of the bilinear quadratic forms $\mathcal{L}^{(2)}(H)$, and $x \mapsto D^2_H u(x)$ belongs to $C^\alpha_H(X, \mathcal{L}^{(2)}(H))$. Consequently, all the second order directional derivatives $\partial^2 u/\partial h \partial k$ with $h, k \in H$ exist and belong to $C^\alpha_H(X, \mathbb{R})$.

In the case that $f \in C_b(X)$ only, we prove that u has bounded and continuous H-gradient $\nabla_H u$, such that

$$\sup_{x\in X,\, h\in H\setminus\{0\}} \frac{\|\nabla_H u(x+2h) - 2\nabla_H u(x+h) + \nabla_H u(x)\|_H}{\|h\|_H} < +\infty,$$

namely $\nabla_H u$ satisfies a Zygmund condition along H. This is an infinite dimensional counterpart of the Zygmund regularity of the gradients of solutions to elliptic differential equations in finite dimension.

Schauder type regularity results are proved also for the mild solution to the Cauchy problem

$$\begin{cases} v_t(t,x) = Lv(t,x) + g(t,x), & t \in [0,T],\ x \in X, \\ v(0,\cdot) = f, \end{cases} \tag{1.2}$$

namely for the function

$$v(t,x) = T(t)f(x) + \int_0^t T(t-s)g(s,\cdot)(x)ds, \quad t \in [0,T],\ x \in X, \tag{1.3}$$

when $f \in C^{2+\alpha}_H(X,\mathbb{R})$ and $g \in C_b([0,T]\times X;\mathbb{R})$ such that $\sup_{t\in[0,T]}[g(t,\cdot)]_{C^\alpha_H(X;\mathbb{R})} < +\infty$. However, while in finite dimension with non-degenerate γ the function v defined by (1.3) is a classical solution to (1.2), in infinite dimension it is not differentiable with respect to t in general, even if $g \equiv 0$.

Our main interest is in the infinite dimensional case. However, if $X = \mathbb{R}^n$ the operator $\mathcal{L}$ reads as

$$\mathcal{L}u(x) = \mathrm{Trace}\,[QD^2u(x)] - \langle x, \nabla u(x)\rangle$$

where $Q \geq 0$ is the covariance matrix of γ. If $Q > 0$, namely if γ is non-degenerate, our results are contained in [7,8]. If Q is not invertible the operator $\mathcal{L}$ is not hypoelliptic, and this paper provides new Hölder and Zygmund regularity results along the directions of the range of Q.

In infinite dimension, Schauder regularity results for elliptic equations driven by Ornstein-Uhlenbeck operators are already available in the case that X is a Hilbert space, γ is non-degenerate, and the corresponding Ornstein-Uhlenbeck semigroup is smoothing in all directions ([3,1,5]). Still in the case that X is a Hilbert space and γ is non-degenerate, Schauder regularity results for elliptic equations driven by the Gross Laplacian and some of its perturbations are also available ([4,1]). See section 4 for details.

2. Notation and preliminaries

Throughout the paper we use notations, definitions and results of [2] concerning Gaussian measures in Banach spaces.

We consider a separable Banach space X endowed with a centered Gaussian measure γ, and we denote by H the corresponding Cameron-Martin space. It consists of the elements $h \in X$ such that

$$\|h\|_H := \sup\{f(h) : \ f \in X^*, \ \|f\|_{L^2(X,\gamma)} \le 1\} < +\infty$$

and it is isometric to the closure of X^* in $L^2(X, \gamma)$, denoted by X^*_γ. The isometry $R_\gamma : X^*_\gamma \mapsto H$ is defined as follows: $R_\gamma f$ is the unique $h \in X$ such that $\int_X f(x)g(x)\gamma(dx) = g(h)$, for every $g \in X^*$. For every $h \in H$, $R_\gamma^{-1}h$ is usually denoted by $\hat{h}$.

We recall the Cameron-Martin formula: for every $h \in H$, the translated measure $\gamma_h(B) := \gamma(B - h)$ is absolutely continuous with respect to γ, with density $\rho(x) = \exp \hat{h}(x) - \|h\|_H^2/2$. So, for every continuous and bounded φ we have

$$\int_X \varphi(x+h)\gamma(dx) = \int_X \varphi(x)e^{\hat{h}(x)-\|h\|_H^2/2}\gamma(dx). \tag{2.1}$$

We also recall that for every $h \in H$, the function $\hat{h}$ is a Gaussian random variable with law $\mathcal{N}(0, \|h\|_H^2)$ in $\mathbb{R}$. Therefore, for every $p \in [1, +\infty)$ we have

$$\|\hat{h}\|_{L^p(X,\gamma)} = \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\xi|^p \exp(-\xi^2/2) d\xi \right)^{1/p} \|h\|_H := k_p \|h\|_H. \tag{2.2}$$

We recall that if X is a Hilbert space and Q is the covariance of γ, then H is just the range of $Q^{1/2}$, and the norm $h \mapsto \|Q^{-1/2}h\|$ (were $Q^{-1/2}$ is the pseudo-inverse of $Q^{1/2}$, defined on $Q^{1/2}(X)$) coincides with the norm of H. Moreover, the functions $\hat{h}$ have a simple representation formula. Denoting by $\{e_j : \ j \in \mathbb{N}\}$ any orthonormal basis of X consisting of eigenvectors of Q, $Qe_j = \lambda_j e_j$, we have $\hat{h}(x) = \sum_{j\in I} \langle x, e_j\rangle \langle h, e_j\rangle \lambda_j^{-1}$, where $\langle \cdot, \cdot \rangle$ is the scalar product in X and $I = \{j \in \mathbb{N} : \lambda_j \neq 0\}$. A function $f : X \mapsto \mathbb{R}$ is called H-differentiable at $x \in X$ if there exists a (unique) linear bounded operator $\ell : H \mapsto \mathbb{R}$ such that

$$\lim_{\|h\|_H \to 0} \frac{f(x+h) - f(x) - \ell(h)}{\|h\|_H} = 0.$$

We set $\ell := D_H f(x)$. Since H is a Hilbert space, there exists a unique $y \in H$ such that $D_H f(x)(h) = \langle h, y\rangle_H$. Such y is denoted by $\nabla_H f(x)$.

Since H is continuously embedded in X, if f is Frechét differentiable at x it is also H-differentiable at x, and $f'(x)(h) = \langle \nabla_H f(x), h\rangle_H$, for every $h \in H$. In particular, if X is a Hilbert space, γ has covariance Q, and $\nabla f(x)$ is the gradient of f at x, we have $\nabla_H f(x) = Q\nabla f(x)$ if Q is one to one, namely if γ is non-degenerate.

