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On a class of degenerate elliptic operators arising from Fleming-Viot processes

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Dedicated to the memory of Ralph Phillips

Abstract. We are dealing with the solvability of an elliptic problem related to a class of degenerate second order operators which arise from the theory of Fleming-Viot processes in population genetics. In the one dimensional case the problem is solved in the space of continuous functions. In higher dimension we study the problem in L^2 spaces with respect to an explicit measure which, under suitable assumptions, can be taken invariant and symmetrizing for the operators. We prove the existence and uniqueness of weak solutions and we show that the closure of the operator in such L^2 spaces generates an analytic C_0-semigroup.

1. Introduction

The aim of this paper is to study the solvability of the degenerate elliptic problem

$$\lambda\varphi(x) - \gamma(x)\,\mathrm{Tr}\,[C(x)D^2\varphi(x)] - \langle \omega(x) - |\widetilde{\omega}(x)|x, D\varphi(x)\rangle = f(x), \quad x \in S, \quad (1.1)$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^d$, λ is any positive constant, S is the simplex of $\mathbb{R}^d$ consisting of all $x \in [0,1]^d$ such that $x_1 + \cdots + x_d \leq 1$, $C(x)$ is the $d \times d$ matrix of components

$$c_{i,j}(x) = x_i(\delta_{ij} - x_j), \quad i, j = 1, \ldots, d,$$

which degenerates on the boundary of S, $\omega(x)$ is the vector of $\mathbb{R}^d$ of components $\omega_1(x), \ldots, \omega_d(x)$ and $\widetilde{\omega}(x)$ is the vector of $\mathbb{R}^{d+1}$ of components $\omega_0(x), \omega_1(x), \ldots, \omega_d(x)$. Notice that here and in what follows we denote $|\widetilde{\omega}(x)| = \sum_{i=0}^d \omega_i(x)$. The functions γ and $\omega_0, \omega_1, \ldots, \omega_d$ are supposed to be continuous and strictly positive on S.

The operator

$$\mathcal{L}\varphi(x) = \gamma(x)\,\mathrm{Tr}\,[C(x)D^2\varphi(x)] + \langle \omega(x) - |\widetilde{\omega}(x)|x, D\varphi(x)\rangle, \quad x \in S, \quad (1.2)$$

arises in the theory of Fleming-Viot processes as the generator of a Markov C_0-semigroup defined on $C(S)$, the space of continuous functions on S. Fleming-Viot processes are

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measure-valued processes that describe the genetic evolution of a population and are defined as the limit in distribution of empirical processes associated with suitable sequences of Markov chains. The operator (1.2) is the generator corresponding to some diffusion model in population dynamics in which each individual is of some *type* and the *type space* E is given by a finite number $d+1$ of elements. In this case the state space is

$$\Delta_E = \left\{ x \in [0,1]^{d+1}\,;\ \sum_{i=0}^{d} x_i = 1 \right\},$$

where x_i denotes the proportion of the population that is of type i. Moreover, the first order term corresponds to a *mutation* operator A which depends on $x \in \Delta_E$ and which is given by

$$A_x f(j) = \int_E (f(j) - f(i))\, p(x, di), \qquad j \in E, \quad x \in \Delta_E,$$

where $f : E \to \mathbb{R}$, and $p\,(x, \{i\}) = \omega_i(x)$, for any $i \in E$.

There are models in which the number of different types is infinite, like in the Ohta and Kimura's model, where $E = \mathbb{Z}$. By using a different approach it is also possible to construct models in which E is an arbitrary complete separable metric space, like for example $\mathbb{R}$ or $[0,1]^{\mathbb{Z}_+}$. In this case Fleming and Viot in [6] had proposed to topologize E and replace Δ_E by $\mathcal{P}(E)$, the space of probability measure on E, with the topology of the weak convergence. For a complete description of such models we refer to the papers by Ethier and Kurtz [5] and Dawson and March [2].

It is known that if the domain of the operator $\mathcal{L}$ defined by (1.2) is $C^2(S)$, then it is dissipative in $C(S)$ and hence, as densely defined, is closable in $C(S)$ with dissipative closure. By the Lumer-Phillips theorem its closure $\bar{\mathcal{L}}$ is the generator of C_0-semigroup of contractions if and only if the range of $\lambda I - \mathcal{L}$ is dense in $C(S)$, for some $\lambda > 0$. This is the reason we are led to study the solvability of the problem (1.1). When the coefficients γ and ω_i are constant, it is easy to show that the image of $I - \mathcal{L}$ contains a dense subset of $C(S)$ and the generation of a semigroup follows (see [12] for the case of a finite type space and [2], [5] for the case of general type spaces E). When the coefficients are not constant the situation is more delicate. However, in [3bis], by using an argument based on the Trotter product formula, it is proved that $\bar{\mathcal{L}}$ generates a non-negative C_0-semigroup of contractions on $C(S)$.

In Section 3 we consider the problem (1.1) when the population is divided into only two different types, that is when $d = 1$ and $S = [0,1]$. In this case by assuming that the functions γ and ω_i are continuous on $[0,1]$ and satisfy some Hölder condition at the boundary points, we show that $C^2[0,1]$ is a core for the operator $\mathcal{L}$ with domain

$$\mathcal{D} = \left\{ \varphi \in C^1[0,1] \cap C^2(0,1) \;:\; \lim_{t \to 0^+, 1^-} x(1-x)\varphi''(x) = 0 \right\}. \tag{1.3}$$

Thus, by applying a recent result by Metafune [8] based on some earlier results of Angenent [1], we can extend the result proved in [3bis] and we can conclude that $\bar{\mathcal{L}}$ is the generator of a C_0-semigroup of contractions on $C[0,1]$, which is positive and analytic and its domain coincides with the set $\mathcal{D}$ in (1.3).

In the section 4 we study the problem (1.1) in a L^2 context. Introduce a measure ν on S which is absolutely continuous with respect to the Lebesgue measure and for which we give an explicit formula of the density. It is important to notice that when γ and ω_i fulfill suitable assumptions, the measure ν can be chosen *invariant* and even *symmetrizing* and when γ and ω_i are constant, it coincides with a Dirichlet distribution $D(q)$ on S, of some parameters q_i related to the coefficients (see e.g. [9] and [12]). Notice that in general the measure ν is neither invariant nor symmetrizing. Nevertheless, we are able to prove an *integration by parts* formula which allows to show that the operator $\mathcal{L}$ is closable in $L^2(S,\nu)$ and its closure $\bar{\mathcal{L}}$ generates a C_0-semigroup. Such a formula also provides a characterization of the domain of $\bar{\mathcal{L}}$ in $L^2(S,\nu)$ in terms of appropriate Sobolev spaces, and a factorization of the symmetric part of $\bar{\mathcal{L}}$ in terms of its square root (which coincides with $\bar{\mathcal{L}}$ when there exists a symmetrizing invariant measure).

Moreover, due to the Lax-Milgram theorem we can define an operator $\mathcal{A}$ in $L^2(S,\nu)$ which is the generator of an analytic semigroup in $L^2(S,\nu)$ with dense domain, which extends the operator $\bar{\mathcal{L}}$. Therefore, as $\bar{\mathcal{L}}$ generates a C_0-semigroup, $\mathcal{A}$ coincides with $\bar{\mathcal{L}}$ and we have generation of analytic semigroup.

2. Notations and preliminary results

We are here concerned with the operator

$$\mathcal{L}\varphi(x) = \gamma(x)\,\mathrm{Tr}\,[C(x)D^2\varphi(x)] + \langle \omega(x) - |\widetilde{\omega}(x)|x, D\varphi(x)\rangle, \qquad x \in S,$$

where $C(x)$ is the matrix of components $c_{ij}(x)$ defined by

$$c_{ij}(x) = x_i(\delta_{ij} - x_j), \quad 1 \le i, j \le d$$

and S is the simplex of $\mathbb{R}^d$ defined by

$$S = \left\{ x \in [0,1]^d \; ; \; \sum_{i=1}^{d} x_i \le 1 \right\}.$$

We recall that in what follows we shall denote

$$|\widetilde{\omega}(x)| = \sum_{i=0}^{d} \omega_i(x).$$

Moreover, we shall denote by $\overset{\circ}{S}$ the interior of S.

HYPOTHESIS 2.1. *The functions γ and ω_i, $i = 0, 1, \dots, d$, are continuous and strictly positive on S.*

Due to the previous condition, the operator $\mathcal{L}$ can be written as MA, with

$$\begin{aligned} M\varphi(x) &= \gamma(x)\varphi(x), \\ A\varphi(x) &= \mathrm{Tr}\,[C(x)D^2\varphi(x)] + \frac{1}{\gamma(x)}\langle \omega(x) - |\widetilde{\omega}(x)|x, D\varphi(x)\rangle. \end{aligned} \tag{2.1}$$

We have the following general result.

LEMMA 2.2. *Let $C(X)$ be the Banach space of continuous functions on a compact set X, equipped with the supremum norm $\|\cdot\|_0$ and let A be a m-dissipative operator in $C(X)$. Moreover, let B be the operator defined by*

$$D(B) = D(A), \quad B = MA,$$

where

$$Mu(x) = m(x)u(x), \quad x \in X,$$

for some continuous and strictly positive function m defined in X. Then, if B is dissipative B is m-dissipative.

Proof. Let f be any function in $C(X)$. We have to prove that there exists $u \in D(A)$ such that $u - MAu = f$. Set

$$\lambda_0 = \min_{x\in X} \frac{1}{m(x)}, \quad p(x) = \lambda_0 - \frac{1}{m(x)} \leq 0$$

and set $Cu = pu$, for any $u \in C(X)$. Then C is m-dissipative and bounded, so that also $A + C$ is m-dissipative (for a proof of this fact see for example [10]-Chapter 3, Corollary 3.3). Hence there exists one and only one $u \in D(A)$ such that $\lambda_0 u - (Au + pu) = f/m$, and this is equivalent to $u = MAu + f$. □

Whence due to the previous lemma, once one proves that $\bar{\mathcal{L}}$ is dissipative, it follows that the m-dissipativity of $\bar{\mathcal{L}}$ is equivalent to m-dissipativity of $\bar{A}$. This is the reason why in what follows, without any loss of generality, we can write the operator $\mathcal{L}$ in the form

$$\mathcal{L}\varphi(x) = \mathrm{Tr}\,[C(x)D^2\varphi(x)] + \langle \sigma(x) - |\widetilde{\sigma}(x)|x, D\varphi(x)\rangle, \quad x \in S, \tag{2.2}$$

where $\sigma_i(x) = \omega_i(x)/\gamma(x)$, for any $x \in S$ and $i = 0, 1, \dots, d$. Notice that all σ_i are continuous and strictly positive on S for each i, so that the Hypothesis 2.1 is fulfilled by σ_i as well.

Thus, next step is showing that $\bar{\mathcal{L}}$ is dissipative in $C(S)$. To this purpose, we first recall some well known facts concerning the matrix $C(x)$, $x \in S$.

LEMMA 2.3. *For any* $h \in \mathbb{R}^d$, $h \neq 0$, *it holds*

$$\langle C(x)h, h\rangle \geq 0, \quad \textit{if } x \in S \quad \textit{and} \quad \langle C(x)h, h\rangle > 0, \quad \textit{if } x \in \overset{\circ}{S}. \tag{2.3}$$

Proof. We have

$$\begin{aligned}\langle C(x)h, h\rangle &= \sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)h_i h_j = \sum_{i=1}^{d} x_i h_i^2 - \sum_{i,j=1}^{d} x_i x_j h_i h_j \\ &= \sum_{i=1}^{d} x_i h_i^2 - \left(\sum_{i=1}^{d} x_i h_i\right)^2.\end{aligned}$$

If we take x in S, it holds

$$\left(\sum_{i=1}^{d} x_i h_i\right)^2 \leq \sum_{i=1}^{d} x_i h_i^2 \sum_{i=1}^{d} x_i \leq \sum_{i=1}^{d} x_i h_i^2,$$

and then

$$\langle C(x)h, h\rangle \geq \sum_{i=1}^{d} x_i h_i^2 - \sum_{i=1}^{d} x_i h_i^2 = 0.$$

Furthermore, if $x \in \overset{\circ}{S}$, then $x_1 + \cdots + x_d < 1$, so that the strict inequality holds in (2.3). □

LEMMA 2.4. *For any* $x \in S$ *it holds*

$$\det C(x) = \prod_{i=0}^{d} x_i,$$

where $x_0 = 1 - \sum_{m=1}^{d} x_m$. *Thus* $\det C(x) = 0$ *if and only if* $x \in \partial S$.

