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Smoluchowski-Kramers approximation for a general class of SPDEs

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Dedicated to Giuseppe Da Prato on the occasion of his 70th birthday

Abstract. We prove that the so-called Smoluchowski-Kramers approximation holds for a class of partial differential equations perturbed by a non-Gaussian noisy term. Namely, we show that the solution of the one-dimensional semi-linear stochastic damped wave equations $\mu u_{tt}(t,x)+u_t(t,x) = \Delta u(t,x)+b(x,u(t,x))+g(x,u(t,x))\dot{w}(t)$, $u(0) = u_0$, $u_t(0) = v_0$, endowed with Dirichlet boundary conditions, converges as the parameter μ goes to zero to the solution of the semi-linear stochastic heat equation $u_t(t,x) = \Delta u(t,x)+b(x,u(t,x))+g(x,u(t,x))\dot{w}(t)$, $u(0) = u_0$, endowed with Dirichlet boundary conditions.

1. Introduction

After the two previous papers [9] and [4], devoted respectively to the study of systems with a finite and an infinite number of degrees of freedom, in this paper we are again concerned with the Smoluchowski-Kramers approximation, this time for a more general class of damped wave equations perturbed by a stochastic term of multiplicative type and having a non-linear term which is possibly non-Lipschitz.

Namely, we are dealing with the following stochastic damped wave equation

$$\begin{cases} \mu \dfrac{\partial^2 u}{\partial t^2}(t,x) + \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + b(x,u(t,x)) \\ \quad +g(x,u(t,x))\frac{\partial w}{\partial t}(t,x), \quad x \in [0,L], \\ \\ u(0,x) = u_0, \quad \dfrac{\partial u}{\partial t}(0,x) = v_0, \quad u(t,0) = u(t,L) = 0, \end{cases} \tag{1.1}$$

where $w(t,x)$ is a Wiener process defined on some stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, which is white both in time and in space. The coefficients b and g are measurable from $[0,L] \times \mathbb{R}$ with values in $\mathbb{R}$ and g is Lipschitz-continuous in the second variable, uniformly with respect the first one. The coefficient b is either assumed to be Lipschitz-continuous in the second variable, uniformly with respect the first one, or satisfying some polynomial growth and

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dissipation conditions in the spirit of the Klein-Gordon model (and in this second case g is taken bounded).

As in [4], where the case of Lipschitz continuous b and additive noise in any space dimension is studied, we show that the so-called Smoluchowski-Kramers approximation of equation (1.1) holds. This means that as the parameter μ goes to zero the solution u^μ of equation (1.1) converges in probability to the solution of the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + b(x,u(t,x)) + g(x,u(t,x))\dfrac{\partial w}{\partial t}(t,x), \quad x \in [0,L], \\ u(0,x) = u_0, \quad u(t,0) = u(t,L) = 0. \end{cases} \tag{1.2}$$

More precisely, we show that for any $\epsilon > 0$ and $T > 0$

$$\lim_{\mu \to 0} \mathbb{P}\left(\sup_{t \in [0,T]} \left| u^\mu(t) - u(t) \right|_{L^2(0,L)} > \epsilon \right) = 0. \tag{1.3}$$

This statement is crucial in applications. For example, in the case of a system with a finite number of degrees of freedom (see [9]), it provides a valuable justification of the description of the motion of a particle of a small mass μ, subject to the force field $b(q) + g(q)\dot{w}$ with the damping proportional to the speed, by the first order equation

$$\dot{q}(t) = b(q(t)) + g(q(t))\dot{w}(t), \quad q(0) = q_0 \in \mathbb{R}^n,$$

instead of the second order equation arising from the Newton law

$$\mu\,\ddot{q}(t) + \dot{q}(t) = b(q(t)) + g(q(t))\dot{w}(t), \quad q(0) = q_0 \in \mathbb{R}^n, \quad \dot{q}(0) = p_0 \in \mathbb{R}^n.$$

Besides, from a purely mathematical point of view, limit (1.3) represents an extension to the field of stochastic partial differential equations of the celebrated theory of singular perturbations of deterministic partial differential equations (for a comprehensive treatment of it see for example the book by J.L. Lions [13]).

The proof of the convergence result (1.3) essentially follows the arguments used in [9] for the finite-dimensional case and in [4] for the infinite-dimensional case. But here the presence of a diffusion coefficient g in front of the stochastic perturbation which is non-constant makes the proof of the Smoluchowski-Kramers approximation (1.3) definitely more delicate and requires some extra work which is not necessary in the case of a Gaussian perturbation.

The main difficulties arise in the proof of uniform bounds for the Hölder norm in space and time of the so-called *stochastic convolution* $\Gamma_\mu(u)$, which is the solution of the problem

$$\begin{cases} \mu\dfrac{\partial^2 \eta}{\partial t^2}(t,x) + \dfrac{\partial \eta}{\partial t}(t,x) = \dfrac{\partial^2 \eta}{\partial x^2}(t,x) + g(x,u(t,x))\dfrac{\partial w}{\partial t}(t,x), \quad x \in [0,L], \\ \eta(0,x) = 0, \quad \dfrac{\partial \eta}{\partial t}(0,x) = 0, \quad \eta(t,0) = \eta(t,L) = 0, \end{cases}$$

for some fixed process $u \in L^p(\Omega; C([0,T]; H^\delta(0,L)))$, and suitable $p \geq 1$ and $\delta \in \mathbb{R}$. Actually, unlike in the Gaussian case studied in [4], where the stochastic convolution is independent of the solution u^μ and it is explicitly given as an infinite sum, here, due to the presence of the non constant coefficient g, there is no explicit expression for $\Gamma_\mu(u)$ and all estimates are more delicate to get.

Moreover, the treatment of a coefficient b which has super-linear growth (this case is important in several applications), concerted with the presence of a noisy perturbation of multiplicative type, causes some problems in proving the required a-priori estimates and in several cases does not allow to obtain all the results that we can obtain when b is Lipschitz continuous. For example in some cases mean-square convergence does not hold and we can only prove convergence in probability (see for example Lemma 4.7.). Nevertheless, here we are able to adapt our arguments, in order to get (1.3) in full generality.

2. Notations and preliminaries

In what follows we shall denote by H the Hilbert space $L^2(0,L)$, endowed with the usual scalar product $\langle \cdot, \cdot \rangle_H$ and the induced norm $|\cdot|_H$.

It is well known that the operator Δ in the interval $[0,L]$, endowed with Dirichlet boundary conditions, is diagonal with respect to the complete orthonormal basis $\{e_k\}$ of H defined by

$$e_k(x) = \sqrt{\frac{2}{L}} \sin \frac{k\pi}{L} x, \quad x \in [0,L], \quad k \in \mathbb{N}.$$

Moreover $\Delta e_k = -\alpha_k e_k$, where $\alpha_k = (k\pi/L)^2$, for any $k \in \mathbb{N}$. Notice that the sup-norm of the eigenfunctions e_k are equi-bounded, that is

$$\sup_{k \in \mathbb{N}} |e_k|_\infty = \sqrt{\frac{2}{L}}. \tag{2.1}$$

For any $\delta \in \mathbb{R}$, we shall denote by $H^\delta(0,L)$ the completion of $C_0^\infty([0,L])$ with respect to the Hilbertian norm

$$\|h\|_{H^\delta}^2 = \sum_{i=1}^\infty \alpha_i^\delta \, \langle h, e_i \rangle_H^2 = \sum_{i=1}^\infty \alpha_i^\delta h_i^2.$$

Furthermore we shall denote by $\mathcal{H}_\delta$ the Hilbert space $H^\delta(0,L) \times H^{\delta-1}(0,L)$, endowed with the natural scalar product and norm inherited from each component.

2.1. *The damped wave semigroup*

For any $\mu > 0$ and $\delta \in \mathbb{R}$ we define on $\mathcal{H}_\delta$ the unbounded operator A_μ by setting

$$A_\mu(h,k) = \frac{1}{\mu}\,(\mu k, \Delta h - k)\,, \qquad (h,k) \in D(A_\mu) := \mathcal{H}_{\delta+1}.$$

Notice that the operators A_μ defined on different $\mathcal{H}_\delta$ are all consistent. As proved for example in [18, section 7.4] A_μ is the generator of a group of bounded linear transformations $\{S_\mu(t)\}_{t\in\mathbb{R}}$ on $\mathcal{H}_\delta$ which is strongly continuous. Moreover, for each $\mu > 0$ the semigroup $S_\mu(t)$ is of negative type on $\mathcal{H}_\delta$, that is

$$\|S_\mu(t)\|_{\mathcal{L}(\mathcal{H}_\delta)} \leq M_\mu\, e^{-\omega_\mu t}, \quad t \geq 0,$$

for some positive constants M_μ and ω_μ (for a proof see [4, Proposition 2.4]).

By setting $\Pi_1(u,v) := u$ and $\Pi_2(u,v) := v$, we have

$$\Pi_1 S_\mu(t)(u,v) = \sum_{k=1}^{\infty} \eta_k^\mu(u,v)(t) e_k, \qquad \Pi_2 S_\mu(t)(u,v) = \sum_{k=1}^{\infty} \vartheta_k^\mu(u,v)(t) e_k, \tag{2.2}$$

where for each $k \in \mathbb{N}$ the pair $(\eta_k^\mu(u,v)(t), \vartheta_k^\mu(u,v)(t))$ is the solution of the system

$$\left\{\begin{array}{l} \eta'(t) = \vartheta(t), \qquad \eta(0) = u_k \\[2ex] \mu\, \vartheta'(t) = -\alpha_k\, \eta(t) - \vartheta(t), \qquad \vartheta(0) = v_k, \end{array}\right. \tag{2.3}$$

(here u_k and v_k denote respectively the k-th Fourier coefficient of u and v). Multiplying the second equation in (2.3) by $\alpha_k^{\delta-1}\, \vartheta_k^\mu(u,v)(t)$ and integrating with respect to $t \geq 0$ we get

$$\begin{aligned} &\mu\, \alpha_k^{\delta-1}\, |\vartheta_k^\mu(u,v)(t)|^2 + \alpha_k^\delta\, |\eta_k^\mu(u,v)(t)|^2 + 2\,\alpha_k^{\delta-1} \int_0^t |\vartheta_k^\mu(u,v)(s)|^2\, ds \\ &\quad = \mu\, \alpha_k^{\delta-1}\, |v_k|^2 + \alpha_k^\delta\, |u_k|^2. \end{aligned} \tag{2.4}$$

In particular, for any $(u,v) \in \mathcal{H}_\delta$ we have

$$|\Pi_1 S_\mu(t)(u,v)|_{H^\delta}^2 \leq |u|_{H^\delta}^2 + \mu\, |v|_{H^{\delta-1}}^2. \tag{2.5}$$

In [4, Proposition 2.2] we have given explicit expressions both for $\eta_k^\mu(u,v)(t)$ and for $\vartheta_k^\mu(u,v)(t)$. In the particular case $u = 0$ we have the following formulas.

PROPOSITION 2.1. *For any $\mu > 0$ and $k \in \mathbb{N}$, let us define $\gamma_k^\mu := \sqrt{1 - 4\alpha_k \mu}/2\mu$. Then, we have*

$$\eta_k^\mu(0,v)(t) = \frac{1}{2}\exp\left(-\frac{t}{2\mu}\right) \frac{1}{\gamma_k^\mu} [\exp\left(\gamma_k^\mu t\right) - \exp\left(-\gamma_k^\mu t\right)]\, v_k, \tag{2.6}$$

and

$$\vartheta_k^\mu(0,v)(t) = \frac{1}{2}\exp\left(-\frac{t}{2\mu}\right)$$
$$\left[\left(1-\frac{1}{2\mu\gamma_k^\mu}\right)\exp\left(\gamma_k^\mu t\right)+\left(1+\frac{1}{2\mu\gamma_k^\mu}\right)\exp\left(-\gamma_k^\mu t\right)\right] v_k, \qquad (2.7)$$

where, in the case $\gamma_k^\mu = 0$, we have set

$$\frac{1}{\gamma_k^\mu}\left[\exp\left(\gamma_k^\mu t\right)-\exp\left(-\gamma_k^\mu t\right)\right] = 2t.$$

We provide an integral representation for the semigroup $S_\mu(t)$ and we prove some regularity properties of the kernel.

