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Small Mass Asymptotics for a Charged Particle in a Magnetic Field and Long-Time Influence of Small Perturbations

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Abstract We consider small mass asymptotics of the motion of a charged particle in a potential combined with a magnetic field. After an appropriate regularization, a Smoluchowski-Kramers type approximation is established. This approximation allows to study long-time influence on the motion of various perturbations, deterministic and stochastic. In particular, even in the case of pure deterministic perturbations, the long-time evolution of the perturbed system can be stochastic.

Keywords Smoluchowski-Kramers approximation · Averaging · Stochasticity in dynamical systems

1 Introduction

Consider the two-dimensional motion of a particle of a small mass μ in a force field which has a random part

$$\begin{cases} \mu\, \ddot{q}^\mu(t) = b(q^\mu(t)) + A_0\dot{q}^\mu(t) + \sigma(q^\mu(t))\dot{w}(t), \\ q^\mu(0) = q \in \mathbb{R}^2, \quad \dot{q}^\mu(0) = p \in \mathbb{R}^2. \end{cases} \tag{1.1}$$

Here b is a vector field in $\mathbb{R}^2$, $\sigma(q)$ is a 2×2 matrix, $w(t)$ is the standard Brownian motion in $\mathbb{R}^2$. If

$$A_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.2}$$

this is the Langevin equation and the term $-\dot{q}(t)$ describes the *friction* the particle has to overcome in its motion.

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It is well known (see [2] for all details) that the so-called Smoluchowski-Kramers approximation holds, as μ goes to zero, that is when the mass μ is negligible the solution $q^\mu(t)$ of the Langevin equation can be approximated by the solution of the first order equation

$$dq(t) = b(q(t))\,dt + \sigma(q(t))\,dw(t), \quad q(0) = q. \tag{1.3}$$

More precisely, for any fixed $T > 0$ and $k \geq 1$

$$\lim_{\mu\to 0} \mathbb{E} \max_{t\in[0,T]} |q^\mu(t) - q(t)|^k = 0.$$

A similar result can be obtained if the eigenvalues of A_0 have negative real parts. Namely it is possible to prove that q^μ converges in probability to the solution of the problem

$$dq(t) = -A_0^{-1} b(q(t))\,dt - A_0^{-1}\sigma(q(t))\,dw(t), \quad q(0) = q. \tag{1.4}$$

Suppose now that the particle is charged, and instead of a friction the particle is subjected to a constant strength magnetic field orthogonal to the plane where the particle moves. The motion of the particle is governed then by (1.1), with

$$A_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.5}$$

For brevity, we assume that the strength of the magnetic field is 1.

The eigenvalues of the matrix A_0 are now purely imaginary and our question is: does an analog of the Smoluchowski-Kramers approximation hold in this case?

One can prove that if the stochastic term in (1.1) is replaced by a continuous function, then q^μ converges uniformly in $[0, T]$ to the solution of (1.4). But if we have the white noise term, we do not have the convergence of q^μ, as $\mu \downarrow 0$. This is, roughly speaking, a consequence of the fact that

$$\lim_{\mu\to 0} \int_0^t \sin\frac{s}{\mu}\varphi(s)\,ds = 0,$$

for any continuous function $\varphi(s)$, and

$$\lim_{\mu\to 0} \int_0^t \sin\frac{s}{\mu}\,dw(s) \neq 0,$$

since

$$\mathrm{Var}\left(\int_0^t \sin\frac{s}{\mu}\,dw(s)\right) = \int_0^t \sin^2\frac{s}{\mu}\,ds \to \frac{t}{2}, \quad \text{as } \mu \downarrow 0.$$

Nevertheless, the problem can be regularized, so that in some sense a counterpart of the Smoluchowski-Kramers approximation is still valid.

There are various ways to regularize the problem. One of them consists in introducing in (1.1) a small friction proportional to the velocity. Namely, we consider the equation

$$\begin{cases} \mu\ddot{q}^{\mu,\epsilon}(t) = b(q^{\mu,\epsilon}(t)) + A_\epsilon \dot{q}^{\mu,\epsilon}(t) + \sigma(q^{\mu,\epsilon}(t))\dot{w}(t), \\ q^{\mu,\epsilon}(0) = q \in \mathbb{R}^2, \quad \dot{q}^{\mu,\epsilon}(0) = p \in \mathbb{R}^2, \end{cases}$$

where

$$A_\epsilon = A_0 - \epsilon I = \begin{pmatrix} -\epsilon & -1 \\ 1 & -\epsilon \end{pmatrix},$$

and $\epsilon > 0$ is a small parameter. We show that for any $T > 0$ and $k \geq 1$

$$\lim_{\mu\to 0} \mathbb{E} \max_{t\in[0,T]} |q^{\mu,\epsilon}(t) - q^\epsilon(t)|^k_{\mathbb{R}^2} = 0 \tag{1.6}$$

where $q^\epsilon(t)$ is the solution of the problem

$$dq(t) = -A_\epsilon^{-1} b(q(t))\, dt - A_\epsilon^{-1}\sigma(q(t))\, dw(t), \quad q(0) = q.$$

Next, we take the limit as ϵ goes to zero and we prove that

$$\lim_{\epsilon\to 0} \mathbb{E} \max_{t\in[0,T]} |q^\epsilon(t) - q(t)|^k_{\mathbb{R}^2} = 0,$$

where $q(t)$ is the solution of the problem

$$dq(t) = A_0 b(q(t))\, dt + A_0\sigma(q(t))\, dw(t), \quad q(0) = q. \tag{1.7}$$

The stochastic integral in (1.7) has to be interpreted in the Itô sense. Notice however that, assuming continuous differentiability of $\sigma(q)$, Itô's and Stratonovich's interpretation of the stochastic terms in (1.1) do coincide, both when A_0 is given by (1.2) (Langevin equation) and when A_0 is given by (1.5).

Another approach to regularization uses the fact that the white noise $\dot{w}(t)$ can be considered as an idealization of an isotropic δ-correlated smooth mean-zero Gaussian process $\dot{w}^\delta(t)$, with $0 < \delta \ll 1$, which converges to the standard white noise $w(t)$, as $\delta \downarrow 0$. In this case, we prove that if $q^{\mu,\delta}(t)$ is the solution of (1.1), with A_0 given by (1.5) and with $\dot{w}(t)$ replaced by $\dot{w}^\delta(t)$, then

$$\lim_{\mu\to 0} \mathbb{E} \max_{t\in[0,T]} |q^{\mu,\delta}(t) - q^\delta(t)|_{\mathbb{R}^2} = 0,$$

where $q^\delta(t)$ solves the equation

$$\dot{q}(t) = A_0 b(q(t)) + A_0\sigma(q(t))\dot{w}^\delta(t), \quad q(0) = q.$$

Next, we take the limit as $\delta \downarrow 0$, and we prove that $q^\delta(t)$ converges to the solution $\hat{q}(t)$ of the problem

$$d\hat{q}(t) = A_0 b(\hat{q}(t))\, dt + A_0\sigma(\hat{q}(t)) \circ dw(t), \quad \hat{q}(0) = q,$$

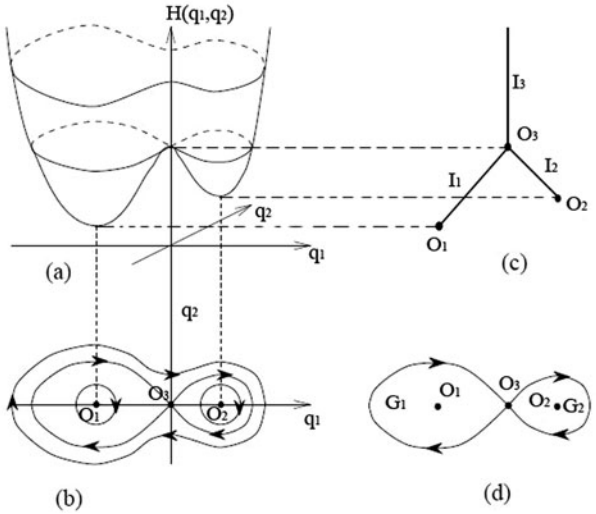
where the stochastic term has to be interpreted in Stratonovich sense.

This small mass approximation allows to calculate long-time influence of deterministic and stochastic perturbations on the motion of a charged particle, in a potential field $b(q) = -\nabla H(q)$ subjected to a magnetic field.

In Sect. 4, we first consider pure deterministic perturbations of the deterministic motion of a charged particle in the potential $-\nabla H(q)$ and the constant magnetic field orthogonal to the plane. To be specific, let the perturbations be caused by a small friction

$$\begin{cases} \mu\ddot{q}^{\mu,\epsilon}(t) = -\nabla H(q^{\mu,\epsilon}(t)) + A_0\dot{q}^{\mu,\epsilon}(t) - \epsilon\dot{q}^{\mu,\epsilon}(t), \\ q^{\mu,\epsilon}(0) = q \in \mathbb{R}^n, \quad \dot{q}^{\mu,\epsilon}(0) = p \in \mathbb{R}^n. \end{cases}$$

Fig. 1 The identification map



Since

$$A_0^{-1}\nabla H(q) = -\overline{\nabla} H(q),$$

we derive from the small mass asymptotics that the limit of $q^{\mu,\epsilon}(t)$, as $\mu \downarrow 0$, exists and coincides with the solution of the problem

$$\dot{q}^{\epsilon}(t) = \frac{1}{1+\epsilon^2}\overline{\nabla} H(q^{\epsilon}(t)) - \frac{\epsilon}{1+\epsilon^2}\nabla H(q^{\epsilon}(t)), \quad q^{\epsilon}(0) = q.$$

Now, let Γ be the graph obtained by the identification of points of each connected component for every level set of $H(q)$ (compare with [3, Chap. 8]). Moreover, let $\mathcal{Y}: \mathbb{R}^2 \to \Gamma$ be the identification map, that is let $\mathcal{Y}(q)$ be the point of Γ corresponding to the trajectory of the Hamiltonian system

$$\dot{q}(t) = \overline{\nabla} H(q(t)), \quad q(0) = q$$

(see Fig. 1). Then $\mathcal{Y}(q^{\epsilon}(t(1+\epsilon^2)/\epsilon))$ is the slow component of $q^{\epsilon}(t)$ and the limit $Y(t)$ of $\mathcal{Y}(q^{\epsilon}(t(1+\epsilon^2)/\epsilon))$ exists and is a Markovian process on Γ (compare with [1]).

