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Schauder estimates for elliptic equations in Banach spaces associated with stochastic reaction–diffusion equations

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Abstract. We consider some reaction–diffusion equations perturbed by white noise and prove Schauder estimates for the elliptic problem associated with the generator of the corresponding transition semigroup, defined in the Banach space of continuous functions. This requires the proof of some new interpolation result.

1. Introduction

Elliptic and parabolic equations can be studied in several spaces, one of the most important is the space of Hölder continuous functions. Here an important role is played by maximal regularity estimates called Schauder estimates. For an overview of classical results concerning elliptic operators with Hölder continuous and bounded coefficients, we refer to [11] or [12].

Recently, motivated also by the relationship between second-order elliptic operators and stochastic differential equations, a new interest in elliptic operators with unbounded coefficients arose.

Concerning Schauder estimates, they were proven in [8] for Ornstein–Uhlenbeck operators and in [6] and [15] for operators with coefficients having arbitrary growth and fulfilling suitable assumptions: in [15], analytical methods were used and in [6] probabilistic ones.

A great attention was also payed to elliptic operators applied to functions defined in a separable Hilbert space H that is in a space with infinitely many variables (see [10] for general results on this topic and references therein). Even in this case, the use of Hölder continuous functions reveals to be useful in several problems as uniqueness in law, pathwise uniqueness and uniqueness of the martingale problem for some stochastic partial differential equations.

Among the results in this direction, we quote Schauder estimates for the Gross Laplacian (see [3]) and for the Ornstein–Uhlenbeck operators (see [1,2,4,18]), with applications to the martingale problem from measure-valued branching diffusions and pathwise uniqueness for some SPDEs (see [7]).

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The present paper is devoted to the study of Schauder estimates for a class of reaction–diffusion equations perturbed by an additive noise, having a polynomially growing reaction term that fulfills suitable dissipativity conditions. An example is given by the following problem

$$\begin{cases} dX(t,\xi) = \left[D_\xi^2 X(t,\xi) - p(X(t,\xi))\right] dt + dW(t,\xi), & t>0,\ \xi\in[0,1], \\ D_\xi X(t,0) = D_\xi X(t,1) = 0, & t>0. \\ X(0,\xi) = x(\xi), \end{cases} \tag{1.1}$$

where p is a polynomial of odd degree with positive leading coefficient and W is the Brownian sheet. It is well known that equation (1.1) can be solved both in the Hilbert space $H = L^2(0,1)$ and in the Banach space $E = C([0,1])$ (see [6,9]). In particular, the corresponding transition semigroup P_t can be defined in both spaces, as

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad t\ge 0,\ x\in X, \tag{1.2}$$

for any bounded Borel mapping $\varphi : X \to \mathbb{R}$, with $X = H$ or $X = E$.

To prove Schauder estimates, we shall follow the method introduced in [15], in the finite-dimensional case. To this purpose, our basic tool is given by the estimates

$$\sup_{x\in E} |D^k P_t\varphi(x)| \le c(t\wedge 1)^{-k/2} \sup_{x\in E} |\varphi(x)|, \tag{1.3}$$

for $k = 1, 2, 3$, combined together with some interpolatory machinery.

Notice that, in the case of the reaction–diffusion equations we are dealing with in the present paper, these estimates do not hold in the space H of square integrable functions (contrary to the case of the Ornstein–Uhlenbeck semigroups) but, thanks to some results proved in [6], they do hold in the Banach space E of continuous function. On the other hand, for interpolation a technical difficulty arises when working in E, as we do not know whether the following interpolatory result holds

$$(C_b(E), C_b^1(E))_{\theta,\infty} = C_b^\theta(E), \quad \theta\in(0,1).$$

Actually, we can only prove that

$$(C_b(E), \mathrm{Lip}_b(E))_{\theta,\infty} = C_b^\theta(E), \quad \theta\in(0,1),$$

(see Appendix A). But we will show that this result is enough for our purposes.

The main result of the paper concerns Schauder estimates for the elliptic equation

$$\lambda\varphi - \mathscr{M}\varphi = f, \tag{1.4}$$

where $\mathscr{M}$ is the *weak* infinitesimal generator of the transition semigroup (1.2) (see [5] for all definitions). More precisely, we show that for any $\lambda > 0$ and any $f \in C_b^\theta(E)$, with $\theta\in(0,1)$, we have $\varphi \in C_b^{2+\theta}(E)$.

We believe that this result could be applied to more general situations where estimates (1.3) are available and also to prove the uniqueness in law and the uniqueness of the martingale problem for suitable non-Lipschitz perturbations of (1.1). This will be the object of a forthcoming research.

In Sect. 2, we will review some of the main properties of the solution $X(t, x)$ of the stochastic reaction–diffusion equation. In Sect. 3, we will apply those results to the study of the associated transition semigroup P_t and to the proof of estimates (1.3). In Sect. 4, we will prove the validity of Schauder estimates for the elliptic problem. In Appendix A, we will prove the required interpolatory identities.