LEMMA 2.2. *Fix $\mu > 0$ and $\delta < 1/2$. For any $v \in H^{-\delta}(0,L)$ it holds*

$$\Pi_1 S_\mu(t)(0,v)(x) = \int_0^L K_\mu(t,x,y)v(y)\,dy, \qquad (t,x)\in[0,\infty)\times[0,L], \qquad (2.8)$$

where $K_\mu : [0,\infty)\times[0,L]^2 \to \mathbb{R}$ is the measurable mapping defined by

$$K_\mu(t,x,y) := \sum_{k=1}^\infty \eta_k^\mu(0,e_k)(t)e_k(x)e_k(y). \qquad (2.9)$$

Proof. According to (2.2) and (2.3), for any $t \geq 0$ we have

$$\Pi_1 S_\mu(t)(0,v) = \sum_{k=1}^\infty \eta_k^\mu(0,e_k)(t)\,\langle v, e_k\rangle\, e_k = \sum_{k=1}^\infty \eta_k^\mu(0,e_k)(t)\int_0^L e_k(y)v(y)\,dy\, e_k.$$

Thus, if we show that for any $t \geq 0$ and $x \in [0,L]$ the mapping

$$y \in [0,L] \mapsto \sum_{k=1}^\infty \eta_k^\mu(0,e_k)(t)e_k(x)e_k(y),$$

is well defined in $H^\delta(0,L)$, we obtain (2.8) and (2.9).

Thanks to (2.1) and (2.4), for any $n,m \in \mathbb{N}$ and $v \in H^{-\delta}(0,L)$ we have

$$\left|\left\langle \sum_{k=n+1}^{n+m} \eta_k^\mu(0,e_k)(t)e_k(x)e_k, v\right\rangle\right|^2 = \left|\sum_{k=n+1}^{n+m} \eta_k^\mu(0,e_k)(t)e_k(x)\,\langle e_k, v\rangle\right|^2$$
$$\leq \sum_{k=n+1}^{n+m}\left|\eta_k^\mu(0,e_k)(t)\right|^2\alpha_k^\delta \sum_{k=n+1}^{n+m}\frac{|\langle e_k, v\rangle|^2}{\alpha_k^\delta}\sup_{k\in\mathbb{N}}|e_k|_\infty^2 \leq c\,|v|_{H^{-\delta}}^2 \sum_{k=n+1}^{n+m}\mu\,\alpha_k^{\delta-1}.$$

Then, as $\delta < 1/2$, we have

$$\lim_{n\to\infty}\left|\left\langle \sum_{k=n+1}^{n+m} \eta_k^\mu(0,e_k)(t)e_k(x)e_k, v\right\rangle\right|^2 = 0.$$

This implies that the mapping $K_\mu(t,x,\cdot)$ is well defined in $H^\delta(0,L)$ and (2.8) holds. □

Next, we prove some regularity properties of the kernel $K_\mu(t,x,y)$.

LEMMA 2.3. *Let $\delta < 1/2$. Then, for any $\rho < 1/2-\delta$ there exists some constant $c_\rho > 0$ such that for any $0 \leq r \leq t$ and $x, y \in [0,L]$*

$$\int_0^t \left|K_\mu(s,x,\cdot) - K_\mu(s,y,\cdot)\right|_{H^\delta}^2\, ds \leq c_\rho\, \mu^2\, |x-y|^{2\rho} \tag{2.10}$$

and

$$\int_0^r \left|K_\mu(t-s,x,\cdot) - K_\mu(r-s,x,\cdot)\right|_{H^\delta}^2\, ds + \int_r^t \left|K_\mu(t-s,x,\cdot)\right|_{H^\delta}^2\, ds \leq c_\rho\, \mu^2\, |t-r|^\rho. \tag{2.11}$$

Proof. In order to prove (2.10), we notice that for any $v \in H^{-\delta}(0,L)$

$$\left|\left\langle K_\mu(s,x,\cdot) - K_\mu(s,y,\cdot), v\right\rangle\right|^2 = \left|\sum_{k=1}^\infty \eta_k^\mu(0,e_k)(s)\,\langle v, e_k\rangle\,[e_k(x)-e_k(y)]\right|^2.$$

As for any $k \in \mathbb{N}$ and $\rho > 0$ there exists some $c_\rho > 0$ such that

$$|e_k(x)-e_k(y)| \leq c_\rho\,|x-y|^\rho\,\alpha_k^{\rho/2}\,|e_k|_\infty, \qquad x, y \in [0,L],$$

we obtain

$$\begin{aligned}
&|\langle K_\mu(s,x,\cdot) - K_\mu(s,y,\cdot), v\rangle|^2 \leq \\
&\qquad \sum_{k=1}^\infty |\eta_k^\mu(0,e_k)(s)|^2\alpha_k^\delta|e_k(x)-e_k(y)|^2 \sum_{k=1}^\infty |\langle v, e_k\rangle|^2\alpha_k^{-\delta}\\
&\quad \leq c_\rho\,|x-y|^{2\rho}\sum_{k=1}^\infty |\eta_k^\mu(0,e_k)(s)|^2\alpha_k^{\delta+\rho}|v|_{H^{-\delta}}^2,
\end{aligned}$$

and then

$$\left|K_\mu(s,x,\cdot) - K_\mu(s,y,\cdot)\right|_{H^\delta}^2 \leq c_\rho\,|x-y|^{2\rho}\sum_{k=1}^\infty |\eta_k^\mu(0,e_k)(s)|^2\alpha_k^{\delta+\rho}.$$

This implies that

$$\int_0^t \left|K_\mu(s,x,\cdot)-K_\mu(s,y,\cdot)\right|_{H^\delta}^2\,ds \le c_\rho\,|x-y|^{2\rho}\sum_{k=1}^\infty \alpha_k^{\delta+\rho}\int_0^t |\eta_k^\mu(0,e_k)(s)|^2\,ds,$$

so that (2.10) follows, recalling that, as proved in [4, Lemma 3.2, (3.12)],

$$\int_0^t |\eta_k^\mu(0,e_k)(s)|^2\,ds \le c\,\mu^2\,\alpha_k^{-1},\quad k\in\mathbb{N}.$$

By similar arguments we have

$$\int_0^r \left|K_\mu(t-s,x,\cdot)-K_\mu(r-s,x,\cdot)\right|_{H^\delta}^2\,ds$$

$$\le \sup_{k\in\mathbb{N}}|e_k|_\infty^2\sum_{k=1}^\infty \alpha_k^\delta\int_0^r \left|\eta_k^\mu(0,e_k)(t-s)-\eta_k^\mu(0,e_k)(r-s)\right|^2\,ds$$

and

$$\int_r^t \left|K_\mu(t-s,x,\cdot)\right|_{H^\delta}^2\,ds \le \sup_{k\in\mathbb{N}}|e_k|_\infty^2\sum_{k=1}^\infty \alpha_k^\delta\int_0^{t-r}\left|\eta_k^\mu(0,e_k)(s)\right|^2\,ds.$$

Therefore, (2.11) follows recalling that, as proved in [4, Lemma 3.2, (3.16), (3.18) and (3.20)], for any $\rho>0$ there exists some constant $c_\rho>0$ such that

$$\int_0^{t-r}\left|\eta_k^\mu(0,e_k)(s)\right|^2\,ds+\int_0^r\left|\eta_k^\mu(0,e_k)(t-s)-\eta_k^\mu(0,e_k)(r-s)\right|^2\,ds$$
$$\le c_\rho\,\mu^2\,|t-r|^\rho\,\alpha_k^{\rho-1}.$$

□

2.2. *The coefficients b and g*

As far as the coefficients b and g are concerned, we shall consider two different cases. Here is described the first one.

HYPOTHESES 1. 1. *The mapping* $b:[0,L]\times\mathbb{R}\to\mathbb{R}$ *is measurable and there exists* $M>0$ *such that*

$$\sup_{x\in[0,L]}|b(x,\sigma)-b(x,\rho)|\le M\,|\sigma-\rho|,$$

for any $\sigma,\rho\in\mathbb{R}$. *Moreover,* $\sup_{x\in[0,L]}|b(x,0)|=:b_0<\infty$.

2. *The mapping* $g : [0, L] \times \mathbb{R} \to \mathbb{R}$ *is measurable and there exists* $M > 0$ *such that*

$$\sup_{x\in[0,L]} |g(x,\sigma) - g(x,\rho)| \leq M\,|\sigma - \rho|,$$

for any $\sigma, \rho \in \mathbb{R}$. *Moreover,* $\sup_{x\in[0,L]} |g(x,0)| =: g_0 < \infty$.

In particular from the assumptions above we have that both b and g have linear growth in the second variable, uniformly with respect to the first. Namely

$$\sup_{x\in[0,L]} |b(x,\sigma)| \leq c\,(1+|\sigma|)\,, \qquad \sup_{x\in[0,L]} |g(x,\sigma)| \leq c\,(1+|\sigma|)\,, \tag{2.12}$$

for some constant $c > 0$.

For any $\mu > 0$ and $\delta \in [0, 1]$, we set

$$B_\mu(u,v)(x) := \frac{1}{\mu}\,(0, b(x,u(x))), \qquad x \in [0,L], \quad (u,v) \in \mathcal{H}_\delta.$$

As we are assuming that $b(x,\cdot)$ is Lipschitz continuous, uniformly with respect to $x \in [0, L]$, we have that B_μ is Lipschitz continuous as a mapping from $\mathcal{H}_\delta$ into itself. Actually, since $\delta \in [0, 1]$, for any $z_1 = (u_1, v_1)$ and $z_2 = (u_2, v_2)$ in $\mathcal{H}_\delta$

$$|B_\mu(z_1) - B_\mu(z_2)|_{\mathcal{H}_\delta} = \frac{1}{\mu}\,|b(\cdot,u_1) - b(\cdot,u_2)|_{H^{\delta-1}} \leq \frac{c}{\mu}\,|b(\cdot,u_1) - b(\cdot,u_2)|_H,$$

and then, thanks to the Lipschitz assumption, we have

$$|B_\mu(z_1) - B_\mu(z_2)|_{\mathcal{H}_\delta} \leq \frac{c\,M}{\mu}|u_1 - u_2|_H \leq \frac{c\,M}{\mu}|z_1 - z_2|_{\mathcal{H}_\delta}.$$

Analogously, for any $\mu > 0$ and $\delta \in [0, 1]$ we define

$$[G_\mu(u,v)h](x) := \frac{1}{\mu}\,(0, g(x,u(x)))h(x),\ x \in [0,L],\ (u,v) \in \mathcal{H}_\delta,\ h \in L^\infty(0,L).$$

Due to Hypothesis 1, the mapping $G_\mu(\cdot)h : \mathcal{H}_\delta \to \mathcal{H}_\delta$ is Lipschitz continuous, for any fixed $h \in L^\infty(0, L)$. Actually, as $\delta \in [0, 1]$, we have

$$\begin{aligned} |G_\mu(z_1)h - G_\mu(z_2)h|_{\mathcal{H}_\delta} &= \frac{1}{\mu}\,|(g(u_1) - g(u_2))h|_{H^{\delta-1}} \leq \frac{c}{\mu}\,|(g(u_1) - g(u_2))h|_H \\ &\leq \frac{cM}{\mu}\,|u_1 - u_2|_H\,|h|_\infty \leq \frac{cM}{\mu}\,|z_1 - z_2|_{\mathcal{H}_\delta}\,|h|_\infty. \end{aligned} \tag{2.13}$$

The second case that we shall consider in the present paper is described below.

HYPOTHESES 2. 1. *The mapping* $b : [0, L] \times \mathbb{R} \to \mathbb{R}$ *is measurable and* $b(x, \cdot) : \mathbb{R} \to \mathbb{R}$ *is of class* C^1, *for almost all* $x \in [0, L]$. *Moreover,*

(a) *there exist* $\lambda \in (1, 3]$ *and* $c_1 > 0$ *such that for any* $\sigma \in [0, L]$

$$\sup_{x\in[0,L]} |b(x,\sigma)| \leq c_1\,(1+|\sigma|^{\lambda}),$$

$$\sup_{x\in[0,L]} |\partial_\sigma\, b(x,\sigma)| \leq c_1\,(1+|\sigma|^{\lambda-1}); \tag{2.14}$$

(b) *there exists* $c_2 > 0$ *such that for any* $\sigma \in \mathbb{R}$

$$\sup_{x\in[0,L]} \beta(x,\sigma) \leq c_2\,(1-|\sigma|^{\lambda+1}), \tag{2.15}$$

where $\beta(x,\cdot)$ *is the antiderivative of* $b(x,\cdot)$ *vanishing at* $\sigma = 0$*;*

(c) *for any* $(x,\sigma) \in [0, L]\times\mathbb{R}$

$$\partial_\sigma\, b(x,\sigma) \leq 0. \tag{2.16}$$

2. *The mapping* $g : [0, L]\times\mathbb{R} \to \mathbb{R}$ *is measurable and there exists* $M > 0$ *such that*

$$\sup_{x\in[0,L]} |g(x,\sigma) - g(x,\rho)| \leq M\,|\sigma-\rho|,$$

for any $\sigma, \rho \in \mathbb{R}$*. Moreover,*

$$\sup_{(x,\sigma)\in[0,L]\times\mathbb{R}} |g(x,\sigma)| =: g_0 < \infty. \tag{2.17}$$

REMARK. 1. A typical example of a function b fulfilling conditions (2.14), (2.15) and (2.16) is

$$b(x,\sigma) = -\alpha\,|\sigma|^{\lambda-1}\sigma,$$

for any strictly positive constant α (the Klein-Gordon equation).