Moreover, we show that the process $Y(t)$ on Γ is deterministic inside the edges and, in general, has stochastic behavior at the interior vertices of Γ. More precisely, when the trajectory $Y(t)$ comes to the vertex $O_3 \in \Gamma$ it goes without any delay to one of the edges I_1 or I_2 of Γ with certain probabilities. We calculate these probabilities and the motion inside the edges, so that the process $Y(t)$ on Γ characterizes long-time evolution of $\mathcal{Y}(q^{\mu,\epsilon}(t))$, as $0 < \mu \ll \epsilon \ll 1$.

Next, we consider small white noise perturbations

$$\begin{cases} \mu \ddot{q}^{\mu,\epsilon}(t) = -\nabla H(q^{\mu,\epsilon}(t)) + A_0 \dot{q}^{\mu,\epsilon}(t) - \epsilon \dot{q}^{\mu,\epsilon}(t) + \sqrt{\epsilon}\,\dot{w}(t), \\ q^{\mu,\epsilon}(0) = q \in \mathbb{R}^2, \quad \dot{q}^{\mu,\epsilon}(0) = p \in \mathbb{R}^2. \end{cases} \tag{1.8}$$

If $0 < \mu \ll \epsilon \ll 1$, then the evolution of the system on long time intervals (of order ϵ^{-1}) is defined by a diffusion process $Y(t)$ on the graph Γ related to the potential $\nabla H(q)$, $q \in \mathbb{R}^2$.

We will show that the projection $Y^{\mu,\epsilon}(t)$ of $q^{\mu,\epsilon}(t(1+\epsilon^2)/\epsilon)$ on Γ converges to a limit $Y(t)$ as first μ goes to zero and then ϵ goes to zero. The process $Y(t)$ in this case is a diffusion process on the graph Γ and is defined by a family of second order ordinary differential operators, one on each edge, and by gluing conditions on the vertices of Γ.

In the next section we consider regularization of (1.1) by adding a small friction. The regularization by replacing the white noise with a smooth δ-correlated process is considered in Sect. 3. Long-time influence of small deterministic and stochastic perturbations is studied in Sect. 4. The 3-dimensional case is briefly considered in Sect. 5.

2 The Regularization by Adding a Friction Term

For every fixed $\epsilon > 0$, we consider the following second order SDE in $\mathbb{R}^2$

$$\begin{cases} \mu\ddot{q}^{\mu,\epsilon}(t) = b(q^{\mu,\epsilon}(t)) + A_\epsilon \dot{q}^{\mu,\epsilon}(t) + \sigma(q^{\mu,\epsilon}(t))\dot{w}(t), \\ q^{\mu,\epsilon}(0) = q \in \mathbb{R}^2, \quad \dot{q}^{\mu,\epsilon}(0) = p \in \mathbb{R}^2, \end{cases} \tag{2.1}$$

where, for any $\epsilon \geq 0$, A_ϵ is the 2×2 matrix

$$A_\epsilon = A_0 - \epsilon I = \begin{pmatrix} -\epsilon & -1 \\ 1 & -\epsilon \end{pmatrix}.$$

It is immediate to check that for any $v \in \mathbb{R}^2$

$$|A_\epsilon v|_{\mathbb{R}^2} = \sqrt{1+\epsilon^2}|v|_{\mathbb{R}^2}. \tag{2.2}$$

Moreover, A_ϵ is invertible for any $\epsilon \geq 0$ and

$$A_\epsilon^{-1} = \frac{1}{1+\epsilon^2}\begin{pmatrix} -\epsilon & 1 \\ -1 & -\epsilon \end{pmatrix}. \tag{2.3}$$

In particular, for any $v \in \mathbb{R}^2$

$$|A_\epsilon^{-1} v|_{\mathbb{R}^2} = \frac{1}{\sqrt{1+\epsilon^2}}|v|_{\mathbb{R}^2}. \tag{2.4}$$

Notice that (2.1) can be rewritten as the following system of two variables q and p in $\mathbb{R}^2$

$$\begin{cases} dq^{\mu,\epsilon}(t) = p^{\mu,\epsilon}(t)\,dt, & q^{\mu,\epsilon}(0) = q, \\ dp^{\mu,\epsilon}(t) = \frac{1}{\mu}[b(q^{\mu,\epsilon}(t)) + A_\epsilon p^{\mu,\epsilon}(t)]\,dt + \frac{1}{\mu}\sigma(q^{\mu,\epsilon}(t))\,dw(t), & p^{\mu,\epsilon}(0) = p. \end{cases} \tag{2.5}$$

We first study the limiting behavior of the first component $q^{\mu,\epsilon}$ in $C([0,T];\mathbb{R}^2)$, as μ goes to zero, for any fixed $\epsilon > 0$.

Theorem 2.1 *Assume that the mappings $b : \mathbb{R}^2 \to \mathbb{R}^2$ and $\sigma : \mathbb{R}^2 \to \mathcal{L}(\mathbb{R}^2)$ are Lipschitz-continuous. Then, for any $T > 0$ and $k > 2$*

$$\mathbb{E} \sup_{t\in[0,T]} |q^{\mu,\epsilon}(t) - q^\epsilon(t)|_{\mathbb{R}^2}^k \leq c_k(T)\big(1 + |p|_{\mathbb{R}^2}^k + |q|_{\mathbb{R}^2}^k\big)\left(\frac{\mu}{\epsilon} \wedge 1\right)^{\frac{k}{2}-1}, \tag{2.6}$$

where $q^\epsilon(t)$ is the solution of the problem

$$dq(t) = -A_\epsilon^{-1} b(q(t))\,dt - A_\epsilon^{-1}\sigma(q(t))\,dw(t), \quad q(0) = q. \tag{2.7}$$

In particular, for any $\epsilon_0 > 0$ and $k \geq 1$

$$\lim_{\mu\to 0} \sup_{\epsilon\geq\epsilon_0} \mathbb{E} \sup_{t\in[0,T]} |q^{\mu,\epsilon}(t) - q^\epsilon(t)|^k_{\mathbb{R}^2} = 0.$$

Proof From the second equation in (2.5), we have

$$\begin{aligned} p^{\mu,\epsilon}(t) &= e^{t\frac{A_\epsilon}{\mu}} p + \frac{1}{\mu}\int_0^t e^{(t-s)\frac{A_\epsilon}{\mu}} b(q^{\mu,\epsilon}(s))\,ds \\ &\quad + \frac{1}{\mu}\int_0^t e^{(t-s)\frac{A_\epsilon}{\mu}} \sigma(q^{\mu,\epsilon}(s))\,dw(s). \end{aligned} \tag{2.8}$$

Note that for any $t \in \mathbb{R}$ and $\epsilon > 0$

$$e^{tA_\epsilon} = e^{-\epsilon t}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{2.9}$$

The matrix e^{tA_0} is orthogonal, and then for any $v \in \mathbb{R}^2$

$$|e^{tA_\epsilon} v|_{\mathbb{R}^2} = e^{-\epsilon t}|v|_{\mathbb{R}^2}, \quad t \geq 0. \tag{2.10}$$

By integrating with respect to t in (2.8), we get

$$\begin{aligned} q^{\mu,\epsilon}(t) &= q + \int_0^t e^{s\frac{A_\epsilon}{\mu}} p\,ds + \frac{1}{\mu}\int_0^t\int_0^s e^{(s-r)\frac{A_\epsilon}{\mu}} b(q^{\mu,\epsilon}(r))\,dr\,ds \\ &\quad + \frac{1}{\mu}\int_0^t\int_0^s e^{(s-r)\frac{A_\epsilon}{\mu}} \sigma(q^{\mu,\epsilon}(r))\,dw(r)\,ds \\ &= q - \int_0^t A_\epsilon^{-1} b(q^{\mu,\epsilon}(s))\,ds - \int_0^t A_\epsilon^{-1}\sigma(q^{\mu,\epsilon}(s))\,dw(s) + R_{\mu,\epsilon}(t), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} R_{\mu,\epsilon}(t) &:= \mu A_\epsilon^{-1}\big[e^{t\frac{A_\epsilon}{\mu}} - I\big]p + \int_0^t A_\epsilon^{-1} e^{(t-s)\frac{A_\epsilon}{\mu}} b(q^{\mu,\epsilon}(s))\,ds \\ &\quad + \int_0^t A_\epsilon^{-1} e^{(t-s)\frac{A_\epsilon}{\mu}} \sigma(q^{\mu,\epsilon}(s))\,dw(s), \end{aligned}$$

so that

$$\begin{aligned} q^{\mu,\epsilon}(t) - q^\epsilon(t) &= -\int_0^t A_\epsilon^{-1}[b(q^{\mu,\epsilon}(s)) - b(q^\epsilon(s))]\,ds \\ &\quad - \int_0^t A_\epsilon^{-1}[\sigma(q^{\mu,\epsilon}(s)) - \sigma(q^\epsilon(s))]\,dw(s) + R_{\mu,\epsilon}(t). \end{aligned}$$

Due to the Lipschitz-continuity of b and to (2.4), this yields, for $t \in [0, T]$,

$$|q^{\mu,\epsilon}(t) - q^{\epsilon}(t)|_{\mathbb{R}^2}^k \leq c_k T^{k-1} \int_0^t |q^{\mu,\epsilon}(s) - q^{\epsilon}(s)|_{\mathbb{R}^2}^k \, ds \\ + c_k \left| \int_0^t [\sigma(q^{\mu,\epsilon}(s)) - \sigma(q^{\epsilon}(s))] \, dw(s) \right|_{\mathbb{R}^2}^k + c_k |R_{\mu,\epsilon}(t)|_{\mathbb{R}^2}^k.$$