1.1. Notations

We conclude this section by giving some notations. Let X be a separable Banach space. In what follows, we shall denote by $B_b(X)$ the Banach space of bounded Borel function $\varphi : X \to \mathbb{R}$, endowed with the sup-norm

$$\|\varphi\|_0 := \sup_{x\in X} |\varphi(x)|,$$

and by $C_b(X)$ the subspace of uniformly continuous mappings. $\mathrm{Lip}_b(X)$ is the subspace of Lipschitz-continuous mappings, endowed with the norm

$$\|\varphi\|_{\mathrm{Lip}_b(X)} := \|\varphi\|_0 + \sup_{x,y\in E,\, x\neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|_X} =: \|\varphi\|_0 + [\varphi]_{\mathrm{Lip}_b(X)}.$$

For any $\theta \in (0, 1)$, we denote by $C_b^\theta(X)$ the Banach space of all θ-Hölder continuous mappings $\varphi \in C_b(X)$, endowed with the norm

$$\|\varphi\|_{C^\theta(X)} = \|\varphi\|_0 + \sup_{x,y\in X,\, x\neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|_X^\theta}.$$

Finally, for any integer $k \geq 1$, we denote by $C_b^k(X)$ the space of all mappings $\varphi : X \to \mathbb{R}$ which are k times differentiable, with uniformly continuous and bounded derivatives. $C_b^k(X)$ is a Banach space, endowed with the norm

$$\|\varphi\|_{C_b^k(X)} =: \|\varphi\|_0 + \sum_{j=1}^k \sup_{x\in X} \|D^j\varphi(x)\|_{\mathscr{L}^j(X)}.$$

Spaces $C_b^{\theta+k}(X)$, with $k \in \mathbb{N}$ and $\theta \in (0, 1)$, are defined similarly.

2. The reaction–diffusion equation

We are here concerned with the following stochastic reaction–diffusion equation in the Banach space $C([0,1])$,

$$\begin{cases} dX(t,\xi) = [D_\xi^2 X(t,\xi) + b(\xi, X(t,\xi))]dt + dW(t,\xi), \quad \xi \in (0,1), \\ \mathscr{N} X(t,0) = \mathscr{N} X(t,1) = 0, \quad t \geq 0, \\ X(0,\xi) = x(\xi), \quad \xi \in [0,1], \ x \in C([0,1]), \end{cases} \tag{2.1}$$

where $b : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given function, W is a cylindrical Wiener process in $L^2(0,1)$, defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, and either $\mathscr{N} u = u$ (Dirichlet boundary condition) or $\mathscr{N} u = u'$ (Neumann boundary condition).

If we denote by A the realization in $C([0,1])$ of the operator D_ξ^2, endowed with the boundary condition $\mathscr{N}$, and if we denote by B the Nemytski operator associated with b, namely

$$B(x)(\xi) = b(\xi, x(\xi)), \quad x \in C([0,1]), \quad \xi \in [0,1],$$

then problem (2.1) can be written as the following stochastic differential equation in $C([0,1])$

$$\begin{cases} dX(t) = [AX(t) + B(X(t))]dt + dW(t), \\ X(0) = x. \end{cases} \tag{2.2}$$

In what follows, we shall denote $E := \overline{D(A)}$, the closure of $C([0,1])$ under the uniform norm. Notice that in the case of Neumann boundary conditions we have $E = C([0,1])$, but in the case of Dirichlet boundary conditions $E \subsetneq C([0,1])$. However, with this choice of E, the semigroup e^{tA} generated by A, which is analytic in $C([0,1])$, turns out to be also strongly continuous in E.

In what follows, we shall assume that the mapping $b : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions.

HYPOTHESIS 2.1. *1. For any $\xi \in [0,1]$, the mapping $b(\xi, \cdot) : \mathbb{R} \to \mathbb{R}$ is of class C^3 and there exists an integer $m \geq 0$ such that*

$$\sup_{\xi \in [0,1]} \sup_{s \in \mathbb{R}} \frac{|D_s^j b(\xi, s)|}{1 + |s|^{2m+1-j}} < \infty, \quad j = 0, 1, 2, 3. \tag{2.3}$$

Moreover, the mappings $D_s^j b : [0,1] \times \mathbb{R} \to \mathbb{R}$ are all continuous.

2. If $m \geq 1$, then there exist $\alpha > 0$, $\gamma \geq 0$ and $c \in \mathbb{R}$ such that

$$\sup_{\xi \in [0,1]} (b(\xi, s+h) - b(\xi, s))\, h \leq -\alpha h^{2(m+1)} + c \left(1 + |s|^\gamma\right).$$

A simple example of a function b fulfilling all conditions in Hypothesis 2.1 is

$$b(\xi, s) = -\alpha(\xi)\, s^{2m+1} + \sum_{j=0}^{2m} c_j(\xi)\, s^j, \qquad (\xi, s) \in [0,1] \times \mathbb{R},$$

for some continuous functions $\alpha, c_j : [0,1] \to \mathbb{R}$, with

$$\inf_{\xi \in [0,1]} \alpha(\xi) =: \alpha_0 > 0.$$

DEFINITION 2.2. Let $x \in E$. We say that an adapted process X is a *mild* solution of problem (2.1) if $X(t) \in E$, for all $t \geq 0$, and fulfills the integral equation