2. Here we are assuming the condition $\partial_\sigma\, b(x,\sigma) \leq 0$, for any $(x,\sigma) \in [0, L]\times\mathbb{R}$, just for simplicity of notations. Actually we could also treat the case

$$\sup_{(x,\sigma)\in[0,L]\times\mathbb{R}} \partial_\sigma\, b(x,\sigma) \leq c,$$

for some constant c, by setting $b(x,\sigma) = b_1(x,\sigma) + b_2(x,\sigma)$, where $b_1(x,\sigma) = b(x,\sigma) - c\,\sigma$ fulfills conditions (2.14), (2.15) and (2.16), and $b_2(x,\sigma) = c\,\sigma$ is a Lipschitz perturbation.

In this second case the mapping $b(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is no more Lipschitz continuous and has not necessarily sublinear growth. From (2.16) we obtain $\sigma b(x,\sigma) \leq \beta(x,\sigma)$, for any $(x,\sigma) \in [0, L]\times\mathbb{R}$, and then according to (2.15) it follows

$$\lim_{|\sigma|\to\infty} \sigma b(x,\sigma) = -\infty, \quad x \in [0, L]. \tag{2.18}$$

Moreover, due to (2.15), for any $(x,\sigma) \in [0, L]\times\mathbb{R}$ we have

$$|\sigma|^{\lambda+1} \leq 1 - \frac{1}{c_2}\beta(x,\sigma) =: -\hat{\beta}(x,\sigma). \tag{2.19}$$

Now, by proceeding as in [15], where the particular case of the Klein-Gordon equation is considered, we approximate b by means of Lipschitz continuous mappings, by setting for any $n \in \mathbb{N}$

$$b_n(x,\sigma) := \begin{cases} b(x,n) + (\sigma - n)\,\partial_\sigma\, b(x,n), & \text{if } \sigma \geq n, \\ \\ b(x,\sigma), & \text{if } |\sigma| \leq n, \\ \\ b(x,-n) + (\sigma + n)\,\partial_\sigma\, b(x,-n), & \text{if } \sigma \leq -n. \end{cases}$$

Clearly, $b_n(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, uniformly with respect to $x \in [0, L]$, and $b_n(x,\sigma) \equiv b(x,\sigma)$ on $[0, L] \times [-n, n]$, so that the mapping $B_{n,\mu}$ defined by

$$B_{n,\mu}(u,v)(x) = \frac{1}{\mu}\,(0, b_n(x, u(x))), \qquad x \in [0, L], \quad (u, v) \in \mathcal{H}_\delta, \tag{2.20}$$

is Lipschitz continuous from $\mathcal{H}_\delta$ into itself, for any $\delta \in [0, 1]$, and for any $(u, v) \in \mathcal{H}_\delta$ it holds

$$|u|_\infty \leq n \Rightarrow B_{n,\mu}(u,v) = B_\mu(u,v). \tag{2.21}$$

Moreover, for any $\sigma \in \mathbb{R}$

$$\sup_{x\in[0,L]} |b_n(x,\sigma)| \leq c\,(1 + |\sigma|^\lambda), \qquad \sup_{x\in[0,L]} |\partial_\sigma\, b_n(x,\sigma)| \leq c\,(1 + |\sigma|^{\lambda-1}), \tag{2.22}$$

for some constant c not depending on n.

Next, for any $x \in [0, L]$ we denote by $\beta_n(x,\cdot)$ the antiderivative of $b_n(x,\cdot)$ vanishing at zero. If we set

$$\hat{\beta}_n(x,\sigma) := \frac{1}{c_2}\beta_n(x,\sigma) - 1, \tag{2.23}$$

due to (2.16), (2.18) and (2.19) it is possible to show that there exists some $n_0 > 0$ such that for any $n \geq n_0$ and $(x,\sigma) \in [0, L] \times \mathbb{R}$

$$-\hat{\beta}_n(x,\sigma) \geq n^{\lambda+1} > 0. \tag{2.24}$$

Now, since $\lambda \leq 3$, it is possible to adapt the arguments used in [15, Lemma A1] for the Klein-Gordon case and prove the following inequality.

LEMMA 2.4. *Assume that b fulfills Hypothesis 2.-1. Then, there exists $n_0 > 0$ such that for any $n \geq n_0$, $x \in [0, L]$ and $\sigma, \rho \in \mathbb{R}$ it holds*

$$|\partial_\sigma\, b_n(x,\sigma+\rho)|^2 \leq c\,(1 - \hat{\beta}_n(x,\sigma) + |\rho|^{2(\lambda-1)}),$$

for some constant c not depending on n. In particular, as b_n coincides with b on $[0, L] \times [-n, n]$, for any $n > 0$, an analogous inequality is fulfilled by b and β on $[0, L] \times \mathbb{R}$.

As far as the diffusion coefficient G_μ is concerned, due to (2.17) in this second case the mapping $G_\mu(\cdot)h : \mathcal{H}_\delta \to \mathcal{H}_\delta$ is bounded, for any fixed $h \in L^\infty(0, L)$.

We conclude this section by recalling that the cylindrical Wiener process $w(t)$, $t \geq 0$, can be written formally as

$$w(t) = \sum_{k=1}^{\infty} e_k \beta_k(t), \qquad t \geq 0,$$

where $\{\beta_k\}_{k\in\mathbb{N}}$ is a sequence of mutually independent standard Brownian motions defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. More rigorously, for any $t \geq 0$ and $u \in H$ the random variable $\langle w(t), u\rangle_H$ is well defined in $L^2(\Omega)$ and for any $s, t \geq 0$ and $u, v \in H$ it holds

$$\mathbb{E}\, \langle w(s), u\rangle_H \, \langle w(t), v\rangle_H = (s \wedge t)\, \langle u, v\rangle_H \, .$$

3. Uniform bounds for the stochastic convolution

Fix $\mu > 0$ and $T > 0$. For any $z \in L^p(\Omega; C([0, T]; \mathcal{H}_\delta))$ we define

$$\Gamma_\mu(z)(t) := \int_0^t S_\mu(t-s) G_\mu(z(s))\, dw(s), \qquad t \in [0, T].$$

$\Gamma_\mu(z)$ is known as the *stochastic convolution* associated with the semigroup $S_\mu(t)$, the non-linear operator G_μ and the process $z(t)$.

Our aim in this section is proving some a-priori estimates for $\Gamma_\mu(z)$, which are uniform with respect to $\mu \in (0, 1]$. These estimates represent the key point in the proof of the tightness of the family of probability measures $\{\mathcal{L}(u^\mu)\}_{\mu\in(0,1]}$ in $C([0, T]; H)$ and as, a consequence, of the Smoluchowski-Kramers approximation.

We first prove uniform bounds for $\Pi_1\Gamma_\mu(z)$ as a function of $t \in [0, T]$ with values in $H^\delta(0, L)$.

PROPOSITION 3.1. *Fix $T > 0$. Then for any $p > 4$, $z \in L^p(\Omega; L^p(0, T; \mathcal{H}_0))$ and $\delta < 1/2 - 2/p$ we have*

$$\sup_{\mu>0} E\, |\Pi_1\Gamma_\mu(z)|^p_{C([0,T]; H^\delta(0,L))} \leq c_p(T) \left(1 + \mathbb{E} \int_0^T |\Pi_1 z(t)|^p_H \, dt\right). \tag{3.1}$$

Proof. Due to the conditions on p and δ, we can fix $\alpha \in (1/p, (1-2\delta)/4)$ in such a way that $1 - \delta - 2\alpha > 1/2$. By using the stochastic factorization formula (for a proof see e.g. [8, Theorem 8.3]) we have

$$\Pi_1\Gamma_\mu(z)(t) = \frac{\sin \pi\alpha}{\pi} \int_0^t (t-s)^{\alpha-1} \Pi_1\, S_\mu(t-s) Y_\alpha^\mu(s)\, ds,$$

where

$$Y_\alpha^\mu(s) := \int_0^s (s-r)^{-\alpha} S_\mu(s-r) G_\mu(z(r))\, dw(r). \tag{3.2}$$

Since $p > 1/\alpha$, due to (2.5) this yields

$$\begin{aligned}
\left|\Pi_1 \Gamma_\mu(z)(t)\right|_{H^\delta}^p &\leq c_p \left(\int_0^t (t-s)^{(\alpha-1)\frac{p}{p-1}}\, ds\right)^{p-1} \int_0^t |\Pi_1 S_\mu(t-s) Y_\alpha^\mu(s)|_{H^\delta}^p\, ds \\
&\leq c_p(T) \int_0^t \left(\left|\Pi_1\, Y_\alpha^\mu(s)\right|_{H^\delta}^p + \mu^{p/2}\, \left|\Pi_2\, Y_\alpha^\mu(s)\right|_{H^{\delta-1}}^p\right) ds,
\end{aligned}$$

and hence

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \left|\Pi_1 \Gamma_\mu(z)(t)\right|_{H^\delta}^p \leq \\
&\qquad c_p(T)\, \mathbb{E} \int_0^T (|\Pi_1\, Y_\alpha^\mu(s)|_{H^\delta}^p + \mu^{p/2}\, |\Pi_2\, Y_\alpha^\mu(s)|_{H^{\delta-1}}^p)\, ds.
\end{aligned} \tag{3.3}$$

If we set $u := \Pi_1 z$, according to the Burkholder-Davies-Gundy inequality we have

$$\begin{aligned}
\mathbb{E} \left|\Pi_1\, Y_\alpha^\mu(s)\right|_{H^\delta}^p &= \frac{1}{\mu^p} \mathbb{E} \left|\int_0^s (s-r)^{-\alpha} \sum_{k=1}^\infty \Pi_1\, S_\mu(s-r)(0, g(u(r))e_k)\, d\beta_k(r)\right|_{H^\delta}^p \\
&\leq \frac{c_p}{\mu^p}\, \mathbb{E} \left(\int_0^s (s-r)^{-2\alpha} \sum_{k=1}^\infty \left|\Pi_1\, S_\mu(s-r)(0, g(u(r))e_k)\right|_{H^\delta}^2\, dr\right)^{p/2},
\end{aligned}$$

so that

$$\begin{aligned}
\mathbb{E} \left|\Pi_1\, Y_\alpha^\mu(s)\right|_{H^\delta}^p &\leq \frac{c_p}{\mu^p}\, \mathbb{E} \left(\int_0^s (s-r)^{-2\alpha} \sum_{k=1}^\infty \sum_{h=1}^\infty \alpha_h^\delta\, \left|\eta_h^\mu(0, g(u(r))e_k)(s-r)\right|^2 dr\right)^{p/2} \\
&= \frac{c_p}{\mu^p}\, \mathbb{E} \left(\int_0^s (s-r)^{-2\alpha} \sum_{k=1}^\infty \sum_{h=1}^\infty \alpha_h^\delta\, \left|\eta_h^\mu(0, e_h)(s-r)\right|^2 \left|\langle e_k, g(u(r))e_h\rangle_H\right|^2 dr\right)^{p/2} \\
&= \frac{c_p}{\mu^p}\, \mathbb{E} \left(\int_0^s (s-r)^{-2\alpha} \sum_{h=1}^\infty \alpha_h^\delta\, \left|\eta_h^\mu(0, e_h)(s-r)\right|^2\, |g(u(r))e_h|_H^2\, dr\right)^{p/2}.
\end{aligned}$$

Therefore, by using the Young inequality, thanks to (2.1) and (2.12) we obtain

$$\begin{aligned}
&\mathbb{E} \int_0^T \left|\Pi_1\, Y_\alpha^\mu(s)\right|_{H^\delta}^p\, ds \leq c_p\, (1 + \mathbb{E}\, |u|_{L^p(0,T;H)}^p) \\
&\left(\sum_{h=1}^\infty \frac{\alpha_h^\delta}{\mu^2} \int_0^T s^{-2\alpha} \left|\eta_h^\mu(0, e_h)(s)\right|^2\, ds\right)^{p/2}.
\end{aligned} \tag{3.4}$$

