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Large Deviations for the Langevin Equation with Strong Damping

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Abstract We study large deviations in the Langevin dynamics, with damping of order ϵ^{-1} and noise of order 1, as $\epsilon \downarrow 0$. The damping coefficient is assumed to be state dependent. We proceed first with a change of time and then we use a weak convergence approach to large deviations and its equivalent formulation in terms of the Laplace principle, to determine the good action functional. Some applications of these results to the exit problem from a domain and to the wave front propagation for a suitable class of reaction diffusion equations are considered.

Keywords Large deviations · Laplace principle · Over damped stochastic differential equations

1 Introduction

For every $\epsilon > 0$, let us consider the Langevin equation

$$\begin{cases} \ddot{q}^{\,\epsilon}(t) = b(q^\epsilon(t)) - \dfrac{\alpha(q^\epsilon(t))}{\epsilon}\dot{q}^{\,\epsilon}(t) + \sigma(q^\epsilon(t))\dot{B}(t), \\ q^\epsilon(0) = q \in \mathbb{R}^d, \quad \dot{q}^\epsilon(0) = p \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

Here $B(t)$ is a r-dimensional standard Wiener process, defined on some complete stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. In what follows, we shall assume that b is Lipschitz continuous and α and σ are bounded and continuously differentiable, with bounded derivative. Moreover, σ is invertible and there exist two constants $0 < \alpha_0 < \alpha_1$ such that $a_0 \leq \alpha(q) \leq \alpha_1$, for all $q \in \mathbb{R}^d$. Equation (1.1) can be rewritten as the following system in $\mathbb{R}^{2d}$

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$$\begin{cases} \dot{q^\epsilon}(t) = p^\epsilon(t), & q^\epsilon(0) = q \in \mathbb{R}^d, \\ \dot{p^\epsilon}(t) = b(q^\epsilon(t)) - \dfrac{\alpha(q^\epsilon(t))}{\epsilon} p^\epsilon(t) + \sigma(q^\epsilon(t))\dot{B}(t), & p^\epsilon(0) = p \in \mathbb{R}^d, \end{cases}$$

and, due to our assumptions on the coefficients, for any $\epsilon > 0$, $T > 0$ and $k \geq 1$, the system above admits a unique solution $z^\epsilon = (q^\epsilon, p^\epsilon) \in L^k(\Omega, C([0, T]; \mathbb{R}^{2d}))$, which is a Markov process.

Now, if we do a change of time and define $q_\epsilon(t) := q^\epsilon(t/\epsilon)$, $t \geq 0$, we have

$$\begin{cases} \epsilon^2 \ddot{q_\epsilon}(t) = b(q_\epsilon(t)) - \alpha(q_\epsilon(t))\dot{q}_\epsilon(t) + \sqrt{\epsilon}\,\sigma(q_\epsilon(t))\dot{w}(t), \\ q_\epsilon(0) = q \in \mathbb{R}^d, \quad \dot{q_\epsilon}(0) = \dfrac{p}{\epsilon} \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

where $w(t) = \sqrt{\epsilon} B(t/\epsilon)$, $t \geq 0$, is another $\mathbb{R}^r$-valued Wiener process, defined on the same stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

In the present paper, we are interested in studying the large deviation principle for Eq. (1.2), as $\epsilon \downarrow 0$. Namely, we want to prove that the family $\{q_\epsilon\}_{\epsilon>0}$ satisfies a large deviation principle in $C([0, T]; \mathbb{R}^d)$, with the same action functional I and the same normalizing factor ϵ that describe the large deviation principle for the first order equation

$$\dot{g_\epsilon}(t) = \frac{b(g_\epsilon(t))}{\alpha(g_\epsilon(t))} + \sqrt{\epsilon}\,\frac{\sigma(g_\epsilon(t))}{\alpha(g_\epsilon(t))}\dot{w}(t), \quad g_\epsilon(0) = q \in \mathbb{R}^d. \tag{1.3}$$

In particular, as shown in Sect. 4, this implies that the asymptotic behavior of the exit time from a basin of attraction for the over damped Langevin dynamics (1.1) can be described by the quasi potential V associated with I, as well as the asymptotic behavior of the solutions of the degenerate parabolic and elliptic problems associated with the Langevin dynamics.

Moreover, in Sect. 4, we will show how these results allow to prove that in reaction–diffusion equations with non-linearities of KPP type, where the transport is described by the Langevin dynamics itself, the interface separating the areas where u^ϵ is close to 1 and to 0, as $\epsilon \downarrow 0$, is given in terms of the action functional I, as in the classical case, when the vanishing mass approximation is considered.