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} B(X(s))\mathrm{d}s + W_A(t), \quad t \geq 0, \tag{2.4}$$

where $W_A(t)$ is the stochastic convolution

$$W_A(t) = \int_0^t e^{(t-s)A}\mathrm{d}W(s), \quad t \geq 0.$$

In [6, Proposition 6.2.2], it is proved that for any $x \in E$, problem (2.1) admits a unique adapted mild solution $X(t,x) \in L^p(\Omega; C([0,T]; E))$, for any $T > 0$ and $p \geq 1$, such that

$$|X(t,x)|_E \leq e^{ct}|x|_E + h(t), \qquad \mathbb{P} - \text{a.s.} \tag{2.5}$$

where

$$h(t) := c\, e^{ct} \int_0^t \left(1 + |W_A(s)|_E^{2m+1}\right) \mathrm{d}s + \sup_{s \in [0,t]} |W_A(s)|_E. \tag{2.6}$$

Moreover, in [6, Theorem 6.2.3], it has been proved that for any $t > 0$

$$\sup_{x \in E} |X(t,x)|_E \leq k(t)\, t^{-\frac{1}{2m}}, \qquad \mathbb{P} - \text{a.s.} \tag{2.7}$$

where

$$k(t) := c\left(1 + \sup_{s \in [0,t]} |W^A(s)|_E^{\frac{\gamma}{2m+1} \wedge 1}\right), \tag{2.8}$$

and the constants γ and m are those introduced in Hypothesis 2.1.

Notice that for any $p \geq 1$ and $T > 0$, we have

$$\mathbb{E} \sup_{t \in [0,T]} |W^A(t)|_E^p < \infty. \tag{2.9}$$

Moreover,

$$\lim_{t \to 0} \mathbb{E}\, |W^A(t)|_E^p = 0. \tag{2.10}$$

Now, we define

$$\mathscr{C}_p(T) := L^p(\Omega; C([0,T]; E)),$$

and we introduce the mapping

$$x \in E \mapsto X(\cdot, x) \in \mathscr{C}_p(T).$$

As proved in [6, Theorem 6.3.3], as we are assuming in Hypothesis 2.1 that the mapping $b(\xi, \cdot)$ is of class C^3 and estimates (2.3) hold, the mapping above is three times differentiable, for any $T > 0$ and $p \geq 1$ fixed. Moreover, for any $q \geq 1$, $j = 1, 2, 3$ and $h_1, \ldots, h_j \in E$

$$\sup_{\substack{t \in [0,T] \\ x \in E}} |D_x^j X(t,x)(h_1, \ldots, h_j)|_E^q \leq k_{j,q}(T) \prod_{i=1}^{j} |h_i|_E^q, \quad \mathbb{P} - \text{a.s.}, \tag{2.11}$$

for some random variable $k_{j,q}(T)$ having finite moments of any order.

3. The transition semigroup

In what follows we shall denote by P_t the Markov transition semigroup associated with equation (2.2). P_t is defined by

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad x \in E, \tag{3.1}$$

for any $\varphi \in B_b(E)$. In the previous section, we have seen that the mapping $x \in E \mapsto X(\cdot, x) \in \mathscr{C}_p(T)$ is three times differentiable and estimates (2.11) hold. Then, by differentiating under the sign of expectation, it is immediate to check that for any integer $j \leq 3$ the semigroup P_t maps $C_b^j(E)$ into itself and

$$\|P_t\varphi\|_{C_b^j(E)} \leq c_j(T)\|\varphi\|_{C_b^j(E)}, \quad t \in [0,T].$$

In particular, the semigroup P_t is Feller that is it maps $C_b(E)$ into $C_b(E)$, for any $t \geq 0$.

Once we have the semigroup P_t, by proceeding as in [5] we want to define its infinitesimal generator $\mathscr{M}$. To this purpose, we need to recall the notion of $\mathscr{K}$-convergence and of *weakly continuous semigroup* in the space $C_b(E)$, as introduced in [5] (see also [6, Appendix B]).

DEFINITION 3.1. A sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_b(E)$ is *$\mathscr{K}$-convergent* to $\varphi \in C_b(E)$ if

$$\sup_{n\in\mathbb{N}} \|\varphi_n\|_0 < \infty, \quad \lim_{n\to\infty} \sup_{x\in K} |\varphi_n(x) - \varphi(x)| = 0, \tag{3.2}$$

for any compact set $K \subset E$. Moreover, given a family $\{\varphi_t\}_{t\in I} \subset C_b(E)$ and t_0, accumulation point for I, we say

$$\mathscr{K} - \lim_{t\to t_0} \varphi_t = \varphi,$$

if φ_{t_n} is $\mathscr{K}$-convergent to φ, as $n \to \infty$, for any sequence $\{t_n\}_{n\in\mathbb{N}}$ converging to t_0.

DEFINITION 3.2. A semigroup of bounded linear operators P_t, defined on $C_b(E)$, is said to be *weakly continuous* if

1. the family $\{P_t\varphi\,;\ t \in [0,T]\} \subset C_b(E)$ is equi-uniformly continuous, for any $\varphi \in C_b(E)$ and $T > 0$;
2. there exist $M > 0$ and $\omega \in \mathbb{R}$ such that for every $t \geq 0$
$$\|P_t\|_{\mathscr{L}(C_b(E))} \leq M\,e^{\omega t};$$
3. for every $\varphi \in C_b(E)$ it holds
$$\mathscr{K} - \lim_{t\to 0^+} P_t\varphi = \varphi;$$
4. for every $\varphi \in C_b(E)$ and for every sequence $\{\varphi_n\} \subset C_b(E)$ which is $\mathscr{K}$-convergent to φ, for any $t \geq 0$ we have
$$\mathscr{K} - \lim_{n\to\infty} P_t\varphi_n = P_t\varphi,$$
and the limit is uniform in $t \in [0,T]$, for every $T > 0$.