More generally, if Y is a Banach space, a function $F : X \mapsto Y$ is called H-differentiable at $x \in X$ if there exists a linear bounded operator $L : H \mapsto Y$ such that

$$Y - \lim_{\|h\|_H \to 0} \frac{F(x+h) - F(x) - L(h)}{\|h\|_H} = 0.$$

n times H-differentiable functions are defined by recurrence, in a canonical way. Here we are interested in $n = 2, 3$. So, if $f : X \mapsto \mathbb{R}$ is H-differentiable in X, we say that it is twice H-differentiable at x if $D_H f : X \mapsto H'$ is differentiable at x, (equivalently, $\nabla_H f : X \mapsto H$ is differentiable at x) and we define the Hessian operator $D_H^2 f(x) \in \mathcal{L}^{(2)}(H)$ (the space of the bounded bilinear forms from H^2 to $\mathbb{R}$), by $D_H^2 f(x)(k, h) := (Lh)(k)$, where L is the operator in the definition, with $F(x) = D_H f(x)$, $Y = H'$. Similarly, if f is twice H-differentiable in X, we say that it is thrice H-differentiable at x if $D_H^2 f : X \mapsto \mathcal{L}^{(2)}(H)$ is H-differentiable at x; in this case the third order derivative $D_H^3 f(x) \in \mathcal{L}^{(3)}(H)$ is defined as $D_H^3 f(x)(h, k, l) := (Lh)(k, l)$, where L is the operator in the definition, with now $F(x) = D_H^2 f(x)$, $Y = \mathcal{L}^{(2)}(H)$.

Definition 2.1. For $k \in \mathbb{N}$ we denote by $C_H^k(X)$ the subspace of $C_b(X)$ consisting of functions k times H-differentiable at any point, with $D_H^j f$ continuous and bounded in $\mathcal{L}^{(j)}(H)$ for $j \le k$. $C_H^k(X)$ is endowed with the norm

$$\|f\|_{C_H^k(X)} := \sup_{x \in X} |f(x)| + \sum_{j=1}^{k} \sup_{x \in X} \|D_H^j f(x)\|_{\mathcal{L}^{(j)}(H)}.$$

The Ornstein–Uhlenbeck semigroup is defined by

$$T(t)f(x) := \int_X f(e^{-t}x + \sqrt{1 - e^{-2t}}\, y)\gamma(dy), \quad t > 0,\ f \in C_b(X). \tag{2.3}$$

Then $T(t)$ maps $C_b(X)$ into itself for every $t > 0$, and

$$\|T(t)f\|_\infty \le \|f\|_\infty, \quad t > 0,\ f \in C_b(X). \tag{2.4}$$

Nevertheless, $T(t)$ is not strongly continuous in $C_b(X)$, and not even in the subspace $BUC(X)$ of the bounded and uniformly continuous functions. Indeed, for $f \in BUC(X)$ it is easy to see that

$$\lim_{t \to 0^+} \|T(t)f - f\|_\infty = 0 \Longleftrightarrow \lim_{t \to 0^+} \|f(e^{-t}\cdot) - f\|_\infty = 0.$$

However, for every fixed $x \in X$ the function $t \mapsto T(t)f(x)$ is continuous in $[0, +\infty)$ by the Dominated Convergence Theorem. It follows that for every $\lambda > 0$ the linear operator $F(\lambda)$ defined by

$$F(\lambda)f(x) := \int_0^{+\infty} e^{-\lambda t} T(t)f(x)\, dt, \quad \lambda > 0,\ f \in C_b(X),\ x \in X,$$

belongs to $\mathcal{L}(C_b(X))$ and it is one to one. Moreover, since $T(t)$ is a semigroup, the family $\{F(\lambda) : \lambda > 0\}$ satisfies the resolvent identity. Therefore there exists a linear operator $L : D(L) \mapsto X$ such that $F(\lambda) = R(\lambda, L)$ for every $\lambda > 0$.

The operator L is called *generator* of $T(t)$ in $C_b(X)$, although it is not an infinitesimal generator in the usual sense. So, as in the case of strongly continuous semigroups, we have

$$(R(\lambda, L)f)(x) = \int_0^{+\infty} e^{-\lambda t} T(t) f(x)\, dt, \quad \lambda > 0,\ f \in C_b(X),\ x \in X, \tag{2.5}$$

and by (2.4) we obtain

$$\|R(\lambda, L)f\|_\infty \le \|f\|_\infty, \quad \lambda > 0,\ f \in C_b(X). \tag{2.6}$$

Let us recall that the realization $T_p(t)$ of $T(t)$ in $L^p(X, \gamma)$ is a strongly continuous, contraction, analytic semigroup, for every $p \in (1, +\infty)$. The domain of its infinitesimal generator L_p is equal to the Sobolev space $W^{2,p}(X, \gamma)$, and the graph norm of L_p is equivalent to the Sobolev norm. Moreover,

$$L_p u = \mathrm{div}_\gamma \nabla_H u = \sum_{j=1}^{\infty} \left(\frac{\partial}{\partial h_j} - \hat{h}_j \right) \frac{\partial u}{\partial h_j},$$

where div_γ is the Gaussian divergence, $\{h_j : j \in \mathbb{N}\}$ is any orthonormal basis of H, and the series converges in $L^p(X, \gamma)$. See e.g. [2, Ch. 5]. If X is a Hilbert space, γ is a non-degenerate Gaussian measure with covariance Q, and $\{e_j : j \in \mathbb{N}\}$ is any orthonormal basis of X consisting of eigenvectors of Q, $Qe_j = \lambda_j e_j$, then $\{\sqrt{\lambda_j} e_j : j \in \mathbb{N}\}$ is an orthonormal basis of H and the above series reads as

$$L_p u(x) = \sum_{j=1}^{\infty} \left(\lambda_j \frac{\partial^2 u}{\partial e_j^2}(x) - x_j \frac{\partial u}{\partial e_j}(x) \right),$$

where $x_j := \langle x, e_j \rangle$.

Using the characterizations $D(L_p) = W^{2,p}(X, \gamma)$ for $p > 1$, we obtain a characterization of $D(L)$, as follows.

Lemma 2.2.

$$D(L) = \{u \in \bigcap_{p>1} W^{2,p}(X, \gamma) : u,\ \mathcal{L}u \in C_b(X)\} = \{u \in \bigcup_{p>1} W^{2,p}(X, \gamma) : u,\ \mathcal{L}u \in C_b(X)\}.$$

Moreover, for every $u \in D(L)$, Lu is a continuous and bounded version of $\mathrm{div}_\gamma \nabla_H u$.

Proof. For $u \in D(L)$ and $\lambda > 0$ set $f := \lambda u - Lu$, so that u is given by (2.5). Since $T_p(t)$ agrees with $T(t)$ on $C_b(X)$ for every $p > 1$, we have $u = \int_0^{+\infty} e^{-\lambda t} T_p(t) f\, dt = R(\lambda, L_p) f$. Therefore, $u \in W^{2,p}(X, \gamma)$ for every $p > 1$ and $Lu = L_p u$, γ-a.e. So, Lu is a continuous and bounded version of $L_p u = \mathrm{div}_\gamma \nabla_H u$.

Conversely, if $u \in W^{2,p}(X, \gamma)$ for some $p > 1$ we have $u = \int_0^{+\infty} e^{-\lambda t} T_p(t)(\lambda u - L_p u)\, dt$ for every $\lambda > 0$. If u, $\mathcal{L}u = L_p u \in C_b(X)$ we obtain $u = R(\lambda, L) f$, with $f = \lambda u - \mathcal{L}u$, and therefore $u \in D(L)$. □

The following smoothing properties are easily shown.