Thanks to (2.6), with a change of variable, we have

$$I_h^\mu(s) := \frac{1}{\mu^2}\int_0^s r^{-2\alpha}|\eta_h^\mu(0,e_h)(r)|^2\,dr$$
$$= \frac{\mu^{1-2\alpha}}{|1-4\alpha_h\mu|}\int_0^{s/\mu} r^{-2\alpha}\exp(-(1-\sqrt{(1-4\alpha_h\mu)^+})r)|1-\exp(-\sqrt{1-4\alpha_h\mu}\,r)|^2 dr.$$

If $4\alpha_h\mu > 1$, then

$$I_h^\mu(s) = 2\,\mu^{1-2\alpha}\int_0^{s/\mu} r^{-2\alpha}\exp(-r)\frac{(1-\cos\sqrt{(1-4\alpha_h\mu)^-}r)}{(1-4\alpha_h\mu)^- r^2}\,r^2\,dr.$$

Now, for any $\delta \in [0,2]$ there exists $c_\delta > 0$ such that

$$1-\cos\beta \leq c_\delta\frac{\beta^\delta}{\beta^\delta\vee 1},\qquad \beta > 0.$$

Hence, since we are assuming $4\alpha_h\mu > 1$, by taking $\delta = 2$ we obtain

$$I_h^\mu(s) \leq \frac{c}{\mu^{2\alpha}\alpha_h}\int_0^{s/\mu} r^{-2\alpha}\exp(-r)\,\frac{4\alpha_h\mu r^2}{(4\alpha_h\mu-1)r^2\vee 1}\,dr$$
$$\leq \frac{c}{\mu^{2\alpha}\alpha_h}\int_0^\infty r^{-2\alpha}\exp(-r)\,(1+r^2)\,dr \leq c\,\alpha_h^{2\alpha-1}.$$

In the case $4\alpha_h\mu < 1$ we distinguish the case $1-4\alpha_h\mu \leq 1/4$ and the case $1-4\alpha_h\mu > 1/4$. Assume first that $1-4\alpha_h\mu \leq 1/4$. For any $\delta \in [0,1]$ there exists $c_\delta > 0$ such that

$$1-\exp(-\beta) \leq c_\delta\beta^\delta,\qquad \beta > 0. \tag{3.5}$$

Then, as $\mu < 1/4\alpha_h$ we have

$$I_h^\mu(s) \leq c\,\mu^{1-2\alpha}\int_0^{s/\mu} r^{-2\alpha}\exp\left(-\frac{r}{2}\right)r^2\,dr \leq c\,\alpha_h^{2\alpha-1}.$$

If $1-4\alpha_h\mu > 1/4$, we have

$$I_h^\mu(s) \leq c\,\mu^{1-2\alpha}\int_0^{s/\mu} r^{-2\alpha}\exp\left(-\left(1-\sqrt{1-4\alpha_h\mu}\right)r\right)\,dr$$

and then, by proceeding with a change of variable it follows

$$I_h^\mu(s) \leq c\,\mu^{1-2\alpha}\left(1-\sqrt{1-4\alpha_h\mu}\right)^{2\alpha-1}\int_0^\infty r^{-2\alpha}\exp(-r)\,dr \leq c\,\alpha_h^{2\alpha-1}.$$

Therefore, in all the cases we have considered $I_h^\mu(s) \leq c\,\alpha_h^{2\alpha-1}$ and due to (3.4) this implies

$$\mathbb{E}\int_0^T \left|\Pi_1\,Y_\alpha^\mu(s)\right|_{H^\delta}^p\,ds \leq c_p\,\left(1+\mathbb{E}\,|u|_{L^p(0,T;H)}^p\right)\left(\sum_{h=1}^\infty \alpha_h^{\delta+2\alpha-1}\right)^{p/2}. \tag{3.6}$$

By proceeding analogously for $\Pi_2 Y^\mu_\alpha$ we have

$$\mathbb{E}\int_0^T \left|\Pi_2 Y^\mu_\alpha(s)\right|^p_{H^{\delta-1}}\,ds$$

$$\leq c_p\,\left(1+\mathbb{E}\,|u|^p_{L^p(0,T;H)}\right)\left(\sum_{h=1}^\infty \frac{\alpha_h^{\delta-1}}{\mu^2}\int_0^s r^{-2\alpha}\left|\vartheta^\mu_h(0,e_h)(r)\right|^2\,dr\right)^{p/2}.$$

Therefore, if we show that

$$\int_0^s r^{-2\alpha}\left|\vartheta^\mu_h(0,e_h)(r)\right|^2\,dr \leq c\,\mu\,\alpha_h^{2\alpha}, \tag{3.7}$$

we have

$$\mathbb{E}\int_0^T \left|\Pi_2 Y^\mu_\alpha(s)\right|^p_{H^{\delta-1}}\,ds \leq \frac{c_p}{\mu^{p/2}}\,\left(1+\mathbb{E}\,|u|^p_{L^p(0,T;H)}\right)\left(\sum_{h=1}^\infty \alpha_h^{\delta+2\alpha-1}\right)^{p/2},$$

and hence, from (3.3) and (3.6) we obtain (3.1), for some $c_p(T) > 0$ independent of $\mu \in (0,1]$.

According to (2.7), with the usual change of variable we have

$$J^\mu_h(s) := \int_0^s r^{-2\alpha}\left|\vartheta^\mu_h(0,e_h)(r)\right|^2\,dr$$

$$= \frac{1}{4}\int_0^s r^{-2\alpha}\exp\left(-\frac{r}{\mu}\right)\left|\left(1-\frac{1}{2\mu\gamma^\mu_k}\right)\exp\left(\gamma^\mu_k r\right)+\left(1+\frac{1}{2\mu\gamma^\mu_k}\right)\exp\left(-\gamma^\mu_k r\right)\right|^2\,dr$$

$$= \frac{\mu^{1-2\alpha}}{4\,|1-4\alpha_h\mu|}\int_0^{s/\mu} r^{-2\alpha}\exp\left(-\left(1-\sqrt{(1-4\alpha_h\mu)^+}\right)r\right)$$

$$\times\left|\left(\sqrt{1-4\alpha_h\mu}-1\right)+\left(\sqrt{1-4\alpha_h\mu}+1\right)\exp\left(-\sqrt{1-4\alpha_h\mu}\,r\right)\right|^2\,dr.$$

If $4\alpha_h\mu > 1$, we have

$$J^\mu_h(s) \leq c\,\mu^{1-2\alpha}\int_0^\infty r^{-2\alpha}\exp(-r)\left(1+\frac{\left(1-\cos\sqrt{(1-4\alpha_h\mu)^-}r\right)}{(1-4\alpha_h\mu)^-r^2}\,r^2\right)\,dr$$

and then, by proceeding as above for $I^\mu_h(s)$, we get

$$J^\mu_h(s) \leq c\,\mu^{1-2\alpha} \leq c\,\mu\,\alpha_h^{2\alpha}. \tag{3.8}$$

Analogously, if $1/4 \leq 4\alpha_h\mu < 1$ due to (3.5) we have

$$J_h^\mu(s) \leq c\,\mu^{1-2\alpha}\int_0^\infty r^{-2\alpha}\exp\left(-\frac{r}{2}\right)\left(1+r^2\right)\,dr \leq c\,\mu\,\alpha_h^{2\alpha}.$$

Finally, if $4\alpha_h\mu < 1/4$ we have

$$J_k^\mu(s) \leq c\,\mu^{1-2\alpha}\left(1-\sqrt{1-4\alpha_h\mu}\right)^2\int_0^\infty r^{-2\alpha}\exp\left(-\left(1-\sqrt{1-4\alpha_h\mu}\right)r\right)\,dr$$

$$+c\,\mu^{1-2\alpha}\int_0^\infty r^{-2\alpha}\exp\left(-\left(1+\sqrt{1-4\alpha_h\mu}\right)r\right)\,dr$$

$$+c\,\mu^{1-2\alpha}\left(1-\sqrt{1-4\alpha_h\mu}\right)^2\int_0^\infty r^{-2\alpha}\exp\left(-r\right)\,dr =: J_{1,h}^\mu(s)+J_{2,h}^\mu(s)+J_{3,h}^\mu(s).$$

By a change of variable we obtain

$$\begin{aligned}J_{1,h}^\mu(s) &= c\,\mu^{1-2\alpha}\left(1-\sqrt{1-4\alpha_h\mu}\right)^{1+2\alpha}\int_0^\infty r^{-2\alpha}\exp\left(-r\right)\,dr\\ &\leq c\,\mu^2\,\alpha_h^{1+2\alpha} \leq c\,\mu\,\alpha_h^{2\alpha}.\end{aligned}$$

Analogously

$$J_{2,h}^\mu(s) \leq c\,\mu^{1-2\alpha}\left(1+\sqrt{1-4\alpha_h\mu}\right)^{2\alpha-1}\int_0^\infty r^{-2\alpha}\exp\left(-r\right)\,dr$$

$$\leq c\,\mu^{1-2\alpha}\frac{(4\alpha_h\,\mu)^{2\alpha}}{\left(1+\sqrt{1-4\alpha_h\mu}\right)\left(1-\sqrt{1-4\alpha_h\mu}\right)^{2\alpha}} \leq c\,\mu\,\alpha_h^{2\alpha}.$$

Finally, for the last term we have

$$J_{3,h}^\mu(s) \leq c\,\mu^{1-2\alpha}\left(1-\sqrt{1-4\alpha_h\mu}\right)^{1-2\alpha}\frac{(4\alpha_h\,\mu)^{2\alpha}}{\left(1+\sqrt{1-4\alpha_h\mu}\right)^{2\alpha}} \leq c\,\mu\,\alpha_h^{2\alpha}.$$

This means that (3.8) holds in all possible cases and hence (3.7) is proved. This concludes the proof of (3.1).

□

The estimates above for the $H^\delta(0,L)$-norm of $\Pi_1\Gamma_\mu(z)$, are obtained only for $\delta < 1/2$. This means that they do not provide any bound in the space of continuous functions. In what follows we shall prove some pointwise uniform bounds, both in the space and in the time variables, which will lead us to uniform bounds in the space of Hölder continuous functions.

LEMMA. *Assume that $g : [0, L] \times \mathbb{R} \to \mathbb{R}$ is a measurable mapping such that*

$$\sup_{x\in[0,L]} |g(x,\sigma)| \leq c\left(1+|\sigma|^\kappa\right), \quad \sigma \in \mathbb{R}, \tag{3.9}$$

for some $\kappa \in [0,1]$. Then, for any $\epsilon \in (0, 1/2\kappa)$, $\rho \in (0, 1/2 - \kappa(1/2 - \kappa\epsilon))$ and $p \geq 1$ there exists a constant $c = c(\epsilon, \rho, p)$ such that

$$\sup_{\mu>0} E\,|\Pi_1\,\Gamma_\mu(z)(t,x) - \Pi_1\,\Gamma_\mu(z)(s,y)|^p$$
$$\leq c\,(1+\mathbb{E}\,|\Pi_1 z|^{p\kappa}_{C([0,T];H^{\kappa\epsilon})})\,(\,|t-s|^{\frac{\rho}{2}} + |x-y|^\rho)^p,$$

for any $z \in L^p(\Omega; C([0,T]; \mathcal{H}_{\kappa\epsilon}))$, $x, y \in [0, L]$ and $t, s \in [0, T]$.

Proof. First step: space bounds. We show that for any $z \in L^p(\Omega; C([0,T]; \mathcal{H}_{\kappa\epsilon}))$, $x, y \in [0, L]$ and $t \in [0,T]$

$$\mathbb{E}\,|\Pi_1\,\Gamma_\mu(z)(t,x) - \Pi_1\,\Gamma_\mu(z)(t,y)|^p$$
$$\leq c(1+\mathbb{E}\,|\Pi_1 z|^{p\kappa}_{C([0,T];H^{\kappa\epsilon}(0,L))})|x-y|^{\rho p}, \tag{3.10}$$

for some constat c depending on ϵ, ρ and p and not on $\mu > 0$.