PROPOSITION 3.3. *The semigroup P_t defined in (3.1) is weakly continuous.*

Proof. As $X(t,x)$ is differentiable with respect to $x \in E$, for any $x, y \in E$ and $t \geq 0$ we have

$$X(t,x) - X(t,y) = \int_0^1 D_x X(t, \theta x + (1-\theta)y)(x-y)\,\mathrm{d}\theta,$$

so that thanks to estimate (2.11), for $j = 1$, we have

$$\mathbb{E}\,|X(t,x) - X(t,y)|_E \leq c(T)|x-y|_E, \qquad t \in [0,T]. \tag{3.3}$$

Then, if $\varphi \in C_b(E)$, for any fixed $\varepsilon > 0$ and $T > 0$ there exists $\delta_{\varepsilon,T}(\varphi)$ such that for any $t \in [0,T]$

$$|x-y|_E < \delta_{\varepsilon,T}(\varphi) \Longrightarrow |P_t\varphi(x) - P_t\varphi(y)| \leq \mathbb{E}\,|\varphi(X(t,x)) - \varphi(X(t,y))| < \varepsilon,$$

so that condition 1 in Hypothesis 3.2 is satisfied.

Moreover, P_t is clearly a contraction in $C_b(E)$, and hence condition 2 is true, with $M = 1$ and $\omega = 0$.

Next, let us check that condition 3 in Definition 3.2 is satisfied. For any $\varphi \in \mathrm{Lip}_b(E)$ we have, see [5, Proposition 1.4.1],

$$
\begin{aligned}
|P_t\varphi(x) - \varphi(x)| &\leq [\varphi]_{Lip_b(E)}\mathbb{E}\,|X(t,x) - x|_E \\
&\leq [\varphi]_{Lip_b(E)}\left(|e^{tA}x - x|_E + \mathbb{E}\left|\int_0^t e^{(t-s)A}B(X(s,x))\,\mathrm{d}s\right|_E + \mathbb{E}|W^A(t)|_E\right).
\end{aligned} \tag{3.4}
$$

Now, for any $K \subset E$ compact we clearly have

$$\lim_{t\to 0}\sup_{x\in K}|e^{tA}x - x|_E = 0. \tag{3.5}$$

Moreover,

$$\mathbb{E}\left|\int_0^t e^{(t-s)A}B(X(s,x))\,\mathrm{d}s\right|_E \leq c(T)\int_0^t\left(1 + \mathbb{E}\,|X(s,x)|_E^{2m+1}\right)\mathrm{d}s,$$

so that, thanks to (2.5) and (2.9) we get

$$\lim_{t\to 0}\sup_{x\in K}\mathbb{E}\left|\int_0^t e^{(t-s)A}B(X(s,x))\,\mathrm{d}s\right|_E = 0.$$

Together with (3.5) and (2.10), as a consequence of (3.4), this implies

$$\lim_{t\to 0}\sup_{x\in K}|P_t\varphi(x) - \varphi(x)| = 0.$$

Therefore, as P_t is a contraction, property 3 is satisfied for any $\varphi \in \mathrm{Lip}_b(E)$. Next, as $\mathrm{Lip}_b(E)$ is dense in $C_b(E)$ (see Proposition A.4 for a proof), condition 3 follows for any $\varphi \in C_b(E)$.

Finally, condition 4, that is continuity of P_t with respect to $\mathscr{K}$-convergence, can be proved as in [6, Proposition 4.3.1]. □

Next, by proceeding as in [5] (see also [6, Appendix B]), we introduce the generator of P_t. For any $\lambda > 0$ and $\varphi \in C_b(E)$, we define

$$F(\lambda)\varphi(x) = \int_0^\infty e^{-\lambda t}P_t\varphi(x)\mathrm{d}t, \quad x \in E. \tag{3.6}$$

As proved for example in [6, Proposition B.1.4], there exists a unique m–dissipative operator $\mathscr{M}$ in $C_b(E)$ such that

$$(\lambda - \mathscr{M})^{-1} = F(\lambda), \quad \forall\,\lambda > 0.$$

$\mathscr{M} : D(\mathscr{M}) \subseteq C_b(E) \to C_b(E)$ is the *infinitesimal generator* of P_t.

By using estimates (2.7) and (2.11), the Bismut–Elworty–Li and the semigroup law with a suitable approximation procedure the following estimates for the derivatives of $P_t\varphi$, have been proved (see [6, Theorem 6.5.1]).

PROPOSITION 3.4. *For any* $\varphi \in B_b(E)$ *and* $t > 0$*, we have that* $P_t\varphi \in C_b^3(E)$ *and for any* $0 \leq i \leq j \leq 3$

$$\|D^j(P_t\varphi)\|_0 \leq c_j\,(t \wedge 1)^{-\frac{j-i}{2}}\,\|\varphi\|_i. \tag{3.7}$$

In what follows we shall need the following result.