Proposition 2.3. *For every $f \in C_b(X)$ and $t > 0$, $T(t)f$ is infinitely times H-differentiable at every $x \in X$. Setting*

$$c(t) := \frac{e^{-t}}{\sqrt{1-e^{-2t}}}, \quad t > 0,$$

we have

$$D_H T(t)f(x)(h) = \langle \nabla_H T(t)f(x), h\rangle_H = c(t)\int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)\hat{h}(y)\gamma(dy), \tag{2.7}$$

$$D_H^2 T(t)f(x)(h,k) = c(t)^2 \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)(\hat{h}(y)\hat{k}(y) - \langle h,k\rangle_H)\gamma(dy), \tag{2.8}$$

$$\begin{aligned} D_H^3 T(t)f(x)(h,k,l) \;=\; & -c(t)^3 \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)(\hat{l}(y)\langle h,k\rangle_H \\ & +\hat{h}(y)\langle k,l\rangle_H + \hat{k}(y)\langle h,l\rangle_H)\gamma(dy) \\ & +c(t)^3 \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)\hat{h}(y)\hat{k}(y)\hat{l}(y)\,\gamma(dy), \end{aligned} \tag{2.9}$$

for every h, k, $l \in H$. The function $(t,x) \mapsto T(t)f(x)$ is continuous in $[0,+\infty) \times X$, and the functions $(t,x) \mapsto D_H^j T(t)f(x)$ ($j = 1,2,3$) are continuous in $(0,+\infty) \times X$, with values in $\mathcal{L}^{(j)}(X)$, respectively. Moreover, for every $x \in X$ and $t > 0$ we have

$$\begin{aligned} &(i) \quad |\nabla_H T(t)f(x)|_H \le c(t)\|f\|_\infty, \\ &(ii) \quad \|D_H^2 T(t)f(x)\|_{\mathcal{L}^{(2)}(H)} \le 2c(t)^2\|f\|_\infty, \\ &(iii) \quad \|D_H^3 T(t)f(x)\|_{\mathcal{L}^{(3)}(H)} \le (3+k_3^3)c(t)^3\|f\|_\infty. \end{aligned} \tag{2.10}$$

Proof. Formulae (2.7), (2.8), (2.9) are easily proved using the Cameron-Martin formula. For instance concerning (2.7), using (2.1) we get

$$T(t)f(x+h) - T(t)f(x) = \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)[\exp(c(t)\hat{h}(y) - c(t)^2\|h\|_H^2/2) - 1]\,\gamma(dy)$$

which yields (2.7). (2.8), (2.9) are proved in the same way. Estimates (2.10) are consequence of (2.7), (2.8), (2.9) through the Hölder inequality and (2.2) (in particular, the constant k_3^3 in the right hand side of (2.10) comes from estimating $\|\hat{h}\hat{k}\hat{l}\|_{L^1(X,\gamma)} \le \|\hat{h}\|_{L^3(X,\gamma)}\|\hat{k}\|_{L^3(X,\gamma)}\|\hat{l}\|_{L^3(X,\gamma)}$). Also the continuity of $(t,x) \mapsto T(t)f(x)$ in $[0,+\infty) \times X$ and of $(t,x) \mapsto D_H^j T(t)f(x)$ in $(0,+\infty) \times X$ for $j = 1,2,3$ is a consequence of the respective representation formulae, through the Dominated Convergence Theorem and the Hölder inequality. □

For functions in $C_H^1(X)$ the estimates in (2.10) may be improved. The proof is similar, and it is omitted.

Proposition 2.4. *For every $f \in C^1_H(X)$, for any $t \geq 0$, and for every $x \in X$ we have*

$$\langle \nabla_H T(t)f(x), h\rangle_H = e^{-t}\int_X \langle \nabla_H f(e^{-t}x + \sqrt{1-e^{-2t}}y), h\rangle_H\, \gamma(dy), \tag{2.11}$$

$$\langle D^2_H T(t)f(x)(h,k) = e^{-t}\int_X \langle \nabla_H f(e^{-t}x + \sqrt{1-e^{-2t}}y), h\rangle_H \hat{k}(y)\, \gamma(dy), \tag{2.12}$$

$$\langle D^3_H T(t)f(x)(h,k,l) = e^{-t}\int_X \langle \nabla_H f(e^{-t}x + \sqrt{1-e^{-2t}}y), h\rangle_H (\hat{k}(y)\hat{l}(y) - \langle k,l\rangle_H)\, \gamma(dy). \tag{2.13}$$

The function $(t,x) \mapsto \nabla_H T(t)f(x)$ is continuous in $[0,+\infty) \times X$ with values in H, and for every $x \in X$ and $t > 0$ we have

$$\begin{aligned} &(i) && \|\nabla_H T(t)f(x)\|_H \leq \|\nabla_H f\|_\infty, \\ &(ii) && \|D^2_H T(t)f(x)\|_{\mathcal{L}^{(2)}(H)} \leq c(t)\|\nabla_H f\|_\infty, \\ &(iii) && \|D^3_H T(t)f(x)\|_{\mathcal{L}^{(3)}(H)} \leq 2c(t)^2\|\nabla_H f\|_\infty. \end{aligned} \tag{2.14}$$

3. Hölder spaces and Schauder type theorems

We introduce a class of Hölder spaces that arise “naturally” in this setting.

Definition 3.1. If Y is any Banach space and $\alpha \in (0,1)$, the space $C^\alpha_H(X,Y)$ is the subspace of $C_b(X,Y)$ consisting of the functions F such that

$$[F]_\alpha := \sup_{h\in H\setminus\{0\},\, x\in X} \frac{\|F(x+h) - F(x)\|_Y}{\|h\|^\alpha_H} < +\infty.$$

$C^\alpha_H(X,Y)$ is normed by

$$\|F\|_{C^\alpha_H(X,Y)} := \|F\|_\infty + [F]_\alpha.$$

If $Y = \mathbb{R}$ the space $C^\alpha_H(X,\mathbb{R})$ is denoted by $C^\alpha_H(X)$. Moreover, we denote by $C^{1+\alpha}_H(X)$, $C^{2+\alpha}_H(X)$ the subspaces of $C^1_H(X)$, $C^2_H(X)$, consisting of functions f such that $D_H f \in C^\alpha_H(X,\mathcal{L}(H))$, $D^2_H f \in C^\alpha_H(X,\mathcal{L}^{(2)}(H))$, respectively. They are endowed with the norms

$$\begin{aligned} \|f\|_{C^{1+\alpha}_H(X)} &:= \|f\|_{C^1_H(X)} + [D_H f]_\alpha \\ &= \|f\|_{C^1_H(X)} + \sup_{h\in H\setminus\{0\},\, x\in X} \frac{\|D_H f(x+h) - D_H f(x)\|_{\mathcal{L}(H)}}{\|h\|^\alpha_H} \\ \|f\|_{C^{2+\alpha}_H(X)} &:= \|f\|_{C^2_H(X)} + [D^2_H f]_\alpha \\ &= \|f\|_{C^2_H(X)} + \sup_{h\in H\setminus\{0\},\, x\in X} \frac{\|D^2_H f(x+h) - D^2_H f(x)\|_{\mathcal{L}^{(2)}(H)}}{\|h\|^\alpha_H} \end{aligned}$$

Notice that if X is a Hilbert space and γ is non-degenerate, the Hölder seminorm $[F]_\alpha$ of any $F : X \mapsto Y$ is given by

$$[F]_\alpha = \sup_{z \in X\setminus\{0\},\, x\in X} \frac{\|F(x+Q^{1/2}z) - F(x)\|_Y}{\|z\|^\alpha}$$

where Q is the covariance of γ.

The behavior of the semigroup $T(t)$ in the space $C^{k+\alpha}_H(X)$, $k = 0, 1, 2$, is similar to the one in $C_b(X)$. Below, we just state the properties that will be used in the sequel.