In [3,8], the system

$$\begin{cases} \mu \ddot{q_{\mu,\epsilon}}(t) = b(q_{\mu,\epsilon}(t)) - \alpha(q_{\mu,\epsilon}(t))\dot{q}_{\mu,\epsilon}(t) + \sqrt{\epsilon}\,\sigma(q_{\mu,\epsilon}(t))\dot{w}(t), \\ q_{\mu,\epsilon}(0) = q \in \mathbb{R}^d, \quad \dot{q_{\mu,\epsilon}}(0) = \dfrac{p}{\epsilon} \in \mathbb{R}^d, \end{cases} \tag{1.4}$$

for $0 < \mu, \epsilon << 1$, has been studied, under the crucial assumption that the friction coefficient α is independent of q.

It has been proven that, in this case, the so-called Kramers–Smoluchowski approximation holds, that is for any fixed $\epsilon > 0$ the solution $q_{\mu,\epsilon}$ of system (1.4) converges in $L^2(\Omega; C([0, T]; \mathbb{R}^d))$, as $\mu \downarrow 0$, to g_ϵ, the solution of the first order equation (1.3). Moreover, it has been proven that, if $V_\mu(q, p)$ is the quasi-potential associated with the family $\{q_{\mu,\epsilon}\}_{\epsilon>0}$, for $\mu > 0$ fixed, then

$$\lim_{\mu\to 0}\ \inf_{p\in\mathbb{R}^d} V_\mu(q, p) = V(q),$$

where V is the quasi-potential associated with the action-functional I.

In [9] and Eq. 1.4 with non constant friction α has been considered and it has been shown that in this case the situation is considerably more delicate. Actually, in [9] the limit of $q_{\mu,\epsilon}$

to g_ϵ has only been proven via a previous regularization of the noise, which has led to the convergence of $q_{\mu,\epsilon}$ to the solution $\tilde{g}_\epsilon$ of the first order equation with Stratonovich integral. Moreover, in [12] it is shown that $q_{\mu,\varepsilon}$ converges, as μ goes to zero, to the solution of a first order equation of the same type as Eq. (1.3), where an extra drift term is added.

Finally, we would like to mention that in the recent paper [13], by Lyv and Roberts, an analogous problem has been studied for the stochastic damped wave equation in a bounded regular domain $D \subset \mathbb{R}^d$, with $d = 1, 2, 3$,

$$\begin{cases} \epsilon \dfrac{\partial^2 u(t,x)}{\partial t^2} = \Delta u(t,x) + f(u(t,x)) - \dfrac{\partial u(t,x)}{\partial t} + \epsilon^\alpha \dfrac{\partial w(t,x)}{\partial t} \\ u(t,x) = 0, \quad x \in \partial D, \quad u(0,x) = u_0(x), \quad \dfrac{\partial u(0,x)}{\partial t} = v_0, (x), \end{cases}$$

where $\epsilon > 0$ is a small parameter, the friction coefficient is constant ($\alpha = 1$), $w(t,x)$ is a smooth cylindrical Wiener process and f is a cubic non-linearity. By using the weak convergence approach, the authors show that the family $\{u_\epsilon\}_{\epsilon>0}$ satisfies a large deviation principle in $C([0,T]; L^2(D))$, with normalizing factor $\epsilon^{2\alpha}$ and the same action functional that describes the large deviation principle for the stochastic parabolic equation.

As mentioned above, in the present paper we are dealing with the case of non-constant friction α and $\mu = \epsilon^2$. Dealing with a non-constant friction coefficient turns out to be important in applications, as it allows to describes new effects in reaction–diffusion equations and exit problems (see Sect. 4). Here, we will study the large deviation principle for Eq. (1.2) by using the approach of weak convergence (see [1,2]) and we will show the validity of the Laplace principle, which, together with the compactness of level sets, is equivalent to the large deviation principle.

At this point, it is worth mentioning that one major difficulty here is handling the integral

$$\int_0^t \exp\left(-\int_s^t \alpha(q_\epsilon(r))\,dr\right)\sigma(q_\epsilon(s))\,dw(s),$$

and proving that it converges to zero, as $\epsilon \downarrow 0$, in $L^1(\Omega; C([0,T]; \mathbb{R}^d))$. Actually, as α is non-constant, the integral above cannot be interpreted as an Itô's integral and in our estimates we cannot use Itô's isometry. Nevertheless, due to the regularity of $q_\epsilon(t)$, we can consider the integral above as a pathwise integral, and with appropriate integrations by parts, we can get the estimates required to prove the Laplace principle.

2 The Problem and the Method

We are dealing here with the equation

$$\begin{cases} \epsilon^2 \ddot{q}_\epsilon(t) = b(q_\epsilon(t)) - \alpha(q_\epsilon(t))\dot{q}_\epsilon(t) + \sqrt{\epsilon}\,\sigma(q_\epsilon(t))\dot{w}(t), \\ q_\epsilon(0) = q \in \mathbb{R}^d, \quad \dot{q}_\epsilon(0) = \dfrac{p}{\epsilon} \in \mathbb{R}^d. \end{cases} \tag{2.1}$$

Here $w(t)$, $t \geq 0$, is a r-dimensional Brownian motion and the coefficients b, σ and α satisfy the following conditions.