If we set $u := \Pi_1 z$, we have

$$\Pi_1\,\Gamma_\mu(z)(t,x) - \Pi_1\,\Gamma_\mu(z)(t,y) =$$

$$\frac{1}{\mu}\sum_{k=1}^\infty \int_0^t \left[\Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(x) - \Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(y)\right] d\beta_k(r).$$

Then, from the Burkholder-Davies-Gundy inequality it follows

$$\mathbb{E}\,\left|\Pi_1\,\Gamma_\mu(z)(t,x) - \Pi_1\,\Gamma_\mu(z)(t,y)\right|^p \leq \frac{c_p}{\mu^p}\,\mathbb{E}$$
$$\left(\int_0^t \sum_{k=1}^\infty \left|\Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(x) - \Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(y)\right|^2 dr\right)^{p/2}.$$

According to (2.8), for each $k \in \mathbb{N}$ we have

$$\sum_{k=1}^\infty \left|\Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(x) - \Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(y)\right|^2$$

$$= \sum_{k=1}^\infty \left|\left\langle K_\mu(t-r,x,\cdot) - K_\mu(t-r,y,\cdot), g(u(r))e_k\right\rangle\right|^2$$

$$= \left|\left[K_\mu(t-r,x,\cdot) - K_\mu(t-r,y,\cdot)\right] g(u(r))\right|_H^2.$$

Thanks to the Sobolev embedding theorem for any $\epsilon \in (0, 1/2\kappa)$

$$H^{\kappa\epsilon}(0, L) \hookrightarrow L^{\frac{2}{1-2\kappa\epsilon}}(0, L), \quad H^{\kappa\left(\frac{1}{2}-\kappa\epsilon\right)}(0, L) \hookrightarrow L^{\frac{2}{1-\kappa(1-2\kappa\epsilon)}}(0, L).$$

Then, due to (3.9) we have

$$\sum_{k=1}^{\infty} \left|\Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(x) - \Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(y)\right|^2$$

$$\leq \left|K_\mu(t-r, x, \cdot) - K_\mu(t-r, y, \cdot)\right|^2_{L^{2/1-\kappa(1-2\kappa\epsilon)}} |g(u(r))|^2_{L^{2/k(1-2\kappa\epsilon)}}$$

$$\leq c\, \left|K_\mu(t-r, x, \cdot) - K_\mu(t-r, y, \cdot)\right|^2_{L^{2/1-\kappa(1-2\kappa\epsilon)}} (1 + |u(r)|^{2\kappa}_{L^{2/1-2\kappa\epsilon}})$$

$$\leq c\, \left|K_\mu(t-r, x, \cdot) - K_\mu(t-r, y, \cdot)\right|^2_{H^{\kappa(\frac{1}{2}-\kappa\epsilon)}} (1 + |u(r)|^{2\kappa}_{H^{\kappa\epsilon}}).$$

By using (2.10), this allows to conclude that for any $\rho < 1/2 - \kappa(1/2 - \kappa\epsilon)$

$$\mathbb{E}\, \left|\Pi_1\, \Gamma_\mu(z)(t, x) - \Pi_1\, \Gamma_\mu(z)(t, y)\right|^p \leq \frac{c_p}{\mu^p}(1 + \mathbb{E}\, |u|^{p\kappa}_{C([0,T]; H^{\kappa\epsilon}(0,L))}) c_\rho^{p/2}\, \mu^p |x-y|^{\rho p},$$

which implies (3.10).

Second step: time bounds. We show that for any $z \in L^p(\Omega; C([0, T]; \mathcal{H}_{\kappa\epsilon}))$, $x \in [0, L]$ and $t, s \in [0, T]$

$$\mathbb{E}\, \left|\Pi_1\, \Gamma_\mu(z)(t, x) - \Pi_1\, \Gamma_\mu(z)(s, x)\right|^p \leq c(1 + \mathbb{E}\, |\Pi_1 z|^{p\kappa}_{C([0,T]; H^{\kappa\epsilon}(0,L))})|t-s|^{\frac{\rho p}{2}}, \tag{3.11}$$

for some constat c depending on ϵ, ρ and p and independent of $\mu > 0$.

By setting as above $u := \Pi_1 z$, we have

$$\Pi_1\, \Gamma_\mu(z)(t, x) - \Pi_1\, \Gamma_\mu(z)(s, x) = \sum_{k=1}^{\infty} \frac{1}{\mu} \int_0^s \left[\Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(x) - \Pi_1 S_\mu(s-r)(0, g(u(r))e_k)(x)\right] d\beta_k(r)$$
$$+ \sum_{k=1}^{\infty} \frac{1}{\mu} \int_s^t \Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(x)\, d\beta_k(r)$$

and then, according to the Burkholder-Davies-Gundy inequality, for any $p \geq 2$ we get

$$\mathbb{E}\left|\Pi_1 \Gamma_\mu(z)(t,x) - \Pi_1 \Gamma_\mu(z)(s,x)\right|^p$$
$$\leq \frac{c_p}{\mu^p}\mathbb{E}\left(\int_0^s \sum_{k=1}^\infty |\Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(x)\right.$$
$$\left. - \Pi_1 S_\mu(s-r)(0, g(u(r))e_k)(x)|^2\, dr\right)^{p/2}$$
$$+ \frac{c_p}{\mu^p}\mathbb{E}\left(\int_s^t \sum_{k=1}^\infty |\Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(x)|^2\, dr\right)^{p/2}.$$

By proceeding as in the first step, from (2.8) we obtain

$$\sum_{k=1}^\infty \left|\Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(x) - \Pi_1 S_\mu(s-r)(0, g(u(r))e_k)(x)\right|^2$$
$$= \left|\left[K_\mu(t-r,x,\cdot) - K_\mu(s-r,x,\cdot)\right] g(u(r))\right|_H^2$$

and

$$\sum_{k=1}^\infty |\Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(x)|^2 = \left|K_\mu(t-r,x,\cdot) g(u(r))\right|_H^2.$$

Then, as in the first step, we have

$$\sum_{k=1}^\infty \left|\Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(x) - \Pi_1 S_\mu(s-r)(0, g(u(r))e_k)(x)\right|^2$$
$$\leq c\left|K_\mu(t-r,x,\cdot) - K_\mu(s-r,x,\cdot)\right|^2_{H^{\kappa(\frac{1}{2}-\kappa\epsilon)}} (1 + |u(r)|^{2\kappa}_{H^{\kappa\epsilon}})$$

and

$$\sum_{k=1}^\infty |\Pi_1 S_\mu(t-r)(0, g(u(r))e_k)(x)|^2 \leq c\, |K_\mu(t-r,x,\cdot)|^2_{H^{\kappa(\frac{1}{2}-\kappa\epsilon)}} (1 + |u(r)|^{2\kappa}_{H^{\kappa\epsilon}}).$$

Due to (2.11), for any $\rho < 1/2 - \kappa(1/2 - \kappa\epsilon)$ this implies

$$\mathbb{E}\left|\Pi_1 \Gamma_\mu(z)(t,x) - \Pi_1 \Gamma_\mu(z)(s,x)\right|^p \leq \frac{c_p}{\mu^p}$$
$$\left(1 + \mathbb{E}\,|u|^{p\kappa}_{C([0,T];H^{\kappa\epsilon}(0,L))}\right) c_\rho^{p/2}\, \mu^p |t-s|^{\frac{\rho p}{2}},$$

and this yields (3.11). □

As a consequence of the Garcia-Rademich-Rumsey theorem, from the previous lemma we obtain the following result.

PROPOSITION 3.2. *Assume that* $g : [0, L] \times \mathbb{R} \to \mathbb{R}$ *is a measurable mapping such that*

$$\sup_{x\in[0,L]} |g(x,\sigma)| \leq c\,(1+|\sigma|^\kappa), \qquad \sigma \in \mathbb{R},$$

for some $\kappa \in [0,1]$. *Then, for any* $\mu > 0$, $\epsilon \in (0, 1/2\kappa)$, $\rho \in (0, 1/4 - \kappa(1/2 - \kappa\epsilon)/2)$, $p \geq 1$ *and* $z \in L^p(\Omega; C([0,T]; \mathcal{H}_{\kappa\epsilon}))$, *the process* $\Pi_1\,\Gamma_\mu(z)$ *has a version which is* ρ*-Hölder continuous with respect to* $(t, x) \in [0, T] \times [0, L]$. *Moreover,*

$$\sup_{\mu>0} E\,|\Pi_1\,\Gamma_\mu(z)|^p_{C^\rho([0,T]\times[0,L])} \leq c\,\left(1 + \mathbb{E}\,|\Pi_1 z|^{p\kappa}_{C([0,T];H^{\kappa\epsilon})}\right),$$

for some constat $c = c(\epsilon, \rho, p)$.

REMARK. In particular, if Hypothesis 1 is satisfied, then g has linear growth ($\kappa = 1$), so that as a consequence of Proposition 3.2 for any $\epsilon \in (0, 1/2)$, $\rho < \epsilon/2$ and $p \geq 1$ there exists some constant $c = c(\epsilon, \rho, p)$ such that

$$\sup_{\mu>0} E\,|\Pi_1\,\Gamma_\mu(z)|^p_{C^\rho([0,T]\times[0,L])} \leq c\,(1 + \mathbb{E}\,|\Pi_1 z|^p_{C([0,T];H^\epsilon)}), \tag{3.12}$$

for any $z \in L^p(\Omega; C([0,T]; \mathcal{H}_\epsilon))$.

Analogously, if Hypothesis 2 is verified, then g is bounded ($\kappa = 0$), so that for any $\rho < 1/4$ and $p \geq 1$ there exists some constant $c = c(\rho, p)$ such that

$$\sup_{\mu>0} E\,|\Pi_1\,\Gamma_\mu(z)|^p_{C^\rho([0,T]\times[0,L])} \leq c, \tag{3.13}$$

for any $z \in L^p(\Omega; C([0,T]; \mathcal{H}_0))$.

4. The convergence result

With the notations introduced in section 2, if we set $z := (u, \partial u/\partial t)$ equation (1.1) can be written in the following abstract form

$$dz(t) = \left[A_\mu z(t) + B_\mu(z(t))\right] dt + G_\mu(z(t))\,dw(t), \quad z(0) = z_0 = (u_0, v_0). \tag{4.1}$$

First of all we introduce the notion of mild solution for the problem above.

DEFINITION 4.1. Let $\delta \in [0,1]$, $T > 0$ and $p \geq 1$. A process u^μ is a *mild solution* of problem (4.1) in $L^p(\Omega; C([0,T]; \mathcal{H}_\delta))$ if

$$u^\mu \in L^p(\Omega; C([0,T]; H^\delta(0,L))), \quad v^\mu := \frac{\partial u^\mu}{\partial t} \in L^p(\Omega; C([0,T]; H^{\delta-1}(0,L))),$$

and, by setting $z^\mu := (u^\mu, v^\mu)$, it holds

$$z^\mu(t) = S_\mu(t)(u_0, v_0) + \int_0^t S_\mu(t-s)B_\mu(z^\mu(s))\,ds + \int_0^t S_\mu(t-s)G_\mu(z^\mu(s))\,dw(s).$$

The existence and uniqueness of a mild solution of equation (4.1), for any fixed $\mu > 0$, is a well known fact in the literature, both under Hypothesis 1 (see [1]) and under Hypothesis 2 (see [15] in the more delicate case of space dimension $d = 2$, under more restrictive conditions on the noise and on the initial data u_0 and v_0, due to the higher dimension). Here we give a self-contained proof for the reader convenience.

THEOREM 4.2. *Fix $\mu > 0$ and $\delta \in [0, 1/2)$. Then, both under Hypothesis* 1 *and under Hypothesis* 2*, for any $T > 0$ and $p \geq 1$ and for any initial datum $z_0 = (u_0, v_0) \in \mathcal{H}_1$ there exists a unique mild solution z^μ to problem* (4.1) *in $L^p(\Omega; C([0, T]; \mathcal{H}_\delta))$.*

Proof. Case b Lipschitz (Hypothesis 1.) By arguing as in the proof of Proposition 3.1, it is possible to prove that for any $p \geq 1$ and $z_1, z_2 \in L^p(\Omega; L^p(0, T; \mathcal{H}_\delta))$

$$\mathbb{E}\left|\Gamma_\mu(z_1) - \Gamma_\mu(z_2)\right|^p_{C([0,T];\mathcal{H}_\delta)} \leq \frac{c_p(T)}{\mu^p}\,\mathbb{E}\,|z_1 - z_2|^p_{L^p(0,T;\mathcal{H}_\delta)},$$

for some continuous increasing function $c_p(T)$ vanishing at $T = 0$.

Now, as B_μ is Lipschitz continuous on $\mathcal{H}_\delta$, with the same arguments used in the proof of [4, Proposition 4.2], it is immediate to check that the mapping

$$z \mapsto \left(t \mapsto \int_0^t S_\mu(t-s)B_\mu(z(s))\,ds\right),$$

is Lipschitz continuous on $L^p(\Omega; C([0, T]; \mathcal{H}_\delta))$, with the Lipschitz constant vanishing as T goes to zero. Hence the mapping $\mathcal{F}_\mu$ defined by

$$\mathcal{F}_\mu(z)(t) := S_\mu(t)(u_0, v_0) + \int_0^t S_\mu(t-s)B_\mu(z(s))\,ds + \int_0^t S_\mu(t-s)\,G_\mu(z(s))\,dw(s)$$

is Lipschitz continuous on $L^p(\Omega; C([0, T]; \mathcal{H}_\delta))$ and the Lipschitz constant is strictly less than 1 for some $T = T_0$ sufficiently small. By a fixed point argument this implies that there exists a unique mild solution for equation (4.1) in the time interval $[0, T_0]$ and, by repeating the same arguments in the intervals $[T_0, 2T_0]$, $[2T_0, 3T_0]$ and so on, we get a unique solution defined in the whole interval $[0, T]$.