Lemma 3.2. *$T(t) \in \mathcal{L}(C^{k+\alpha}_H(X))$ for every $t > 0$, $k = 0, 1, 2$, $\alpha \in (0, 1)$, and there are $C_k > 0$ such that*

$$\|T(t)\|_{\mathcal{L}(C^{k+\alpha}_H(X))} \le 1, \quad t > 0. \tag{3.1}$$

Moreover, we have

$$[T(t)f]_\alpha \le e^{-\alpha t}[f]_\alpha, \quad t > 0,\ f \in C^\alpha_H(X), \tag{3.2}$$

$$[D_H T(t)f]_{C^\alpha_H(X,H')} \le e^{-\alpha t}c(t)[f]_\alpha, \quad t > 0,\ f \in C^\alpha_H(X), \tag{3.3}$$

whereas

$$[D_H T(t)f]_{C^\alpha_H(X,H')} \le 2e^{-\alpha t}c(t)^{1+\alpha}\|f\|_\infty, \quad t > 0,\ f \in C_b(X). \tag{3.4}$$

Proof. Let $t > 0$ and $f \in C^\alpha_H(X)$. For every $h \in H$ we have

$$\begin{aligned}
&|T(t)f(x+h) - T(t)f(x)| \\
&\quad = \left| \int_X [f(e^{-t}(x+h) + \sqrt{1-e^{-2t}}y) - f(e^{-t}x + \sqrt{1-e^{-2t}}y)]\gamma(dy) \right| \\
&\quad \le (e^{-t}\|h\|_H)^\alpha [f]_\alpha,
\end{aligned}$$

which yields (3.2). (3.1) follows, for $k = 0$.

If $f \in C^{1+\alpha}_H(X)$, $T(t)f \in C^1_H(X)$ by Proposition 2.4, and estimates (2.4) and (2.14)(i) yield

$$\|T(t)f\|_{C^1_H(X)} \le \|f\|_{C^1_H(X)}.$$

By (2.11), for every $t > 0$, $x \in X$ we have

$$D_H T(t)f(x) = e^{-t} \int_X D_H f(e^{-t}x + \sqrt{1-e^{-2t}}y)\gamma(dy),$$

so that for each $h, k \in H$ we have

$$
\begin{aligned}
&|(D_H T(t) f(x+h) - D_H T(t) f(x))(k)| = \\
&= e^{-t}\left|\int_X [D_H f(e^{-t}(x+h) + \sqrt{1-e^{-2t}}\,y) - D_H f(e^{-t}x + \sqrt{1-e^{-2t}}\,y)](k)\gamma(dy)\right| \\
&\leq e^{-t}(e^{-t}\|h\|_H)^\alpha [D_H f]_{C^\alpha_H(X,H')}\|k\|_H,
\end{aligned}
$$

and (3.1) follows, for $k=1$. The statement for $k=2$ is proved in the same way.

Let us prove (3.3). Let $f \in C^\alpha_H(X)$. By (2.7), for every $h, k \in H$ we have

$$
\begin{aligned}
&|(D_H T(t) f(x+h) - D_H T(t) f(x))(k)| \\
&= c(t)\left|\int_X [f(e^{-t}(x+h) + \sqrt{1-e^{-2t}}\,y) - f(e^{-t}x + \sqrt{1-e^{-2t}}\,y)]\hat{k}(y)\gamma(dy)\right| \\
&\leq c(t)(e^{-t}\|h\|_H)^\alpha \|\hat{k}\|_{L^1(X,\gamma)}[f]_\alpha
\end{aligned}
$$

and (3.3) follows, recalling that $\|\hat{k}\|_{L^1(X,\gamma)} \leq \|\hat{k}\|_{L^2(X,\gamma)} = \|k\|_H$.

Estimate (3.4) follows combining (2.10)(i)-(ii): indeed, for every $t>0$, $x \in X$, $h \in H$ we have

$$\|D_H T(t) f(x+h) - D_H T(t) f(x)\|_{H'} \leq 2c(t)\|f\|_\infty$$

by (2.10)(i), and

$$\|D_H T(t) f(x+h) - D_H T(t) f(x)\|_{H'} \leq 2c(t)^2\|h\|_H\|f\|_\infty$$

by (2.10)(ii). Therefore,

$$\|D_H T(t) f(x+h) - D_H T(t) f(x)\|_{H'} \leq (2c(t))^{1-\alpha}(2c(t)^2\|h\|_H)^\alpha\|f\|_\infty$$

and (3.4) is proved. □

The key estimates in what follows are in the next lemma.

Lemma 3.3. *For every $\alpha \in (0,1)$ there is $C_{1,\alpha} > 0$ such that*

$$\|\nabla_H T(t) f(x)\|_H \leq \frac{C_{1,\alpha}}{t^{(1-\alpha)/2}}\|f\|_{C^\alpha_H(X)}, \quad t>0,\ f \in C^\alpha_H(X),\ x \in X. \tag{3.5}$$

Consequently, there are $C_{2,\alpha}$, $C_{3,\alpha} > 0$ such that

$$
\begin{aligned}
&(i) \quad \|D^2_H T(t) f(x)\|_{\mathcal{L}^{(2)}(H)} \leq \frac{C_{2,\alpha}}{t^{1-\alpha/2}}\|f\|_{C^\alpha_H(X)}, \qquad t>0,\ f \in C^\alpha_H(X),\ x \in X, \\
&(ii) \quad \|D^3_H T(t) f(x)\|_{\mathcal{L}^{(3)}(H)} \leq \frac{C_{3,\alpha}}{t^{3/2-\alpha/2}}\|f\|_{C^\alpha_H(X)}, \quad t>0,\ f \in C^\alpha_H(X),\ x \in X.
\end{aligned}
\tag{3.6}
$$

Proof. Let $t > 0$, $f \in C^\alpha_H(X)$, $h \in H \setminus \{0\}$. For every $s > 0$ we have

$$\begin{aligned}|\langle \nabla_H T(t) f(x), h\rangle_H| \le & \left| \langle \nabla_H T(t) f(x), h\rangle_H - \frac{T(t)f(x+sh) - T(t)f(x)}{s}\right| \\ & + \left|\frac{T(t)f(x+sh) - T(t)f(x)}{s}\right| \\ =: & \; I_1(s) + I_2(s).\end{aligned}$$

Using (3.3) we get

$$\begin{aligned}|I_1(s)| &= \left| \frac{1}{s}\int_0^s \Big(\langle \nabla_H T(t) f(x+\sigma h), h\rangle_H - \langle \nabla_H T(t) f(x), h\rangle_H \Big) d\sigma \right| \\ &\le \frac{1}{s}\int_0^s c(t)\sigma^\alpha \|h\|_H^{\alpha+1}[f]_\alpha d\sigma = \frac{1}{\alpha+1} c(t) s^\alpha \|h\|_H^{\alpha+1}[f]_\alpha,\end{aligned}$$

while using (3.2) we get

$$|I_2(s)| \le s^{\alpha-1}\|h\|_H^\alpha [f]_\alpha.$$

Choosing now $s = t^{1/2}/\|h\|_H$ we obtain

$$|\langle \nabla_H T(t) f(x), h\rangle_H| \le \left(\frac{1}{\alpha+1} c(t) t^{\alpha/2} + t^{(\alpha-1)/2}\right)\|h\|_H [f]_\alpha,$$

and this yields (3.5).