PROPOSITION 3.5. *The semigroup* P_t *maps* $\mathrm{Lip}_b(E)$ *into itself, for any* $t \geq 0$*, and*

$$\|P_t\varphi\|_{\mathrm{Lip}_b(E)} \leq c\,\|\varphi\|_{\mathrm{Lip}_b(E)}, \quad t \geq 0. \tag{3.8}$$

Moreover, the semigroup P_t *maps* $\mathrm{Lip}_b(E)$ *into* $C_b^1(E)$*, for any* $t > 0$*, and*

$$\|D(P_t\varphi)\|_0 \leq c\,\|\varphi\|_{\mathrm{Lip}_b(E)}, \qquad t > 0. \tag{3.9}$$

Proof. Due to (3.3), for any $x, y \in E$ and $t \leq 1$ we have

$$\begin{aligned}&|P_t\varphi(x) - P_t\varphi(y)| \leq \mathbb{E}\,|\varphi(X(t,x)) - \varphi(X(t,y))|\\ &\leq [\varphi]_{\mathrm{Lip}_b(E)}\mathbb{E}\,|X(t,x) - X(t,y)|_E \leq c\,[\varphi]_{\mathrm{Lip}_b(E)}|x-y|_E,\end{aligned}$$

so that (3.8) follows for any $t \leq 1$. For $t > 1$, we have

$$[P_t\varphi]_{\mathrm{Lip}_b(E)} = [P_{1/2}(P_{t-1/2}\varphi)]_{\mathrm{Lip}_b(E)} \leq c\,[P_{t-1/2}\varphi]_{\mathrm{Lip}_b(E)} \leq c\,\|P_{t-1/2}\varphi\|_1$$

and then, as $t - 1/2 > 1/2$, by using (3.7) we get (3.8) for any $t \geq 0$.

From (3.7), with $j = 1$ and $i = 0$, it immediately follows that $P_t\varphi \in C_b^1(E)$, for any $\varphi \in \mathrm{Lip}_b(E)$ and $t > 0$. Moreover, as

$$D(P_t\varphi)(x)h = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\,(P_t\varphi(x + \varepsilon h) - P_t\varphi(x)),$$

due to (3.8) we get

$$\begin{aligned}&|D(P_t\varphi)(x)h| = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\,|P_t\varphi(x + \varepsilon h) - P_t\varphi(x)|\\ &\leq [P_t\varphi]_{\mathrm{Lip}_b(E)}\,|h|_E \leq c\,[\varphi]_{\mathrm{Lip}_b(E)}\,|h|_E,\end{aligned}$$

and (3.9) follows. □

By using (3.7) and the semigroup law, (3.9) implies that for any $\varphi \in \mathrm{Lip}_b(E)$ and $t > 0$

$$\|D^j(P_t\varphi)\|_0 \leq c_j\,(t \wedge 1)^{-\frac{j-1}{2}}\,\|\varphi\|_{\mathrm{Lip}_b(E)}, \quad j = 1, 2, 3. \tag{3.10}$$

By interpolation, see Theorem A.3 in Appendix A, from (3.7) and (3.10) we obtain the following result.

COROLLARY 3.6. *For any* $\theta \in (0, 1)$ *and* $j = 2, 3$*, there exists* $c_{\theta,j} > 0$ *such that for all* $\varphi \in C_b^\theta(E)$ *and all* $t > 0$

$$\|D^j(P_t\varphi)\|_0 \leq c_{\theta,j}(t \wedge 1)^{-\frac{j-\theta}{2}}\,\|\varphi\|_\theta. \tag{3.11}$$

4. The main result

We are now ready to prove the main result of the paper. The proof is based on some interpolatory arguments (recalled in Appendix A below) which are well known in the finite-dimensional case, see [14].

THEOREM 4.1. *Let $f \in C_b^\theta(E)$, with $\theta \in (0,1)$, and let $\varphi = (\lambda - \mathscr{M})^{-1} f$, with $\lambda > 0$. Then we have $\varphi \in C_b^{2+\theta}(E)$ and there exists $M > 0$ (independent of f) such that*

$$\|\varphi\|_{C_b^{2+\theta}(E)} \le M\, \|f\|_{C_b^\theta(E)}. \tag{4.1}$$

Proof. Let $f \in C_b^\theta(H)$ and let $\varphi = (\lambda - \mathscr{M})^{-1} f$. We have

$$\varphi(x) = \int_0^{+\infty} e^{-\lambda t} P_t f(x) \mathrm{d}t, \quad x \in E. \tag{4.2}$$

By Corollary 3.6, it follows that $\|D^2 P_r\|_0$ is integrable with respect to $e^{-r}\mathrm{d}r$ over $[0, \infty)$, so that for any $h, k \in E$ we can write,

$$D^2\varphi(x)(h,k) = \int_0^{+\infty} e^{-\lambda s} D^2 P_s f(x)(h,k)\mathrm{d}s, \quad x \in E.$$

Now, given $t > 0$, we look for $a_t \in C_b(E)$ and $b_t \in Lip_b(E)$ such that $D^2\varphi(\cdot)(h,k) = a_t + b_t$ and

$$\|a_t\|_0 \le c\, t^\theta, \quad \|b_t\|_1 \le c\, t^{\theta-1}, \tag{4.3}$$

for a suitable positive constant c (independent of h, k). This will imply that $D^2\varphi(\cdot)(h,k) \in C_b^\theta(E)$ and finally, taking the supremum on $|h|_E \le 1$ and $|k|_E \le 1$, that $D^2\varphi$ is θ-Hölder continuous.