Hence, as also σ is Lipschitz-continuous,

$$\begin{aligned} \mathbb{E} \sup_{s\in[0,t]} |q^{\mu,\epsilon}(s) - q^{\epsilon}(s)|_{\mathbb{R}^2}^k &\leq c_k T^{k-1} \int_0^t \mathbb{E} \sup_{r\in[0,s]} |q^{\mu,\epsilon}(r) - q^{\epsilon}(r)|_{\mathbb{R}^2}^k \, ds \\ &\quad + c_k \mathbb{E} \left(\int_0^t \|\sigma(q^{\mu,\epsilon}(s)) - \sigma(q^{\epsilon}(s))\|_{\mathcal{L}(\mathbb{R}^2)}^2 \, ds \right)^{\frac{k}{2}} \\ &\quad + c_k \mathbb{E} \sup_{s\in[0,t]} |R_{\mu,\epsilon}(s)|_{\mathbb{R}^2}^k \\ &\leq c_k (T^{k-1} + T^{\frac{k}{2}-1}) \int_0^t \mathbb{E} \sup_{r\in[0,s]} |q^{\mu,\epsilon}(r) - q^{\epsilon}(r)|_{\mathbb{R}^2}^k \, ds \\ &\quad + c_k \mathbb{E} \sup_{s\in[0,t]} |R_{\mu,\epsilon}(s)|_{\mathbb{R}^2}^k. \end{aligned}$$

By comparison, this yields

$$\begin{aligned} \mathbb{E} \sup_{s\in[0,T]} |q^{\mu,\epsilon}(s) - q^{\epsilon}(s)|_{\mathbb{R}^2}^k &\leq c_k \mathbb{E} \sup_{s\in[0,T]} |R_{\mu,\epsilon}(s)|_{\mathbb{R}^2}^k + c_k(T) \int_0^T e^{c_k(T-s)} \mathbb{E} \sup_{r\in[0,s]} |R_{\mu,\epsilon}(r)|_{\mathbb{R}^2}^k \\ &\leq c_k(T) \mathbb{E} \sup_{s\in[0,T]} |R_{\mu,\epsilon}(s)|_{\mathbb{R}^2}^k. \end{aligned} \tag{2.12}$$

Hence, in order to obtain (2.6), we have to estimate

$$\mathbb{E} \sup_{s\in[0,T]} |R_{\mu,\epsilon}(s)|_{\mathbb{R}^2}^k = 0.$$

Due to (2.4) and (2.9), we have

$$\left| A_\epsilon^{-1} \left[e^{t \frac{A_\epsilon}{\mu}} - I \right] p \right|_{\mathbb{R}^2}^2 \leq \left(\left| e^{-\frac{\epsilon t}{\mu}} \cos \frac{t}{\mu} - 1 \right| + e^{-\frac{\epsilon t}{\mu}} \left| \sin \frac{t}{\mu} \right| \right)^2 |p|_{\mathbb{R}^2}^2$$

and this implies

$$\sup_{\epsilon\geq 0} \sup_{t\in[0,T]} \mu \left| A_\epsilon^{-1} \left[e^{-t \frac{A_\epsilon}{\mu}} - I \right] p \right|_{\mathbb{R}^2} \leq c\mu |p|_{\mathbb{R}^2}. \tag{2.13}$$

Next, for any $k \geq 1$ we estimate

$$\mathbb{E} \sup_{t\in[0,T]} \left| A_\epsilon^{-1} \int_0^t e^{(t-r) \frac{A_\epsilon}{\mu}} b(q^{\mu,\epsilon}(r)) \, dr \right|_{\mathbb{R}^2}^k.$$

As

$$\left| A_\epsilon^{-1} \int_0^t e^{(t-r) \frac{A_\epsilon}{\mu}} b(q^{\mu,\epsilon}(r)) \, dr \right|_{\mathbb{R}^2}^k \leq \left| \int_0^t e^{(t-r) \frac{A_\epsilon}{\mu}} b(q^{\mu,\epsilon}(r)) \, dr \right|_{\mathbb{R}^2}^k,$$

we have to estimate for any $\epsilon > 0$

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t e^{(t-r)\frac{A_\epsilon}{\mu}}b(q^{\mu,\epsilon}(r))\,dr\right|_{\mathbb{R}^2}^k.$$

To this purpose, we will need the following lemma, whose proof is postponed.

Lemma 2.2 *Under the same conditions of Theorem 2.1, for any $T > 0$ and $k \geq 1$ we have*

$$\sup_{\epsilon,\mu>0}\mathbb{E}|q^{\mu,\epsilon}|_{C([0,T];\mathbb{R}^2)}^k =: c_k(T)\big(1+|q|_{\mathbb{R}^2}^k\big) < \infty. \tag{2.14}$$

Thanks to (2.10), for any $t \in [0, T]$ we have

$$\begin{aligned}\left|\int_0^t e^{(t-r)\frac{A_\epsilon}{\mu}}b(q^{\mu,\epsilon}(r))\,dr\right|_{\mathbb{R}^2}^k &\leq \int_0^t |b(q^{\mu,\epsilon}(r))|_{\mathbb{R}^2}^k\,dr\left(\int_0^t e^{-\frac{\epsilon(t-r)}{\mu}\frac{k}{k-1}}\right)^{k-1}\\ &\leq c_k\int_0^t\big(1+|q^{\mu,\epsilon}(r)|_{\mathbb{R}^2}^k\big)\,dr\left(\frac{\mu}{\epsilon}\wedge 1\right)^{k-1}t^{k-1}. \end{aligned}\tag{2.15}$$

Then, in view of (2.14), we get

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t A_\epsilon^{-1}e^{(t-r)\frac{A_\epsilon}{\mu}}b(q^{\mu,\epsilon}(r))\,dr\right|_{\mathbb{R}^2}^k \leq c_k(T)\left(\frac{\mu}{\epsilon}\wedge 1\right)^{k-1}\big(1+|q|_{\mathbb{R}^2}^k\big). \tag{2.16}$$

Finally, for any $k \geq 1$ we estimate

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t A_\epsilon^{-1}e^{(t-s)\frac{A_\epsilon}{\mu}}\sigma(q^{\mu,\epsilon}(s))\,dw(s)\right|_{\mathbb{R}^2}^k.$$

By using a factorization argument, for any $\alpha \in (0, 1)$ we have

$$\int_0^t A_\epsilon^{-1}e^{(t-s)\frac{A_\epsilon}{\mu}}\sigma(q^{\mu,\epsilon}(s))\,dw(s) = \frac{\sin\pi\alpha}{\pi}A_\epsilon^{-1}\int_0^t (t-s)^{\alpha-1}e^{(t-s)\frac{A_\epsilon}{\mu}}Y_{\alpha,\mu,\epsilon}(s)\,ds,$$

where

$$Y_{\alpha,\mu,\epsilon}(s) = \int_0^s (s-r)^{-\alpha}e^{(s-r)\frac{A_\epsilon}{\mu}}\sigma(q^{\mu,\epsilon}(r))\,dw(r)$$

and then, due to (2.4) and (2.10), for any $k > 2$ and $\alpha \in (1/k, 1/2)$, we get

$$\begin{aligned}&\sup_{t\in[0,T]}\left|\int_0^t A_\epsilon^{-1}e^{(t-s)\frac{A_\epsilon}{\mu}}\sigma(q^{\mu,\epsilon}(s))\,dw(s)\right|_{\mathbb{R}^2}^k\\ &\quad\leq c_{\alpha,k}\left(\int_0^T s^{(\alpha-1)\frac{k}{k-1}}e^{-\frac{\epsilon s}{\mu}\frac{k}{k-1}}\,ds\right)^{k-1}\int_0^T |Y_{\alpha,\mu,\epsilon}(s)|_{\mathbb{R}^2}^k\,ds\\ &\quad\leq c_\alpha(T)\left(\frac{\mu}{\epsilon}\wedge 1\right)^{\alpha k-1}\int_0^T |Y_{\alpha,\mu,\epsilon}(s)|_{\mathbb{R}^2}^k\,ds. \end{aligned}\tag{2.17}$$

Now,

$$\mathbb{E}|Y_{\alpha,\mu,\epsilon}(s)|_{\mathbb{R}^2}^k \leq c_k \mathbb{E}\left(\int_0^s (s-r)^{-2\alpha} e^{-\frac{2\epsilon(s-r)}{\mu}} \|\sigma(q^{\mu,\epsilon}(r))\|^2_{\mathcal{L}(\mathbb{R}^2)} dr\right)^{\frac{k}{2}},$$

so that we obtain

$$\begin{aligned}\mathbb{E}\int_0^T |Y_{\alpha,\mu,\epsilon}(s)|_{\mathbb{R}^2}^k ds &\leq c_k \left(\int_0^T s^{-2\alpha} e^{-\frac{2\epsilon s}{\mu}} ds\right)^{\frac{k}{2}} \mathbb{E}\int_0^T \|\sigma(q^{\mu,\epsilon}(r))\|^k_{\mathcal{L}(\mathbb{R}^2)} dr \\ &\leq c_k \left(\frac{\mu}{\epsilon}\wedge 1\right)^{\frac{k}{2}-\alpha k} \int_0^T \left(1+\mathbb{E}|q^{\mu,\epsilon}(r)|^k_{\mathbb{R}^2}\right) dr.\end{aligned}$$

Hence, thanks to (2.14), for any $k>2$ we get

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t A_\epsilon^{-1} e^{(t-s)\frac{A_\epsilon}{\mu}} \sigma(q^{\mu,\epsilon}(s))\,dw(s)\right|^k_{\mathbb{R}^2} \leq c_k(T)\left(\frac{\mu}{\epsilon}\wedge 1\right)^{\frac{k}{2}-1}\left(1+|q|^k_{\mathbb{R}^2}\right). \tag{2.18}$$