Hypothesis 1 1. The mapping $b : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz-continuous and the mapping $\sigma : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^r, \mathbb{R}^d)$ is continuously differentiable and bounded, together with its derivative. Moreover, the matrix $\sigma(q)$ is invertible, for any $q \in \mathbb{R}^d$, and $\sigma^{-1} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^r, \mathbb{R}^d)$ is bounded.

2. The mapping $\alpha : \mathbb{R}^d \to \mathbb{R}$ belongs to $C_b^1(\mathbb{R}^d)$ and

$$\inf_{x\in\mathbb{R}^d} \alpha(x) =: \alpha_0 > 0. \tag{2.2}$$

In view of the conditions on the coefficients α, b and σ assumed in Hypothesis 1, for every fixed $\epsilon > 0$, Eq. (2.6) admits a unique solution $z_\epsilon = (q_\epsilon, p_\epsilon) \in L^k(0, T; \mathbb{R}^d)$, with $T > 0$ and $k \geq 1$.

Now, for any predictable process u taking values in $L^2((0, T); \mathbb{R}^r)$, we introduce the problem

$$\dot{g}^u(t) = \frac{b(g^u(t))}{\alpha(g^u(t))} + \frac{\sigma(g^u(t))}{\alpha(g^u(t))}u(t), \quad g^u(0) = q \in \mathbb{R}^d. \tag{2.3}$$

The existence and uniqueness of a pathwise solution g^u to Problem (2.3) in $C([0, T]; \mathbb{R}^d)$ is an immediate consequence of the conditions on the coefficients b, σ and α that we have assumed in Hypothesis 1.

In what follows, we shall denote by $\mathcal{G}$ the mapping

$$\mathcal{G} : L^2((0, T); \mathbb{R}^r) \to C([0, T]; \mathbb{R}^d), \quad u \mapsto \mathcal{G}(u) = g^u.$$

Moreover, for any $f \in C([0, T]; \mathbb{R}^d)$ we shall define

$$I(f) = \frac{1}{2} \inf \left\{ \int_0^T |u(t)|^2\, dt \;:\; f = \mathcal{G}(u),\; u \in L^2((0, T); \mathbb{R}^r) \right\},$$

with the usual convention $\inf \emptyset = +\infty$. This means that

$$I(f) = \frac{1}{2} \int_0^T \left| \alpha(f(s))\sigma^{-1}(f(s)) \left(\dot{f}(s) - \frac{b(f(s))}{\alpha(f(s))} \right) \right|^2 ds, \tag{2.4}$$

for all $f \in W^{1,2}(0, T; \mathbb{R}^d)$.

If we denote by g_ϵ the solution of the stochastic equation

$$\dot{g_\epsilon}(t) = \frac{b(g_\epsilon(t))}{\alpha(g_\epsilon(t))} + \sqrt{\epsilon}\, \frac{\sigma(g_\epsilon(t))}{\alpha(g_\epsilon(t))}\dot{w}(t), \quad g_\epsilon(0) = q \in \mathbb{R}^d, \tag{2.5}$$

we have that I is the large deviation action functional for the family $\{g_\epsilon\}_{\epsilon>0}$ in the space of continuous trajectories $C([0, T]; \mathbb{R}^d)$ (for a proof see e.g. [11]). This means that the level sets $\{I(f) \leq c\}$ are compact in $C([0, T]; \mathbb{R}^d)$, for any $c > 0$, and for any closed subset $F \subset C([0, T]; \mathbb{R}^d)$ and any open set $G \subset C([0, T]; \mathbb{R}^d)$ it holds

$$\limsup_{\epsilon\to 0^+} \epsilon \log \mathbb{P}(g_\epsilon \in F) \leq -I(F),$$
$$\liminf_{\epsilon\to 0^+} \epsilon \log \mathbb{P}(g_\epsilon \in G) \geq -I(G),$$

where, for any subset $A \subset C([0, T]; \mathbb{R}^d)$, we have denoted

$$I(A) = \inf_{f\in A} I(f).$$

The main result of the present paper is to prove that in fact the family of solutions q_ϵ of Eq. (1.2) satisfies a large deviation principle with the same action functional I that describes the large deviation principle for the family of solutions g_ϵ of Eq. (2.5). And, due to the fact that $q^\epsilon(t) = q_\epsilon(\epsilon t)$, $t \geq 0$, this allows to describe the behavior of the over damped Langevin dynamics (1.1) (see Sect. 4 for all details).

Theorem 2.1 *Under Hypothesis 1, the family of probability measures* $\{\mathcal{L}(q_\epsilon)\}_{\epsilon>0}$*, in the space of continuous paths* $C([0,T];\mathbb{R}^d)$*, satisfies a large deviation principle with action functional* I.