Let us set

$$a_t(x) = \int_0^{t^2} e^{-\lambda s} D^2 P_s f(x)(h,k)\mathrm{d}s, \quad x \in E,$$

and

$$b_t(x) = \int_{t^2}^{+\infty} e^{-\lambda s} D^2 P_s f(x)(h,k)\mathrm{d}s, \quad x \in E.$$

As

$$|a_t(x)| \le \int_0^{t^2} e^{-\lambda s} |D^2 P_s f(x)(h,k)|\mathrm{d}s, \quad x \in E,$$

by (3.11) we deduce

$$\|a_t\|_0 \le c_\theta \|f\|_\theta |h|_E\, |k|_E \int_0^{t^2} e^{-\lambda s}(s \wedge 1)^{\theta/2-1}\mathrm{d}s \le \frac{2\, c_\theta}{\theta} \|f\|_\theta\, t^\theta. \tag{4.4}$$

In the same way, since

$$Db_t(x) = \int_{t^2}^{+\infty} e^{-\lambda s} D^3 P_s f(x)(h,k)ds, \quad x \in E,$$

we deduce by (3.11) (with $j = 3$) that

$$\begin{aligned} |Db_t(x)| &\le c_{1,\theta} \|f\|_\theta \, |h|_E \, |k|_E \int_{t^2}^{+\infty} e^{-\lambda s} (s \wedge 1)^{(\theta-3)/2} \mathrm{d}s \\ &= \frac{2c_{1,\theta}}{1-\theta} \, \|f\|_\theta \, t^{\theta-1}, \quad x \in E. \end{aligned} \tag{4.5}$$

Therefore, $D^2 P_t \varphi(h,k)$ belongs to $\big(C_b(E), \mathrm{Lip}_b(E)\big)_{\theta,\infty} \subset C_b^\theta(E)$ and estimate (4.1) holds true. □

Appendix A. Interpolation spaces

We shall use the K method for real interpolation spaces, see e. g. [16]. Let X and Y be Banach spaces such that $Y \subset X$, with continuous embedding. For any $t > 0$ and any $x \in H$, define

$$K(t,x) = \inf\{\|a\|_X + t\|b\|_Y : \; x = a + b, \; a \in X, \; b \in Y\}.$$

Then, for arbitrary $\theta \in (0,1)$, set

$$\begin{aligned} \|x\|_{(X,Y)_{\theta,\infty}} &= \sup_{t>0} t^{-\theta} K(t,x), \\ (X,Y)_{\theta,\infty} &= \{x \in X : \|x\|_{(X,Y)_{\theta,\infty}} < +\infty\}. \end{aligned}$$

As is easily seen, $(X,Y)_{\theta,\infty}$, endowed with the norm

$$\|x\|_{(X,Y)_{\theta,\infty}},$$

is a Banach space.

REMARK A.1. It is not difficult to check that the following statement

(i) for all $t > 0$, there exist $a_t \in X$ and $b_t \in Y$ such that $x = a_t + b_t$ and

$$\|a_t\|_X + t\|b_t\|_Y \le Lt^\theta,$$

implies that

(ii) $x \in (X,Y)_{\theta,\infty}$ and $\|x\|_{(X,Y)_{\theta,\infty}} \le L$.

Conversely, statement (ii) implies that for all $\varepsilon > 0$ and all $t > 0$ there exist $a_t \in X$ and $b_t \in Y$ such that $x = a_t + b_t$ and

$$\|a_t\|_X + t\|b_t\|_Y \le (L+\varepsilon)t^\theta.$$

Let us recall the basic interpolation theorem, see e. g. [16].

THEOREM A.2. *Let X, X_1, Y, Y_1 be Banach spaces such that $Y \subset X$, $Y_1 \subset X_1$ with continuous embeddings. Let moreover T be a linear mapping $T : X \to X_1$, $T : Y \to Y_1$, such that for some $M, N > 0$*

$$\|Tx\|_{X_1} \le M\|x\|_X, \ \|Ty\|_{Y_1} \le N\|y\|_Y.$$

Then T maps $(X, Y)_{\theta,\infty}$ into $(X_1, Y_1)_{\theta,\infty}$, and

$$\|Tx\|_{(X_1,Y_1)_{\theta,\infty}} \le M^{1-\theta} N^{\theta} \ \|x\|_{(X,Y)_{\theta,\infty}}, \quad x \in (X, Y)_{\theta,\infty}.$$

Now, we show that $C_b^\theta(X)$ is an interpolation space.

THEOREM A.3. *Let X be a separable Banach space. Then we have*

$$\big(C_b(X), \mathrm{Lip}_b(X)\big)_{\theta,\infty} = C_b^\theta(X), \quad \theta \in (0, 1). \tag{A.1}$$

Proof. We proceed in several steps.

Step 1. $\big(C_b(X), \mathrm{Lip}_b(X)\big)_{\theta,\infty}$ is continuously embedded in $C_b^\theta(X)$.

The proof of Step 1 is similar to [14, page 37]. We give the proof for the reader's convenience.