Therefore, collecting together (2.13), (2.16) and (2.18), for any $k>2$ we get

$$\mathbb{E}\sup_{s\in[0,T]}|R_{\mu,\epsilon}(s)|^k_{\mathbb{R}^2} \leq c_k(T)\left(\frac{\mu}{\epsilon}\wedge 1\right)^{\frac{k}{2}-1}\left(1+|p|^k_{\mathbb{R}^2}+|q|^k_{\mathbb{R}^2}\right),$$

so that, thanks to (2.12), we obtain (2.6). □

Proof of Lemma 2.2 Due to (2.11) and to (2.4) we have

$$\begin{aligned}|q^{\mu,\epsilon}(t)|^k_{\mathbb{R}^2} \leq c_k|q|^k_{\mathbb{R}^2} &+ c_{k,T}\int_0^t \left(1+|q^{\mu,\epsilon}(s)|^k_{\mathbb{R}^2}\right) ds \\ &+ c_k\left|\int_0^t \sigma(q^{\mu,\epsilon}(s))\,dw(s)\right|^k_{\mathbb{R}^2} + c_k|R_{\mu,\epsilon}(t)|^k_{\mathbb{R}^2}.\end{aligned}$$

By proceeding as in the proof of (2.15) and (2.18), it is possible to show that for any $t\in[0,T]$ and $k\geq 1$

$$\mathbb{E}\sup_{s\in[0,t]}|R_{\mu,\epsilon}(s)|^k_{\mathbb{R}^2} \leq c_k\mu^k|p|^k_{\mathbb{R}^2} + c_{k,T}\left(1+\int_0^t \mathbb{E}\sup_{r\in[0,s]}|q^{\mu,\epsilon}(r)|^k_{\mathbb{R}^2} ds\right),$$

and then

$$\mathbb{E}\sup_{t\in[0,T]}|q^{\mu,\epsilon}(t)|^k_{\mathbb{R}^2} \leq c_k\left(|q|^k_{\mathbb{R}^2}+\mu^k|p|^k_{\mathbb{R}^2}\right) + c_{k,T}\left(1+\int_0^t \mathbb{E}\sup_{r\in[0,s]}|q^{\mu,\epsilon}(r)|^k_{\mathbb{R}^2} ds\right).$$

By using the Gronwall Lemma this yields (2.14). □

Once we have proved that $q^{\mu,\epsilon}$ converges to q^ϵ in $C([0,T];\mathbb{R}^2)$, as μ goes to zero, for any fixed $\epsilon>0$, we study the asymptotic behavior of q^ϵ in $C([0,T];\mathbb{R}^2)$, as ϵ goes to zero.

Theorem 2.3 *Assume that the mappings $b:\mathbb{R}^2\to\mathbb{R}^2$ and $\sigma:\mathbb{R}^2\to\mathcal{L}(\mathbb{R}^2)$ are Lipschitz-continuous. Then, for any $T>0$ and $k\geq 1$ we have*

$$\lim_{\epsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}|q^\epsilon(t)-q(t)|^k_{\mathbb{R}^2} = 0, \tag{2.19}$$

where $q(t)$ is the solution of the problem

$$dq(t) = A_0 b(q(t))\,dt + A_0\sigma(q(t))\,dw(t), \quad q(0) = q. \tag{2.20}$$

Proof First of all we notice that, as $q^\epsilon(t)$ solves (2.7) and the matrix A_ϵ^{-1} satisfies (2.4), for any $T > 0$ and $k \geq 1$ we have

$$\sup_{\epsilon \geq 0} \mathbb{E} \sup_{t\in[0,T]} |q^\epsilon(t)|_{\mathbb{R}^2}^k =: c_k(T)\big(1 + |q|_{\mathbb{R}^2}^k\big) < \infty. \tag{2.21}$$

It is immediate to check that

$$q^\epsilon(t) - q(t) = A_0 \int_0^t [b(q^\epsilon(s)) - b(q(s))]\,ds + A_0 \int_0^t [\sigma(q^\epsilon(s)) - \sigma(q(s))]\,dw(s) + R_\epsilon(t),$$

where

$$R_\epsilon(t) = \frac{1}{1+\epsilon^2}(\epsilon - \epsilon^2 A_0)\left[\int_0^t b(q^\epsilon(s))\,ds + \int_0^t \sigma(q^\epsilon(s))\,dw(s)\right].$$

Hence, for any $t \in [0, T]$ we have

$$\mathbb{E} \sup_{s\in[0,t]} |q^\epsilon(s) - q(s)|_{\mathbb{R}^2}^k \leq c_k(T) \int_0^t \mathbb{E} \sup_{r\in[0,s]} |q^\epsilon(r) - q(r)|_{\mathbb{R}^2}^k\,ds + c_k \mathbb{E} \sup_{s\in[0,T]} |R_\epsilon(s)|_{\mathbb{R}^2}^k$$

and by the Gronwall lemma we get

$$\mathbb{E} \sup_{s\in[0,T]} |q^\epsilon(s) - q(s)|_{\mathbb{R}^2}^k \leq c_k(T) \mathbb{E} \sup_{s\in[0,T]} |R_\epsilon(s)|_{\mathbb{R}^2}^k. \tag{2.22}$$

Now, by arguing as in the proof of Theorem 2.1, it is possible to prove that for any $\epsilon \in [0, 1]$

$$\mathbb{E} \sup_{t\in[0,T]} |R_\epsilon(t)|_{\mathbb{R}^2}^k \leq \epsilon^k c_k(T)\Big(1 + \mathbb{E} \sup_{t\in[0,T]} |q^\epsilon(t)|_{\mathbb{R}^2}^k\Big).$$

Then, as a consequence of (2.21) we get

$$\mathbb{E} \sup_{t\in[0,T]} |R_\epsilon(t)|_{\mathbb{R}^2}^k \leq \epsilon^k c_k(T)\big(1 + |q|_{\mathbb{R}^2}^k\big),$$

so that from (2.22) we obtain (2.19). □

3 The Regularization by Approximation of the Noise

In Theorems 2.1 and 2.3, we have shown that under quite general assumptions on the coefficients b and σ, the position component $q^{\mu,\epsilon}(t)$ converges to the solution $q(t)$ of problem (2.20), when $0 < \mu \ll \epsilon \ll 1$ (first μ converges to zero and then ϵ converges to zero).

Now, we introduce a sequence of regularized noises $w^\delta(t)$ and we denote by $q^{\mu,\delta}(t)$ the solution of the corresponding problem. Our aim in this section is to prove that, for $0<\mu\ll\delta\ll 1$, $q^{\mu,\delta}(t)$ converges to a process $\hat{q}(t)$ satisfying as $q(t)$ problem (2.20), with the Itô integral replaced by the Stratonovich integral.

To this purpose, for any $\delta>0$ we construct a smooth process $w^\delta(t)$ such that

$$\lim_{\delta\to 0}\mathbb{E}\max_{t\in[0,T]}|w^\delta(t)-w(t)|^2_{\mathbb{R}^2}=0. \tag{3.1}$$

Let ρ be a C^∞ function whose support is contained in the interval [0, 1] and such that

$$\int_0^1\rho(s)\,ds=1.$$

By following [5, Example 7.3, Chap. VI], we define

$$\begin{aligned} w_i^\delta(t) &:= \frac{1}{\delta}\int_0^\infty w_i(s)\rho((s-t)/\delta)\,ds \\ &= \frac{1}{\delta}\int_0^\delta w_i(s+t)\rho(s/\delta)\,ds, \quad i=1,2, \end{aligned}$$

and we can show that the process $w^\delta(t)$ is an *approximation of the Wiener process* $w(t)$ in the sense of (3.1).

As

$$\dot{w}_i^\delta(t)=-\frac{1}{\delta}\int_0^1 w_i(t+\delta r)\dot{\rho}(r)\,dr, \quad i=1,2,$$

for any $k\geq 1$ and $T>0$ we have

$$\mathbb{E}\sup_{t\in[0,T]}|\dot{w}^\delta(t)|^k\leq c_{\delta,k}\mathbb{E}\sup_{t\in[0,T+\delta]}|w(t)|^k\leq c_{k,\delta,T}. \tag{3.2}$$

Analogously, as

$$\ddot{w}_i^\delta(t)=\frac{1}{\delta^2}\int_0^1 w_i(t+\delta r)\ddot{\rho}(r)\,dr, \quad i=1,2,$$

for any $k\geq 1$ and $T>0$ we have

$$\mathbb{E}\sup_{t\in[0,T]}|\ddot{w}^\delta(t)|^k\leq c_{\delta,k,T}. \tag{3.3}$$

Now, for any $\delta>0$, we introduce the problems

$$\begin{cases} \dot{q}(t)=p(t), & q(0)=q, \\ \dot{p}(t)=\frac{1}{\mu}[b(q(t))+A_0p(t)]+\frac{1}{\mu}\sigma(q(t))\dot{w}^\delta(t), & p(0)=p \end{cases} \tag{3.4}$$

and

$$\dot{q}(t)=-A_0^{-1}b(q(t))-A_0^{-1}\sigma(q(t))\dot{w}^\delta(t), \quad q(0)=q. \tag{3.5}$$

The solution of the second order problem (3.4) will be denoted by $(q^{\mu,\delta},p^{\mu,\delta})$ and the solution of the first order problem (3.5) will be denoted by q^δ.