Case b not Lipschitz (Hypothesis 2). For any $n > 0$ we consider the problem

$$dz(t) = \left[A_\mu z(t) + B_{n,\mu}(z(t))\right]dt + G_\mu(z(t))\,dw(t), \quad z(0) = z_0 = (u_0, v_0),$$

where $B_{n,\mu}$ is the mapping introduced in (2.20). As $B_{n,\mu}$ is Lipschitz continuous, due to the previous step for any $n > 0$ the problem above admits a unique mild solution $z^\mu_n = (u^\mu_n, v^\mu_n)$ in $L^p(\Omega; C([0, T]; \mathcal{H}_\delta))$, for any $p \geq 1$ and $\delta < 1/2$.

If we show that for any $T > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} |u_n^\mu(t)|_{C([0,L])} \geq n\right) = 0, \tag{4.2}$$

by setting $\tau_n := \inf\left\{ t \geq 0 \,:\, |u_n^\mu(t)|_{C([0,L])} \geq n \right\}$, with $\inf \emptyset = +\infty$, we have

$$\lim_{n\to\infty} \mathbb{P}(\tau_n \leq T) = \lim_{n\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} |u_n^\mu(t)|_{C([0,L])} \geq n\right) = 0,$$

so that, by defining $\tau := \sup_{n>0} \tau_n$ we obtain

$$\mathbb{P}(\tau < \infty) = \lim_{T\to\infty} \mathbb{P}(\tau \leq T) = \lim_{T\to\infty}\lim_{n\to\infty} \mathbb{P}(\tau_n \leq T) = 0. \tag{4.3}$$

In particular, this means that for any $t \geq 0$ and $\omega \in \{\tau = \infty\}$ there exists $n > 0$ such that $t \leq \tau_n(\omega)$ and then we can define

$$z^\mu(t)(\omega) = z_n^\mu(t)(\omega). \tag{4.4}$$

As $t \leq \tau_n(\omega)$, we have $|u_n^\mu(t)(\omega)|_{C([0,L])} \leq n$ and then, due to (2.21), we have $B_{n,\mu}(z^\mu(t)) = B_\mu(z^\mu(t))$. By proceeding as e.g. in [3, Proof of Theorem 5.3] this implies that (4.4) is a good definition of solution in $L^p(\Omega; C([0,T]; \mathcal{H}_\delta))$ and such solution is unique.

Thus, in order to conclude our proof it remains to prove (4.2). We may assume that z_n^μ is a strict solution; otherwise we can approximate it by means of suitable approximating problems (for all details see for example [2, Proposition 6.2.2]). If we set

$$\rho_n^\mu(t) := u_n^\mu(t) - \Pi_1\Gamma_\mu(z_n^\mu)(t), \quad t \in [0,T],$$

we have that ρ_n^μ is the solution of the problem

$$\begin{cases} \mu\dfrac{\partial^2\rho_n^\mu}{\partial t^2}(t,x) + \dfrac{\partial\rho_n^\mu}{\partial t}(t,x) = \Delta\rho_n^\mu(t,x) + b_n(x, u_n^\mu(t,x)), & t > 0, \;\; x \in [0,L], \\[2ex] \rho_n^\mu(0) = u_0, \quad \dfrac{\partial\rho_n^\mu}{\partial t}(0) = u_1, \quad \rho_n^\mu(t,0) = \rho_n^\mu(t,L) = 0, & t \geq 0. \end{cases}$$

If we multiply both sides in the equation above by $\partial\rho_n^\mu/\partial t$ and then integrate with respect to $x \in [0, L]$, we obtain

$$\mu \frac{d}{dt} \left|\frac{\partial \rho_n^\mu}{\partial t}(t)\right|_H^2 + \frac{d}{dt} \left|\rho_n^\mu(t)\right|_{H^1}^2$$

$$+2 \left|\frac{\partial \rho_n^\mu}{\partial t}(t)\right|_H^2 = 2 \left\langle b_n(\cdot, u_n^\mu(t)), \frac{\partial \rho_n^\mu}{\partial t}(t)\right\rangle_H$$

$$= 2 \left\langle b_n(\cdot, \rho_n^\mu(t)), \frac{\partial \rho_n^\mu}{\partial t}(t)\right\rangle_H$$

$$+2 \left\langle b_n(\cdot, \rho_n^\mu(t) + \Pi_1 \Gamma_\mu(z_n^\mu)(t)) - b_n(\cdot, \rho_n^\mu(t)), \frac{\partial \rho_n^\mu}{\partial t}(t)\right\rangle_H$$

$$\leq 2\, c_2 \frac{d}{dt} \int_0^L \hat{\beta}_n(x, \rho_n^\mu(x,t))\, dx + |b_n(\cdot, \rho_n^\mu(t)$$

$$+ \Pi_1 \Gamma_\mu(z_n^\mu)(t)) - b_n(\cdot, \rho_n^\mu(t))|_H^2 + |\frac{\partial \rho_n^\mu}{\partial t}(t)|_H^2$$

where $\hat{\beta}_n$ is the function defined in (2.23). Now, we remark that

$$\left|b_n(\cdot, \rho_n^\mu(t) + \Pi_1 \Gamma_\mu(z_n^\mu)(t)) - b_n(\cdot, \rho_n^\mu(t))\right|_H^2$$

$$= \int_0^L \left| \int_0^1 \partial_\sigma\, b_n(x, \rho_n^\mu(x,t) + r\, \Pi_1 \Gamma_\mu(z_n^\mu)(x,t))\, dr\;\; \Pi_1 \Gamma_\mu(z_n^\mu)(x,t)\right|^2 dx,$$

and then, according to Lemma 2.4., there exists some $n_0 > 0$ such that for any $n > n_0$

$$\left|b_n(\cdot, \rho_n^\mu(t) + \Pi_1 \Gamma_\mu(z_n^\mu)(t)) - b_n(\cdot, \rho_n^\mu(t))\right|_H^2$$

$$\leq c \int_0^L \left(1 - \hat{\beta}_n(x, \rho_n^\mu(x,t)) + |\Pi_1 \Gamma_\mu(z_n^\mu)(x,t))|^{2(\lambda-1)}\right) \left|\Pi_1 \Gamma_\mu(z_n^\mu)(x,t)\right|^2 dx.$$

If we set

$$\Lambda_n^\mu := \sup_{t \in [0,T]} |\Pi_1 \Gamma_\mu(z_n^\mu)(t)|_{C([0,L])},$$

since thanks to (2.24) $\hat{\beta}_n > 0$ for n large enough, this implies

$$\left|b_n(\cdot, \rho_n^\mu(t) + \Pi_1 \Gamma_\mu(z_n^\mu)(t)) - b_n(\cdot, \rho_n^\mu(t))\right|_H^2$$

$$\leq c \left(1 + (\Lambda_n^\mu)^{2\lambda}\right) + c\, (\Lambda_n^\mu)^2 \left(-\int_0^L \hat{\beta}_n(x, \rho_n^\mu(x,t))\, dx\right),$$

so that

$$\mu\,\frac{d}{dt}\,\left|\frac{\partial\rho_n^\mu}{\partial t}(t)\right|_H^2+\frac{d}{dt}\,\left|\rho_n^\mu(t)\right|_{H^1}^2-2\,c_2\,\frac{d}{dt}\int_0^L\hat{\beta}_n(x,\rho_n^\mu(x,t))\,dx+\left|\frac{\partial\rho_n^\mu}{\partial t}(t)\right|_H^2$$

$$\leq c\,\left(1+(\Lambda_n^\mu)^{2\lambda}\right)+c\,(\Lambda_n^\mu)^2\left(-\int_0^L\hat{\beta}_n(x,\rho_n^\mu(x,t))\,dx\right).$$

By integrating with respect to $t\in[0,T]$ it follows

$$\mu\,\left|\frac{\partial\rho_n^\mu}{\partial t}(t)\right|_H^2+\left|\rho_n^\mu(t)\right|_{H^1}^2-2\,c_2\int_0^L\hat{\beta}_n(x,\rho_n^\mu(x,t))\,dx+\int_0^t\left|\frac{\partial\rho_n^\mu}{\partial t}(s)\right|_H^2\,ds$$

$$\leq\mu\,|v_0|_H^2+|u_0|_{H^1}^2-2\,c_2\int_0^L\hat{\beta}_n(x,u_0(x))\,dx+ct\,\left(1+(\Lambda_n^\mu)^{2\lambda}\right)$$

$$+c\,(\Lambda_n^\mu)^2\int_0^t\left(-\int_0^L\hat{\beta}_n(x,\rho_n^\mu(x,s))\,dx\right)ds.$$

According to (2.22), we immediately have

$$\int_0^L\hat{\beta}_n(x,u_0(x))\,dx\leq c\,\left(1+|u_0|_{C([0,L])}^{\lambda+1}\right)\leq c\,\left(1+|u_0|_{H^1}^{\lambda+1}\right),$$

and then, from the Gronwall lemma we obtain

$$\sup_{t\leq T}\left(\mu\,\left|\frac{\partial\rho_n^\mu}{\partial t}(t)\right|_H^2+\left|\rho_n^\mu(t)\right|_{H^1}^2-2\,c_2\int_0^L\hat{\beta}_n(x,\rho_n^\mu(x,t))\,dx\right)+\int_0^T\left|\frac{\partial\rho_n^\mu}{\partial t}(s)\right|_H^2 ds$$

$$\leq c\left(1+(\Lambda_n^\mu)^{2\lambda}\right)\exp\left(c\,(\Lambda_n^\mu)^2T\right).$$

In particular, for any $\theta\in[0,1/2)$ we have

$$\sup_{t\leq T}\left|\rho_n^\mu(t)\right|_{C^\theta([0,L])}^2\leq c_\theta\,\sup_{t\leq T}\left|\rho_n^\mu(t)\right|_{H^1}^2\leq c_\theta\,\left(1+(\Lambda_n^\mu)^{2\lambda}\right)\exp\left(c\,(\Lambda_n^\mu)^2T\right).$$

Now, recalling that $u_n^\mu=\rho_n^\mu+\Pi_1\Gamma_\mu(z_n^\mu)$, for any $n\geq n_0$ we have

$$\mathbb{P}\left(\sup_{t\leq T}|u_n^\mu(t)|_{C([0,L])}\geq n\right)\leq\mathbb{P}\,\left(c_0\,\left(1+(\Lambda_n^\mu)^{2\lambda}\right)\exp\left(c\,(\Lambda_n^\mu)^2T\right)\geq n/2\right)$$

$$+\mathbb{P}\,\left(\Lambda_n^\mu\geq n/2\right)=\mathbb{P}\,\left(\Lambda_n^\mu\geq f_T(n)\right)+\mathbb{P}\,\left(\Lambda_n^\mu\geq n/2\right)\leq\left(\frac{1}{f_T(n)}+\frac{2}{n}\right)\mathbb{E}\,\Lambda_n^\mu,$$

where $f_T : [0,\infty) \to [0,\infty)$ is the inverse of the function $x \mapsto c_0(1+x^{2\lambda})\exp(cx^2T)$. As $f_T(n)$ diverges to $+\infty$, as $n \to +\infty$, this allows us to conclude the proof of (4.2), and hence the proof of the theorem in the case of non-Lipschitz b, as due to Proposition 3.2 and to (3.13) we have

$$\mathbb{E}\,\Lambda_n^\mu \leq c,$$

for a constant c not depending on n (and neither of $\mu \in (0,1]$). □

REMARK 4.3. 1. By looking at the proof of the previous theorem, one sees that in the case of Lipschitz continuous b in order to have solutions in $L^p(\Omega; C([0,T];\mathcal{H}_\delta))$ it is not necessary to take the initial data $z_0 = (u_0, v_0)$ in $\mathcal{H}_1$, but it is sufficient to take them in $\mathcal{H}_\delta$.
2. From the proof we see that the solution is also unique in $L^p(\Omega; L^p(0,T;\mathcal{H}_\delta))$.

Now, the key point in the proof of our convergence result is showing that if $z^\mu = (u^\mu, v^\mu)$ denotes the unique mild solution in $L^p(\Omega; C([0,T];\mathcal{H}_\delta))$ to equation (4.1), then the family of probabilities $\{\mathcal{L}(u^\mu)\}_{\mu\in(0,1]}$ is tight in $C([0,T];H)$.