In order to prove Theorem 2.1, we follow the weak convergence approach, as developed in [1], (see also [2]). To this purpose, we need to introduce some notations. We denote by $\mathcal{P}_T$ the set of predictable processes in $L^2(\Omega\times[0,T];\mathbb{R}^r)$, and for any $T>0$ and $\gamma>0$, we define the sets

$$\mathcal{S}_T^\gamma=\left\{f\in L^2((0,T);\mathbb{R}^d)\,:\,\int_0^T|f(s)|^2\,ds\leq\gamma\right\}$$
$$\mathcal{A}_T^\gamma=\left\{u\in\mathcal{P}_T\;:\;u\in\mathcal{S}_T^\gamma,\;\mathbb{P}-\text{a.s.}\right\}.$$

Next, for any predictable process u taking values in $L^2((0,T);\mathbb{R}^r)$, we denote by $q_\epsilon^u(t)$ the solution of the problem

$$\begin{cases}\epsilon^2\ddot{q}_\epsilon^u(t)=b(q_\epsilon^u(t))-\alpha(q_\epsilon^u(t))\dot{q}_\epsilon^u(t)+\sqrt{\epsilon}\,\sigma(q_\epsilon^u(t))\dot{w}(t)+\sigma(q_\epsilon^u(t))u(t),\\ q_\epsilon^u(0)=q\in\mathbb{R}^d,\quad\dot{q}_\epsilon^u(0)=\dfrac{p}{\epsilon}\in\mathbb{R}^d.\end{cases}\tag{2.6}$$

As well known, for any fixed $\epsilon>0$ and for any $T>0$ and $k\geq1$, this equation admits a unique solution q_ϵ^u in $L^k(\Omega;C([0,T];\mathbb{R}^d))$.

By proceeding as in the proof of [2, Theorem 4.3], the following result can be proven.

Theorem 2.2 *Let* $\{u_\epsilon\}_{\epsilon>0}$ *be a family of processes in* $\mathcal{S}_T^\gamma$ *that converge in distribution, as* $\epsilon\downarrow0$*, to some* $u\in\mathcal{S}_T^\gamma$*, as random variables taking values in the space* $L^2((0,T);\mathbb{R}^d)$*, endowed with the weak topology.*

If the sequence $\{q_\epsilon^{u_\epsilon}\}_{\epsilon>0}$ *converges in distribution to* g^u*, as* $\epsilon\downarrow0$*, in the space of continuous paths* $C([0,T];\mathbb{R}^d)$*, then the family* $\{\mathcal{L}(q_\epsilon)\}_{\epsilon>0}$ *satisfies a large deviation principle in* $C([0,T];\mathbb{R}^d)$*, with action functional* I.

Actually, as shown in [2], the convergence of $q_\epsilon^{u_\epsilon}$ to g^u implies the validity of the Laplace principle with rate functional I. This means that, for any continuous mapping $\Lambda:C([0,T];\mathbb{R}^d)\to\mathbb{R}$ it holds

$$\lim_{\epsilon\to0}-\epsilon\log\mathbb{E}\exp\left(-\frac{1}{\epsilon}\,\Lambda(q_\epsilon)\right)=\inf_{f\in C([0,T];\mathbb{R}^d)}\left(\Lambda(f)+I(f)\right).$$

And, as the level sets of I are compact, this is equivalent to say that $\{\mathcal{L}(q_\epsilon)\}_{\epsilon>0}$ satisfies a large deviation principle in $C([0,T];\mathbb{R}^d)$, with action functional I.

3 Proof of Theorem 2.1

As we have seen in the previous section, in order to prove Theorem 2.1, we have to show that if $\{u_\epsilon\}_{\epsilon>0}$ is a family of processes in $\mathcal{S}_T^\gamma$ that converge in distribution, as $\epsilon\downarrow0$, to some $u\in\mathcal{S}_T^\gamma$, as random variables taking values in the space $L^2((0,T);\mathbb{R}^d)$, endowed with the weak topology, then the sequence $\{q_\epsilon^{u_\epsilon}\}_{\epsilon>0}$ converges in distribution to g^u, as $\epsilon\downarrow0$, in the space $C([0,T];\mathbb{R}^d)$.

In view of the Skorohod representation theorem, we can rephrase such a condition in the following way. On some probability space $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}})$, consider a Brownian motion $\bar{w}_t$, $t\geq0$, along with the corresponding natural filtration $\{\bar{\mathcal{F}}_t\}_{t\geq0}$. Moreover, consider a family of $\{\bar{\mathcal{F}}_t\}$-predictable processes $\{\bar{u}_\epsilon,\;\bar{u}\}_{\epsilon>0}$ in $L^2(\bar{\Omega}\times[0,T];\mathbb{R}^d)$, taking values in $\mathcal{S}_T^\gamma$, $\bar{\mathbb{P}}$-a.s., such that the joint law of $(\bar{u}^\epsilon, \bar{u}, \bar{w})$, under $\bar{\mathbb{P}}$, coincides with the joint law of (u^ϵ, u, w), under $\mathbb{P}$, and such that

$$\lim_{\epsilon\to 0} \bar{u}_\epsilon = \bar{u}, \quad \bar{\mathbb{P}} - \text{a.s.} \tag{3.1}$$

as $L^2((0,T);\mathbb{R}^d)$-valued random variables, endowed with the weak topology. Let $\bar{q}_\epsilon^{\bar{u}_\epsilon}$ be the solution of a problem analogous to (2.6), with u and w replaced respectively by $\bar{u}_\epsilon$ and $\bar{w}$.