To prove (3.6) it is sufficient to split $D^2_H T(t) f = D^2_H T(t/2) T(t/2) f$, $D^3_H T(t) f = D^3_H T(t/2) T(t/2) f$, and to use estimates (3.5) and (2.14)(ii) and (iii). □

Theorem 3.4. *Let $\lambda > 0$, $f \in C^\alpha_H(X)$ with $0 < \alpha < 1$. Then the unique solution to*

$$\lambda u - Lu = f$$

belongs to $C^{2+\alpha}_H(X)$, and there is $C = C(\lambda, \alpha) > 0$ such that

$$\|u\|_{C^{2+\alpha}_H(X)} \le C \|f\|_{C^\alpha_H(X)}. \tag{3.7}$$

Proof. Recalling that u is given by the representation formula (2.5) it is not difficult to see that $u \in C^2_H(X)$, and that

$$\nabla_H u(x) = \int_0^{+\infty} e^{-\lambda t} \nabla_H T(t) f(x)\, dt, \tag{3.8}$$

$$D_H^2 u(x) = \int_0^{+\infty} e^{-\lambda t} D_H^2 T(t) f(x)\, dt. \tag{3.9}$$

Notice that the right-hand sides of (3.8) and (3.9) are meaningful, since $t \mapsto \nabla_H T(t) f(x)$, $t \mapsto D_H^2 T(t) f(x)$, are continuous for $t > 0$ with values in H, $\mathcal{L}^{(2)}(H)$, respectively, by Proposition 2.3, and their norms are bounded by $C_{1,\alpha} t^{-1/2+\alpha/2} \|f\|_{C_H^\alpha(X)}$, $C_{2,\alpha} t^{-1+\alpha/2} \|f\|_{C_H^\alpha(X)}$, respectively, by Lemma 3.3. Then, (3.8) and (3.9) follow in a standard way. They yield that $\nabla_H u$, $D_H^2 u$ are continuous and bounded, with

$$\begin{aligned} &\|\nabla_H u(x)\|_H \le C_{1,\alpha} \lambda^{-1/2-\alpha/2} \Gamma(1/2+\alpha/2) \|f\|_{C_H^\alpha(X)}, \\ &\|D_H^2 u(x)\|_{\mathcal{L}^{(2)}(H)} \le C_{2,\alpha} \lambda^{-\alpha/2} \Gamma(\alpha/2) \|f\|_{C_H^\alpha(X)}, \end{aligned} \tag{3.10}$$

for every x, where $\Gamma(\theta) = \int_0^\infty e^{-t} t^{\theta-1} dt$ is the Euler function, and the constants $C_{1,\alpha}$, $C_{2,\alpha}$ are given by (3.6).

To prove that $D_H^2 u \in C_H^\alpha(X, \mathcal{L}^{(2)}(H))$ we use an interpolation argument. For every $x \in X$ and $h \in H$ we split $D_H^2 u(x+h) - D_H^2 u(x)$ as $a(x+h) - a(x) + b(x+h) - b(x)$, where

Then,

$$\|a(x+h) - a(x)\|_{\mathcal{L}^{(2)}(H)} \le \int_0^{\|h\|_H^2} e^{-\lambda t} \|D_H^2 T(t) f(x+h) - D_H^2 T(t) f(x)\|_{\mathcal{L}^{(2)}(H)} dt,$$

$$a(y) := \int_0^{\|h\|_H^2} e^{-\lambda t} D_H^2 T(t) f(y)\, dt, \quad b(y) := \int_{\|h\|_H^2}^{\infty} e^{-\lambda t} D_H^2 T(t) f(y)\, dt, \tag{3.11}$$

where, for every $t > 0$,

$$\begin{aligned} \|D_H^2 T(t) f(x+h) - D_H^2 T(t) f(x)\|_{\mathcal{L}^{(2)}(H)} &\le 2 \sup_{y \in X} \|D_H^2 T(t) f(y)\|_{\mathcal{L}^{(2)}(H)} \\ &\le 2 C_{2,\alpha} t^{-1+\alpha/2} \|f\|_{C_H^\alpha(X)}, \end{aligned}$$

by (3.6)(i). Therefore,

$$\|a(x+h) - a(x)\|_{\mathcal{L}^{(2)}(H)} \le \frac{4 C_{2,\alpha}}{\alpha} \|f\|_{C_H^\alpha(X)} \|h\|_H^\alpha.$$

Moreover,

$$\|b(x+h) - b(x)\|_{\mathcal{L}^{(2)}(H)} \le \int_{\|h\|_H^2}^{\infty} e^{-\lambda t} \|D_H^2 T(t) f(x+h) - D_H^2 T(t) f(x)\|_{\mathcal{L}^{(2)}(H)} dt,$$

where, for every $t > 0$,

$$\|D^2_H T(t)f(x+h) - D^2_H T(t)f(x)\|_{\mathcal{L}^{(2)}(H)} = \|\int_0^1 D^3_H T(t)f(x+\sigma h)(h,\cdot,\cdot)d\sigma\|_{\mathcal{L}^{(2)}(H)}$$
$$\le C_{3,\alpha} t^{-3/2+\alpha/2}\|f\|_{C^\alpha_H(X)}\|h\|_H,$$

by (3.6)(ii). Therefore,

$$\|b(x+h) - b(x)\|_{\mathcal{L}^{(2)}(H)} \le \frac{2C_{3,\alpha}}{1-\alpha}\|f\|_{C^\alpha_H(X)}\|h\|^\alpha_H.$$

Summing up we obtain that $D^2_H u$ is H-Hölder continuous and

$$[D^2_H u]_{C^\alpha_H(X,\mathcal{L}^{(2)}(H))} \le \left(\frac{4C_{2,\alpha}}{\alpha} + \frac{2C_{3,\alpha}}{1-\alpha}\right)\|f\|_{C^\alpha_H(X)}.$$

Such estimate and (2.6), (3.10) yield (3.7). □

The procedure of Theorem 3.4 fails for $\alpha = 0$ from the very beginning, since the (optimal) estimate $\|D^2 T(t)f\|_{\mathcal{L}^{(2)}(H)} \le ct^{-1}\|f\|_\infty$ is not enough to guarantee that the right hand side of (3.9) is meaningful for general $f \in C_b(X)$. This is not due to our technique, but to the general lack of maximal regularity results for elliptic differential equations in spaces of continuous functions: even in finite dimension it is known that the domain of Ornstein-Uhlenbeck operators (as well as the domains of the Laplacian and of other second order elliptic differential operators) in $C_b(\mathbb{R}^d)$ is not contained in $C^2(\mathbb{R}^d)$, for $d \ge 2$.

Of course, estimate (3.4) and the procedure of Theorem 3.4 give, for $u = R(\lambda, L)f$,

$$\nabla_H u \in C^\theta_H(X, H), \quad \|\nabla_H u\|_{C^\theta_H(X,H)} \le K\|f\|_\infty,$$

for every $\theta \in (0, 1)$, with $K = K(\lambda, \theta)$ independent of f. However, $K(\lambda, \theta)$ blows up as θ goes to 1.

Still, a modification of Theorem 3.4 gives an embedding of the domain of L that is similar to known embeddings in the finite dimensional case. To this aim we have to introduce Zygmund spaces along H, as follows.