Proof. For any $x \in S$ we have

$$C(x) = \begin{pmatrix} x_1(1-x_1) & -x_1x_2 & \cdots & -x_1x_d \\ -x_2x_1 & x_2(1-x_2) & \cdots & -x_2x_d \\ \vdots & \vdots & \vdots & \vdots \\ -x_dx_1 & -x_dx_2 & \cdots & x_d(1-x_d) \end{pmatrix}$$

By easy calculations

$$\det C(x) = \prod_{i=1}^{d} x_i \begin{vmatrix} 1-x_1 & -x_1 & \cdots & -x_1 \\ -x_2 & 1-x_2 & \cdots & -x_2 \\ \vdots & \vdots & \vdots & \vdots \\ -x_d & -x_d & \cdots & 1-x_d \end{vmatrix} = \prod_{i=1}^{d} x_i \begin{vmatrix} 1-x_1 & -1 & -1 & \cdots & -1 \\ -x_2 & 1 & 0 & \cdots & 0 \\ -x_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_d & 0 & 0 & \cdots & 1 \end{vmatrix}$$

and developing along the first row we get

$$\begin{aligned} \det C(x) &= \prod_{i=1}^{d} x_i \left(1 - x_1 + \sum_{i=2}^{d} -(-1)^{i+1}(-1)^i(-x_i)\right) \\ &= \prod_{i=1}^{d} x_i \left(1 - x_1 - \sum_{i=2}^{d} x_i\right) = \prod_{i=0}^{d} x_i. \end{aligned}$$

□

LEMMA 2.5. *If we denote* $\Lambda(x) = diag\,\{x_1, \ldots, x_d\}$ *and* $x \otimes x = xx^t$, *we have*

$$C(x) = \Lambda(x) - x \otimes x, \quad x \in S. \tag{2.4}$$

Moreover, $C(x)$ *is invertible for any* $x \in \overset{\circ}{S}$ *and it holds*

$$C^{-1}(x) = \Lambda^{-1}(x) + \frac{e \otimes e}{x_0}, \quad x \in \overset{\circ}{S}, \tag{2.5}$$

where e *is the vector of* $\mathbb{R}^d$ *having all components equal* 1.

Proof. For any $h, k \in \mathbb{R}^d$ and $x \in S$ we have

$$\begin{aligned} \langle h, C(x)k \rangle &= \sum_{i=1}^{d} h_i \sum_{j=1}^{d} x_i(\delta_{ij} - x_j)k_j \\ &= \sum_{i=1}^{d} x_i h_i k_i - \sum_{i,j=1}^{d} x_i h_i x_j k_j = \langle h, \Lambda(x)k \rangle - \langle h, (x \otimes x)k \rangle, \end{aligned}$$

so that (2.4) follows. Concerning $C^{-1}(x)$, let us solve for any $x \in \overset{\circ}{S}$ and $v \in \mathbb{R}^d$ the equation

$$\Lambda(x)u - \langle x, u \rangle\, x = v.$$

If we set $\lambda = \langle x, u \rangle$, we have $x_i u_i - \lambda x_i = v_i$, for any $i = 1, \ldots, d$ and then

$$u_i = \frac{v_i + \lambda x_i}{x_i} = \frac{v_i}{x_i} + \lambda.$$

This implies that

$$\lambda = \sum_{i=1}^{d} x_i u_i = \sum_{i=1}^{d} v_i + \lambda \sum_{i=1}^{d} x_i = \sum_{i=1}^{d} v_i + \lambda(1 - x_0),$$

so that $\lambda = \langle v, e\rangle / x_0$. Therefore

$$C^{-1}(x) v = u = \Lambda^{-1}(x)\, v + \frac{\langle v, e\rangle\, e}{x_0} = \left(\Lambda^{-1}(x) + \frac{e \otimes e}{x_0}\right) v.$$

□

Next proposition is the main result in order to have the dissipativity of the operator $\mathcal{L}$.

PROPOSITION 2.6. *Let $\varphi \in C^2(S)$ and let $\bar{x} \in S$ be a point where φ achieves its minimum. Then*

$$\mathrm{Tr}\,[C(\bar{x}) D^2\varphi(\bar{x})] \geq 0, \quad \langle \omega(\bar{x}) - |\widetilde{\omega}(\bar{x})|\bar{x}, D\varphi(\bar{x})\rangle \geq 0. \tag{2.6}$$

Proof. It will be convenient to rewrite the operator $\mathcal{L}$ in y-coordinates

$$y_0(x) = 1 - \sum_{m=1}^{d} x_m, \quad y_i(x) = x_i, \quad 1 \leq i \leq d.$$

In this case, for any $x \in S$ we have that $y(x)$ belongs to the set

$$\Delta = \left\{ y \in [0, 1]^{d+1}\, ;\ \sum_{j=0}^{d} y_j = 1 \right\}.$$

If $\psi \in C^2(\Delta)$ and $y = y(x)$, for some $x \in S$, we define

$$\varphi(x_1, \ldots, x_d) = \psi(y_0(x), y_1(x), \ldots, y_d(x)).$$

Then

$$\frac{\partial \varphi(x)}{\partial x_i} = \sum_{j=0}^{d} \frac{\partial \psi(y)}{\partial y_j} \frac{\partial y_j}{\partial x_i} = -\left(\frac{\partial}{\partial y_0} - \frac{\partial}{\partial y_i}\right) \psi(y)$$

and

$$\begin{aligned} \frac{\partial^2 \varphi(x)}{\partial x_i \partial x_j} &= \left(\frac{\partial}{\partial y_0} - \frac{\partial}{\partial y_i}\right)\left(\frac{\partial}{\partial y_0} - \frac{\partial}{\partial y_j}\right) \psi(y) \\ &= \frac{\partial^2 \psi(y)}{\partial y_i \partial y_j} + \frac{\partial^2 \psi(y)}{\partial y_0^2} - \frac{\partial^2 \psi(y)}{\partial y_0 \partial y_j} - \frac{\partial^2 \psi(y)}{\partial y_0 \partial y_i}. \end{aligned}$$

Therefore we get

$$\begin{aligned}\mathrm{Tr}\,[C(x)D^2\varphi(x)] \;&=\; \sum_{i,j=1}^{d} y_i(\delta_{ij}-y_j)\frac{\partial^2\psi(y)}{\partial y_i\partial y_j} - \sum_{i,j=1}^{d} y_i(\delta_{ij}-y_j)\frac{\partial^2\psi(y)}{\partial y_i\partial y_0}\\ &\quad - \sum_{i,j=1}^{d} y_i(\delta_{ij}-y_j)\frac{\partial^2\psi(y)}{\partial y_j\partial y_0} + \sum_{i,j=1}^{d} y_i(\delta_{ij}-y_j)\frac{\partial^2\psi(y)}{\partial y_0^2}\\ &= \sum_{i,j=1}^{d} y_i(\delta_{ij}-y_j)\frac{\partial^2\psi(y)}{\partial y_i\partial y_j} - \sum_{i=1}^{d} y_i y_0\frac{\partial^2\psi(y)}{\partial y_i\partial y_0}\\ &\quad - \sum_{j=1}^{d} y_j y_0\frac{\partial^2\psi(y)}{\partial y_j\partial y_0} + y_0(1-y_0)\frac{\partial^2\psi(y)}{\partial y_0^2},\end{aligned}$$

so that

$$\mathrm{Tr}\,[C(x)D^2\varphi(x)] = \sum_{i,j=0}^{d} y_i(x)(\delta_{ij}-y_j(x))\frac{\partial^2\psi(y(x))}{\partial y_i\partial y_j}. \tag{2.7}$$

Similarly, it is possible to show that

$$\sum_{i=1}^{d}(\omega_i(x)-|\widetilde{\omega}(x)|x_i)\,\frac{\partial\varphi(x)}{\partial x_i} = \sum_{i=0}^{d}(\omega_i(x)-|\widetilde{\omega}(x)|y_i(x))\,\frac{\partial\psi(y(x))}{\partial y_i}.$$

Now, if $\bar{x}$ is a point in S where φ has its minimum, let us denote by $\bar{y}$ the corresponding point in Δ where ψ achieves its minimum. We first suppose that $\bar{y}$ is an extremal point, that is all coordinates $\bar{y}_i$'s are zero but one $\bar{y}_{i_0}=1$. It is easy to check that in this case the matrix of components $\bar{y}_i(\delta_{ij}-\bar{y}_j)$, $0\le i,\,j\le d$, is identically zero, indeed the coefficients can be non zero only if $i=j=i_0$ and in this case we have $\bar{y}_{i_0}(1-\bar{y}_{i_0})=0$. Thus, due to (2.7) we have

$$\mathrm{Tr}\,[C(\bar{x})D^2\varphi(\bar{x})]=0.$$

Now, we consider the case when $\bar{y}_i<1$ for each $i=0,1,\dots,d$. If we denote

$$I(\bar{y})=\{\,i\in\{0,1,\dots,d\}\,;\;\bar{y}_i\in(0,1)\},$$

we have that $\bar{y}$ is an interior point of the set

$$\{\,y\in\Delta\,;\;y_i=0,\;i\in I^c(\bar{y})\}.$$

It follows that the submatrix $[D^2_{ij}\psi(\bar{y})]_{i,j\in I(\bar{y})}$ is nonnegative definite and then

$$\sum_{i,j=0}^{d}\bar{y}_i(\delta_{ij}-\bar{y}_j)\frac{\partial^2\psi(\bar{y})}{\partial y_i\partial y_j} = \sum_{i,j\in I(\bar{y})}\bar{y}_i(\delta_{ij}-\bar{y}_j)\frac{\partial^2\psi(\bar{y})}{\partial y_i\partial y_j}\ge 0.$$

Due to (2.7) this implies the first inequality in (2.6).

Concerning the second inequality, we observe that for any $\theta \in S$ it holds

$$\langle \theta - \bar{x}, D\varphi(\bar{x})\rangle \geq 0.$$

Actually, since S is convex, the function

$$[0,1] \to S, \quad t \mapsto \varphi((1-t)\bar{x} + t\theta)$$

is in $C^1(S)$ and attains its minimum at $t = 0$. This implies that

$$\langle \theta - \bar{x}, D\varphi(\bar{x})\rangle = \frac{d}{dt}^{+} \varphi((1-t)\bar{x} + t\theta)_{|t=0} \geq 0. \tag{2.8}$$

Next, we remark that

$$\langle \omega(\bar{x}) - |\widetilde{\omega}(\bar{x})|\bar{x}, D\varphi(\bar{x})\rangle = |\widetilde{\omega}(\bar{x})| \left\langle \frac{\omega(\bar{x})}{|\widetilde{\omega}(\bar{x})|} - \bar{x}, D\varphi(\bar{x}) \right\rangle.$$

Thus, since

$$|\widetilde{\omega}(\bar{x})| > 0, \quad \frac{1}{|\widetilde{\omega}(\bar{x})|}\,\omega(\bar{x}) \in S,$$

from (2.8) we conclude that the second inequality in (2.6) holds. □

As an immediate consequence of the previous proposition we have the following *maximum principle* for the operator $\mathcal{L}$.

COROLLARY 2.7. *Let m_1 and m_2 be two nonnegative functions in $C(S)$ and let $\lambda > 0$ and $\varphi \in C^2(S)$ be such that*

$$\lambda\varphi(x) - m_1(x)\mathrm{Tr}\,[C(x)D^2\varphi(x)] - m_2(x)\,\langle \omega(x) - |\widetilde{\omega}(x)|x, D\varphi(x)\rangle \geq 0, \quad x \in S.$$

Then $\varphi(x) \geq 0$, for any $x \in S$.

In particular, the operator $\mathcal{L} : D(\mathcal{L}) = C^2(S) \to C(S)$ defined by

$$\mathcal{L}\varphi(x) = \gamma(x)\,\mathrm{Tr}\,[C(x)D^2\varphi(x)] + \langle \omega(x) - |\widetilde{\omega}(x)|x, D\varphi(x)\rangle, \quad x \in S,$$

is dissipative and closable in $C(S)$, with dissipative closure. Moreover, if for any $i = 0, 1, \ldots, d$

$$\sigma_i(x) = \frac{\omega_i(x)}{\gamma(x)} \equiv \bar{\sigma}_i, \quad x \in S,$$

$\bar{\mathcal{L}}$ generates a C_0-semigroup of positive contractions in $C(S)$.

Proof. The first part of the corollary immediately follows from the Proposition 2.6. As $\mathcal{L}\,1 = 0$, this implies the dissipativity of $\mathcal{L}$ in $C(S)$ and, since $D(\mathcal{L})$ is dense in $C(S)$, $\mathcal{L}$ is closable with dissipative closure.