Let $\varphi \in \big(C_b(X), \mathrm{Lip}_b(X)\big)_{\theta,\infty}$, and set $L = \|\varphi\|_{(C_b(X),\mathrm{Lip}_b(X))_{\theta,\infty}}$. Then for any $\epsilon > 0$, $t > 0$ there exist $a_t \in C_b(X)$, $b_t \in \mathrm{Lip}_b(X)$, such that $\varphi = a_t + b_t$ and

$$\|a_t\|_0 + t\|b_t\|_1 \le (L + \epsilon)t^\theta, \quad t > 0.$$

Then, if $x, y \in X$, we have

$$\varphi(x) - \varphi(y) = a_t(x) - a_t(y) + b_t(x) - b_t(y),$$

from which

$$|\varphi(x) - \varphi(y)| \le |a_t(x)| + |a_t(y)| + |b_t(x) - b_t(y)|$$

It follows that

$$|\varphi(x) - \varphi(y)| \le 2(L + \epsilon)t^\theta + (L + \epsilon)t^{\theta-1}|x - y|, \quad \epsilon > 0, \ t > 0, \ x, y \in X.$$

Now, if $|x - y| \le 1$, setting $t = |x - y|$, we have

$$|\varphi(x) - \varphi(y)| \le 3L|x - y|^\theta, \quad x, y \in X,$$

whereas if $|x - y| \ge 1$, we have

$$|\varphi(x) - \varphi(y)| \le 2\|\varphi\|_0|x - y|^\theta, \quad x, y \in X.$$

Consequently,

$$|\varphi(x) - \varphi(y)| \le (3L + 2\|\varphi\|_0)|x - y|^\theta, \quad x, y \in X,$$

and the statement of Step 1 is proved.

It remains to show that $C_b^\theta(X)$ is continuously embedded in

$$\big(C_b(X), \mathrm{Lip}_b(X)\big)_{\theta,\infty}.$$

To prove this, we need to use some properties of the following Hamilton–Jacobi semigroup

$$U_t\varphi(x) := \inf\left\{\varphi(y) + \frac{1}{2t}\,|x - y|^2,\ y \in X\right\}, \quad \varphi \in C_b(X). \tag{A.2}$$

Let us first notice that, given $t > 0$, $x \in X$ and $\varphi \in C_b(X)$, for any $\varepsilon > 0$ there exists $y_\varepsilon \in X$ such that

$$U_t\varphi(x) > \varphi(y_\varepsilon) + \frac{1}{2t}\,|x - y_\varepsilon|^2 - \varepsilon. \tag{A.3}$$

Therefore, since $U_t\varphi(x) \le \varphi(x)$, we have

$$0 \le \varphi(x) - U_t\varphi(x) \le \varphi(x) - \varphi(y_\varepsilon) - \frac{1}{2t}\,|x - y_\varepsilon|^2 + \varepsilon \tag{A.4}$$

which yields the following estimate for $|x - y_\varepsilon|^2$

$$\frac{1}{2t}\,|x - y_\varepsilon|^2 \le \varphi(x) - \varphi(y_\varepsilon) + \varepsilon. \tag{A.5}$$

Step 2. Let $\theta \in (0, 1)$, $\varphi \in C_b^\theta(X)$ and $t > 0$. Then there exists $c_\theta > 0$ such that

$$\sup_{x\in E}\,|\varphi(x) - U_t\varphi(x)| \le c_\theta\, t^{\frac{\theta}{2-\theta}}. \tag{A.6}$$

By (A.4), it follows that

$$\frac{|x - y_\varepsilon|^2}{2t} \le \varphi(x) - \varphi(y_\varepsilon) + \varepsilon \le [\varphi]_\theta |x - y_\varepsilon|^\theta + \varepsilon,$$

which implies

$$|x - y_\varepsilon|^2 \le 2t[\varphi]_\theta |x - y_\varepsilon|^\theta + 2\varepsilon t.$$

By Young's inequality, this yields for any $c > 0$

$$|x - y_\varepsilon|^2 \le \frac{\theta}{2} c^{\frac{2}{\theta}} |x - y_\varepsilon|^2 + \left(1 - \frac{\theta}{2}\right)\left[\frac{2t[\varphi]_\theta}{c}\right]^{\frac{2}{2-\theta}} + 2\varepsilon t.$$

Setting $c = (1/\theta)^{\theta/2}$ we find that, for a suitable constant c_θ,

$$|x - y_\varepsilon|^2 \le c_\theta^2\, [\varphi]_\theta^{\frac{2}{2-\theta}}\, t^{\frac{2}{2-\theta}} + 4\varepsilon t,$$

and, by (A.4), it follows that

$$\begin{aligned}\varphi(x) - U_t\varphi(x) &\le [\varphi]_\theta |x - y_\varepsilon|^\theta + \varepsilon\\ &\le \left(c_\theta^2\,[\varphi]_\theta^{\frac{2}{2-\theta}}\,t^{\frac{2}{2-\theta}} + 4\varepsilon t\right)^{\frac{\theta}{2}} + \varepsilon.\end{aligned}$$

The conclusion follows again from the arbitrariness of ε.