Definition 3.5. If Y is any Banach space, we denote by $Z_H(X, Y)$ the set of continuous and bounded functions $F : X \mapsto Y$ such that

$$[F]_{Z_H(X,Y)} := \sup_{x\in X,\, h\in H\setminus\{0\}} \frac{\|F(x+2h) - 2F(x+h) + F(x)\|_Y}{\|h\|_H} < +\infty. \tag{3.12}$$

$Z_H(X, Y)$ is normed by

$$\|F\|_{Z_H(X,Y)} := \sup_{x\in X}\|F(x)\|_Y + [F]_{Z_H(X,Y)}.$$

It is easy to see that continuous and bounded H-Lipschitz functions from X to Y belong to $Z_H(X, Y)$. Even in the one dimensional case (with $X = Y = H = \mathbb{R}$), there are continuous and bounded functions satisfying condition (3.12) that are not locally Lipschitz continuous.

Theorem 3.6. *Let $\lambda > 0$, $f \in C_b(X)$. Then the unique solution to*

$$\lambda u - Lu = f$$

satisfies $\nabla_H u \in Z_H(X, H)$. Moreover there is $C > 0$ such that

$$\|\nabla_H u\|_{Z_H(X,H)} \leq C\|f\|_\infty. \tag{3.13}$$

Proof. We already know that $u \in C_H^{1+\theta}(X)$ for every $\theta \in (0, 1)$ by the above considerations; in particular $\nabla_H u$ is continuous and bounded.

To prove that $\nabla_H u \in Z_H(X, H)$, for every $h \in H$ we consider the functions a and b defined in (3.11). Using (2.10)(i) we get, for every $x \in X$,

$$\begin{aligned}
&\|a(x+2h) - 2a(x+h) + a(x)\|_H \\
&\leq \int_0^{\|h\|_H^2} e^{-\lambda t}\|\nabla_H T(t)f(x+2h) - 2\nabla_H T(t)f(x+h) + \nabla_H T(t)f(x)\|_H dt \\
&\leq 4 \int_0^{\|h\|_H^2} e^{-\lambda t} c(t)\|f\|_\infty dt,
\end{aligned}$$

and setting $c_0 := \sup_{t>0} t^{1/2}c(t)$ we obtain

$$\|a(x+2h) - 2a(x+h) + a(x)\|_H \leq 2c_0\|f\|_\infty\|h\|_H, \quad x \in X.$$

From the obvious equalities

$$\langle\nabla_H T(t)f(x+2h) - \nabla_H T(t)f(x+h), k\rangle_H = \int_0^1 D_H^2 T(t)f(x+(1+\sigma)h)(h,k)\, d\sigma,$$

$$\langle\nabla_H T(t)f(x+h) - \nabla_H T(t)f(x), k\rangle_H = \int_0^1 D_H^2 T(t)f(x+\sigma h)(h,k)\, d\sigma, \quad k \in H,\ x \in X,$$

we obtain, using (2.10)(iii)

$$\begin{aligned}
&|\langle\nabla_H T(t)f(x+2h) - 2\nabla_H T(t)f(x+h) + \nabla_H T(t)f(x), k\rangle_H| \\
&= \left|\int_0^1 (D_H^2 T(t)f(x+(1+\sigma)h) - D_H^2 T(t)f(x+\sigma h))(h,k) d\sigma\right| \\
&\leq \sup_{y\in X} \|D_H^3 T(t)f(y)\|_{\mathcal{L}^{(3)}(H)}\|h\|_H^2\|k\|_H \leq (3+k_3^3)c(t)^3\|f\|_\infty\|h\|_H^2\|k\|_H,
\end{aligned}$$

so that, for every $x \in X$,

$$\|\nabla_H T(t)f(x+2h) - 2\nabla_H T(t)f(x+h) + \nabla_H T(t)f(x)\|_H \le c_1^3 t^{-3/2}\|f\|_\infty \|h\|_H^2, \quad t>0,$$

with $c_1 = (3+k_3^3)c_0^3$, and therefore

$$\begin{aligned}
&\|b(x+2h) - 2b(x+h) + b(x)\|_H \\
&\le \int_{\|h\|^2}^{\infty} e^{-\lambda t}\|\nabla_H T(t)f(x+2h) - 2\nabla_H T(t)f(x+h) + \nabla_H T(t)f(x)\|_H dt \\
&\le c_1 \int_{\|h\|^2}^{\infty} e^{-\lambda t} t^{-3/2} dt\, \|f\|_\infty \|h\|_H^2 \\
&\le 2c_1 \|f\|_\infty \|h\|_H .
\end{aligned}$$

Summing up, we obtain

$$\|\nabla_H u(x+2h) - 2\nabla_H u(x+h) + \nabla_H u(x)\|_H \le (2c_0 + 2c_1)\|f\|_\infty \|h\|_H ,$$

and the statement follows. □

A similar procedure yields maximal Hölder regularity results for the mild solutions to evolution problems such as (1.2), namely for the functions given by (1.3), for suitable f and g. Precisely, we consider the function spaces defined as follows.

Definition 3.7. Let Y be any Banach space. For $\alpha \in (0,1)$ we denote by $C_H^{0,\alpha}([0,T]\times X;Y)$ the space of the functions $g \in C_b([0,T]\times X;Y)$ such that $g(t,\cdot) \in C_H^\alpha(X;Y)$ for every $t \in [0,T]$, and

$$\|g\|_{C_H^{0,\alpha}([0,T]\times X;Y)} := \sup_{t\in[0,T]} \|g(t,\cdot)\|_{C_H^\alpha(X;Y)} < +\infty.$$

If $Y = \mathbb{R}$ we set $C_H^{0,\alpha}([0,T]\times X;\mathbb{R}) = C_H^{0,\alpha}([0,T]\times X)$. Moreover, we denote by $C_H^{0,2+\alpha}([0,T]\times X)$ the subspace of $C_b([0,T]\times X)$ consisting of the functions g such that $g(t,\cdot) \in C_H^{2+\alpha}(X)$ for every $t\in[0,T]$, and

$$\|g\|_{C_H^{0,2+\alpha}([0,T]\times X)} := \sup_{t\in[0,T]} \|g(t,\cdot)\|_{C_H^{2+\alpha}(X)} < +\infty.$$

Theorem 3.8. *Let $f \in C_H^{2+\alpha}(X)$, $g \in C_H^{0,\alpha}([0,T]\times X)$ with $\alpha\in(0,1)$, and let v be defined by (1.3). Then $v \in C_H^{0,2+\alpha}([0,T]\times X)$, and there is $C = C(T) > 0$, independent of f and g, such that*

$$\|v\|_{C_H^{0,2+\alpha}([0,T]\times X)} \le C(\|f\|_{C_H^{2+\alpha}(X)} + \|g\|_{C_H^{0,\alpha}([0,T]\times X)}). \tag{3.14}$$