Now, thanks to the Lemma 2.2, if we show that, under the assumption that all σ_i are constant, the closure of the operator A introduced in (2.1) is m-dissipative, then $\bar{\mathcal{L}}$ is m-dissipative as well and, due to the Lumer-Phillips Theorem, $\bar{\mathcal{L}}$ is the generator of a C_0-semigroup of contractions in $C(S)$. Since $\bar{A}$ is dissipative, we have only to show that Range$(\lambda I - \bar{A}) = C(S)$, for some $\lambda > 0$, or, equivalently, Range$(\lambda I - A)$ is dense in $C(S)$, for some $\lambda > 0$. As we are assuming all σ_i constant, the operator $\lambda I - A$ maps the subset $\mathcal{P}_k(S)$ of polynomials on S of degree less or equal to k into itself. Moreover, the operator $\lambda I - A : \mathcal{P}_k(S) \to \mathcal{P}_k(S)$ is injective and then, as $\dim(\mathcal{P}_k(S)) < \infty$, it is also surjective. This implies that

$$\text{Range}(\lambda I - A) \supset \bigcup_{k \geq 0} \mathcal{P}_k(S) = \mathcal{P}(S),$$

so that Range $(\lambda I - A)$ is dense in $C(S)$.

The positivity of the semigroup follows from the fact that

$$\lambda\varphi(x) - \mathcal{L}\varphi(x) \geq 0, \quad \text{on } S \Longrightarrow \varphi(x) \geq 0 \quad \text{on } S.$$

□

Notice that in [3bis] the same result is also proved when the coefficients ω_i are not constant. But in this case the proof is more delicate, as the argument based on polynomials fails.

3. The one-dimensional case

In the 1-d case, the operator $\mathcal{L}$ is

$$\mathcal{L}\varphi(x) = \gamma(x)\,x(1-x)\,\varphi''(x) + (\omega_1(x) - (\omega_0(x) + \omega_1(x))\,x)\,\varphi'(x), \qquad x \in [0,1].$$

For simplicity we shall assume that for some $\theta \in (0,1)$

$$\omega_1(x) = \frac{\theta}{1-\theta}\omega_0(x), \qquad x \in S,$$

so that the operator $\mathcal{L}$: $D\mathcal{L} = C^2(S) \to C(S)$ can be rewritten as

$$\mathcal{L}\varphi(x) = \gamma(x)\,x(1-x)\,\varphi''(x) + \frac{\omega_0(x)}{1-\theta}(\theta - x)\varphi'(x), \qquad x \in [0,1].$$

Besides to the Hypothesis 2.1, in what follows we shall assume that the functions γ and ω_0 fulfill the following conditions.

HYPOTHESIS 3.1. *If we denote*

$$\sigma(x) = \frac{\omega_0(x)}{\gamma(x)}(\theta - x), \quad x \in [0, 1],$$

it holds

$$\int_0^1 \left(\frac{|\sigma(x) - \sigma(0)|}{x} + \frac{|\sigma(x) - \sigma(1)|}{1 - x} \right) dx < \infty. \tag{3.1}$$

Then the following results holds.

THEOREM 3.2. *Assume the Hypotheses* 2.1 *and* 3.1. *Then* $\mathcal{L}$ *is closable and its closure* $\bar{\mathcal{L}}$ *is the infinitesimal generator of a* C_0*-semigroup of contractions on* $C[0, 1]$*, which is positive and analytic of angle* $\pi/2$*. Moreover*

$$D(\bar{\mathcal{L}}) = \left\{ \varphi \in C^1[0, 1] \cap C^2(0, 1) \, ; \, \lim_{x \to 0^+, 1^-} x(1 - x)\varphi''(x) = 0 \right\}.$$

Before proceeding with the proof of the theorem, we recall a result proved by Metafune in [8].

THEOREM 3.3. *Let* m *be a continuous and strictly positive function on* $[0, 1]$ *and let* b *be any continuous function which satisfies* (3.1) *and such that* $b(0) > 0$ *and* $b(1) < 0$*. Set*

$$D(\mathcal{A}) = \left\{ \varphi \in C^1[0, 1] \cap C^2(0, 1) \, ; \, \lim_{x \to 0^+, 1^-} x(1 - x)\varphi''(x) = 0 \right\}$$

$$\mathcal{A}\varphi(x) = m(x)(x(1 - x)\varphi''(x) + b(x)\varphi'(x)), \quad x \in (0, 1).$$

Then $\mathcal{A}$ *generates a* C_0*-semigroup of contractions on* $C[0, 1]$ *which is positive and analytic.*

Proof of Theorem 3.2. We introduce the following operator

$$D(\widetilde{\mathcal{L}}) = \left\{ \varphi \in C^1[0, 1] \cap C^2(0, 1) \, ; \, \lim_{x \to 0^+, 1^-} x(1 - x)\varphi''(x) = 0 \right\},$$

$$\widetilde{\mathcal{L}}\varphi(x) = \mathcal{L}\varphi(x) \quad x \in (0, 1).$$

Due to Metafune's theorem, the operator $\widetilde{\mathcal{L}} : D(\widetilde{\mathcal{L}}) \subset C[0, 1] \to C[0, 1]$ is m-dissipative, densely defined and generates a positive analytic semigroup of angle $\pi/2$. Therefore, it is enough to show that $\bar{\mathcal{L}} = \widetilde{\mathcal{L}}$. Clearly $C^2[0, 1] \subset D(\widetilde{\mathcal{L}})$, hence $\bar{\mathcal{L}} \subseteq \widetilde{\mathcal{L}}$, as the graph norms of $D(\mathcal{L})$ and $D(\widetilde{\mathcal{L}})$ coincide on $C^2[0, 1]$ and $\widetilde{\mathcal{L}}$ is closed. The opposite inclusion follows once one proves that $C^2[0, 1]$ is a core for $\widetilde{\mathcal{L}}$.

We fix $\varphi \in D(\widetilde{\mathcal{L}})$. Our aim is to show that there exists a sequence $\{\varphi_n\} \subset C^2[0,1]$ such that

$$\lim_{n\to+\infty} \varphi_n = \varphi \quad \text{and} \quad \lim_{n\to+\infty} \widetilde{\mathcal{L}}_1 \varphi_n = \widetilde{\mathcal{L}}_1 \varphi, \quad \text{in } C[0,1], \tag{3.2}$$

where

$$\widetilde{\mathcal{L}}_1 \varphi(x) = \frac{\gamma(x)}{\omega_0(x)} x(1-x)\varphi''(x) + \frac{1}{1-\theta}(\theta - x)\varphi'(x), \qquad x \in S.$$

As

$$\widetilde{\mathcal{L}}\varphi(x) = \omega_0(x)\widetilde{\mathcal{L}}_1\varphi(x),$$

this immediately implies that $C^2(S)$ is a core for $D(\widetilde{\mathcal{L}})$.

We fix a sequence $\{f_n\} \subset C^1[0,1]$ which converges to $f := \varphi - \widetilde{\mathcal{L}}_1 \varphi$ in $C[0,1]$ and a sequence $\{\eta_n\}$ in $C^2[0,1]$, which converges to γ/ω_0 in $C[0,1]$, such that

$$l = \inf_{n\in\mathbb{N}} \min_{x\in S} \eta_n(x) > 0. \tag{3.3}$$

We denote by $\widetilde{\mathcal{L}}_{1,n}$ the operator defined as $\widetilde{\mathcal{L}}_1$, with γ/ω_0 replaced by η_n. As the functions η_n can be chosen in such a way that $1/\eta_n$ fulfills the Hypothesis 3.1, for each $n \in \mathbb{N}$ there exists a unique solution φ_n to the problem

$$\varphi = \widetilde{\mathcal{L}}_{1,n} \varphi + f_n. \tag{3.4}$$

We claim that $\varphi_n \in C^2[0,1]$ and (3.2) holds.

STEP 1. We prove that $\varphi_n \in C^2[0,1]$, for each $n \in \mathbb{N}$. Since η_n and f_n are in $C^1[0,1]$, we can consider the problem

$$\frac{2-\theta}{1-\theta}\,\psi = \alpha_n \psi'' + (\beta + \alpha_n')\psi' + f_n',$$

where

$$\alpha_n(x) = \eta_n(x)\, x(1-x), \qquad \beta(x) = \frac{1}{1-\theta}(\theta - x).$$

Notice that formally we get such a problem by differentiating each side of (3.4) and by setting $\psi = \varphi'$. Moreover, it can be written as

$$\frac{2-\theta}{1-\theta}\,\psi(x) = \eta_n(x)\,(x(1-x)\psi''(x) + b_n(x)\psi'(x)) + f_n'(x), \tag{3.5}$$

where

$$b_n(x) = \frac{1}{\eta_n(x)}\left(\frac{1}{1-\theta}(\theta - x) + \eta_n'(x)\, x(1-x)\right) + (1-2x).$$

Now, since the function b_n fulfills the conditions assumed for b in the Theorem 3.3, it follows that the equation (3.5) has a unique solution ψ_n in $D(\widetilde{\mathcal{L}})$. By setting

$$\chi_n(x) = \int_0^x \psi_n(t)\, dt$$

and by integrating each side of (3.5) on [0, 1], it follows

$$\chi_n = \alpha_n \chi_n'' + \beta\chi_n' + f_n - f_n(0) = \widetilde{\mathcal{L}}_{1,n}\chi_n + f_n - f_n(0), \quad x \in [0, 1].$$

By uniqueness this implies that $\varphi_n = \chi_n + f_n(0)$ and then $\varphi_n' = \psi_n$. As $\psi_n \in D(\widetilde{\mathcal{L}}) \subset C^1[0, 1]$, we have that $\varphi_n \in C^2[0, 1]$.

STEP 2. We prove an a priori estimate for the derivative of φ_n. More precisely we show that there exists a constant $c > 0$, independent of $n \in \mathbb{N}$, such that for every $\varphi \in C^2[0, 1]$

$$\|\varphi'\|_0 \le c\,(\|\varphi\|_0 + \|\widetilde{\mathcal{L}}_{1,n}\,\varphi\|_0), \quad n \in \mathbb{N}. \tag{3.6}$$

For $x \in [0, 1/2]$ we have

$$\begin{aligned} x\varphi''(x) + \frac{\theta}{\eta_n(0)(1-\theta)}\varphi'(x) &= \frac{\mathcal{L}_{1,n}\,\varphi(x)}{\eta_n(x)(1-x)} \\ &+ \left(\frac{\theta}{\eta_n(0)(1-\theta)} - \frac{\beta(x)}{\eta_n(x)(1-x)}\right)\varphi'(x) = g(x). \end{aligned}$$

If we set $c_n = \theta/\eta_n(0)(1-\theta) - 1$, by multiplying each side by x^{c_n}, we get

$$(x^{c_n+1}\varphi'(x))' = x^{c_n} g(x), \quad x \in [0, 1/2].$$

Hence, since $c_n + 1 > 0$ and $\varphi' \in C[0, 1/2]$, for any $x \in (0, 1/2]$ we easily get

$$\varphi'(x) = \frac{1}{x^{c_n+1}} \int_0^x t^{c_n} g(t)\, dt.$$

Thus, due to (3.3), for any $\delta \le 1/2$ and $x \in [0, \delta]$

$$\begin{aligned} |\varphi'(x)| &\le \frac{1}{c_n+1} \sup_{t\in[0,\delta]} |g(t)| \\ &\le \frac{2}{c_n+1} \sup_{t\in[0,\delta]} \left(\frac{1}{l}\,|\widetilde{\mathcal{L}}_{1,n}\,\varphi(t)| + \left|\frac{\theta}{\eta_n(0)(1-\theta)} - \frac{\beta(t)}{\eta_n(t)}\right| |\varphi'(t)|\right). \end{aligned}$$

Due to the continuity of η_n and β, recalling that $c_n + 1 = \theta/\eta_n(0)(1-\theta) \ge c$, for some positive constant c, we can find $\delta_1 \le 1/2$ such that

$$\sup_{t\in[0,\delta_1]} |\varphi'(t)| \le \frac{1}{cl} \sup_{t\in[0,\delta_1]} |\widetilde{\mathcal{L}}_{1,n}\,\varphi(t)|.$$

Similarly we can find $\delta_2 \leq 1/2$ such that

$$\sup_{t\in[1-\delta_2,1]} |\varphi'(t)| \leq \frac{1}{cl} \sup_{t\in[1-\delta_2,1]} |\widetilde{\mathcal{L}}_{1,n}\, \varphi(t)|.$$

Moreover, from standard estimates for the regular Sturm-Liouville problem, there exists $c > 0$ such that

$$\sup_{t\in[\delta_1,1-\delta_2]} |\varphi'(t)| \leq c \sup_{t\in[\delta_1,1-\delta_2]} (|\widetilde{\mathcal{L}}_{1,n}\, \varphi(t)| + |\varphi(t)|),$$

and this completes the proof.