Theorem 3.1 *Assume that* $b : \mathbb{R}^2 \to \mathbb{R}^2$ *and* $\sigma : \mathbb{R}^2 \to \mathcal{L}(\mathbb{R}^2)$ *are Lipschitz-continuous. Moreover, assume that* σ *is bounded. Then, for any* $T > 0$ *and* $\delta > 0$

$$\lim_{\mu\to 0} \mathbb{E} \sup_{t\in[0,T]} |q^{\mu,\delta}(t) - q^{\delta}(t)|_{\mathbb{R}^2} = 0. \tag{3.6}$$

Proof By proceeding as in the proof of Theorem 2.1, we can prove that

$$\begin{aligned} q^{\mu,\delta}(t) &= q - \int_0^t A_0^{-1} b(q^{\mu,\delta}(s))\,ds - \int_0^t A_0^{-1}\sigma(q^{\mu,\delta}(s))\dot{w}^{\delta}(s)\,ds \\ &\quad + \int_0^t A_0^{-1} e^{(t-s)\frac{A_0}{\mu}} \big(b(q^{\mu,\delta}(s)) - b(q^{\delta}(s))\big)\,ds \\ &\quad \times \int_0^t A_0^{-1} e^{(t-s)\frac{A_0}{\mu}} \big(\sigma(q^{\mu,\delta}(s)) - \sigma(q^{\delta}(s))\big)\dot{w}^{\delta}(s)\,ds + R_{\mu,\delta}(t), \end{aligned}$$

where

$$\begin{aligned} R_{\mu,\delta}(t) &:= \mu A_0^{-1}\big[e^{t\frac{A_0}{\mu}} - I\big]p + \int_0^t A_0^{-1} e^{(t-s)\frac{A_0}{\mu}} b(q^{\delta}(s))\,ds \\ &\quad + \int_0^t A_0^{-1} e^{(t-s)\frac{A_0}{\mu}} \sigma(q^{\delta}(s))\dot{w}^{\delta}(s)\,ds. \end{aligned}$$

Thus, by adapting the arguments used in the proof of Theorem 2.1 to this case, (3.6) follows once we show that for any fixed $\delta > 0$

$$\lim_{\mu\to 0} \mathbb{E} \sup_{t\in[0,T]} |R_{\mu,\delta}(t)|_{\mathbb{R}^2} = 0. \tag{3.7}$$

Due to (2.9), for any $f = (f_1, f_2) : [0,\infty) \to \mathbb{R}^2$ we have

$$\left[\int_0^t e^{(t-r)\frac{A_0}{\mu}} f(r)\,dr\right]_1 = \int_0^t \cos\left(\frac{t-r}{\mu}\right) f_1(r)\,dr - \int_0^t \sin\left(\frac{t-r}{\mu}\right) f_2(r)\,dr, \tag{3.8}$$

and

$$\left[\int_0^t e^{(t-r)\frac{A_0}{\mu}} f(r)\,dr\right]_2 = \int_0^t \sin\left(\frac{t-r}{\mu}\right) f_1(r)\,dr + \int_0^t \cos\left(\frac{t-r}{\mu}\right) f_2(r)\,dr. \tag{3.9}$$

If we assume that f is continuous, for any $t \in [0,T]$ we easily obtain

$$\begin{aligned} &\left|\int_0^t \cos\left(\frac{t-r}{\mu}\right) f_i(r)\,dr\right| + \left|\int_0^t \sin\left(\frac{t-r}{\mu}\right) f_i(r)\,dr\right| \\ &\quad \le T\rho_{f,T}(2\pi\mu) + 2\pi\mu\|f\|_T, \end{aligned}$$

where for any $T > 0$ $\rho_{f,T}$ denotes the continuity modulus of f in $[0,T]$, that is

$$\rho_{f,T}(\lambda) := \sup_{\substack{s,r\in[0,T]\\ |s-r|\le\lambda}} |f(s) - f(r)|_{\mathbb{R}^2}, \quad \lambda > 0,$$

and

$$\|f\|_T = \max_{t\in[0,T]} |f(t)|.$$

This implies that for any continuous function $f=(f_1,f_2):[0,\infty)\to\mathbb{R}^2$

$$\left|\int_0^t e^{(t-r)\frac{A_0}{\mu}} f(r)\,dr\right|_{\mathbb{R}^2} \le c_T(\rho_{f,T}(2\pi\mu)+\mu\|f\|_T). \tag{3.10}$$

In what follows, we shall need the following lemma, whose proof is postponed to the end of this section.

Lemma 3.2 *Let q^δ be the solution of problem (3.5), with $\delta>0$. Then, for any $k\ge 1$ and $T>0$ we have*

$$\mathbb{E}\|q^\delta\|_T^k \le c_{k,T,\delta}\big(1+|q|^k\big) \tag{3.11}$$

and

$$\mathbb{E}\rho^k_{q^\delta,T}(\lambda)\le c_{k,T,\delta}\lambda^k. \tag{3.12}$$

Once we have the above estimates for the sup-norm and the continuity modulus of q^δ, we can proceed with the estimate of $R_{\mu,\delta}$.

Due to (3.10), for any $t\in[0,T]$ we have

$$\left|\int_0^t e^{(t-r)\frac{A_0}{\mu}} b(q^\delta(r))\,dr\right|_{\mathbb{R}^2} \le c_T\big(\rho_{b(q^\delta),T}(2\pi\mu)+2\pi\mu\|b(q^\delta)\|_T\big),\quad \mathbb{P}\text{-a.s.}$$

and then, as b is Lipschitz-continuous, in view of (3.11) and (3.12) we get

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t A_0^{-1}e^{(t-r)\frac{A_0}{\mu}} b(q^\delta(r))\,dr\right|_{\mathbb{R}^2} \le c_{T,\delta}\mu. \tag{3.13}$$

Next, let us estimate

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t A_0^{-1}e^{(t-s)\frac{A_0}{\mu}} \sigma(q^\delta(s))\dot{w}^\delta(s)\,ds\right|_{\mathbb{R}^2}.$$

Due to (3.10), we have

$$\sup_{t\in[0,T]}\left|\int_0^t A_0^{-1}e^{(t-s)\frac{A_0}{\mu}} \sigma(q^\delta(s))\dot{w}^\delta(s)\,ds\right|_{\mathbb{R}^2} \le c_T\big(\rho_{\sigma(q^\delta)\dot{w}^\delta,T}(2\pi\mu)+2\pi\mu\|\sigma(q^\delta)\dot{w}^\delta\|_T\big).$$

Therefore, as

$$\rho_{\sigma(q^\delta)\dot{w}^\delta,T}(\lambda)\le \rho_{\dot{w}^\delta,T}(\lambda)\big(1+\|q^\delta\|_T\big)+\rho_{q^\delta,T}(\lambda)\|\dot{w}^\delta\|_T,$$

thanks to (3.2) and (3.3) and thanks to (3.11) and (3.12), we can conclude that

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t A_0^{-1}e^{(t-s)\frac{A_0}{\mu}} \sigma(q^\delta(s))\dot{w}^\delta(s)\,ds\right|_{\mathbb{R}^2} \le c_{T,\delta}(T)\mu.$$

Together with (3.13), this implies (3.7) and hence (3.6) follows. □

Now we can conclude with the following result, whose proof can be found e.g. in [5, Theorem 7.2, Chap. VI].

Theorem 3.3 *Under the same assumptions of Theorem 3.1, we have*

$$\lim_{\delta\to 0}\mathbb{E}\sup_{t\in[0,T]}|q^\delta(t)-\hat{q}(t)|_{\mathbb{R}^2}=0,$$

where $\hat{q}(t)$ is the solution of the problem

$$d\hat{q}(t)=A_0 b(\hat{q}(t))\,dt+A_0\sigma(\hat{q}(t))\circ dw(t),\quad \hat{q}(0)=q. \tag{3.14}$$

Proof of Lemma 3.2 As q^δ solves the random equation (3.5), due to the boundedness of σ, for any $k\geq 1$ we have

$$|q^\delta(t)|^k\leq c_k|q|^k+c_{k,T}\int_0^t\big(1+|q^\delta(s)|^k\big)\,ds+c_{k,T}\int_0^t|\dot{w}^\delta(s)|^k\,ds,\quad t\in[0,T].$$

This implies that

$$\mathbb{E}\|q^\delta\|_T^k\leq c_{k,T}\big(1+|q|^k+\mathbb{E}\|\dot{w}^\delta\|_T^k\big)$$

and due to (3.2) we can conclude that (3.11) holds.

Moreover,

$$|\dot{q}^\delta(t)|\leq|b(q^\delta(t))|+|\sigma(q^\delta(t))\dot{w}^\delta(t)|\leq c\big(1+|q^\delta(t)|+|\dot{w}^\delta(t)|\big),$$

so that, from (3.11) we conclude

$$\mathbb{E}\|\dot{q}^\delta\|_T^k\leq c_{k,T,\delta}\big(1+|q|^k\big). \tag{3.15}$$

In particular, as

$$\rho_{q^\delta,T}(\lambda)=\sup_{\substack{s,r\in[0,T]\\|r-s|\leq\lambda}}|q^\delta(s)-q^\delta(r)|\leq\|\dot{q}^\delta\|_T\lambda,$$

from (3.15) we can conclude that for any $k\geq 1$ (3.12) is verified. □

4 Long-Time Influence of Small Perturbations

In this section we consider applications of small mass asymptotics to the long-time motion of a particle in a potential combined with a magnetic field.