Proof. We already know that $(t,x) \mapsto T(t)f(x)$ is in $C_H^{0,2+\alpha}([0,T]\times X)$, by Lemma 3.2. So, we consider the function

$$v_0(t,x) := \int_0^t T(s)g(t-s,\cdot)(x)ds, \quad t\in[0,T],\ x\in X.$$

The same arguments used in the proof of Theorem 3.4 show that $v(t,\cdot)\in C_H^2(X)$ for every $t\in[0,T]$, that

$$D_H^2 v_0(t,x) = \int_0^t D_H^2 T(s)g(t-s,\cdot)(x)ds, \quad t\in[0,T],\ x\in X,$$

and that there is $C = C(T) > 0$, independent of g, such that

$$\|v_0(t,\cdot)\|_{C_H^2(X)} \le C\|g\|_{C_H^{0,\alpha}([0,T]\times X)}.$$

Let us prove that $D_H^2 v_0$ is continuous at any (t_0,x_0). If $t > t_0$, $x\in X$, we split

$$\begin{aligned}
&\|D_H^2 v_0(t,x) - D_H^2 v_0(t_0,x_0)\|_{\mathcal{L}^{(2)}(H)} \le \\
&\le \int_0^{t_0} \|D_H^2 T(s)g(t-s,\cdot)(x) - D_H^2 T(s)g(t_0-s,\cdot)(x_0))\|_{\mathcal{L}^{(2)}(H)}ds \\
&\quad + \int_{t_0}^t \|D_H^2 T(s)g(t-s,\cdot)(x)\|_{\mathcal{L}^{(2)}(H)}ds \\
&=: I_1(t,x) + I_2(t,x).
\end{aligned} \tag{3.15}$$

Estimate (3.6)(i) yields

$$I_2(t,x) \le \int_{t_0}^t \frac{C_{2,\alpha}}{s^{1-\alpha/2}}ds \sup_{0\le r\le T}\|g(r,\cdot)\|_{C_H^\alpha(X)},$$

so that $\lim_{t\to t_0^+, x\to x_0} I_2(t,x) = 0$. Concerning $I_1(t,x)$, for every $s\in[0,t_0]$ and $h,k\in H$, formulae (2.8) and (2.2) yield

$$\begin{aligned}
&|(D_H^2 T(s)g(t-s,\cdot)(x) - D_H^2 T(s)g(t_0-s,\cdot)(x_0))(h,k)| \le \\
&\le c(s)^2\Big(\int_X |g(t-s, e^{-s}x + \sqrt{1-e^{-2s}}y) - g(t_0-s, e^{-s}x_0 + \sqrt{1-e^{-2s}}y)|^2\gamma(dy)\Big)^{1/2}\cdot \\
&\quad \cdot \|\hat{h}\hat{k} - \langle h,k\rangle_H\|_{L^2(X,\gamma)}
\end{aligned}$$

$$\leq c(s)^2\left(\int_X |g(t-s,e^{-s}x+\sqrt{1-e^{-2s}}y)-g(t_0-s,e^{-s}x_0+\sqrt{1-e^{-2s}}y)|^2\gamma(dy)\right)^{1/2}\cdot$$
$$\cdot(k_4^2+1)\|h\|_H\|k\|_H$$

and since g is continuous and bounded, by the Dominated Convergence Theorem we get

$$\lim_{t\to t_0, x\to x_0}\|D_H^2T(s)g(t-s,\cdot)(x)-D_H^2T(s)g(t_0-s,\cdot)(x_0)\|_{\mathcal{L}^{(2)}(H)}=0.$$

Moreover, estimate (3.6)(i) yields

$$\|D_H^2T(s)g(t-s,\cdot)(x)-D_H^2T(s)g(t_0-s,\cdot)(x_0))\|_{\mathcal{L}^{(2)}(H)}\leq\frac{2C_{2,\alpha}}{s^{1-\alpha/2}}\sup_{0\leq r\leq T}\|g(r,\cdot)\|_{C_H^\alpha(X)},$$
$$0<s<t.$$

Therefore, still by the Dominated Convergence Theorem, $\lim_{t\to t_0^+, x\to x_0} I_1(t,x)=0$. Summing up, we get $\lim_{t\to t_0^+, x\to x_0} D^2v_0(t,x)=D^2v_0(t_0,x_0)$. If $t<t_0$, changing the roles of t and t_0 in the splitting (3.15), we obtain $\lim_{t\to t_0^-, x\to x_0} D^2v_0(t,x)=D^2v_0(t_0,x_0)$, and continuity of D^2v_0 is proved.

To prove that $D^2v_0(t,\cdot)\in C_H^\alpha(X,\mathcal{L}^{(2)}(H))$ for every $t\in[0,T]$ we argue as in Theorem 3.4, namely we split $D^2v_0(t,\cdot)(x+h)-D^2v_0(t,\cdot)=a(x+h)-a(x)+b(x+h)-b(x)$, where now

$$a(y)=\int_0^{\min\{t,\|h\|^2\}} D_H^2T(s)g(t-s,\cdot)(y)\,ds,$$
$$b(y)=\int_{\min\{t,\|h\|^2\}}^t D_H^2T(s)g(t-s,\cdot)(y)\,ds,\quad y\in X,$$

and we proceed as in the proof of Theorem 3.4, to get

$$[D^2v_0(t,\cdot)]_{C_H^\alpha(X,\mathcal{L}^{(2)}(H))}\leq\left(\frac{4C_{2,\alpha}}{\alpha}+\frac{2C_{3,\alpha}}{1-\alpha}\right)\sup_{r\in[0,T]}\|g(r,\cdot)\|_{C_H^\alpha(X)}.\quad\square$$

4. Open problems and bibliographical remarks

Although many of our proofs rely on typical arguments from interpolation theory, interpolation spaces are not explicitly mentioned. If $X=\mathbb{R}^d$, Schauder theorems for non-degenerate Ornstein-Uhlenbeck operators were first proved in [7], relying on other interpolation techniques. It was shown that for every $f\in C_b^\alpha(\mathbb{R}^d)$ the function $R(\lambda,L)f$ defined in (2.5) is the unique bounded classical solution to (1.1), that its second order derivatives belong to the interpolation space

$$(C_b(\mathbb{R}^d),D(L))_{\alpha/2,\infty}=\{f\in C_b(\mathbb{R}^d):\ \sup_{t>0}t^{-\alpha/2}\|T(t)f-f\|_\infty<+\infty\},$$

where $T(t)$ is the corresponding Ornstein-Uhlenbeck semigroup, and the latter space was characterized as

$$\{f \in C_b^\alpha(\mathbb{R}^d) : \sup_{t>0} t^{-\alpha/2} \| f(e^{-t}\cdot) - f\|_\infty < +\infty\}.$$

A similar characterization is open in infinite dimension. Even the simpler characterization

$$(C_b(X), C_H^1(X))_{\alpha,\infty} = C_H^\alpha(X), \quad 0 < \alpha < 1, \tag{4.1}$$

is not clear in general Banach spaces. In the next lemma we only prove embeddings, through (by now) standard methods.

Lemma 4.1. *For every $\alpha \in (0, 1)$ we have*

$$\begin{aligned} &(i) \quad (C_b(X), C_H^1(X))_{\alpha,\infty} \subset C_H^\alpha(X), \\ &(ii) \quad (C_b(X), D(L))_{\alpha/2,\infty} \subset C_H^\alpha(X). \end{aligned}$$

Proof. We recall that, given two Banach spaces $\mathcal{Y} \subset \mathcal{X}$ with continuous embedding and $\alpha \in (0, 1)$, the interpolation space $(\mathcal{X}, \mathcal{Y})_{\alpha,\infty}$ consists of all $u \in \mathcal{X}$ such that $\|u\|_{(\mathcal{X},\mathcal{Y})_{\alpha,\infty}} := \sup_{t>0} t^{-\theta} K(t, u) < +\infty$, where $K(t, u) := \inf\{\|a\|_{\mathcal{X}} + t\|b\|_{\mathcal{Y}} : u = a + b,\ a \in \mathcal{X},\ b \in \mathcal{Y}\}$. We also recall that $(\mathcal{X}, \mathcal{Y})_{\alpha,\infty} \subset \mathcal{X}$, with continuous embedding.