PROPOSITION 4.4. *Let $z_0 \in \mathcal{H}_1$ and assume that either Hypothesis* 1 *or Hypothesis* 2 *hold. Then the family of probability measures $\{\mathcal{L}(u^\mu)\}_{\mu\in(0,1]}$ is tight in $C([0,T];H)$.*

Proof. As in the proof of Theorem 4.2. we consider separately the case b Lipschitz and the case b non-Lipschitz.

First case: b Lipschitz. If we set $\rho^\mu := u^\mu - \Pi_1\Gamma_\mu(z^\mu)$, by proceeding as in the proof of Theorem 4.2., for any $\theta \in [0,1]$ we have

$$\begin{aligned}\mu\,\frac{d}{dt}\left|\frac{\partial\rho^\mu}{\partial t}(t)\right|^2_{H^{\theta-1}} &+ \frac{d}{dt}\left|\rho^\mu(t)\right|^2_{H^\theta} + 2\left|\frac{\partial\rho^\mu}{\partial t}(t)\right|^2_{H^{\theta-1}}\\ &= 2\left\langle b(\cdot,u^\mu(t)),(-\Delta)^{\theta-1}\frac{\partial\rho^\mu}{\partial t}(t)\right\rangle_H.\end{aligned}$$

Then, due to (2.12), we obtain

$$\begin{aligned}\mu\,\frac{d}{dt}\left|\frac{\partial\rho^\mu}{\partial t}(t)\right|^2_{H^{\theta-1}} &+ \frac{d}{dt}\left|\rho^\mu(t)\right|^2_{H^\theta} + 2\left|\frac{\partial\rho^\mu}{\partial t}(t)\right|^2_{H^{\theta-1}}\\ &\leq c\left(1+|u^\mu(t)|^2_H\right) + \left|\frac{\partial\rho^\mu}{\partial t}(t)\right|^2_{H^{\theta-1}}.\end{aligned}$$

Integrating with respect to time and taking the supremum with respect to $s \in [0, t]$, this yields for $\mu \in (0, 1]$

$$\sup_{s\in[0,t]} \left|\rho^\mu(s)\right|^2_{H^\theta} + \int_0^t \left|\frac{\partial \rho^\mu}{\partial t}(s)\right|^2_{H^{\theta-1}} ds$$
$$\leq |v_0|^2_{H^{\theta-1}} + |u_0|^2_{H^\theta} + ct + \int_0^t |u^\mu(s)|^2_H\, ds. \tag{4.5}$$

Then, by taking expectation of both sides, thanks to estimate (3.1) for any $p > 4$ and $\delta < 1/2 - 2/p$ we have

$$\mathbb{E} \sup_{s\in[0,t]} |u^\mu(s)|^p_{H^\delta} \leq c_p \left(\mathbb{E} \sup_{s\in[0,t]} |\rho^\mu(s)|^p_{H^\delta} + \mathbb{E} \sup_{s\in[0,t]} |\Pi_1 \Gamma_\mu(u^\mu)(s)|^p_{H^\delta}\right)$$
$$\leq c_p \,(|v_0|^p_{H^{\delta-1}} + |u_0|^p_{H^\delta} + t^{p/2} + t^{p/2} \int_0^t \mathbb{E} \sup_{r\in[0,s]} |u^\mu(r)|^p_H\, ds$$
$$+\mathbb{E} \sup_{s\in[0,t]} \left|\Pi_1 \Gamma_\mu(u^\mu)(s)\right|^p_{H^\delta})$$
$$\leq c_p(T) \left(|v_0|^p_{H^{\delta-1}} + |u_0|^p_{H^\delta} + 1 + \int_0^t \mathbb{E} \sup_{r\in[0,s]} |u^\mu(r)|^p_H\, ds\right),$$

and the Gronwall inequality implies

$$\sup_{\mu\in(0,1]} \mathbb{E} \sup_{s\in[0,T]} |u^\mu(s)|^p_{H^\delta} \leq c_p(T) \left(1 + |v_0|^p_{H^{\delta-1}} + |u_0|^p_{H^\delta}\right). \tag{4.6}$$

In particular, this means that for any $\epsilon > 0$ there exists $R_\epsilon > 0$ independent of $\mu \in (0, 1]$ such that

$$\mathbb{P}\left(\sup_{s\in[0,T]} |u^\mu(s)|_{H^\delta} \leq R_\epsilon\right) \geq 1 - \epsilon, \quad \mu \in (0, 1]. \tag{4.7}$$

Now, since $(u_0, v_0) \in \mathcal{H}_1$, according to estimate (4.5) with $\theta = 1$, we have

$$u^\mu \in B_{C([0,T];H^\delta)}(R_\epsilon) \Longrightarrow \rho^\mu \in \mathcal{K}^1_\epsilon,$$

where, thanks to the Ascoli-Arzelà theorem, $\mathcal{K}^1_\epsilon$ is the compact subset of $C([0, T]; H)$ defined by

$$\mathcal{K}^1_\epsilon := \left\{ f \ :\ \sup_{t\in[0,T]} |f(t)|_{H^1} + \int_0^T \left|\frac{\partial f}{\partial t}(t)\right|^2_H dt \leq |v_0|^2_H + |u_0|^2_{H^1} + T(c + R^2_\epsilon), \right\}.$$

Moreover, according to Proposition 3.2 and to (3.12)

$$u^\mu \in B_{C([0,T];H^\delta)}(R_\epsilon) \Longrightarrow \Gamma_\mu(u^\mu) \in \mathcal{K}^2_\epsilon,$$

where, thanks again to the Ascoli-Arzelà theorem, $\mathcal{K}^2_\epsilon$ is the compact subset of $C([0,T];H)$ defined by

$$\mathcal{K}^2_\epsilon := \left\{ f \ : \ |f|_{C^\rho([0,T]\times[0,L])} \leq c(T)\,(1+R_\epsilon) \right\},$$

for some $\rho < \delta/2$ and some positive constant $c(T)$.

Therefore, as $u^\mu = \rho^\mu + \Gamma_\mu(u^\mu)$, we have

$$u^\mu \in B_{C([0,T];H^\delta)}(R_\epsilon) \Longrightarrow u^\mu \in \mathcal{K}_\epsilon := \mathcal{K}^1_\epsilon + \mathcal{K}^2_\epsilon,$$

and hence, by using (4.7) we can conclude that for the compact subset $\mathcal{K}_\epsilon \subset C([0,T];H)$ it holds $\mathbb{P}\,(u^\mu \in \mathcal{K}_\epsilon) \geq 1-\epsilon$. This implies the tightness of the family $\{\mathcal{L}(u^\mu)\}_{\mu\in(0,1]}$.

Second case: b non-Lipschitz. By repeating for u^μ, ρ^μ and $\Pi_1\Gamma(z^\mu)$ the same arguments used in the second step of the proof of Theorem 4.2 for u^μ_n, ρ^μ_n and $\Pi_1\Gamma(z^\mu_n)$, we have

$$\sup_{t\leq T}\left|\rho^\mu(t)\right|^2_{H^1} + \int_0^T \left|\frac{\partial\rho^\mu}{\partial t}(s)\right|^2_H ds \leq c\left(1+(\Lambda^\mu)^{2\lambda}\right)\exp\left(c\,(\Lambda^\mu)^2 T\right).$$

where

$$\Lambda^\mu := \sup_{t\in[0,T]} |\Pi_1\Gamma(z^\mu)(t)|_{C([0,L])}.$$

This means that for any $R>0$ we have

$$\mathbb{P}(\rho^\mu \in \mathcal{K}^1_R) \geq \mathbb{P}(c(1+(\Lambda^\mu)^{2\lambda})\exp(c\,(\Lambda^\mu)^2T) \leq R),$$

where $\mathcal{K}^1_R$ is the compact subset of $C([0,T];H)$ defined by

$$\mathcal{K}^1_R := \left\{ f \ : \ \sup_{t\leq T}|f(t)|^2_{H^1} + \int_0^T \left|\frac{\partial f}{\partial t}(s)\right|^2_H ds \leq R \right\}.$$

Now, as in the proof of Theorem 4.2. we have

$$\mathbb{P}(c(1+(\Lambda^\mu)^{2\lambda})\exp(c\,(\Lambda^\mu)^2T) > R) = \mathbb{P}(\Lambda^\mu > f_T(R)),$$

for some function f_T diverging to $+\infty$, as $R\to+\infty$. According to Proposition 3.2 and to (3.13), we have

$$\mathbb{P}(\Lambda^\mu > f_T(R)) \leq \frac{1}{f_T(R)}\,\mathbb{E}\sup_{t\in[0,T]}|\Pi_1\Gamma(z^\mu)(t)|_{C([0,L])} \leq \frac{c}{f_T(R)},$$

for some constant c independent of μ. Therefore, for any $\epsilon > 0$ there exists $R_{1,\epsilon} > 0$ independent of $\mu \in (0, 1]$ such that

$$\mathbb{P}(\rho^\mu \in \mathcal{K}^1_{R_{1,\epsilon}}) \geq \mathbb{P}(c(1 + (\Lambda^\mu)^{2\lambda})\exp(c\,(\Lambda^\mu)^2 T \leq R_{1,\epsilon})) \geq 1 - \epsilon/2. \tag{4.8}$$

Moreover, if we denote by $\mathcal{K}^2_R$ the compact subset of $C([0, T]; H)$ defined by

$$\mathcal{K}^2_R := \left\{ f \,:\, |f|_{C^\theta([0,T]\times[0,L])} \leq R \right\},$$

for some $\theta < 1/4$, by using again Proposition 3.2 and (3.13), we have

$$\mathbb{P}\left(\Pi_1\Gamma(z^\mu) \in (\mathcal{K}^2_R)^c\right) \leq \frac{1}{R}\,\mathbb{E}\,\left|\Pi_1\Gamma(z^\mu)\right|_{C^\theta([0,T]\times[0,L])} \leq \frac{c}{R},$$

for some constant c independent of $\mu \in (0, 1]$, so that for any $\epsilon > 0$ we can fix $R_{2,\epsilon} > 0$ independent of $\mu \in (0, 1]$ such that

$$\mathbb{P}\left(\Pi_1\Gamma(z^\mu) \in \mathcal{K}^2_{R_{2,\epsilon}}\right) \geq 1 - \epsilon/2, \tag{4.9}$$

Hence we can conclude as in the previous step. Actually, due to (4.8) and (4.9) for any $\epsilon > 0$ we have

$$\mathbb{P}\left(u^\mu \in \mathcal{K}_\epsilon := \mathcal{K}^1_{R_{1,\epsilon}} + \mathcal{K}^2_{R_{2,\epsilon}}\right) \geq 1 - \epsilon.$$

□

REMARK 4.5. According to inequality (4.6), if b is Lipschitz continuous, then the family $\{u^\mu\}_{\mu\in(0,1]}$ is bounded in $L^p(\Omega; C([0, T]; H^\delta))$, for any $p \geq 1$ and $\delta < 1/2$. In the case that b is not Lipschitz continuous we cannot prove that. Nevertheless, from the proof of the proposition above we have that for any $\theta < 1/4$ the family $\{u^\mu\}_{\mu\in(0,1]}$ is uniformly integrable in $C([0, T]; C^\theta([0, L]))$, that is

$$\lim_{R\to\infty}\ \sup_{\mu\in(0,1]} \mathbb{P}\left(\sup_{t\in[0,T]} |u^\mu(t)|_{C^\theta([0,L])} > R\right) = 0. \tag{4.10}$$

Actually, if we denote by f_T the inverse of the function $x \mapsto c(1 + x^{2\lambda})\exp(cx^2T)$, by following the proof of the second step in the proposition above we have

$$\mathbb{P}\left(\sup_{t\in[0,T]} |u^\mu(t)|_{C^\theta([0,L])} > R\right) \leq \mathbb{P}\left(\sup_{t\in[0,T]} |\Pi_1\Gamma_\mu(z^\mu)(t)|_{C([0,L])} > f_T(R/2)\right)$$

$$+\mathbb{P}\left(\sup_{t\in[0,T]} |\Pi_1\Gamma_\mu(z^\mu)(t)|_{C^\theta([0,L])} > R/2\right)$$

$$\leq \left(\frac{1}{f_T(R/2)} + \frac{2}{R}\right)\mathbb{E}\sup_{t\in[0,T]} |\Pi_1\Gamma_\mu(z^\mu)(t)|_{C^\theta([0,L])}.$$

Hence, as $f_T(R)$ diverges to $+\infty$ as $R \to \infty$, due to (3.13) we obtain (4.10).

Next, by proceeding as in [4, Lemma 4.4] it is possible to prove the following integration by part formula.