Then, we have to prove that

$$\lim_{\epsilon\to 0} \bar{q}_\epsilon^{\bar{u}_\epsilon} = g^{\bar{u}}, \qquad \bar{\mathbb{P}} - \text{a.s.}$$

in $C([0,T];\mathbb{R}^d)$. In fact, we will prove more. Actually, we will show that

$$\lim_{\epsilon\to 0} \bar{\mathbb{E}} \sup_{t\in[0,T]} |\bar{q}_\epsilon^{\bar{u}_\epsilon}(t) - g^{\bar{u}}(t)| = 0. \tag{3.2}$$

In order to prove (3.2), we will need some preliminary estimates. For any $\epsilon > 0$, we define the process

$$H_\epsilon(t) = \sqrt{\epsilon}\, e^{-A_\epsilon(t)} \int_0^t e^{A_\epsilon(s)} \sigma(q_\epsilon^u(s))\, dw(s), \quad t \geq 0. \tag{3.3}$$

where, for any $0 \leq s \leq t$ and $\varepsilon > 0$ we define

$$A_\epsilon(t,s) := \frac{1}{\epsilon^2} \int_s^t \alpha(q_\epsilon^u(r))\, dr, \quad A_\epsilon(t) := A_\epsilon(t,0).$$

Lemma 3.1 *Under Hypothesis 1, for any $T > 0$, $k \geq 1$ and $\gamma > 0$, there exists $\epsilon_0 > 0$ such that for any $u \in S_T^\gamma$ and $\epsilon \in (0, \epsilon_0]$*

$$\sup_{s\leq t} \mathbb{E}\, |H_\epsilon(t)|^k \leq c_{k,\gamma}(T)(|q|^k + |p|^k + 1)\epsilon^{\frac{3k}{2}} + c_k\, \epsilon^{\frac{k}{2}}\, t^{\frac{k}{2}} e^{-\frac{k\alpha_0 t}{\epsilon^2}}. \tag{3.4}$$

Moreover, we have

$$\mathbb{E} \sup_{t\in[0,T]} |H_\epsilon(t)| \leq \sqrt{\epsilon}\, c_\gamma(T)(1 + |q| + |p|). \tag{3.5}$$

Proof Equation (2.6) can be rewritten as the system

$$\begin{cases} \dot{q}_\epsilon^u(t) = p_\epsilon^u(t), \quad q_\epsilon^u(0) = q \\ \epsilon^2 \dot{p}_\epsilon^u(t) = b(q_\epsilon^u(t)) - \alpha(q_\epsilon^u(t)) p_\epsilon^u(t) + \sqrt{\epsilon}\, \sigma(q_\epsilon^u(t))\dot{w}(t) + \sigma(q_\epsilon^u(t))u(t), \quad p_\epsilon^u(0) = \dfrac{p}{\epsilon}. \end{cases}$$

Thus, we have

$$\begin{aligned} p_\epsilon^u(t) = {} & \frac{1}{\epsilon} e^{-A_\epsilon(t)} p + \frac{1}{\epsilon^2} \int_0^t e^{-A_\epsilon(t,s)} b(q_\epsilon^u(s))\, ds \\ & + \frac{1}{\epsilon^2} \int_0^t e^{-A_\epsilon(t,s)} \sigma(q_\epsilon^u(s))\, u(s)\, ds + \frac{1}{\epsilon^2} H_\epsilon(t). \end{aligned} \tag{3.6}$$

Integrating with respect to t, this yields

$$\begin{aligned} q_\epsilon^u(t) = {} & q + \frac{1}{\epsilon} \int_0^t e^{-A_\epsilon(s)} p\, ds + \frac{1}{\epsilon^2} \int_0^t \int_0^s e^{-A_\epsilon(s,r)} b(q_\epsilon^u(r))\, dr\, ds \\ & + \frac{1}{\epsilon^2} \int_0^t \int_0^s e^{-A_\epsilon(s,r)} \sigma(q_\epsilon^u(r))\, u(r)\, dr\, ds + \frac{1}{\epsilon^2} \int_0^t H_\epsilon(s)\, ds. \end{aligned} \tag{3.7}$$