4.1 Pure Deterministic Perturbations

Consider a charged particle motion in a potential, combined with the magnetic field, perturbed by a small friction. The corresponding equation has the form

$$\begin{cases}\mu\ddot{q}^{\mu,\epsilon}(t)=-\nabla H(q^{\mu,\epsilon}(t))+A_0\dot{q}^{\mu,\epsilon}(t)-\epsilon\dot{q}^{\mu,\epsilon}(t),\\ q^{\mu,\epsilon}(0)=q\in\mathbb{R}^2,\quad \dot{q}^{\mu,\epsilon}(0)=p\in\mathbb{R}^2,\end{cases} \tag{4.1}$$

for some smooth function $H:\mathbb{R}^2\to\mathbb{R}$ such that

$$\lim_{|q|_{\mathbb{R}^2}\to+\infty}H(q)=+\infty.$$

As follows from Theorem 2.1, for any fixed $M, T, \epsilon > 0$

$$\lim_{\mu\to 0} \sup_{q,p\in B_{\mathbb{R}^2}(M)} \sup_{t\in[0,T]} |q^{\mu,\epsilon}(t) - q^\epsilon(t)|_{\mathbb{R}^2} = 0,$$

where q^ϵ is the solution of the equation

$$\dot{q}^\epsilon(t) = A_\epsilon^{-1}\nabla H(q^\epsilon(t)), \quad q^\epsilon(0) = q \in \mathbb{R}^2. \tag{4.2}$$

By taking into account that

$$A_\epsilon^{-1} = (A_0 - \epsilon I)^{-1} = -\frac{1}{1+\epsilon^2}(A_0 + \epsilon I),$$

and

$$-A_0\nabla H = \left(\frac{\partial H}{\partial q_2}, -\frac{\partial H}{\partial q_1}\right) = \overline{\nabla} H,$$

one can rewrite (4.2) in the form

$$\dot{q}^\epsilon(t) = \frac{1}{1+\epsilon^2}\overline{\nabla} H(q^\epsilon(t)) - \frac{\epsilon}{1+\epsilon^2}\nabla H(q^\epsilon(t)), \quad q^\epsilon(0) = q \in \mathbb{R}^2. \tag{4.3}$$

For $0 < \epsilon \ll 1$, (4.3) can be considered as the result of small perturbations of the Hamiltonian system

$$\dot{q}^0(t) = \overline{\nabla} H(q^0(t)), \quad q^0(0) = q \in \mathbb{R}^2,$$

with one degree of freedom. To study the long time behavior of such a perturbed system, it is convenient to define

$$\tilde{q}^\epsilon(t) := q^\epsilon(t(1+\epsilon^2)/\epsilon), \quad t \geq 0.$$

With this time change, $\tilde{q}^\epsilon(t)$ solves the problem

$$\dot{\tilde{q}}^\epsilon(t) = \frac{1}{\epsilon}\overline{\nabla} H(\tilde{q}^\epsilon(t)) - \nabla H(\tilde{q}^\epsilon(t)), \quad \tilde{q}^\epsilon(0) = q \in \mathbb{R}^2. \tag{4.4}$$

In what follows, when we need to emphasize that $\tilde{q}^\epsilon(0) = q$, we will denote $\tilde{q}^\epsilon(t)$ by $\tilde{q}^{\epsilon,q}(t)$.

The planar motion described by (4.4) has a fast component, which is actually the motion along the non-perturbed trajectory $q^0(t(1+\epsilon^2)/\epsilon)$ and a slow component. The latter is the projection $\mathcal{Y}(q^\epsilon(t(1+\epsilon^2)/\epsilon))$ of $q^\epsilon(t(1+\epsilon^2)/\epsilon)$ on the graph Γ corresponding to $H(q)$ (Fig. 1).

To be more specific, assume that $H(q)$ has just two local minima at O_1 and O_2 and a saddle point at O_3. All the three critical points are assumed to be non-degenerate. Let Γ be the graph corresponding to $H(q)$ and let $\mathcal{Y} : \mathbb{R}^2 \to \Gamma$ be the corresponding identification mapping. If we number the edges of Γ, each point $y \in \Gamma$ can be characterized by two coordinates (z, k), where k is the number of the edge containing y and $z = H(\mathcal{Y}^{-1}(y))$. Then, the slow component $\mathcal{Y}(\tilde{q}^\epsilon(t))$ of (4.4) can be written as

$$Y^\epsilon(t) = (z^\epsilon(t), k^\epsilon(t)).$$

Now, let

$$C_k(z) = \mathcal{Y}^{-1}(z,k),$$

and let $G_k(z)$ be the domain in $\mathbb{R}^2$ bounded by $C_k(z)$. The ∞-shaped curve

$$C(H(O_3)) = \{q \in \mathbb{R}^2 : H(q) = H(O_3)\}$$

bounds two domains G_1 and G_2, with $G_i = \mathcal{Y}^{-1}(I_i)$. Define, for $k = 1, 2, 3$

$$T_k(z) = \oint_{C_k(z)} \frac{dl}{|\nabla H(q)|}, \qquad \beta_k(z) = -\int_{G_k(z)} \Delta H(q)\,dq, \tag{4.5}$$

where dl is the length element on $C_k(z)$, and for $i = 1, 2$

$$p_i = \left(\int_{G_1 \cup G_2} \Delta H(q)\,dq\right)^{-1} \int_{G_i} \Delta H(q)\,dq.$$

Assume for brevity that

$$\int_{G_i} \Delta H(q)\,dq > 0, \quad i = 1, 2.$$

Suppose now that the initial point q_δ is random and uniformly distributed in the δ-neighborhood $\mathcal{E}_\delta(q_0)$ of a point $q_0 \in \mathbb{R}^2$. Then, for each $\epsilon > 0$ we can consider $\tilde{q}^{\epsilon,\delta}(t) := \tilde{q}^{\epsilon,q_\delta}(t)$ as a stochastic process depending on the parameter $\delta > 0$. It follows from [1, Theorem 3.2] that for each $T > 0$ and $q_0 \in \mathbb{R}^2$ the process $\mathcal{Y}(\tilde{q}^{\epsilon,\delta}(t))$ converges weakly in $C([0,T];\Gamma)$, the space of continuous functions $\varphi : [0,T] \to \Gamma$, to a Markov continuous process $Y(t) = (z(t), k(t))$ on Γ, when first ϵ and then δ tend to zero. The process $Y(t) = (z(t), k(t))$, the limiting slow motion, is defined as follows

$$\begin{cases} \dot{z}_k(t) = \frac{\beta_k(z_k(t))}{T_k(z_k(t))}, & \text{inside each edge } I_k \subset \Gamma, \\ Y(0) = (z(0), k(0)) = \mathcal{Y}(q_0). \end{cases}$$

If $H(q_0) > H(O_3)$, then $Y(t)$ reaches $\mathcal{Y}(O_3)$ in a finite time and then, without any delay, it goes to edges I_1 and I_2 with probabilities p_1 and p_2, respectively. These properties define the process $Y(t)$ on Γ in a unique way. The process $Y(t)$ is deterministic inside each edge of Γ and its stochasticity becomes apparent just at interior vertices of Γ.

Combining all these arguments, we come to the following result.

Theorem 4.1 *Let $q^{\mu,\epsilon,\delta}(t)$ be the solution of* (4.1), *with random initial position $q^{\mu,\epsilon,\delta}(0) = q_\delta$, uniformly distributed in the δ-neighborhood of $q_0 \in \mathbb{R}^2$, and deterministic initial velocity $\dot{q}^{\mu,\epsilon,\delta}(0) = p$.*

Then, the slow component $\mathcal{Y}(q^{\mu,\epsilon,\delta}(t(1+\epsilon^2)/\epsilon))$ converges weakly in the space $C([0,T];\Gamma)$ to the Markov process $Y(t)$, described above, with $Y(0) = \mathcal{Y}(q_0)$, as first $\mu \downarrow 0$, then $\epsilon \downarrow 0$ and then $\delta \downarrow 0$.

One should emphasize that the stochasticity of the limiting slow motion is an intrinsic property of (4.1). This is the consequence of the instability of the system near the saddle point O_3.

Random perturbations of the initial point were introduced just as a regularization of the problem. Other types of regularizations lead to the same stochastic process $Y(t)$ on Γ (compare with [1]). For instance, one can regularize (4.1) by adding a small additive noise

$$\begin{cases} \mu\ddot{q}^{\mu,\epsilon,\delta}(t) = -\nabla H(q^{\mu,\epsilon,\delta}(t)) + A_0\dot{q}^{\mu,\epsilon,\delta}(t) - \epsilon\dot{q}^{\mu,\epsilon,\delta}(t) + \sqrt{\epsilon\delta}\dot{w}(t), \\ q^{\mu,\epsilon,\delta}(0) = q, \quad \dot{q}^{\mu,\epsilon,\delta}(0) = p. \end{cases} \tag{4.6}$$

Here $w(t)$ is the Wiener process in $\mathbb{R}^2$ and $0 < \mu \ll \epsilon \ll \delta \ll 1$.

As proved in Theorem 2.1, for any $\epsilon > 0$ and $\delta > 0$ fixed we have

$$\lim_{\mu\to 0} \mathbb{E}|q^{\mu,\epsilon,\delta} - q^{\epsilon,\delta}|_{C([0,T];\mathbb{R}^2)} = 0,$$

where the limit process $q^{\epsilon,\delta}$ solves the problem

$$dq(t) = A_\epsilon^{-1}\nabla H(q(t))\,dt - \sqrt{\epsilon\delta}A_\epsilon^{-1}\,dw(t), \quad q(0) = q.$$

Now, recalling that

$$A_\epsilon^{-1} = -\frac{1}{1+\epsilon^2}(A_0 + \epsilon I),$$

we have that

$$\begin{aligned} dq^{\epsilon,\delta}(t) &= \frac{1}{1+\epsilon^2}\overline{\nabla} H(q^{\epsilon,\delta}(t))\,dt - \frac{\epsilon}{1+\epsilon^2}\nabla H(q^{\epsilon,\delta}(t))\,dt \\ &\quad + \sqrt{\frac{\epsilon\delta}{1+\epsilon^2}}\,d\hat{w}(t), \quad q^{\epsilon,\delta}(0) = q, \end{aligned}$$

where $\hat{w}(t) = (1+\epsilon^2)^{-1/2}(A_0 + \epsilon I)w(t)$ is just another Wiener process in $\mathbb{R}^2$. Thus, if we set

$$\tilde{q}^{\epsilon,\delta}(t) = q^{\epsilon,\delta}(t(1+\epsilon^2)/\epsilon),$$

we have

$$d\tilde{q}^{\epsilon,\delta}(t) = \frac{1}{\epsilon}\overline{\nabla} H(\tilde{q}^{\epsilon,\delta}(t))\,dt - \nabla H(\tilde{q}^{\epsilon,\delta}(t))\,dt + \sqrt{\delta}d\hat{w}(t), \quad \tilde{q}^{\epsilon,\delta}(0) = q. \tag{4.7}$$

Now, for any $\delta > 0$ fixed, we introduce the following diffusion $Y_\delta(t)$ on Γ. Inside any edge I_k, the generator $\mathcal{A}_\delta$ of the process $Y_\delta(t)$ is given by

$$L_k^\delta u_k(z) = \frac{\delta}{2}\frac{d^2u_k}{dz^2}(z) + \frac{\beta_k(z)}{T_k(z)}\frac{du_k}{dz}(z),$$

where T_k and β_k are the functions defined in (4.5). Moreover, the following gluing condition is satisfied at the interior vertex $\mathcal{O}_3$

$$\beta_1 D_1 u(\mathcal{O}_3) + \beta_2 D_2 u(\mathcal{O}_3) = \beta_3 D_3 u(\mathcal{O}_3),$$

where D_j is the derivative along the edge I_j.