Step 3. Let $\varphi \in C_b^\theta(X)$ and $t > 0$. Then

$$|U_t\varphi(\overline{x}) - U_t\varphi(x)| \le c_\theta |x - \overline{x}|\, t^{\frac{\theta-1}{2-\theta}}, \quad x,\ \overline{x} \in X. \tag{A.7}$$

Let us first show that

$$|x - y_\varepsilon|^2 \le c_\theta (t^{\frac{2}{2-\theta}} + \varepsilon^{\frac{2}{\theta}}). \tag{A.8}$$

From (A.5) it follows that

$$|x - y_\varepsilon|^2 \le 2t(\varphi(x) - \varphi(y_\varepsilon)) + 2t\varepsilon \le 2t[\varphi]_\theta |x - y_\varepsilon|^\theta + 2t\varepsilon.$$

On the other hand, from Young's inequality, we get

$$2t[\varphi]_\theta |x - y_\varepsilon|^\theta \le \frac{\theta}{2}[|x - y_\varepsilon|^\theta]^{\frac{2}{\theta}} + \frac{2-\theta}{2}\,(2t[\varphi]_\theta)^{\frac{2}{2-\theta}}$$

and

$$2t\varepsilon \le \frac{2-\theta}{2}\,(2t)^{\frac{2}{2-\theta}} + \frac{\theta}{2}\varepsilon^{\frac{2}{\theta}},$$

so that (A.8) is proved. Now we can estimate

$$U_t\varphi(\overline{x}) - U_t\varphi(x)$$

By taking into account (A.3) and the definition of $U_t\varphi$, for any $y \in X$, we can write

$$U_t\varphi(\overline{x}) - U_t\varphi(x) \le \varphi(y) - \varphi(y_\varepsilon) + \frac{1}{2t}\,|\overline{x} - y|^2 - \frac{1}{2t}\,|x - y_\varepsilon|^2 + \varepsilon,$$

from which, setting $y = y_\varepsilon$, we deduce

$$\begin{aligned}U_t\varphi(\overline{x}) - U_t\varphi(x) &\le \frac{1}{2t}\,|\overline{x} - y_\varepsilon|^2 - \frac{1}{2t}\,|x - y_\varepsilon|^2 + \varepsilon\\ &\le \frac{1}{2t}\left(|x - \overline{x}|^2 + 2\,|x - \overline{x}|\,|y_\varepsilon - x|\right) + \varepsilon,\end{aligned} \tag{A.9}$$

and then, thanks to (A.8), we get

$$U_t\varphi(\overline{x}) - U_t\varphi(x) \le \frac{1}{2t}\left(|x - \overline{x}|^2 + 2\,|x - \overline{x}|\,c_\theta t^{\frac{1}{2-\theta}}\right) + \varepsilon.$$

This concludes the proof of (A.7).

Step 4 Conclusion.

Let $\varphi \in C_b^\theta(X)$, define $\varphi_t = U_t\varphi$ and

$$a_t = \varphi - \varphi_{t^{2-\theta}}, \quad b_t = \varphi_{t^{2-\theta}}.$$

Then by (A.6) we have

$$\|a_t\|_0 = \|\varphi - U_{t^{2-\theta}}\varphi\|_0 \le c_\theta\, t^\theta$$

and by (A.7)

$$\|b_t\|_{\mathrm{Lip}_b(X)} \le c_\theta\; t^{\theta-1}.$$

The theorem is thus proved. □

By using once again the Hamilton-Jacobi semigroup, we obtain the following fact (see [17]).

PROPOSITION A.4. *The space* $\mathrm{Lip}_b(X)$ *is dense in* $C_b(X)$.

Proof. From (A.5) we have

$$|x - y_\varepsilon| \le \sqrt{2t\,\|\varphi\|_0 + 2t\,\varepsilon}.$$

Therefore, if we denote by ω_φ the continuity modulus of φ, due to (A.4) we have

$$\begin{aligned} 0 &\le \varphi(x) - U_t\varphi(x) \le \varphi(x) - \varphi(y_\varepsilon) + \varepsilon \\ &\le \omega_\varphi\left(\sqrt{2t\,\|\varphi\|_0 + 2t\,\varepsilon}\right) + \varepsilon. \end{aligned}$$

From the arbitrariness of ε, this allows us to conclude that

$$|\varphi(x) - U_t\varphi(x)| \le \omega_\varphi\left(\sqrt{2t\,\|\varphi\|_0}\right),$$

so that $U_t\varphi$ converges to φ uniformly, as $t \downarrow 0$. Therefore, since $U_t\varphi \in \mathrm{Lip}_b(X)$, for any $\varphi \in C_b(X)$ and $t > 0$ (see Theorem A.3), we can conclude that $\mathrm{Lip}_b(E)$ is dense in $C_b(E)$. □

REMARK A.5. When X is a Hilbert space one can show that

$$\left(C_b(X), C_b^1(X)\right)_{\theta,\infty} = C_b^\theta(X), \quad \theta \in (0, 1).$$

This result was proved in [3] using the inf-sup convolutions introduced in [13], which provide a uniform approximation of a function φ from $C_b(X)$ by $C^{1,1}$ functions. Though one can define inf-sup convolutions also in separable Banach spaces, it does not seem easy to show this regularity result. For this reason, we used a different method in order to characterize $\left(C_b(X), \mathrm{Lip}_b(X)\right)_{\theta,\infty}$ rather than $\left(C_b(X), C_b^1(X)\right)_{\theta,\infty}$.

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