STEP 3. Finally, we can prove (3.2). We denote by $\psi_n \in D(\widetilde{\mathcal{L}}_1) = D(\widetilde{\mathcal{L}})$ the unique solution of the problem

$$\psi = \widetilde{\mathcal{L}}_1 \psi + f_n, \tag{3.7}$$

for any $n \in \mathbb{N}$. Recalling that $\varphi_n = \widetilde{\mathcal{L}}_{1,n}\, \varphi_n + f_n$, by easy calculations we have

$$\varphi_n = \widetilde{\mathcal{L}}_1\, \varphi_n + f_n + h_n,$$

with

$$h_n(x) = (\eta_n(x) - \eta(x))\, x(1-x)\varphi_n''(x).$$

Then, as ψ_n is the solution of the problem (3.7), due to the dissipativity of $\widetilde{\mathcal{L}}_1$ we get

$$\|\varphi_n - \psi_n\|_0 \leq \|h_n\|_0.$$

Observe that

$$\begin{aligned} h_n(x) &= (\eta_n(x) - \eta(x))\, \frac{1}{\eta_n(x)} \left(\widetilde{\mathcal{L}}_{1,n}\varphi_n(x) - \frac{1}{1-\theta}(\theta - x)\varphi_n'(x) \right) \\ &= (\eta_n(x) - \eta(x))\, \frac{1}{\eta_n(x)} \left(\varphi_n(x) - f_n(x) - \frac{1}{1-\theta}(\theta - x)\varphi_n'(x) \right). \end{aligned}$$

Hence

$$\|h_n\|_0 \leq \frac{1}{l}\, \|\eta_n - \eta\|_0\, (\|\varphi_n\|_0 + \|f_n\|_0 + c\, \|\varphi_n'\|_0).$$

Recalling (3.6), as $\|\widetilde{\mathcal{L}}_1\varphi_n\|_0 \leq \|f_n\|_0 + \|\varphi_n\|_o$ and $\|\varphi_n\|_0 \leq \|f_n\|_0 \leq c$ we have

$$\|h_n\|_0 \leq c\, \|\eta_n - \eta\|_0 \sup_{n\in\mathbb{N}} (\|f_n\|_0 + \|\widetilde{\mathcal{L}}_{1,n}\, \varphi_n\|_0) \leq c\, \|\eta_n - \eta\|_0\, .$$

This implies that h_n converges to 0 in $C\,[0,1]$. Moreover, as $\varphi-\psi_n = \widetilde{\mathcal{L}}_1\,(\varphi-\psi_n)+f-f_n$, we have

$$\|\varphi-\psi_n\|_0 \le \|f-f_n\|_0 \to 0,$$

as n goes to infinity, so that

$$\|\varphi-\varphi_n\|_0 \le \|\varphi_n-\psi_n\|_0 + \|\varphi-\psi_n\|_0 \le \|h_n\|_0 + \|f-f_n\|_0 \to 0,$$

as n goes to infinity.

Finally, we can conclude, as we have

$$\lim_{n\to+\infty} \widetilde{\mathcal{L}}_1\,\varphi_n = \lim_{n\to+\infty} \varphi_n - f_n - h_n = \varphi - f = \widetilde{\mathcal{L}}_1\,\varphi.$$

□

REMARK 3.4. It appears that in dimension $d = 1$ the domain of the closure of $\mathcal{L}$ is neither

$$\left\{\varphi \in\ C\,[0,1]\cap C^2\,(0,1)\ ;\ \lim_{x\to 0^+,1^-} \mathcal{L}\varphi(x) \text{ exists}\right\}, \qquad \text{maximal domain}$$

nor

$$\left\{\varphi \in\ C\,[0,1]\cap C^2\,(0,1)\ ;\ \lim_{x\to 0^+,1^-} \mathcal{L}\varphi(x) = 0\right\}, \qquad \text{Ventcel's boundary conditions.}$$

4. The multidimensional case in L^2 spaces

In this section we study the operator $\mathcal{L}$ in the space of square integrable functions, with respect to a suitable measure ν. Such a measure is chosen absolutely continuous with respect to the Lebesgue measure and an explicit formula for its density ρ is given. Moreover, under suitable conditions on the coefficients σ_i, the measure ν can be taken invariant and symmetrizing for the operator $\mathcal{L}$.

For each $i = 0, 1, \ldots, d$, let α_i be a continuous function on S such that

$$\bar{\alpha}_i := \min_{x\in S} \alpha_i(x) > -1. \tag{4.1}$$

For any $x \in \overset{\circ}{S}$ we define the function $\rho_\alpha(x)$ by setting

$$\rho_\alpha(x) = \prod_{i=0}^{d} x_i^{\alpha_i(x)} = \exp\left(\sum_{i=0}^{d} \alpha_i(x)\log x_i\right), \tag{4.2}$$

where we recall $x_0 = 1 - \sum_{i=1}^d x_i$. If $\bar{\alpha}_i$ is as in (4.1), we have

$$\int_S \rho_\alpha(x)\,dx = \int_S \prod_{i=0}^d x_i^{\alpha_i(x)}\,dx \leq \int_S \prod_{i=0}^d x_i^{\bar{\alpha}_i}\,dx \leq \prod_{i=0}^d \int_0^1 t^{\bar{\alpha}_i}\,dt = \prod_{i=0}^d \frac{1}{\bar{\alpha}_i + 1},$$

so that $\rho_\alpha \in L^1(S, dx)$. Hence, as $\rho_\alpha(x) > 0$, we can introduce the probability measure ν_α on S defined by

$$\nu_\alpha(dx) = \langle \rho_\alpha \rangle^{-1}\ \rho_\alpha(x)\,dx = \left(\int_S \rho_\alpha(y)\,dy \right)^{-1} \rho_\alpha(x)\,dx.$$

The measure ν_α is a generalization of the *Dirichlet distribution* with parameters $\alpha_i(x) - 1$, which depend on $x \in S$.

Our aim is to study the operator $\mathcal{L}$ in the Hilbert space $H = L^2(S, \nu_\alpha)$, endowed with the inner product

$$\langle \varphi, \psi \rangle_H = \int_S \varphi(x)\psi(x)\,d\nu_\alpha(x) = \langle \rho_\alpha \rangle^{-1} \int_S \varphi(x)\psi(x)\,\rho_\alpha(x)\,dx,$$

and the corresponding norm $\|\cdot\|_H$.

In what follows we shall assume that the following assumptions hold.

HYPOTHESIS 4.1. If $b(x)$ is the vector of components

$$b_i(x) = \frac{1 - \sigma_i(x)}{x_i} - \frac{1 - \sigma_0(x)}{x_0}, \quad x \in \mathring{S}, \tag{4.3}$$

for each $i = 0, 1, \ldots, d$ there exists a function $\alpha_i \in C^1(S)$ which fulfills (4.1), such that

$$\langle C(D \log \rho_\alpha + b), D \log \rho_\alpha + b \rangle \in L^\infty(S). \tag{4.4}$$

4.1. *An integration by parts formula*

The first important step is given by the following identity.

THEOREM 4.2. *Assume that the Hypotheses* 2.1 *and* 4.1 *hold. Then, if $\mathcal{L}$ is the operator defined by* (2.2), *for any $\varphi \in C^2(S)$ and $\psi \in C^1(S)$ it holds*

$$\int_S \langle C\,D\varphi, D\psi \rangle\ d\nu_\alpha + \int_S \langle C\,D\varphi, D \log \rho_\alpha + b \rangle\,\psi\,d\nu_\alpha = -\langle \mathcal{L}\varphi, \psi \rangle_H. \tag{4.5}$$

Proof. For any $\delta \in (0, 1/(d+1))$ we define

$$S_\delta = \{x \in S\,;\ x \in [\delta, 1-\delta]^d,\ x_0 \in [\delta, 1-\delta]\}.$$

Notice that since we have taken $\delta \in (0, 1/(d+1))$, the definition above is meaningful and

$$\overset{\circ}{S} = \bigcup_{n>d+1} S_{\frac{1}{n}}.$$

Moreover, it is immediate to check that

$$\partial S_\delta = \bigcup_{i=0}^{d} \{x \in S\,;\; x_i = \delta\} = \bigcup_{i=0}^{d} \partial S_{\delta,i}.$$

This implies that if we denote by n^k the exterior normal to $\partial S_{\delta,k}$, we have

$$n^k(x) = -e_k, \quad 1 \le k \le d, \qquad n^0(x) = \frac{1}{\sqrt{d}}(1, \dots, 1)^t.$$

Now, for any $\delta < 1/(d+1)$ we have

$$\int_{S_\delta} \langle CD\varphi, D\psi\rangle \, d\nu_\alpha = \int_{S_\delta} \sum_{i,j=1}^{d} c_{ij}(x) D_i\varphi(x) D_j\psi(x)\, d\nu_\alpha = I_{\delta,0} - I_{\delta,1} - I_{\delta,2},$$

where

$$I_{\delta,0} = \langle\rho_\alpha\rangle^{-1} \int_{S_\delta} \sum_{i,j=1}^{d} D_j(\rho_\alpha\, c_{ij} D_i\varphi\, \psi)\, dx$$

$$I_{\delta,1} = \langle\rho_\alpha\rangle^{-1} \int_{S_\delta} \sum_{i,j=1}^{d} c_{ij} D_{ij}\varphi\, \psi\rho_\alpha\, dx$$

$$I_{\delta,2} = \langle\rho_\alpha\rangle^{-1} \int_{S_\delta} \sum_{i,j=1}^{d} D_i\varphi\, \psi\, D_j(c_{ij}\rho_\alpha)\, dx.$$

By the divergence theorem we have

$$\begin{aligned} I_{\delta,0} &= \langle\rho_a\rangle^{-1} \sum_{i,j=1}^{d} \int_{\partial S_\delta} n_j c_{ij}\rho_\alpha D_i\varphi\, \psi\, d\sigma \\ &= \langle\rho_\alpha\rangle^{-1} \sum_{k=0}^{d} \sum_{i,j=1}^{d} \int_{\partial S_{\delta,k}} n_j^k c_{ij}\rho_\alpha D_i\varphi\, \psi\, d\sigma, \end{aligned}$$

where n_j denotes the j-th component of the exterior normal to ∂S_δ and n_j^k denotes the j-th component of the exterior normal to $\partial S_{\delta,k}$. Our aim is to show that

$$\lim_{\delta\to 0^+} I_{\delta,0} = 0.$$

To this purpose it is sufficient to show that for each $k = 0, \dots, d$ and $i = 1, \dots, d$

$$\lim_{\delta \to 0^+} \int_{\partial S_{\delta,k}} \sum_{j=1}^d n_j^k c_{ij} \rho_\alpha D_i \varphi \, \psi \, d\sigma = 0. \tag{4.6}$$

If $x \in \partial S_{\delta,k}$, for some $1 \leq k \leq d$, we have

$$\sum_{j=1}^d c_{ij}(x) n_j^k(x) \rho_\alpha(x) = -\sum_{j=1}^d c_{ij}(x) \delta_{kj} \rho_\alpha(x) = -c_{ik}(x)\rho_\alpha(x)$$

$$= -x_i(\delta_{ik} - x_k) \prod_{h=0}^d x_h^{\alpha_h(x)} = -\delta_{ik} \prod_{h=0}^d x_h^{\alpha_h(x)+\delta_{hi}} + \prod_{h=0}^d x_h^{\alpha_h(x)+\delta_{hi}+\delta_{hk}}.$$

Now, if $x \in \partial S_{\delta,k}$ we have that x_k converges to 0, as δ goes to 0. Thus, recalling that $\varphi \in C^2(S)$ and $\psi \in C^1(S)$, from the dominated convergence theorem it immediately follows (4.6). Next, if $x \in \partial S_{\delta,0}$ we have

$$\sum_{j=1}^d c_{ij}(x) n_j^0(x) \rho_\alpha(x) = \frac{1}{\sqrt{d}} \sum_{j=1}^d x_i(\delta_{ij} - x_j) \prod_{h=0}^d x_h^{\alpha_h(x)}$$

$$= \frac{1}{\sqrt{d}} \prod_{h=0}^d x_h^{\alpha_h(x)+\delta_{hi}} - \frac{1}{\sqrt{d}} \prod_{h=0}^d x_h^{\alpha_h(x)+\delta_{hi}}(1 - x_0) = \frac{1}{\sqrt{d}} \prod_{h=0}^d x_h^{\alpha_h(x)+\delta_{hi}+\delta_{h0}},$$

and then by arguing as above (4.6) follows also for $k = 0$.