Let $u \in (C_b(X), C_H^1(X))_{\alpha,\infty}$. For every decomposition $u = a + b$, with $a \in C_b(X)$, $b \in C_H^1(X)$, we have

$$|u(x + h) - u(x)| \le |a(x + h) - a(x)| + |b(x + h) - b(x)| \le 2\|a\|_\infty + \|\nabla_H b\|_\infty \|h\|_H, \quad x \in X,\ h \in H,$$

so that, taking the infimum over all such decompositions,

$$|u(x + h) - u(x)| \le 2K(\|h\|_H, u) \le 2\|h\|_H^\alpha \|u\|_{(C_b(X),C_H^1(X))_{\alpha,\infty}}, \quad x \in X,\ h \in H,$$

and (i) follows.

To prove statement (ii) we use (2.10)(i), that yields, for every $u \in D(L)$ and $\lambda > 0$, $x \in X$,

$$\begin{aligned} \|\nabla_H u(x)\|_H &\le \int_0^\infty e^{-\lambda t} \|\nabla_H T(t)(\lambda u - Lu)(x)\|_H dt \\ &\le \int_0^\infty e^{-\lambda t} c(t) dt\, (\lambda\|u\|_\infty + \|Lu\|_\infty) \\ &\le c_0 \Gamma(1/2)(\lambda^{1/2}\|u\|_\infty + \lambda^{-1/2}\|Lu\|_\infty), \end{aligned}$$

where $c_0 = \sup_{t>0} t^{1/2} c(t)$. Taking the minimum over λ we get

$$\sup_{x\in X} \|\nabla_H u(x)\|_H \le C\|u\|_\infty^{1/2} \|Lu\|_\infty^{1/2},$$

for some $C > 0$, independent of u. This implies that the space $C^1_H(X)$ belongs to the class $J_{1/2}$ between $C_b(X)$ and $D(L)$ (e.g., [9, Sect. 1.10.1]). The Reiteration Theorem ([9, Sect. 1.10.2]) yields

$$(C_b(X), D(L))_{\alpha/2,\infty} \subset (C_b(X); C^1_H(X))_{\alpha,\infty},$$

and (ii) follows from (i). □

Going back to (4.1), in the case where X is a Hilbert space and γ is non-degenerate, the similar equality

$$(BUC(X), BUC^1_H(X))_{\alpha,\infty} = C^\alpha_H(X) \cap BUC(X)$$

was stated in [4].

Concerning Schauder estimates in infinite dimension, if X is an infinite dimensional Hilbert space, smoothing Ornstein-Uhlenbeck semigroups such as

$$T(t)f(x) = \int_X f(e^{tA}x + y)\mathcal{N}_{0,Q_t}(dy)$$

were considered in [5,3], under the assumptions that A is the infinitesimal generator of a strongly continuous semigroup e^{tA} in X, $Q \in \mathcal{L}(X)$ is a self-adjoint positive operator, the operators $Q_t := \int_0^t e^{sA}Qe^{sA^*}ds$ have finite trace for every $t > 0$, $e^{tA}(X) \subset Q_t^{1/2}(X)$ for every t, and moreover $\sup_{t>0} t^{1/2}\|Q_t^{-1/2}e^{tA}\|_{\mathcal{L}(X)} < +\infty$. The generator of $T(t)$ is a realization of the operator

$$\mathcal{L}u(x) = \frac{1}{2}\mathrm{Tr}(QD^2u(x)) + \langle x, A^*\nabla u(x)\rangle$$

and $T(t)$ is a smoothing operator in all directions, not only along a subspace. In this case, a Schauder theorem in the usual Hölder spaces holds: namely, if f is any bounded function belonging to $C^\alpha(X)$ for some $\alpha \in (0, 1)$, then the function

$$u(x) = \int_0^\infty e^{-\lambda t}T(t)f(x)dt \tag{4.2}$$

belongs to $C^2(X)$, it has bounded first and second order derivatives, $D^2u \in C^\alpha(X, \mathcal{L}^{(2)}(X))$. This was proved in [3] in the case $Q = I$ and in [5, Ch. 5] in the case that $T(t)$ is the transition semigroup of a suitable linear stochastic PDE with $X = L^2(\Omega)$, Ω being an open bounded subset of $\mathbb{R}^d$ with smooth boundary.

We would like to remind that there are relevant situations in which Schauder estimates cannot be proved for Hilbert spaces, but only for Banach spaces. This is the case considered in [6], where the transition semigroup $T(t)$ associated with a class of stochastic reaction-diffusion equations defined on a bounded interval $[0, 1]$, with polynomially growing coefficients, is studied in the space $X = C([0, 1])$. Actually, for that class of equations the analysis of $T(t)$ in $X = L^2(0, 1)$ is considerably more delicate than in $X = C([0, 1])$ and it is not possible to prove that when

$f \in C^\alpha(L^2(0,1))$, for some $\alpha \in (0,1)$, the function u defined in (4.2) belongs to $C^2(L^2(0,1))$. Notice, in particular, that working in $C([0,1])$ prevents from using the interpolatory identity (4.1).

Under assumptions similar to [3] a related result is in [1], where the space $L^\infty(X,\gamma)$ is considered instead of $C_b(X)$. Regularity results were stated in terms of the spaces $\{f \in L^\infty(X,\gamma) : \sup_{t>0} t^{-\alpha/2}\|T(t)f - f\|_\infty < +\infty\}$, called S^α and endowed with their natural norm

$$\|f\|_\infty + \sup_{t>0} \frac{\|T(t)f - f\|_\infty}{t^{\alpha/2}}.$$

However, since $T(t)$ is strong Feller, we have $S^\alpha = (C_b(X), D(L))_{\alpha/2,\infty}$, with equivalence of the respective norms. In [1] it is proved that if $f \in S^\alpha$, then u and its first and second order derivatives along any direction belong to S^α.

Schauder type theorems for the Gross Laplacian and of some of its perturbations were established in [4]. Here, the semigroup is given by

$$S(t)f(x) = \int_X \varphi(x + \sqrt{t}y)\gamma(dy),$$

where γ is again a centered non-degenerate Gaussian measure in a separable Hilbert space X. In contrast with Ornstein-Uhlenbeck semigroups, $S(t)$ is strongly continuous in $BUC(X)$; similarly to our Ornstein-Uhlenbeck semigroup $S(t)$ is not strong Feller, and it is smoothing only along the directions of the Cameron-Martin space. The infinitesimal generator of $S(t)$ in $BUC(X)$ is a realization A of the operator

$$\mathcal{A}u(x) = \frac{1}{2}\text{Trace}\,(QD^2u(x)),$$

in the space $BUC(X)$. A result similar to Theorem 1 was stated, when $f \in C^\alpha_H(X) \cap BUC(X)$ (the latter space is called $C^\alpha_Q(X)$ in [4], Q being the covariance of γ). Moreover, in [1] it was proved that if $f \in S^\alpha$, where now

$$S^\alpha := \{g \in L^\infty(X) : \sup_{t>0} t^{-\alpha/2}\|S(t)g - g\|_\infty < +\infty\},$$

then for every $\lambda > 0$ the function $u(x) = \int_X e^{-\lambda t} S(t)f(x)ds$ possesses first and second order derivatives along the elements of any orthonormal basis of X consisting of eigenvalues of Q, and they belong to S^α.

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