Thanks to the Young inequality, this implies that for any $t \in [0, T]$

$$|q^u_\epsilon(t)| \leq |q| + \epsilon\,|p| + c\int_0^t (1 + |q^u_\epsilon(s)|)\,ds + c\int_0^t |u(s)|\,ds + \frac{1}{\epsilon^2}\int_0^t |H_\epsilon(s)|\,ds$$
$$\leq c_\gamma(T)(|q| + \epsilon\,|p| + 1) + \frac{1}{\epsilon^2}\int_0^t |H_\epsilon(s)|\,ds + c\int_0^t |q^u_\epsilon(s)|\,ds,$$

and from the Gronwall lemma we can conclude that

$$|q^u_\epsilon(t)| \leq c_\gamma(T)\,(1 + |q| + |p|) + c(T)\frac{1}{\epsilon^2}\int_0^t |H_\epsilon(s)|\,ds.$$

This implies that for any $k \geq 1$

$$|q^u_\epsilon(t)|^k \leq c_{k,\gamma}(T)(|q|^k + |p|^k + 1) + c_{k,\gamma}(T)\epsilon^{-2k}\int_0^t |H_\epsilon(s)|^k\,ds, \quad \epsilon \in (0, 1]. \tag{3.8}$$

Now, due to (3.6), we have

$$|\,p^{u_\epsilon}(t)| \leq \frac{1}{\epsilon}e^{-\frac{\alpha_0 t}{\epsilon^2}}|p| + \frac{c}{\epsilon^2}\int_0^t e^{-\frac{\alpha_0(t-s)}{\epsilon^2}}\left(1 + |q^u_\epsilon(s)|\right)\,ds$$
$$+ \frac{c}{\epsilon^2}\int_0^t e^{-\frac{\alpha_0(t-s)}{\epsilon^2}}|u(s)|\,ds + \frac{1}{\epsilon^2}|H_\epsilon(t)|,$$

so that, thanks to (3.8), for any $\epsilon \in (0, 1]$ we get

$$|p^u_\epsilon(t)| \leq \frac{1}{\epsilon}e^{-\frac{\alpha_0 t}{\epsilon^2}}|p| + c_\gamma(T)(|q| + |p| + 1)$$
$$+ \frac{c}{\epsilon^2}\int_0^t e^{-\frac{\alpha_0(t-s)}{\epsilon^2}}|u(s)|\,ds + c(T)\frac{1}{\epsilon^2}|H_\epsilon(t)|. \tag{3.9}$$

As well known, if $f \in C^1([0, t])$ and $g \in C([0, t])$, then the Stiltjies integral

$$\int_0^t f(s)dg(s), \quad t \geq 0,$$

is well defined and, if $g(0) = 0$, the following integration by parts formula holds

$$\int_0^t f(s)dg(s) = \int_0^t (g(t) - g(s))\,f'(s)\,ds + g(t)f(0), \quad t \geq 0. \tag{3.10}$$

Now, the mapping

$$[0, +\infty) \to \mathcal{L}(\mathbb{R}^r, \mathbb{R}^d), \quad s \mapsto e^{A_\epsilon(s)}\sigma(q^u_\epsilon(s)),$$

is differentiable, $\mathbb{P}$-a.s., so that the stochastic integral in (3.3) is in fact a pathwise integral. In particular, we can apply formula (3.10), with

$$f(s) = e^{A_\epsilon(s)}\sigma(q^u_\epsilon(s)), \quad g(s) = w(s),$$

and we get

$$H_\epsilon(t) = \sqrt{\epsilon}\int_0^t (w(t) - w(s))\,e^{-A_\epsilon(t,s)}\left(\frac{\alpha(q^u_\epsilon(s))}{\epsilon^2} + \sigma'(q^u_\epsilon(s))p^u_\epsilon(s)\right)\,ds$$
$$+ \sqrt{\epsilon}w(t)e^{-A_\epsilon(t)}\sigma(q). \tag{3.11}$$

Thanks to (3.9), this yields for any $\epsilon \in (0,1]$

$$
\begin{aligned}
|H_\epsilon(t)| &\leq c\sqrt{\epsilon}\int_0^t |w(t)-w(s)|\frac{e^{-\frac{\alpha_0(t-s)}{\epsilon^2}}}{\epsilon^2}\left(1+\epsilon^2|p^u_\epsilon(s)|\right)ds + c\sqrt{\epsilon}\,|w(t)|e^{-\frac{\alpha_0 t}{\epsilon^2}}\\
&\leq c_\gamma(T)(|q|+|p|+1)\sqrt{\epsilon}\int_0^{\frac{t}{\epsilon^2}}|w(t)-w(t-\epsilon^2 s)|e^{-\alpha_0 s}\,ds\\
&\quad+\sqrt{\epsilon}\,c_\gamma(T)\int_0^{\frac{t}{\epsilon^2}}|w(t)-w(t-\epsilon^2 s)|e^{-\alpha_0 s}|H_\epsilon(t-\epsilon^2 s)|\,ds + c\sqrt{\epsilon}\,|w(t)|e^{-\frac{\alpha_0 t}{\epsilon^2}},
\end{aligned}
$$