Concerning $I_{\delta,1}$ we have

$$I_{\delta,1} = \int_{S_\delta} \mathrm{Tr}\,[C D^2 \varphi] \, \psi \, d\nu_\alpha.$$

Finally, for $I_{\delta,2}$ we have

$$\begin{aligned} I_{\delta,2} &= \int_{S_\delta} \sum_{i,j=1}^d D_j(c_{ij}\rho_\alpha) \psi \, D_i \varphi \, dx \\ &= \int_{S_\delta} \langle C D\varphi, D \log \rho_\alpha \rangle \, \psi \, d\nu_\alpha + \int_{S_\delta} \sum_{i,j=1}^d D_i \varphi D_j c_{ij} \psi \, d\nu_\alpha. \end{aligned}$$

For each $i = 1, \dots, d$ we have

$$\sum_{j=1}^d D_j c_{ij} = \sum_{j=1}^d D_j(x_j(\delta_{ij} - x_i)) = 1 - (d+1)x_i.$$

Hence, we get

$$I_{\delta,2} = \int_{S_\delta} \langle CD\varphi, D\log\rho_\alpha\rangle\, \psi\, d\nu_\alpha + \int_{S_\delta} \langle D\varphi, e-(d+1)x\rangle\, \psi\, d\nu_\alpha.$$

This implies that

$$\int_{S_\delta} \langle CD\varphi, D\psi\rangle\, d\nu_\alpha = I_{\delta,0} - \langle \mathcal{L}\varphi, \psi\rangle_H - \int_{S_\delta} \langle CD\varphi, D\log\rho_\alpha\rangle\, \psi\, d\nu_\alpha$$

$$- \int_{S_\delta} \langle CD\varphi, C^{-1}\,(e-\sigma+(|\widetilde{\sigma}|-(d+1))x)\rangle \psi\, d\nu_\alpha.$$

Thanks to (2.5) we have

$$C^{-1}(x)\,(e-\sigma(x)+(|\widetilde{\sigma}(x)|-(d+1))x)$$

$$= \left(\Lambda^{-1}(x) + \frac{e\otimes e}{x_0}\right)(e+(|\widetilde{\sigma}(x)|-(d+1))\,x-\sigma(x)),$$

so that

$$C^{-1}(x)\,(e-\sigma(x)+(|\widetilde{\sigma}(x)|-(d+1))x)$$

$$= \Lambda^{-1}(x)e + (|\widetilde{\sigma}(x)|-(d+1))\,e - \Lambda^{-1}(x)\sigma(x)$$

$$+ \frac{d}{x_0}e + (|\widetilde{\sigma}(x)|-(d+1))\,\frac{1-x_0}{x_0}e - \frac{\langle\sigma(x), e\rangle}{x_0}e.$$

Rearranging all terms we easily get

$$C^{-1}(x)\,(e-\sigma(x)+(|\widetilde{\sigma}(x)|-(d+1))x) = b(x),$$

where $b(x)$ is the vector defined in (4.3), so that, for any $\delta \in (0, 1/(d+1)$ we have

$$\int_{S_\delta} \langle CD\varphi, D\psi\rangle\, d\nu_\alpha + \int_{S_\delta} \langle CD\varphi, D\log\rho_\alpha + b\rangle\, \psi\, d\nu_\alpha = I_{\delta,0} - \langle \mathcal{L}\varphi, \psi\rangle_H\,.$$

Thus, thanks to the Hypothesis 4.1, if we take the limit as δ goes to zero, (4.5) follows. □

The integration by parts formula (4.5) has the following important consequence.

THEOREM 4.3. *Under the Hypotheses* 2.1 *and* 4.1, *the operator $\mathcal{L}$ is closable and its closure $\bar{\mathcal{L}}$ generates a C_0-semigroup on $L^2(S, \nu_\alpha)$.*

Proof. Thanks to (4.5), if $\varphi \in D(\mathcal{L})$ we have

$$\langle \mathcal{L}\varphi, \varphi\rangle_H = -\int_S \langle CD\varphi, D\varphi\rangle \, d\nu_\alpha - \int_S \langle CD\varphi, D\log\rho_\alpha + b\rangle\, \varphi\, d\nu_\alpha.$$

By using first the Hölder inequality and then the Young inequality, thanks to (4.4) we get

$$\begin{aligned}&\int_S \langle CD\varphi, D\log\rho_\alpha + b\rangle\, \varphi\, d\nu_\alpha\\ &\quad\leq \int_S \langle CD\varphi, D\varphi\rangle^{1/2} \langle C\,(D\log\rho_\alpha + b)\,, D\log\rho_\alpha + b\rangle^{1/2}\, \varphi\, d\nu_\alpha\\ &\quad\leq H_\alpha \left(\int_S \langle CD\varphi, D\varphi\rangle\, d\nu_\alpha\right)^{1/2} \|\varphi\|_H \leq \int_S \langle CD\varphi, D\varphi\rangle\, d\nu_\alpha + \frac{1}{4} H_\alpha^2\, \|\varphi\|_H^2,\end{aligned}$$

where

$$H_\alpha^2 = \sup_{x\in \mathring{S}} \langle C(x)(D\log\rho_\alpha(x) + b(x)), D\log\rho_\alpha(x) + b(x)\rangle. \tag{4.7}$$

This implies that

$$\langle \mathcal{L}\varphi, \varphi\rangle_H \leq \frac{1}{4} H_\alpha^2\, \|\varphi\|_H^2.$$

Thus, by setting $c_0 = H_\alpha^2/4$, it follows that $\mathcal{L} - c_0$ is dissipative. Now, as $D(\mathcal{L})$ is dense in H, we can conclude that $\mathcal{L} - c_0$ is closable and its closure $\overline{\mathcal{L} - c_0} = \overline{\mathcal{L}} - c_0$ is dissipative. Now, as proved in [3bis], the closure of $\mathcal{L}$ in $C(S)$ generates a C_0-semigroup of contractions. Then, due to the Lumer-Phillips theorem the image of $\lambda - \mathcal{L}$ is dense in $C(S)$. As $C(S)$ is dense in $L^2(S, \nu_\alpha)$, this implies that the image of $\lambda - \mathcal{L}$ is dense in $L^2(S, \nu_\alpha)$ and then, by using again the Lumer-Phillips theorem, $\bar{\mathcal{L}}$ generates a C_0-semigroup. □

4.2. *Existence of an invariant measure*

Under stronger conditions on the coefficients σ_i, due to the integration by parts formula (4.5) it is possible to exhibit an invariant measure for the operator $\mathcal{L}$ which is even symmetrizing.

HYPOTHESIS 4.4. *For each* $i = 0, 1, \ldots, d$, *there exists* $\sigma_i \in C([0, 1])$ *such that*

$$\sigma_i(x) = \sigma_i(x_i), \qquad x \in S.$$

THEOREM 4.5. *Assume the Hypotheses* 2.1 *and* 4.4. *If we define for any* $i = 0, 1, \ldots, d$

$$\alpha_i(t) = \frac{\int_t^1 \frac{1-\sigma_i(s)}{s}\, ds}{\log t}, \qquad t \in (0, 1), \tag{4.8}$$

we have that α_i *can be continuously extended to* $[0,1]$ *and* $\alpha_i(t) > -1$*. Moreover, if we define*

$$\rho_\alpha(x) = \prod_{i=0}^{d} x_i^{\alpha_i(x_i)}, \qquad x \in \overset{\circ}{S},$$

the probability measure $\nu_\alpha(dx) = \langle\rho_\alpha\rangle^{-1}\rho_\alpha\, dx$ *is invariant and symmetrizing for the operator* $\mathcal{L}$*.*

Proof. For each $i = 0, 1, \ldots, d$, the function α_i is well defined and continuous in $(0, 1)$. Thanks to the theorem of De L'Hopital, we have

$$\lim_{t\to 0^+} \frac{\int_t^1 \frac{1-\sigma_i(s)}{s}\, ds}{\log t} = \sigma_i(0) - 1$$

and similarly

$$\lim_{t\to 1^-} \frac{\int_t^1 \frac{1-\sigma_i(s)}{s}\, ds}{\log t} = \sigma_i(1) - 1.$$

Thus α_i can be continuously extended at $t = 0$ and $t = 1$. Moreover it is immediate to check that $\alpha_i(t) > -1$, for any $t \in [0, 1]$.

Next, if we show that $D\log\rho_\alpha + b = 0$, due to (4.5) we have that

$$\int_S \langle CD\varphi, D\psi\rangle\, d\nu_\alpha = -\langle\mathcal{L}\varphi, \psi\rangle_H ,$$

and hence ν_α is invariant and symmetrizing. For any $j = 1, \ldots, d$ we have

$$D_j \log\rho_\alpha = \sum_{k=0}^{d} D_j\,(\alpha_k(x)\log x_k) = \frac{d}{dx_j}(\alpha_j(x_j)\log x_j) - \frac{d}{dx_0}\,(\alpha_0(x_0)\log x_0)$$

and then, since for any $i = 0, \ldots, d$

$$\frac{d}{dx_i}\,(\alpha_i(x_i)\log x_i) = \frac{\sigma_i(x_i) - 1}{x_i},$$

we immediately have that $D\log\rho_\alpha + b = 0$. □

REMARK 4.6. By taking the formula (4.8) for α_i as a starting point, we can construct an example of functions α_i which fulfill the condition (4.4).

Assume that for any $i = 0, 1, \ldots, d$

$$\sigma_i(x) = s_i(x_i)f_i(x) + \bar{\sigma}_i,$$

for some $s_i \in C([0,1])$, $f_i \in C^1(S)$ and $\bar{\sigma}_i > 0$. We define

$$\alpha_i(x) = \frac{\int_{x_i}^1 \frac{1-\sigma_i(x_i(s))}{s}\, ds}{\log x_i}, \qquad x \in \overset{\circ}{S},$$

where $(x_i(s))_j = x_j(1-\delta_{ij}) + s\delta_{ij}$. As seen in the proof of the previous theorem, α_i can be extended as a continuous function on S. For any $i, j = 1, \dots, d$ we have

$$D_j\,(\alpha_i \log x_i)\,(x) = \frac{\sigma_j(x) - 1}{x_j}\,\delta_{ij} - D_j f_i(x) \int_{x_i}^1 \frac{s_i(s)}{s}\, ds\,(1-\delta_{ij})$$

and

$$D_j\,(\alpha_0 \log x_0)\,(x) = \frac{1-\sigma_0(x)}{x_0} - D_j f_0(x) \int_{x_0}^1 \frac{s_0(s)}{s}\, ds.$$

If we show that

$$\sup_{x \in \overset{\circ}{S}} \left| \sum_{k=0}^d D_j(\alpha_k(x) \log x_k) + \frac{1-\sigma_j(x)}{x_j} - \frac{1-\sigma_0(x)}{x_0} \right| < \infty,$$

we immediately have (4.2) Therefore, we need to assume that

$$\sup_{x \in \overset{\circ}{S}} \left| \sum_{\substack{i=0 \\ i \neq j}}^d D_j f_i(x) \int_{x_i}^1 \frac{s_i(s)}{s}\, ds \right| < \infty.$$

This is satisfied for example if we assume that $s_i(s)/s \in L^1(0,1)$ or if the functions f_i are constant in a neighborhood of the boundary of S.

4.3. *The variational formulation*

For any $\varphi, \psi \in C^1(S)$, we define the semi-inner product

$$\langle\langle \varphi, \psi \rangle\rangle_{1,C} = \int_S \langle C D\varphi, D\psi \rangle \, d\nu_\alpha \tag{4.9}$$

with the corresponding semi-norm $[\cdot]_{1,C}$. In what follows we shall denote by $W_C^{1,2}(S, \nu_\alpha)$ the completion of $C^1(S)$ with respect to the norm $\|\cdot\|_{1,C}$ induced by the inner product

$$\langle \varphi, \psi \rangle_{1,C} = \langle \varphi, \psi \rangle_H + \langle\langle \varphi, \psi \rangle\rangle_{1,C}\,.$$

Similarly, for any $\varphi, \psi \in C^2(S)$ we define the semi-inner product

$$\langle\langle \varphi, \psi \rangle\rangle_{2,C} = \int_S \mathrm{Tr}\,[(C D^2\varphi)(C D^2\psi)]\, d\nu_\alpha,$$

with the corresponding semi-norm $[\cdot]_{2,C}$. We shall denote by $W_C^{2,2}(S,\nu_\alpha)$ the completion of $C^2(S)$ with respect to the norm $\|\cdot\|_{2,C}$ induced by the inner product

$$\langle\varphi,\psi\rangle_{2,C} = \langle\varphi,\psi\rangle_{1,C} + \langle\!\langle\varphi,\psi\rangle\!\rangle_{2,C}\,.$$

From the definition we are giving for $W_C^{1,2}(S,\nu_\alpha)$ and $W_C^{2,2}(S,\nu_\alpha)$, we can imbed $W_C^{1,2}(S,\nu_\alpha)$ in $L^2(S,\nu_\alpha)$ and $W_C^{2,2}(S,\nu_\alpha)$ in $W_C^{1,2}(S,\nu_\alpha)$, with continuous embeddings. Clearly, we have

$$\|\varphi\|_H \le \|\varphi\|_0, \qquad \varphi \in C(S) \tag{4.10}$$

and for $i = 1, 2$

$$\|\varphi\|_{i,C} \le \|\varphi\|_i, \qquad \varphi \in C^i(S). \tag{4.11}$$

Thus, since $C^1(S)$ is dense in $C(S)$, from (4.10) it follows that $W_C^{1,2}(S,\nu_\alpha)$ is dense in $L^2(S,\nu_\alpha)$. Similarly, since $C^2(S)$ is dense in $C^1(S)$, from (4.11) it follows that $C^2(S)$ is dense in $W_C^{1,2}(S,\nu_\alpha)$, hence $W_C^{2,2}(S,\nu_\alpha)$ is dense in $L^2(S,\nu_\alpha)$.