As proved in [1, Theorem 2.7], the following averaging result holds true.

Theorem 4.2 *Let $q^{\mu,\epsilon,\delta}(t)$ be the solution of problem (4.6). Assume that $\beta_j(\mathcal{O}_3)\neq 0$. Then, the slow component $Y_{\mu,\epsilon,\delta}(t)=\mathcal{Y}(q^{\mu,\epsilon,\delta}(t/\epsilon))$ converges weakly to $Y(t)$ in $C([0,T];\Gamma)$, as first $\mu\downarrow 0$, then $\epsilon\downarrow 0$ and then $\delta\downarrow 0$.*

We should notice that we would have got the same limiting process Y if instead of regularize the problem with and additive noise we would have done that by using a multiplicative noise, with any dispersion coefficient σ.

4.2 Stochastic Perturbations

We consider here a charged particle moving in a potential combined with a magnetic field, perturbed by a small friction and a small noise

$$\begin{cases} \mu\ddot{q}^{\mu,\epsilon}(t)=-\nabla H(q^{\mu,\epsilon}(t))+A_0\dot{q}^{\mu,\epsilon}(t)-\epsilon\dot{q}^{\mu,\epsilon}(t)+\sqrt{\epsilon}\dot{w}(t),\\ q^{\mu,\epsilon}(0)=q,\quad \dot{q}^{\mu,\epsilon}(0)=p.\end{cases}\tag{4.8}$$

Here $w(t)$ is the Wiener process in $\mathbb{R}^2$ and $0<\mu\ll\epsilon\ll 1$.

Due to Theorem 2.1, we have that for any fixed $\epsilon>0$

$$\lim_{\mu\to 0}\mathbb{E}\sup_{t\in[0,T]}|q^{\mu,\epsilon}(t)-\tilde{q}^{\epsilon}(t)|_{\mathbb{R}^2}=0,$$

where $\tilde{q}^{\epsilon}$ is the solution of the problem

$$d\tilde{q}^{\epsilon}(t)=\frac{1}{1+\epsilon^2}\overline{\nabla}H(\tilde{q}^{\epsilon}(t))\,dt-\frac{\epsilon}{1+\epsilon^2}\nabla H(\tilde{q}^{\epsilon}(t))+\sqrt{\frac{\epsilon}{1+\epsilon^2}}\,d\hat{w}(t),\quad \tilde{q}^{\epsilon}(0)=q$$

for $\hat{w}(t)=(1+\epsilon^2)^{-1/2}(A_0+\epsilon I)w(t)$ is just another Wiener process in $\mathbb{R}^2$ (see the previous subsection). Thus, with a change of time, we get the following equation for $q^{\epsilon}(t):=\tilde{q}^{\epsilon}(t(1+\epsilon^2)/\epsilon)$

$$dq^{\epsilon}(t)=\frac{1}{\epsilon}\overline{\nabla}H(q^{\epsilon}(t))-\nabla(q^{\epsilon}(t))+dw(t),\quad q^{\epsilon}(0)=q,\tag{4.9}$$

for some Wiener process $w(t)$.

The process defined by (4.9) has a fast and a slow component. The slow component is the projection $Y^{\epsilon}(t)=\mathcal{Y}(q^{\epsilon}(t))$, on the graph Γ corresponding to $H(q)$, of the solution $q^{\epsilon}(t)$. Note that, in general, $Y^{\epsilon}(t)$ is not a Markov process, but as follows from [3, Theorem 8.2.2] the process $Y^{\epsilon}(t)$ converges weakly in the space $C([0,T];\Gamma)$ to a Markov diffusion process $Y(t)$, with $Y(0)=q$, which is defined as follows. Inside each edge $I_k\subset\Gamma$, $Y(t)$ is governed by the operator l_k,

$$l_k v(z)=\frac{1}{2T_k(z)}\frac{d}{dz}\left(a_k(z)\frac{dv}{dz}\right)(z)+a_k(z)\frac{dv}{dz}(z),$$

where

$$a_k(z)=\int_{G_k(z)}\Delta H(q)\,dq,$$

and $T_k(z)$ and $G_k(z)$ are defined as before in this section.

Exterior vertices O_1 and O_2 are inaccessible for $Y(t)$. The interior vertex O_3 is accessible in a finite time, and the behavior of $Y(t)$ after reaching O_3 is defined by glueing conditions for functions belonging to the domain D_A of the generator A of the process $Y(t)$. Namely, a function $f(z,k)$, continuous on Γ and smooth inside the edges, belongs to D_A if and only if Af is continuous on Γ and the following condition at $\mathcal{Y}(O_3)$ is satisfied

$$\alpha_1 D_1 f(\mathcal{Y}(O_3)) + \alpha_2 D_2 f(\mathcal{Y}(O_2)) = (\alpha_1+\alpha_2) D_3 f(\mathcal{Y}(O_3)).$$

Here D_1 is the differentiation operator along I_i and

$$\alpha_i = \int_{G_i} \Delta H(z)\, dz.$$

These conditions define $Y(t)$ in a unique way.

Combining the convergence of $\mathcal{Y}(q^{\mu,\epsilon}(t(1+\epsilon^2)/\epsilon))$ to $Y^\epsilon(t)$, as $\mu\downarrow 0$, and $Y^\epsilon(t)$ to $Y(t)$, as $\kappa\downarrow 0$, we get the following statement.

Theorem 4.3 *Let $q^{\mu,\epsilon}(t)$ be the solution of (4.6), with $q^{\mu,\epsilon}(0)=q$ and $\dot q^{\mu,\epsilon}(t)=p$. Then, the slow component $\mathcal{Y}(q^{\mu,\epsilon}(t))$ converges weakly in $C([0,T];\Gamma)$ to the process $Y(t)$ on Γ described above, as first $\mu\downarrow 0$ and then $\epsilon\downarrow 0$.*

This approximation allows to calculate explicitly the main terms of important characteristics of the process $q^{\mu,\epsilon}(t)$, as $0<\mu\ll\epsilon\ll 1$. For instance, let $G\subset\mathbb{R}^2$ be a bounded connected domain. We are here interested in the mean value of the first exit time

$$\tau^{\mu,\epsilon} := \inf\{t\geq 0 : q^{\mu,\epsilon}(t)\notin G\}.$$

One can see that the problem can be reduced to the case when G is bounded just by components of the level sets of $H(q)$, so that we can assume that G is bounded by the level sets components. Now, let $\hat G := \mathcal{Y}(G)$. By Theorem 4.3 we have that

$$\lim_{\substack{\mu,\epsilon\downarrow 0\\ \mu\ll\epsilon}} \kappa\mathbb{E}_q \tau^{\mu,\epsilon} = v(\mathcal{Y}(q)),$$

where the function $v(y)=v(z,k)$ on Γ is the unique solution of the problem

$$\begin{cases} l_k v(z,k) = -1, & (z,k)\in \hat G\setminus\{\mathcal{Y}(O_1),\mathcal{Y}(O_2),\mathcal{Y}(O_3)\}\\ v(z,k)_{|_{\partial\hat G}} = 0, & \\ \alpha_1 D_1 v(\mathcal{Y}(O_2)) + \alpha_2 D_2 v(\mathcal{Y}(O_2)) = (\alpha_1+\alpha_2) D_3 v(\mathcal{Y}(O_2)). & \end{cases} \tag{4.10}$$

Notice that the problem can be solved explicitly.

Remark 4.4 The fast component of $q^{\mu,\epsilon}(t(1+\epsilon^2)/\epsilon)$, as $0<\mu\ll\epsilon\ll 1$, can be characterized by its distribution on $\mathcal{Y}^{-1}(Y(t)) = C_{k(t)}(z(t))$. The density of this distribution is equal to

$$(|\nabla H(q)| T_{k(t)}(z(t)))^{-1}, \quad q\in C_{k(t)}.$$

5 The Three Dimensional Case

In this section we consider the motion of a charged particle in $\mathbb{R}^3$. As before, let the particle move in a vector field $b(q)$, $q \in \mathbb{R}^3$, and be subject to a constant magnetic field and a small friction, proportional to the velocity. Moreover, let the axis q_3 be directed along the magnetic field which is assumed to be of strength one.

Then, the motion of the particle is described by the equation

$$\begin{cases} \mu \ddot{q}^{\mu,\epsilon}(t) = b(q^{\mu,\epsilon}(t)) + A_0 \dot{q}^{\mu,\epsilon}(t) - \epsilon \dot{q}^{\mu,\epsilon}(t), \\ q^{\mu,\epsilon}(0) = q \in \mathbb{R}^3, \quad \dot{q}^{\mu,\epsilon}(0) = p \in \mathbb{R}^3, \end{cases} \tag{5.1}$$

where

$$A_0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Notice that now, unlike in the 2-dimensional case, the matrix A_0 is not invertible. This means that we cannot use any argument based on the regularization of the noise and we cannot prove the convergence of $q^{\mu,\delta}$ to q^δ, solution of (3.5), and then the convergence of q^δ to $\hat{q}$, solution of (3.14). Actually, as A_0 is not invertible, (3.5) and (3.14) are not even well defined.