and hence, for any $k \geq 1$, we have

$$
\begin{aligned}
|H_\epsilon(t)|^k &\leq c_{k,\gamma}(T)(|q|^k+|p|^k+1)\epsilon^{\frac{k}{2}}\int_0^{\frac{t}{\epsilon^2}}|w(t)-w(t-\epsilon^2 s)|^k e^{-\alpha_0 s}\,ds\\
&\quad+\epsilon^{\frac{k}{2}}c_{k,\gamma}(T)\int_0^{\frac{t}{\epsilon^2}}|w(t)-w(t-\epsilon^2 s)|^k e^{-\alpha_0 s}|H_\epsilon(t-\epsilon^2 s)|^k\,ds\\
&\quad+c_k\,\epsilon^{\frac{k}{2}}\,|w(t)|^k e^{-\frac{k\alpha_0 t}{\epsilon^2}}.
\end{aligned}
$$

By taking the expectation, due to the independence of $|w(t)-w(t-\epsilon^2 s)|$ with $|H_\epsilon(t-\epsilon^2 s)|$ this implies that for any $\epsilon \in (0,1]$

$$
\begin{aligned}
\mathbb{E}\,|H_\epsilon(t)|^k &\leq c_{k,\gamma}(T)(|q|^k+|p|^k+1)\epsilon^{\frac{3k}{2}}\int_0^{\frac{t}{\epsilon^2}}s^{\frac{k}{2}}e^{-\alpha_0 s}\,ds\\
&\quad+\epsilon^{\frac{3k}{2}}c_{k,\gamma}(T)\int_0^{\frac{t}{\epsilon^2}}s^{\frac{k}{2}}e^{-\alpha_0 s}\mathbb{E}|H_\epsilon(t-\epsilon^2 s)|^k\,ds + c_k\,\epsilon^{\frac{k}{2}}\,t^{\frac{k}{2}}e^{-\frac{k\alpha_0 t}{\epsilon^2}}\\
&\leq c_{k,\gamma}(T)(|q|^k+|p|^k+1)\epsilon^{\frac{3k}{2}} + c_k\,\epsilon^{\frac{k}{2}}\,t^{\frac{k}{2}}e^{-\frac{k\alpha_0 t}{\epsilon^2}} + \epsilon^{\frac{3k}{2}}c_{k,\gamma}(T)\sup_{s\leq t}\mathbb{E}\,|H_\epsilon(s)|^k.
\end{aligned}
$$

Therefore, if we pick $\epsilon_0 \in (0,1]$ such that

$$
\epsilon_0^{\frac{3k}{2}}c_{k,\gamma}(T) < \frac{1}{2},
$$

we get (3.4).

Now, let us prove (3.5). From (3.11), we have

$$
\begin{aligned}
|H_\epsilon(t)| &\leq \sqrt{\epsilon}\,c\sup_{t\in[0,T]}|w(t)|\left(1+\int_0^t e^{-\frac{\alpha_0(t-2)}{\epsilon^2}}|p^u_\epsilon(s)|\,ds\right)\\
&\leq \sqrt{\epsilon}\,c\sup_{t\in[0,T]}|w(t)|\left(1+\epsilon\left(\int_0^t |p^u_\epsilon(s)|^2\,ds\right)^{\frac{1}{2}}\right),
\end{aligned}
$$

and hence

$$
\mathbb{E}\sup_{t\in[0,T]}|H_\epsilon(t)| \leq \sqrt{\epsilon}\,c(T)\left(1+\epsilon\left(\mathbb{E}\int_0^T |p^u_\epsilon(s)|^2\,ds\right)^{\frac{1}{2}}\right).
$$

Thanks to (3.9), as a consequence of the Young inequality, we get

$$
\int_0^T |p^u_\epsilon(s)|^2\,ds \leq c_\gamma(T)(1+|q|^2+|p|^2)+\frac{1}{\epsilon^4}\,c(T)\int_0^T |H_\epsilon(s)|^2\,ds, \tag{3.12}
$$

so that

$$\mathbb{E}\sup_{t\in[0,T]}|H_\epsilon(t)| \leq \sqrt{\epsilon}\,c_\gamma(T)(1+|q|+|p|)+\frac{1}{\sqrt{\epsilon}}\,c(T)\left(\int_0^t \mathbb{E}\,|H_\epsilon(s)|^2\,ds\right)^{\frac{1}{2}}.$$

Therefore, (3.5) follows from (3.4). □

Lemma 3.2 *Under Hypothesis 1, for any $T>0$, $k\geq 1$ and $\gamma>0$ there exists $\epsilon_0>0$ such that for any $u\in S_T^\gamma$ and $\epsilon\in(0,\epsilon_0)$ we have*

$$\mathbb{E}\sup_{t\in[0,T]}|q_\epsilon^u(t)|^k \leq c_{k,\gamma}(T)(|q|^k+|p|^k+1)\,\epsilon^{-\frac{k}{2}}+c_{k,\gamma}(T)\epsilon^{2-\frac{3k}{2}}. \tag{3.13}$$

Proof Estimate (3.13) follows by combining together (3.4) and (3.8). □

Now, we are ready to prove (3.2), that, in view of Theorem 2.2, implies Theorem 2.1.