In the sequel we shall denote by V the Hilbert space $W_C^{1,2}(S,\nu_\alpha)$, and by $\|\cdot\|_V$, $[\cdot]_V$, $\langle\cdot,\cdot\rangle_V$ and $\langle\!\langle\cdot,\cdot\rangle\!\rangle_V$ respectively the norm, the seminorm, the inner product and the semi inner product in $W_C^{1,2}(S,\nu_\alpha)$, as defined above.

For any $\varphi, \psi \in C^1(S)$ we define the bilinear form

$$a_\alpha(\varphi,\psi) = \int_S \langle CD\varphi, D\psi\rangle\, d\nu_\alpha + \int_S \langle CD\varphi, D\log\rho_\alpha + b\rangle\, \psi\, d\nu_\alpha. \tag{4.12}$$

As V is the completion of $C^1(S)$ with respect to the norm induced by the inner product

$$\langle\varphi,\psi\rangle_V = \int_S \varphi\,\psi\, d\nu_\alpha + \int_S \langle CD\varphi, \psi\rangle\, d\nu_\alpha,$$

it is immediate to check that a_α can be extended as a bilinear form on $V \times V$. Moreover, due to (4.5), if $\varphi \in D(\mathcal{L})$ and $\psi \in C^1(S)$ we have

$$a_\alpha(\varphi,\psi) = -\langle\mathcal{L}\varphi,\psi\rangle_H\,.$$

LEMMA 4.7. *Under the Hypotheses* 2.1 *and* 4.1, *the bilinear form* $a_\alpha : V \times V \to \mathbb{R}$ *is continuous, that is there exists a constant* c *such that*

$$|a_\alpha(\varphi,\psi)| \le c\,\|\varphi\|_V\,\|\psi\|_V, \quad \varphi,\psi \in V. \tag{4.13}$$

Moreover, for any $\delta < 1$ *there exists* $c_\delta \in \mathbb{R}$ *such that*

$$|a_\alpha(\varphi,\varphi)| \ge \delta\,\|\varphi\|_V^2 - c_\delta\,\|\varphi\|_H^2, \qquad \varphi \in V. \tag{4.14}$$

Proof. We first prove (4.13). From (4.12) and (4.9) we have

$$a_\alpha(\varphi, \psi) = \langle\langle \varphi, \psi \rangle\rangle_V + \int_S \langle C D\varphi, D \log \rho_\alpha + b \rangle\, \psi \, d\nu_\alpha.$$

Thus, thanks to (4.4) and to the Hölder inequality, we have

$$|a_\alpha(\varphi, \psi)| \leq \|\varphi\|_V \|\psi\|_V + H_\alpha\, [\varphi]_V\, \|\psi\|_H,$$

which easily yields (4.13).

Now, let us prove (4.14). We have

$$a_\alpha(\varphi, \varphi) = [\varphi]_V^2 + \int_S \langle C D\varphi, D \log \rho_\alpha + b \rangle\, \varphi \, d\nu_\alpha.$$

If we fix $\delta < 1$, we have that $(1-\delta)/2 > 0$ and then, by using the Hölder inequality and the Young inequality, we get

$$\left| \int_S \langle C D\varphi, D \log \rho_\alpha + b \rangle\, \varphi \, d\nu_\alpha \right| \leq \frac{1-\delta}{2} [\varphi]_V^2 + \frac{H_\alpha^2}{2(1-\delta)}\, \|\varphi\|_H^2.$$

Hence it follows

$$a_\alpha(\varphi, \varphi) \geq \frac{1+\delta}{2} [\varphi]_V^2 - \frac{H_\alpha^2}{2(1-\delta)} \|\varphi\|_H^2.$$

Thanks to the Young inequality, for any $\epsilon > 0$ we have

$$[\varphi]_V^2 = (\|\varphi\|_V - \|\varphi\|_H)^2 \geq (1-\epsilon)\, \|\varphi\|_V^2 + \left(1 - \frac{1}{\epsilon}\right) \|\varphi\|_H^2$$

and then, if we choose $\epsilon = (1-\delta)/(1+\delta)$, by easy calculations we obtain

$$a_\alpha(\varphi, \varphi) \geq \delta\, \|\varphi\|_V^2 - \frac{H_\alpha^2 + 2\delta(1+\delta)}{2(1-\delta)} \|\varphi\|_H^2.$$

This yields (4.14), with

$$c_\delta = \frac{H_\alpha^2 + 2\delta(1+\delta)}{2(1-\delta)} \tag{4.15}$$

□

Now, due to (4.13), for any fixed $\psi \in V$ the linear mapping $V \to \mathbb{R}$, $\varphi \mapsto a_\alpha(\varphi, \psi)$, is continuous and then there exists some $f_\psi \in V^\star$ such that

$$a_\alpha(\varphi, \psi) = f_\psi(\varphi), \quad \varphi \in V.$$

This allows us to define the operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ in the classical way

$$D(\mathcal{A}) = \{\psi \in V \ : \ f_\psi \in H\}, \quad A\psi = f_\psi. \tag{4.16}$$

For any $\delta < 1$ we define the operator

$$\mathcal{A}_\delta : D(\mathcal{A}) \subset H \to H, \quad A_\delta = \mathcal{A} + c_\delta I,$$

where c_δ is the constant introduced in (4.15). By using the Lemma 4.8 and the Lax-Milgram theorem, it is possible to show that Range$(\mathcal{A}_\delta) = H$ and $0 \in \rho(\mathcal{A}_\delta)$. Moreover $D(\mathcal{A}_\delta) = D(\mathcal{A})$ is dense in V (and hence in H) and $-\mathcal{A}_\delta$ is a maximal dissipative operator which generates an analytic semigroup (for the proofs of these facts see for example [13]-Section 2.2 and Theorem 3.6.1). Thus we have the following result.

PROPOSITION 4.8. *Under the Hypotheses* 2.1 *and* 4.1 *the C_0-semigroup generated by $\bar{\mathcal{L}}$ is analytic and $\rho(\bar{\mathcal{L}}) \supseteq (H_\alpha^2/2, +\infty)$, where H_α is the constant introduced in* (4.7).

Proof. For any $\delta < 1$ we have $\mathcal{A} = \mathcal{A}_\delta - c_\delta I$ and then it immediately follows that $D(-\mathcal{A})$ is dense in H and $-\mathcal{A}$ generates an analytic semigroup in H. Moreover, as $-\mathcal{A}_\delta$ is dissipative and $0 \in \rho(-\mathcal{A}_\delta)$, we have $\rho(-\mathcal{A}_\delta) \supseteq [0, +\infty)$ so that $\rho(-\mathcal{A}) \supseteq [c_\delta, +\infty)$. Now, as our arguments work for any $\delta < 1$ and $c_\delta \downarrow H_\alpha^2/2$, as δ goes to zero, we conclude that $\rho(-\mathcal{A}) \supseteq (H_\alpha^2/2, +\infty)$.

Finally, let us prove that $\overline{\mathcal{L}} \subseteq -\mathcal{A}$. We have already seen that if $\varphi \in D(\mathcal{L})$, then for any $\psi \in C^1(S)$

$$a_\alpha(\varphi, \psi) = -\langle \mathcal{L}\varphi, \psi \rangle_H$$

and, since $C^1(S)$ is dense in V, the same relation holds for any $\psi \in V$. Recalling the definition of $\mathcal{A}$, this implies that $\varphi \in D(\mathcal{A})$ and $\mathcal{A}\varphi = -\mathcal{L}\varphi$, so that $\mathcal{L} \subseteq -\mathcal{A}$. As $\mathcal{A}$ is a closed operator, this yields $\overline{\mathcal{L}} \subseteq -\mathcal{A}$. Moreover, since $\bar{\mathcal{L}}$ generates a C_0-semigroup, it is the only extension of $\mathcal{L}$ and then $\bar{\mathcal{L}} = -\mathcal{A}$.

□

An important consequence of the previous proposition is the existence of a unique solution for the variational problem associated with the operator $\mathcal{L}$.

THEOREM 4.9. *Under the Hypotheses* 2.1 *and* 4.1, *for any $f \in H$ and $\lambda > H_\alpha^2/2$ there exists a unique solution $\varphi \in D(\bar{\mathcal{L}})$ to the problem*

$$\lambda \langle \varphi, \psi \rangle_H + \int_S \langle CD\varphi, D\psi \rangle \, d\nu_\alpha + \int_S \langle CD\varphi, D\log\rho_\alpha + b \rangle \, \psi \, d\nu_\alpha = \langle f, \psi \rangle_H \,,$$

for any $\psi \in V$. Furthermore, if $\varphi \in D(\mathcal{L})$, then it is a solution of (1.1), *that is*

$$\lambda\varphi - \mathcal{L}\varphi = f.$$

4.4. *Characterization of* $D(\overline{\mathcal{L}})$

In this subsection we are giving a characterization of $D(\overline{\mathcal{L}})$ in H. Moreover, in the case the coefficients σ_i fulfill the Hypothesis 4.4 (so that there exists an invariant measure ν_α which is symmetrizing), we give a factorization of $\overline{\mathcal{L}}$ which provides a characterization of the domain of the square root of $-\overline{\mathcal{L}}$.

The crucial step is the following identity.

PROPOSITION 4.10. *Under the Hypotheses* 2.1 *and* 4.1*, for any* $\varphi \in C^2(S)$ *and* $\lambda \in \mathbb{R}$ *it holds*

$$\begin{aligned}
&[\varphi]^2_{2,C} + \int_S (\lambda + |\widetilde{\sigma}|)\, \langle CD\varphi, D\varphi\rangle\, d\nu_\alpha + \int_S \langle D\log\rho_\alpha + b, CD^2\varphi\,(CD\varphi)\rangle d\nu_\alpha \\
&\quad - \int_S \mathrm{Tr}\,[(CD\varphi\otimes D\varphi)D\sigma]\, d\nu_\alpha + \int_S \langle (CD\varphi\otimes D\varphi)\,x, D|\widetilde{\sigma}|\rangle\, d\nu_\alpha \\
&\quad = \|\mathcal{L}\varphi\|^2_H - \lambda\,\langle\mathcal{L}\varphi,\varphi\rangle_H - \int_S \langle CD\varphi, D\log\rho_\alpha + b\rangle\,(\lambda\varphi - \mathcal{L}\varphi)\, d\nu.
\end{aligned}\tag{4.17}$$

Proof. If we prove (4.17) for $\varphi \in C^3(S)$, then the case of $\varphi \in C^2(S)$ follows by a standard approximation argument. Thus, let us fix $\varphi \in C^3(S)$ and $\lambda \in \mathbb{R}$ and let us define $f = \lambda\varphi - \mathcal{L}\varphi$, that is

$$\lambda\varphi(x) - \mathrm{Tr}\,[C(x)D^2\varphi(x)] - \langle\sigma(x) - |\widetilde{\sigma}(x)|x, D\varphi(x)\rangle = f(x), \quad x \in S.$$

If we derive each side with respect to x_j, we get

$$\begin{aligned}
&\lambda D_j\varphi(x) - \mathrm{Tr}\,[C(x)D^2D_j\varphi(x)] - \langle\sigma(x) - |\widetilde{\sigma}(x)|x, DD_j\varphi(x)\rangle \\
&\quad - \mathrm{Tr}\,[D_jC(x)D^2\varphi(x)] - \langle D_j\sigma(x) - D_j|\widetilde{\sigma}(x)|x - |\widetilde{\sigma}(x)|D_jx, D\varphi(x)\rangle \\
&\quad = D_jf(x).
\end{aligned}$$

where $D_j = \partial/\partial x_j$. If we multiply each side by $c_{ij}(x)D_i\varphi(x)$, by integrating with respect to ν_α and by taking the sum over i and j, it follows

$$\begin{aligned}
&\lambda\int_S D_j\varphi\, c_{ij}D_i\varphi\, d\nu_\alpha - \int_S \mathrm{Tr}\,[CD^2D_j\varphi]c_{ij}D_i\varphi\, d\nu_\alpha \\
&\quad - \int_H \langle\sigma - |\widetilde{\sigma}|x, DD_j\varphi\rangle c_{ij}D_i\varphi\, d\nu_\alpha - \int_S \mathrm{Tr}\,[D_jCD^2\varphi]c_{ij}D_i\varphi\, d\nu_\alpha \\
&\quad - \int_S \langle D_j\sigma - D_j|\widetilde{\sigma}|x - |\widetilde{\sigma}|D_jx, D\varphi\rangle c_{ij}D_i\varphi\, d\nu_\alpha = \int_S D_jf\, c_{ij}D_i\varphi\, d\nu_\alpha
\end{aligned}\tag{4.18}$$

(here and in what follows, any time we have the same index twice we are taking the sum on that index).