It is immediate to check that the arguments used in Theorem 2.1 can be adapted to the present situation and for any $\epsilon > 0$, $k \geq 1$ and $T > 0$ we have

$$\lim_{\mu \to 0} \mathbb{E} \sup_{t \in [0,T]} |q^{\mu,\epsilon}(t) - q^\epsilon(t)|^k_{\mathbb{R}^3} = 0. \tag{5.2}$$

But unfortunately we cannot prove an estimate analogous to (2.14). The only estimate we have is

$$\sup_{\mu > 0, \epsilon \geq \epsilon_0} \mathbb{E} |q^{\mu,\epsilon}|^k_{C([0,T];\mathbb{R}^3)} =: c_{k,\epsilon_0}(T)\big(1 + |q|^k_{\mathbb{R}^3}\big) < \infty,$$

for any $\epsilon_0 > 0$.

In what follows, we shall set

$$q(t) = (X(t), Y(t)), \quad X(t) \in \mathbb{R}^2,\ Y(t) \in \mathbb{R},$$

and

$$\hat{b}(q) = (b_1(q), b_2(q)).$$

Moreover, we shall set

$$\hat{A}_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In particular,

$$\hat{A}_\epsilon^{-1} = (-\epsilon I_{\mathbb{R}^2} + \hat{A}_0)^{-1} = \frac{1}{1+\epsilon^2} \begin{pmatrix} \epsilon & 1 \\ -1 & \epsilon \end{pmatrix},$$

so that, for any $v \in \mathbb{R}^2$

$$|(\hat{A}_\epsilon^{-1} - \hat{A}_0^{-1})v|_{\mathbb{R}^2} = \frac{\epsilon}{\sqrt{1+\epsilon^2}}|v|_{\mathbb{R}^2}. \tag{5.3}$$

With this notation, according to (2.7), the equation satisfied by q^ϵ can be rewritten as the following coupled system

$$\begin{cases} dX^\epsilon(t) = -\hat{A}_\epsilon^{-1}\hat{b}(X^\epsilon(t), Y^\epsilon(t))\,dt, & X^\epsilon(0) = x, \\ \dot{Y}^\epsilon(t) = \frac{1}{\epsilon}b_3(X^\epsilon(t), Y^\epsilon(t)), & Y^\epsilon(0) = y, \end{cases} \tag{5.4}$$

in which $X^\epsilon(t)$ describes the evolution of a two dimensional slow motion and $Y^\epsilon(t)$ the evolution of a one dimensional fast motion.

In what follows, for any fixed $x \in \mathbb{R}^2$ and $y \in \mathbb{R}$, we consider the problem

$$\dot{Y}(t) = b_3(x, Y(t)), \quad Y(0) = y. \tag{5.5}$$

Its solution is denoted by $Y^{x,y}(t)$. We shall assume that the following condition satisfied.

Hypothesis 1 *There exists a Lipschitz continuous function* $f : \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$|Y^{x,y}(t) - f(x)| \leq \alpha_{|x|,|y|}(t), \quad t \geq 0,$$

for any $x \in \mathbb{R}^2$ *and* $y \in \mathbb{R}$, *for some non-negative function* $\alpha_{|x|,|y|} \in L^1(0,\infty)$, *such that*

$$\lim_{t\to\infty} \alpha_{|x|,|y|}(t) = 0.$$

Now, due to Hypothesis 1, the measure

$$\mu^x = \delta_{f(x)}$$

is the unique invariant measure for the group P_t^x associated with equation (5.5), which is defined by

$$P_t^x\varphi(y) = \varphi(Y^{x,y}(t)), \quad t \in \mathbb{R},\ y \in \mathbb{R},$$

for any $\varphi : \mathbb{R} \to \mathbb{R}$ bounded and measurable. Moreover, for any Lipschitz-continuous function φ and $y \in \mathbb{R}$ we have

$$\left|P_t^x\varphi(y) - \int_{\mathbb{R}} \varphi(z)\mu^x(dz)\right| \leq [\varphi]_{\text{Lip}}\alpha_{|x|,|y|}(t), \quad t \in \mathbb{R}.$$

In particular, if we define

$$\bar{b}(x) = \int_{\mathbb{R}} \hat{b}(x,z)\mu^x(dz), \tag{5.6}$$

we have

$$
\begin{aligned}
&\left|\frac{1}{T}\int_0^T \hat{A}_0\hat{b}(x,Y^{x,y}(t))\,dt-\hat{A}_0\bar{b}(x)\right|\\
&\quad\le\frac{1}{T}\int_0^T\left|\hat{A}_0\hat{b}(x,Y^{x,y}(t))-\int_{\mathbb{R}}\hat{A}_0\hat{b}(x,z)\mu^x(dz)\right|dt\\
&\quad\le\frac{1}{T}[\hat{A}_0\hat{b}(x,\cdot)]_{\text{Lip}}\int_0^T\alpha_{|x|,|y|}(t)\,dt\le\frac{1}{T}[\hat{A}_0\hat{b}(x,\cdot)]_{\text{Lip}}\|\alpha_{|x|,|y|}\|_{L^1(0,\infty)}.
\end{aligned}
\tag{5.7}
$$

From all this, we can conclude that the following limiting result holds true.

Theorem 5.1 *Assume that $b:\mathbb{R}^3\to\mathbb{R}^3$ is Lipschitz continuous and b_3 satisfies Hypothesis 1. Then, if q^ϵ is the solution of problem (5.4) and $q^\epsilon=(X^\epsilon,Y^\epsilon)$, we have that for any $T>0$*

$$\lim_{\epsilon\to 0}\sup_{t\in[0,T]}|X^\epsilon(t)-X(t)|_{\mathbb{R}^2}=0, \tag{5.8}$$

where $X(t)$ is the solution of the problem

$$\dot{X}(t)=\hat{A}_0\bar{b}(X(t)),\qquad X(0)=x, \tag{5.9}$$

and for any $\delta>0$

$$\lim_{\epsilon\to 0}\sup_{t\in[\delta,T]}|Y^\epsilon(t)-f(X(t))|=0. \tag{5.10}$$

Remark 5.2

1. Assume that:
   - There exists a Lipschitz-continuous mapping $f:\mathbb{R}^2\to\mathbb{R}$ such that
   $$b_3(x,f(x))=0,\qquad x\in\mathbb{R}^2.$$
   - There exists $\alpha>0$ such that for any $x\in\mathbb{R}^2$ and $y_1,y_2\in\mathbb{R}$
   $$(b_3(x,y_1)-b_3(x,y_2))(y_1-y_2)\le-\alpha|y_1-y_2|^2. \tag{5.11}$$

   Then it is immediate to check that the conditions of Hypothesis 1 are satisfied, with
   $$\alpha_{|x|,|y|}(t)=e^{-\alpha t}|y-f(x)|.$$
2. A result completely analogous can be proved also in the following situation
   $$\begin{cases}\mu\ddot{q}^{\mu,\epsilon}(t)=b(q^{\mu,\epsilon}(t))+A_0\dot{q}^{\mu,\epsilon}(t)-\epsilon\dot{q}^{\mu,\epsilon}+\dot{\hat{w}}(t),\\ q^{\mu,\epsilon}(0)=q\in\mathbb{R}^3,\quad \dot{q}^{\mu,\epsilon}(0)=p\in\mathbb{R}^3,\end{cases} \tag{5.12}$$

   where $\hat{w}(t)$ is a 2-dimensional Brownian motion. Actually, in this case we have the convergence in $L^2(\Omega;C([0,T];\mathbb{R}^2))$ of $X^\epsilon(t)$ to the solution $X(t)$ of the problem
   $$dX(t)=\hat{A}_0\bar{b}(X(t))\,dt+d\hat{w}(t),\qquad X(0)=x,$$
   and the convergence in $L^2(\Omega;C([\delta,T];\mathbb{R}))$ of $Y^\epsilon(t)$ to $f(X(t))$, for any $\delta>0$.

Concerning the long-time influence of small perturbations in this three dimensional case, as in Sect. 4, we assume that $b(q) = -\nabla H(q)$, for some $H : \mathbb{R}^3 \to \mathbb{R}$. Then, if $f : \mathbb{R}^2 \to \mathbb{R}$ is the smooth function introduced in Hypothesis 1, we have that

$$\frac{\partial H}{\partial q_3}(q) > 0, \quad \text{if } q_3 > f(q_1, q_2),$$

and

$$\frac{\partial H}{\partial q_3}(q) < 0, \quad \text{if } q_3 < f(q_1, q_2).$$

Let Γ be the graph corresponding to

$$\hat{H}(q_1, q_2) = H(q_1, q_2, f(q_1, q_2)),$$

and $\mathcal{Y}$ the corresponding identification map. Then we define

$$Y^\epsilon(t) = (\mathcal{Y}(q_1^\epsilon(t), q_2^\epsilon(t)), f(q_1^\epsilon(t), q_2^\epsilon(t))),$$

where the vector $q^\epsilon(t)$ is obtained as in Theorem 2.1, by taking the limit as $\mu \downarrow 0$ in (5.1).

One can prove that the trajectories $Y^\epsilon(t/\epsilon)$ in the phase space $\Gamma \times \mathbb{R}$ in a certain sense converge as $\epsilon \downarrow 0$ to a Markov process $(Y(t), f(Y(t)))$ inside the pages of the open book $\Gamma \times \mathbb{R}$ and may have some stochasticity on the binding (compare with [4]). This means that the slow component of $q^{\mu,\epsilon}(t/\epsilon)$ can be approximated as $0 < \mu \ll \epsilon \ll 1$ by the process $(Y(t), f(Y(t)))$ on $\Gamma \times \mathbb{R}$. The fast component can be characterized by the invariant distribution on the corresponding periodic trajectory.

One can also consider white-noise type perturbations of the second order equation. We will consider these problems in detail elsewhere.

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