LEMMA 4.6. *Assume either Hypothesis* 1 *or Hypothesis* 2 *and fix* $z_0 = (u_0, v_0) \in \mathcal{H}_1$. *Then for any* $\mu > 0$ *and for any* $\varphi \in C^2([0, T] \times [0, L])$, *such that* $\varphi(t, 0) = \varphi(t, L) = 0$, *we have*

$$\begin{aligned}\int_0^L u^\mu(t, x)\varphi(t, x)\, dx &= \int_0^L u_0(x)\varphi(0, x)\, dx \\ &+ \int_0^t \int_0^L u^\mu(s, x)\left[\frac{\partial\varphi}{\partial t}(s, x) + \Delta\varphi(s, x)\right] ds\, dx \\ &+ \int_0^t \int_0^L b(x, u^\mu(s, x))\varphi(s, x)\, ds\, dx \\ &+ \int_0^t \int_0^L \varphi(s, x)g(x, u^\mu(s, x))\, w(ds, dx) + R_\mu(t),\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}R_\mu(t) &:= \mu\left(1 - e^{-\frac{t}{\mu}}\right)\int_0^L v_0(x)\varphi(0, x)\, dx - \int_0^t e^{-\frac{t-s}{\mu}}\, M_\mu(s)\, ds \\ &- \int_0^t e^{-\frac{t-s}{\mu}}\left[\int_0^L \left(u_0(x)\frac{\partial\varphi}{\partial t}(0, x) - u^\mu(s, x)\frac{\partial\varphi}{\partial t}(s, x)\right.\right. \\ &+ \left.\left.\int_0^s u^\mu(r, x)\frac{\partial^2\varphi}{\partial t^2}(r, x)\, dr\right) dx\right] ds \\ &- \int_0^t \int_0^L e^{-\frac{t-s}{\mu}}\, \varphi(s, x)g(x, u^\mu(s, x))w(ds, dx),\end{aligned}$$

and

$$M_\mu(t) := \int_0^L \left(u^\mu(t, x)\Delta\varphi(t, x) + b(x, u^\mu(t, x))\varphi(t, x)\right) dx.$$

As in [4, Lemma 4.5] we have to show that the remainder term $R_\mu(t)$ converges to zero, as the parameter μ goes to zero. In [4, Lemma 4.5], where the only case of Lipschitz b and additive noise has been studied, we could prove the mean-square convergence of $R_\mu(t)$ to zero, for any fixed $t \geq 0$. Here, in the case of non-Lipschitz b we can only prove convergence in probability, but, as we will show later on, this is sufficient in order to establish the validity of the Smoluchowski-Kramers approximation.

LEMMA 4.7. *Under either Hypothesis 1. or Hypothesis 2., for any* $\epsilon > 0$

$$\lim_{\mu \to 0} \mathbb{P}\left(|R_\mu(t)| > \epsilon\right) = 0.$$

Proof. We have

$$
\begin{aligned}
|R_\mu(t)|^2 \;&\leq\; 3\,\mu^2\,|\langle v_0, \varphi(0)\rangle|^2 + 3\left|\int_0^t e^{-\frac{t-s}{\mu}}\,\langle \varphi(s) g(\cdot, u^\mu(s)), dw(s)\rangle\right|^2 \\
&\quad +3\left|\int_0^t e^{-\frac{t-s}{\mu}}\left[M_\mu(s) + \left\langle u_0, \frac{\partial\varphi}{\partial t}(0)\right\rangle - \left\langle u^\mu(s), \frac{\partial\varphi}{\partial t}(s)\right\rangle\right.\right. \\
&\quad \left.\left. + \int_0^s \left\langle u^\mu(r), \frac{\partial^2\varphi}{\partial t^2}(r)\right\rangle dr\right] ds\right|^2 \\
&=: I^1_\mu(t) + I^2_\mu(t) + I^3_\mu(t).
\end{aligned}
$$

Concerning I^2_μ we have

$$
\begin{aligned}
\mathbb{E}\, I^2_\mu(t) &= 3\,\mathbb{E}\int_0^t e^{-\frac{2(t-s)}{\mu}}\,|\varphi(s) g(\cdot, u^\mu(s))|^2_H\, ds \\
&\leq c\,|\varphi|^2_{C([0,T]\times[0,L])}\left(1 + \mathbb{E}\sup_{s\in[0,T]}|u^\mu(s))|^{2\kappa}_H\right)\mu\int_0^{\frac{t}{\mu}} e^{-2s}\, ds,
\end{aligned}
$$

where $\kappa = 1$ or $\kappa = 0$, under respectively Hypothesis 1 and Hypothesis 2. Then, trivially if $\kappa = 0$ and according to (4.6) if $\kappa = 1$, we get

$$
\mathbb{E}\, I^2_\mu(t) \leq c\,|\varphi|^2_{C([0,T]\times[0,L])}\,\mu\int_0^\infty e^{-2s}\, ds,
$$

so that

$$
\lim_{\mu\to 0}\mathbb{E}\, I^2_\mu(t) = 0. \tag{4.12}
$$

Concerning I^3_μ we have

$$
\begin{aligned}
&I^3_\mu(t) \leq c\, t\int_0^t e^{-\frac{2(t-s)}{\mu}}\, ds \\
&\sup_{s\in[0,T]}\left(|M_\mu(s)|^2 + \left|\left\langle u_0, \frac{\partial\varphi}{\partial t}(0)\right\rangle\right|^2 + \left|\left\langle u^\mu(s), \frac{\partial\varphi}{\partial t}(s)\right\rangle\right|^2 + s^2\left|\left\langle u^\mu(s), \frac{\partial^2\varphi}{\partial t^2}(s)\right\rangle\right|^2\right).
\end{aligned}
$$

Under Hypothesis 1. we easily get

$$I^3_\mu(t) \le \mu\, c\, t \int_0^\infty e^{-2s}\, ds\, (1+T^2)\left(1+|u^\mu|^2_{C([0,T];H)}\right)\, |\varphi|^2_{C^2([0,T]\times[0,L])},$$

and hence, due to (4.6) we conclude

$$\begin{aligned}\mathbb{E}\, I^3_\mu(t) &\le \mu\, c_T\, (1+\mathbb{E}\,|u^\mu|^2_{C([0,T];H)})|\varphi|^2_{C^2([0,T]\times[0,L])}\\ &\le \mu\, c_T\, |\varphi|^2_{C^2([0,T]\times[0,L])} \to 0,\end{aligned} \tag{4.13}$$

as $\mu \to 0$. The situation is more delicate in the case that Hypothesis 2. holds. Actually in that case we have

$$\sup_{s\in[0,T]}\left(|M_\mu(s)|^2 + \left|\left\langle u_0, \frac{\partial\varphi}{\partial t}(0)\right\rangle\right|^2 + \left|\left\langle u^\mu(s), \frac{\partial\varphi}{\partial t}(s)\right\rangle\right|^2 + s^2\left|\left\langle u^\mu(s), \frac{\partial^2\varphi}{\partial t^2}(s)\right\rangle\right|^2\right)$$

$$\le c\,\left(1+|u^\mu|^{2\lambda}_{C([0,T];H)}\right)\, |\varphi|^2_{C^2([0,T]\times[0,L])},$$

so that

$$I^3_\mu(t) \le \mu\, c_T\, \left(1+|u^\mu|^{2\lambda}_{C([0,T];H)}\right)\, |\varphi|^2_{C^2([0,T]\times[0,L])}.$$

This means that for any $\epsilon > 0$

$$\mathbb{P}\left(\, I^3_\mu(t) > \epsilon\,\right) \le \mathbb{P}\left(\, |u^\mu|^{2\lambda}_{C([0,T];H)} > \frac{\epsilon}{\mu}\, \left(c_T|\varphi|^2_{C^2([0,T]\times[0,L])}\right)^{-1} - 1\,\right),$$

and then, thanks to (4.10) we can conclude that

$$\lim_{\mu\to 0} \mathbb{P}\left(\, I^3_\mu(t) > \epsilon\,\right) = 0.$$

Together with (4.12) and (4.13) and with the trivial fact that $I^1_\mu(t)$ goes to zero, as μ goes to zero, we obtain our thesis. □

REMARK 4.8. In fact, under Hypothesis 1 we have for any fixed $t > 0$ mean-square convergence of $R_\mu(t)$ to zero, as $\mu \to 0$.

Now we can prove the main result of this paper. Actually, once one has proved that the family of measures $\{\mathcal{L}(u^\mu)\}_{\mu\in(0,1]}$ is tight in $C([0,T];H)$ and that the integration by parts formula of Lemma 4.6. and the limit of Lemma 4.7. hold, we proceed in the proof by adapting the arguments used in [4, Theorem 4.5].

THEOREM 4.9. *For any $\mu > 0$, let $u^\mu = \Pi_1 z^\mu$, where z^μ is the mild solution of equation* (4.1). *Then, under either Hypothesis* 1 *or Hypothesis* 2, *for any $z_0 = (u_0, v_0) \in \mathcal{H}_1$, $T > 0$ and $\epsilon > 0$ we have*

$$\lim_{\mu\to 0} \mathbb{P}\left(|u^\mu - u|_{C([0,T];H)} > \epsilon\right) = 0,$$

where u is the solution of the semi-linear stochastic heat equation (1.2).

Proof. As proved in Proposition 4.4, the sequence $\{\mathcal{L}(u^\mu)\}_{\mu\in(0,1]}$ is tight in $C([0,T];H)$. Then, due to the Skorokhod theorem for any two sequences $\{\mu_n\}_n$ and $\{\mu_m\}_m$ converging to zero there exist subsequences $\{\mu_{n(k)}\}_{k\in\mathbb{N}}$ and $\{\mu_{m(k)}\}_{k\in\mathbb{N}}$ and a sequence of random elements

$$\{\rho_k\}_{k\in\mathbb{N}} := \{(u_1^k, u_2^k, \hat{w}_k)\}_{k\in\mathbb{N}},$$

in $C([0,T];L^2(\mathcal{O}))^2 \times C([0,T];\mathcal{D}'(\mathcal{O}))$, defined on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, such that the law of ρ_k coincides with the law of $(u^{\mu_{n(k)}}, u^{\mu_{m(k)}}, w)$, for each $k \in \mathbb{N}$, and ρ_k converges $\hat{\mathbb{P}}$-a.s. to some random element $\rho := (u_1, u_2, \hat{w}) \in C([0,T];L^2(\mathcal{O}))^2 \times C([0,T];\mathcal{D}'(\mathcal{O}))$. As a consequence of a general argument due to Gyöngy and Krylov (see [10] and also [15] and [4]), if we show that $u_1 = u_2$, then the whole sequence $\{u^\mu\}_{\mu>0}$ converges in probability to some $z \in C([0,T];H)$, as $\mu \to 0$.

Since both u_1^k and u_2^k solve equation (4.1), with w replaced by $\hat{w}_k$, they both verify formula (4.11), with R_1^k and R_2^k obtained replacing u^μ respectively with u_1^k and u_2^k and w with $\hat{w}_k$.

It is immediate to check that, both under Hypothesis 1 and under Hypothesis 2, we have

$$\hat{\mathbb{E}}\,|g(\cdot, u_i)|^2_{L^2(0,T;H)} < \infty, \quad i = 1, 2,$$

and

$$\lim_{k\to\infty} \hat{\mathbb{E}}\,|g(\cdot, u_i^k) - g(\cdot, u_i)|^2_{L^2(0,T;H)} = 0, \quad i = 1, 2.$$

Then, by using arguments analogous to those used in [15, Lemma 4.3], for any $\epsilon > 0$ it is possible to prove

$$\lim_{k\to\infty} \hat{\mathbb{P}}\left(\left|\int_0^T \int_0^L g(x, u_i^k(t,x))\,\hat{w}_k(dt,dx) - \int_0^T \int_0^L g(x, u_i(t,x))\,\hat{w}(dt,dx)\right| > \epsilon\right) = 0.$$

Moreover, due to Lemma 4.7. both R_1^k and R_2^k converge to zero in probability. Then, thanks to formula (4.11), by taking the limit in probability as k goes to infinity, we get

$$\begin{aligned}\int_0^L u_i(t,x)\varphi(t,x)\,dx &= \int_0^L u_0(x)\varphi(0,x)\,dx\\
&+\int_0^t\int_0^L u_i(t,x)\left[\frac{\partial\varphi}{\partial t}(s,x)+\Delta\varphi(s,x)\right]ds\,dx\\
&+\int_0^t\int_0^L b(x,u_i(s,x))\varphi(s,x)\,ds\,dx\\
&+\int_0^t\int_0^L g(x,u_i(s,x))\varphi(s,x)\,\hat{w}(ds,dx),\qquad i=1,2,\end{aligned}$$

so that $u_1 = u_2$, as they coincide with the unique solution of equation (1.2) perturbed by the noise $\hat{w}$.

This completes the proof, since by the same arguments we prove that the limit z of the sequence $\{u^\mu\}_{\mu>0}$ is the solution of equation (1.2). □

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