Now let us calculate each term in (4.18). We have

$$\lambda \int_S D_j\varphi\, c_{ij} D_i\varphi\, d\nu_\alpha = \lambda \int_S \langle CD\varphi, D\varphi\rangle\, d\nu_\alpha.$$

By using (4.5), it holds

$$\begin{aligned}
&-\int_S \mathrm{Tr}\,[CD^2 D_j\varphi] c_{ij} D_i\varphi\, d\nu_\alpha - \int_H \langle \sigma - |\widetilde{\sigma}| x,\, DD_j\varphi(x)\rangle c_{ij} D_i\varphi\, d\nu_\alpha \\
&\quad = -\int_S \mathcal{L} D_j\varphi\, c_{ij} D_i\varphi\, d\nu_\alpha = \int_S \langle CDD_j\varphi,\, D(c_{ij} D_i\varphi)\rangle\, d\nu_\alpha \\
&\qquad + \int_S \langle CDD_j\varphi,\, D\log\rho_\alpha + b\rangle\, c_{ij} D_i\varphi\, d\nu_\alpha = \int_S \langle CDD_j\varphi,\, Dc_{ij}\rangle D_i\varphi\, d\nu_\alpha \\
&\qquad + \int_S \langle CDD_j\varphi,\, DD_i\varphi\rangle c_{ij}\, d\nu_\alpha + \int_S \langle CDD_j\varphi,\, D\log\rho_\alpha + b\rangle\, c_{ij} D_i\varphi\, d\nu_\alpha \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}$$

By some calculations it is possible to show that

$$I_2 = \int_S \mathrm{Tr}\,[(CD^2\varphi)^2]\, d\nu_\alpha$$

and

$$I_3 = \int_S \langle D\log\rho_\alpha + b,\, CD^2\varphi\, CD\varphi\rangle\, d\nu_\alpha,$$

so that

$$\begin{aligned}
&-\int_S \mathrm{Tr}\,[CD^2 D_j\varphi] c_{ij} D_i\varphi\, d\nu_\alpha - \int_H \langle \sigma - |\widetilde{\sigma}| x,\, DD_j\varphi\rangle c_{ij} D_i\varphi\, d\nu_\alpha \\
&\quad = \int_S \langle CDD_j\varphi,\, Dc_{ij}\rangle D_i\varphi\, d\nu_\alpha + \int_S \mathrm{Tr}\,[(CD^2\varphi)^2]\, d\nu_\alpha \\
&\qquad + \int_S \langle D\log\rho_\alpha + b,\, CD^2\varphi\, CD\varphi\rangle\, d\nu_\alpha.
\end{aligned}$$

Next step is proving the following identity

LEMMA 4.11. *It holds*

$$\int_S \langle CDD_j\varphi,\, Dc_{ij}\rangle D_i\varphi\, d\nu_\alpha - \int_S \mathrm{Tr}\,[D_j CD^2\varphi] c_{ij} D_i\varphi\, d\nu_\alpha = 0. \tag{4.19}$$

Proof. It is not difficult to check that the left hand side in (4.19) is given by

$$\int_S D_i\varphi\, D_{kj}\varphi(c_{hk}D_h c_{ij} - c_{ih}D_h c_{kj})\, d\nu_\alpha.$$

Thus, if we show that for any $i, j, k = 1, \ldots, d$

$$I(x) = \sum_{h=1}^{d} c_{hk}(x)D_h c_{ij}(x) - c_{ih}(x)D_h c_{kj}(x) = 0, \qquad x \in S,$$

we immediately get (4.19). Recalling that $c_{kj}(x) = x_k(\delta_{kj} - x_j)$, we have

$$D_h c_{kj}(x) = \delta_{hk}(\delta_{kj} - x_j) - \delta_{hj}x_k,$$

and this implies that

$$\begin{aligned} I(x) &= \delta_{ij}c_{ik}(x)(1-2x_i) - (c_{ik}(x)x_j + c_{jk}(x)x_i)(1-\delta_{ij}) \\ &\quad -\delta_{kj}c_{ik}(x)(1-2x_k) + (c_{ik}(x)x_j + c_{ij}(x)x_k)(1-\delta_{kj}). \end{aligned}$$

If $i = j = k$, we immediately have that $I(x) = 0$. If $i = j$ and $k \neq i, j$ we have

$$I(x) = -c_{ik}(x)x_i + c_{ik}(x) + c_{ii}(x)x_k = x_i^2 x_k - x_i x_k + x_i(1-x_i)x_k = 0.$$

If $k = j$ and $i \neq j$ we have

$$I(x) = -c_{ik}(x)(1-2x_k) - x_k c_{ik}(x) - x_i c_{kk}(x) = -x_i x_k^2 + x_i x_k - x_i x_k(1-x_k) = 0.$$

Finally, if $i \neq j \neq k$ we have

$$I(x) = -x_i c_{jk}(x) + c_{ij}(x)x_k = x_i x_j x_k - x_i x_j x_k = 0,$$

so that $I(x) = 0$, for any $x \in S$ and $i, j, k = 1, \ldots, d$. □

Concerning the remaining terms in (4.18), if we denote by $\{e_j\}$ the standard orthonormal basis in $\mathbb{R}^d$, we have

$$\begin{aligned} &-\int_S \langle D_j\sigma, D\varphi\rangle c_{ij} D_i\varphi\, d\nu_\alpha = -\int_S \langle D\sigma e_j, D\varphi\rangle\langle CD\varphi, e_j\rangle\, d\nu_\alpha \\ &\quad= -\int_S \langle (CD\varphi \otimes D\varphi)\, D\sigma e_j, e_j\rangle\, d\nu_\alpha = -\int_S \mathrm{Tr}\,[(CD\varphi \otimes D\varphi)\, D\sigma]\, d\nu_\alpha \end{aligned}$$

and

$$\begin{aligned} &\int_S D_j|\widetilde{\sigma}|\langle x, D\varphi\rangle c_{ij} D_i\varphi\, d\nu_\alpha \\ &\quad= \int_S \langle CD\varphi, D|\widetilde{\sigma}|\rangle\langle x, D\varphi\rangle\, d\nu_\alpha = \int_S \langle (CD\varphi \otimes D\varphi)\, x, D|\widetilde{\sigma}|\rangle\, d\nu_\alpha. \end{aligned}$$

Moreover,

$$\int_S |\widetilde{\sigma}|\langle D_j x, D\varphi\rangle c_{ij} D_i\varphi\, d\nu_\alpha = \int_S |\widetilde{\sigma}|\langle CD\varphi, D\varphi\rangle\, d\nu_\alpha.$$

Finally, as

$$\int_S D_j f c_{ij} D_i\varphi\, d\nu_\alpha = \int_S \langle Df, CD\varphi\rangle d\nu_\alpha,$$

by using (4.5) we have

$$\int_S D_j f c_{ij} D_i\varphi\, d\nu_\alpha = -\langle f, \mathcal{L}\varphi\rangle_H - \int_S \langle CD\varphi, D\log\rho_\alpha + b\rangle\, f\, d\nu_\alpha$$

$$= \|\mathcal{L}\varphi\|_H^2 - \lambda\langle\varphi, \mathcal{L}\varphi\rangle_H - \int_S \langle CD\varphi, D\log\rho_\alpha + b\rangle\,(\lambda\varphi - \mathcal{L}\varphi)\, d\nu_\alpha.$$

Therefore, collecting all terms in (4.18) we get (4.17). □

The identity (4.17) proved in the previous proposition allows us to give a characterization of $D(\overline{\mathcal{L}})$.

THEOREM 4.12. *The domain of* $\overline{\mathcal{L}}$ *in* $H = L^2(S, \nu_\alpha)$ *equals* $W_C^{2,2}(S, \nu_\alpha)$, *with equivalence of norms.*

Proof. In view of the definition of $W_C^{2,2}(S, \nu_\alpha)$ and $\overline{\mathcal{L}}$, it suffices to prove that there exists two constants $c_1, c_2 > 0$ such that for any $\varphi \in C^2(S)$

$$c_1\,\|\varphi\|_{D(\overline{\mathcal{L}})} \le \|\varphi\|_{2,C} \le c_2\,\|\varphi\|_{D(\overline{\mathcal{L}})}. \tag{4.20}$$

Due to (4.17) we have

$$\begin{aligned}\|\mathcal{L}\varphi\|_H^2 = {} & [\varphi]_{2,C}^2 + \int_S (\lambda + |\widetilde{\sigma}|)\,\langle CD\varphi, D\varphi\rangle\, d\nu_\alpha \\ & + \int_S \langle D\log\rho_\alpha + b, CD^2\varphi\,(CD\varphi)\rangle\, d\nu_\alpha - \int_S \mathrm{Tr}\,[(CD\varphi\otimes D\varphi)\, D\sigma]\, d\nu_\alpha \\ & + \int_S \langle (CD\varphi\otimes D\varphi)\, x, D|\widetilde{\sigma}|\rangle\, d\nu_\alpha + \lambda\langle\mathcal{L}\varphi, \varphi\rangle_H \\ & + \int_S \langle CD\varphi, D\log\rho_\alpha + b\rangle\,(\lambda\varphi - \mathcal{L}\varphi)\, d\nu_\alpha.\end{aligned} \tag{4.21}$$

Then, thanks to the Young inequality and to (4.4), after some calculations we have

$$\begin{aligned}\frac{1}{2}\|\mathcal{L}\varphi\|_H^2 &\leq [\varphi]_{2,C}^2 + (\lambda + \|\widetilde{\sigma}\|_0 + 2\,H_\alpha^2)\,[\varphi]_{1,C}^2\\ &+ \int_S |\langle D\log\rho_\alpha + b, CD^2\varphi\,(CD\varphi)\rangle|d\nu_\alpha\\ &+ \frac{5}{4}\lambda^2\|\varphi\|_H^2 + \int_S |\mathrm{Tr}\,[(CD\varphi\otimes D\varphi)\,D\sigma]|\,d\nu_\alpha + \int_S |\langle (CD\varphi\otimes D\varphi)\,x, D|\widetilde{\sigma}|\rangle|\,d\nu_\alpha.\end{aligned}$$

For any $A, B \in \mathcal{L}(\mathbb{R}^d)$ with $A \geq 0$, it holds

$$\|AB\|_{\mathcal{L}(\mathbb{R}^d)} \leq |\mathrm{Tr}\,[AB]| \leq \mathrm{Tr}\,[A]\,\|B\|_{\mathcal{L}(\mathbb{R}^d)}, \tag{4.22}$$

and for any $u, v \in \mathbb{R}^d$

$$\mathrm{Tr}\,[u\otimes v] = \langle u, v\rangle.$$

Therefore we have

$$\begin{aligned}\int_S |\mathrm{Tr}\,[(CD\varphi\otimes D\varphi)\,D\sigma]|\,d\nu_\alpha &\leq \|D\sigma\|_0\int_S \mathrm{Tr}\,[CD\varphi\otimes D\varphi]\,d\nu_\alpha\\ &= \|D\sigma\|_0\int_S\langle CD\varphi, D\varphi\rangle\,d\nu_\alpha = \|D\sigma\|_0\,[\varphi]_{1,C}^2.\end{aligned} \tag{4.23}$$

Moreover, we have

$$\begin{aligned}\int_S |\langle (CD\varphi\otimes D\varphi)\,x, D|\widetilde{\sigma}|\rangle|\,d\nu_\alpha &\leq \|D\sigma\|_0\int_S \|CD\varphi\otimes D\varphi\|_{\mathcal{L}(\mathbb{R}^d)}d\nu_\alpha\\ &= \|D\sigma\|_0\,[\varphi]_{1,C}^2.\end{aligned} \tag{4.24}$$

Finally,

$$\begin{aligned}&\int_S |\langle D\log\rho_\alpha + b, CD^2\varphi\,(CD\varphi)\rangle|\,d\nu_\alpha\\ &\quad\leq H_\alpha\int_S \langle (C^{1/2}D^2\varphi C^{1/2})^2 C^{1/2}D\varphi, C^{1/2}D\varphi\rangle^{1/2}\,d\nu_\alpha\end{aligned}$$

and then, by using once more the Young inequality, from (4.22) it follows

$$\int_S |\langle D\log\rho_\alpha + b, CD^2\varphi\,(CD\varphi)\rangle|\,d\nu_\alpha \leq \frac{H_\alpha^2}{2}\,[\varphi]_{1,C}^2 + \frac{1}{2}[\varphi]_{2,C}^2. \tag{4.25}$$

Then, collecting all terms we have

$$\|\mathcal{L}\varphi\|_H^2 \leq 3[\varphi]_{2,C}^2 + 2\left(\lambda + \|\widetilde{\sigma}\|_0 + \frac{5H_\alpha^2}{2} + 2\|D\sigma\|_0\right)\,[\varphi]_{1,C}^2 + \frac{5}{2}\lambda^2\|\varphi\|_H^2.$$

This implies that $W_C^{2,2}(S,\nu_\alpha) \subset D(\overline{\mathcal{L}})$ and there exists a constant $c_1 > 0$ such that the first inequality in (4.20) holds.

On the other hand, by using (4.21), (4.23), (4.24) and (4.25) and the Young inequality, we have

$$\|\mathcal{L}\varphi\|_H^2 \geq [\varphi]_{2,C}^2 + \lambda\,[\varphi]_{1,C}^2 - \frac{H_\alpha^2}{2}\,[\varphi]_{1,C}^2 - \frac{1}{2}[\varphi]_{2,C}^2 - 2\|D\sigma\|_0\,[\varphi]_{1,C}^2$$

$$-\frac{1}{2}\|\mathcal{L}\varphi\|_H^2 - \frac{\lambda^2}{2}\|\varphi\|_H^2 - H_\alpha^2\,[\varphi]_{1,C}^2 - \frac{\lambda^2}{2}\|\varphi\|_H^2 - \frac{1}{2}\,\|\mathcal{L}\varphi\|_H^2,$$

so that

$$\frac{1}{2}\,\|\mathcal{L}\varphi\|_H^2 \geq \frac{1}{2}[\varphi]_{2,C}^2 + \left(\lambda - \frac{3}{2}H_\alpha^2 - 2\|D\sigma\|_0\right)\,[\varphi]_{1,C}^2 - \lambda^2\|\varphi\|_H^2.$$

Thus, if we fix

$$\lambda > \frac{3}{2}H_\alpha^2 + 2\|D\sigma\|_0,$$

we can conclude that $D(\overline{\mathcal{L}}) \subseteq W_C^{2,2}(S,\nu_\alpha)$ and there exists a constant $c_2 > 0$ such that the second inequality in (4.20) holds. □

Next, we define the operator $\mathcal{L}_s$ as

$$D(\mathcal{L}_s) = D(\mathcal{L}) = C^2(S), \quad \mathcal{L}_s\varphi = \mathcal{L}\varphi + \langle C D\varphi, D\log\rho_\alpha + b\rangle, \quad \varphi \in D(\mathcal{L}). \tag{4.26}$$

Due to (4.5), for any $\varphi, \psi \in D(\mathcal{L})$ we have

$$\int_S \mathcal{L}_s\varphi\,\psi\,d\nu_\alpha = -\int_S \langle C D\varphi, D\psi\rangle\,d\nu_\alpha = \int_s \varphi\,\mathcal{L}_s\psi\,d\nu_\alpha,$$

so that $\mathcal{L}_s$ can be considered the *symmetric* part of $\mathcal{L}$.

Moreover, we introduce the space $L_C^2(S,\nu_\alpha)$, defined as the set of Borel measurable functions $\Phi : S \to \mathbb{R}^d$ such that

$$\int_S \langle C(x)\Phi(x), \Phi(x)\rangle\,d\nu_\alpha(x) < \infty.$$

In view of (2.3), the space $L_C^2(S,\nu_\alpha)$ becomes a Hilbert space if equipped with the inner product

$$\langle\Phi,\Psi\rangle_C = \int_S \langle C(x)\Phi(x), \Psi(x)\rangle\,d\nu_\alpha(x).$$

We shall denote by $\|\cdot\|_C$ the corresponding norm in $L_C^2(S,\nu_\alpha)$.

Finally, we introduce the *gradient* operator from $L^2(S,\nu_\alpha)$ into $L_C^2(S,\nu_\alpha)$ as follows

$$D(D_C) = C^1(S), \qquad D_C\varphi(x) = (D_1\varphi(x),\ldots,D_d\varphi(x))^t\,, \qquad x \in S.$$

We have the following result.

PROPOSITION 4.13. *Assume that the Hypotheses* 2.1 *and* 4.1 *hold. Then*

1. *the operator* $D_C : C^1(S) \subset L^2(S, \nu_\alpha) \to L^2_C(S, \nu_\alpha)$ *is closable and densely defined;*
2. *if* $\mathcal{D}$ *is the closure of* D_C*, then* $D(\mathcal{D}) = W^{1,2}_C(S, \nu_\alpha)$*;*
3. *if* $\mathcal{D}^\star$ *denotes the adjoint of* $\mathcal{D}$ *from* $L^2_C(S, \nu_\alpha)$ *into* $L^2(S, \nu_\alpha)$*, then* $-2\bar{\mathcal{L}}_s \subseteq \mathcal{D}^\star\mathcal{D}$*.*

Proof. We already know that $C^1(S)$ is dense in $L^2(S, \nu_\alpha)$. We show that D_C is closable. Let $\{\varphi_n\}$ be a sequence in $C^1(S)$ converging to 0 in $L^2(S, \nu_\alpha)$ and such that

$$\lim_{n\to+\infty} D_C\,\varphi_n = \Phi, \quad \text{in } L^2_C(S, \nu_\alpha),$$

that is

$$\lim_{n\to+\infty} \int_S \langle C(D\varphi_n - \Phi), D\varphi_n - \Phi\rangle\, d\nu_\alpha = 0. \tag{4.27}$$

Since the integral in (4.27) is non negative, for any $\delta > 0$ small enough we have

$$\lim_{n\to+\infty} \int_{S_\delta} \langle C(D\varphi_n - \Phi), D\varphi_n - \Phi\rangle\, d\nu_\alpha = 0,$$

where S_δ is the set introduced in the proof of the Proposition 4.2

$$S_\delta = \{\, x \in S \;:\; \operatorname{dist}(x, \partial S) \geq \delta \,\}.$$

Thus, since there exists $k_\delta > 0$ such that

$$\inf_{x\in S_\delta} \langle C(x)h, h\rangle \geq k_\delta\, |h|^2, \quad h \in \mathbb{R}^d$$

we have

$$\lim_{n\to+\infty} \int_{S_\delta} |D\varphi_n - \Phi|^2\, d\nu_\alpha = 0.$$

As there exists a constant $\rho_\delta > 0$ such that $\rho_\alpha(x) \geq \rho_\delta$, for $x \in S_\delta$, we have that

$$\begin{cases} \lim_{n\to+\infty}\varphi_n = 0, & \text{in } L^2(S) \\ \lim_{n\to+\infty} D\varphi_n = \Phi, & \text{in } L^2(S_\delta, \mathbb{R}^d). \end{cases}$$

This implies that the restriction of Φ to S_δ is 0, a.s. for any $\delta > 0$, so that $\Phi = 0$ on S, a.s. and then D_C is closable. The domain of its closure $\mathcal{D}$ coincides with $W^{1,2}_C(S, \nu_\alpha)$, as the graph norm of $\mathcal{D}$ equals the norm of $W^{1,2}_C(S, \nu_\alpha)$ on $C^1(S)$.

Since the operator $\mathcal{D}$ is densely defined, the adjoint of $\mathcal{D}$, $\mathcal{D}^\star : D(\mathcal{D}^\star) \subset L^2_C(S, \nu_\alpha) \to L^2(S, \nu_\alpha)$, is well defined and since $\mathcal{D}$ is closed, from [7]-Chapter 5, Theorem 3.24, it follows that the operator $\mathcal{D}^\star\mathcal{D}$, with domain

$$D(\mathcal{D}^\star\mathcal{D}) = \left\{\, \varphi \in D(\mathcal{D}) \;:\; \mathcal{D}\varphi \in D(\mathcal{D}^\star) \,\right\},$$

is positive and self-adjoint in $L^2(S, \nu_\alpha)$, $D((\mathcal{D}^\star\mathcal{D})^{1/2}) = D(\mathcal{D})$ and $D(\mathcal{D}^\star\mathcal{D})$ is a core for $D(\mathcal{D})$.

Next, from the integration by parts formula (4.5), for any $\varphi \in C^2(S)$ and $\psi \in C^1(S)$ we have

$$\langle \mathcal{D}\varphi, \mathcal{D}\psi\rangle_C = -2\langle \mathcal{L}_s\varphi, \psi\rangle_H.$$

As $C^1(S)$ is dense in $W_C^{1,2}(S, \nu_\alpha)$ and the graph norm of $\mathcal{D}$ equals the norm in $W_C^{1,2}(S, \nu_\alpha)$, the same identity holds for $\psi \in W_C^{1,2}(S, \nu_\alpha)$. Moreover, as $W_C^{2,2}(S, \nu_\alpha)$ continuously embeds into $W_C^{1,2}(S, \nu_\alpha)$, thanks to the Theorem 4.12 we can extend the identity above to all $\varphi \in W_C^{2,2}(S, \nu_\alpha)$ and we have

$$\langle \mathcal{D}\varphi, \mathcal{D}\psi\rangle_C = -2\langle \bar{\mathcal{L}}_s\varphi, \psi\rangle_H, \qquad \varphi \in W_C^{2,2}(S, \nu_\alpha), \quad \psi \in W_C^{1,2}(S, \nu_\alpha).$$

Finally, we prove that $-2\bar{\mathcal{L}}_s \subseteq \mathcal{D}^\star\mathcal{D}$. Let $\varphi \in W_C^{2,2}(S, \nu_\alpha) = D(\bar{\mathcal{L}}_s) \subset W_C^{1,2}(S, \nu_\alpha) = D(\mathcal{D})$ and set $\Phi = \mathcal{D}\varphi \in L_C^2(S, \nu_\alpha)$. For any $\psi \in D(\mathcal{D})$ we have

$$|\langle \Phi, \mathcal{D}\psi\rangle_C| = 2|\langle \bar{\mathcal{L}}_s\varphi, \psi\rangle_H| \le 2\, \|\bar{\mathcal{L}}_s\varphi\|_H\, \|\psi\|_H,$$

and hence $\Phi \in D(\mathcal{D}^\star)$ and $\langle \mathcal{D}^\star\Phi, \psi\rangle_H = \langle \Phi, \mathcal{D}\psi\rangle_C$. Therefore $\varphi \in D(\mathcal{D}^\star\mathcal{D})$ and

$$\langle \mathcal{D}^\star\mathcal{D}\varphi, \psi\rangle_H = -2\langle \bar{\mathcal{L}}_s\varphi, \psi\rangle_H.$$

Since $D(\mathcal{D})$ is dense in $L^2(S, \nu_\alpha)$, it follows that $\mathcal{D}^\star\mathcal{D}\varphi = -2\bar{\mathcal{L}}_s\varphi$ and $-2\bar{\mathcal{L}}_s \subseteq \mathcal{D}^\star\mathcal{D}$. □

In the case there exists a symmetrizing measure, the previous results allows us to give a factorization of $-\bar{\mathcal{L}}$ and a characterization of $D((-\bar{\mathcal{L}})^{1/2})$.

COROLLARY 4.14. *Assume the Hypotheses* 2.1 *and* 4.4. *Then* $-2\bar{\mathcal{L}} = \mathcal{D}^\star\mathcal{D}$ *and* $D((-\bar{\mathcal{L}})^{1/2}) = W_C^{1,2}(S, \nu_\alpha)$.

Proof. Since in this case $-2\bar{\mathcal{L}}$ and $\mathcal{D}^\star\mathcal{D}$ are both self-adjoint and $-2\bar{\mathcal{L}}_s \subseteq \mathcal{D}^\star\mathcal{D}$, we have that they coincide. □

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