Theorem 3.3 *Let $\{u_\epsilon\}_{\epsilon>0}$ be a family of predictable processes in $\mathcal{S}_T^\gamma$ that converge $\mathbb{P}$-a.s., as $\epsilon\downarrow 0$, to some $u\in\mathcal{S}_T^\gamma$, with respect to the weak topology of $L^2(0,T;\mathbb{R}^d)$. Then, we have*

$$\lim_{\epsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}|q_\epsilon^{u_\epsilon}(t)-g^u(t)|=0. \tag{3.14}$$

Proof Integrating by parts in (3.7), we obtain

$$q_\epsilon^{u_\epsilon}(t)=q+\int_0^t\frac{b(q_\epsilon^{u_\epsilon}(s))}{\alpha(q_\epsilon^{u_\epsilon}(s))}\,ds+\int_0^t\frac{\sigma(q_\epsilon^{u_\epsilon}(s))}{\alpha(q_\epsilon^{u_\epsilon}(s))}u_\epsilon(s)\,ds+R_\epsilon(t),$$

where

$$\begin{aligned}R_\epsilon(t)&=\frac{p}{\epsilon}\int_0^t e^{-A_\epsilon(s)}\,ds-\frac{1}{\alpha(q_\epsilon^{u_\epsilon}(t))}\int_0^t e^{-A_\epsilon(t,s)}b(q_\epsilon^{u_\epsilon}(s))\,ds+\sqrt{\epsilon}\int_0^t\frac{\sigma(q_\epsilon^{u_\epsilon}(s))}{\alpha(q_\epsilon^{u_\epsilon}(s))}\,dw(s)\\&\quad+\int_0^t\left(\int_0^s e^{-A_\epsilon(s,r)}b(q_\epsilon^{u_\epsilon}(r))\,dr\right)\frac{1}{\alpha^2(q_\epsilon^{u_\epsilon}(s))}\left\langle\nabla\alpha(q_\epsilon^{u_\epsilon}(s)),p_\epsilon^{u_\epsilon}(s)\right\rangle ds\\&\quad-\frac{1}{\alpha(q_\epsilon^{u_\epsilon}(t))}H_\epsilon(t)+\int_0^t\frac{1}{\alpha^2(q_\epsilon^{u_\epsilon}(s))}H_\epsilon(s)\left\langle\nabla\alpha(q_\epsilon^{u_\epsilon}(s)),p_\epsilon^{u_\epsilon}(s)\right\rangle ds=:\sum_{k=1}^6 I_\epsilon^k(t).\end{aligned}$$

This implies that

$$\begin{aligned}q_\epsilon^{u_\epsilon}(t)-g^u(t)&=\int_0^t\left[\frac{b(q_\epsilon^{u_\epsilon}(s))}{\alpha(q_\epsilon^{u_\epsilon}(s))}-\frac{b(g^u(s))}{\alpha(g^u(s))}\right]ds+\int_0^t\left[\frac{\sigma(q_\epsilon^{u_\epsilon}(s))}{\alpha(q_\epsilon^{u_\epsilon}(s))}-\frac{\sigma(g^u(s))}{\alpha(g^u(s))}\right]u_\epsilon(s)\,ds\\&\quad+\int_0^t\frac{\sigma(g^u(s))}{\alpha(g^u(s))}\left[u_\epsilon(s)-u(s)\right]ds+R_\epsilon(t).\end{aligned} \tag{3.15}$$

Due to the Lipschitz-continuity and the boundedness of the functions σ and $1/\alpha$, we have that σ/α is bounded and Lipschitz continuous. Then, as $u_\epsilon, u\in\mathcal{S}_T^\gamma$, we obtain

$$\begin{aligned}&|q_\epsilon^{u_\epsilon}(t)-g^u(t)|^2\\&\leq c\left|\int_0^t\frac{\sigma(g^u(s))}{\alpha(g^u(s))}\left[u_\epsilon(s)-u(s)\right]ds\right|^2+c\,|R_\epsilon(t)|^2+c(T)\int_0^t|q_\epsilon^{u_\epsilon}(s)-g^u(s)|^2\,ds\\&\quad+c(T)\int_0^t|q_\epsilon^{u_\epsilon}(s)-g^u(s)|^2\,ds\left(\int_0^t|u_\epsilon(s)|^2\,ds+\sup_{s\in[0,t]}|g^u(s)|^2\right)\\&\leq c\left|\int_0^t\frac{\sigma(g^u(s))}{\alpha(g^u(s))}\left[u_\epsilon(s)-u(s)\right]ds\right|^2+c\,|R_\epsilon(t)|^2+c_\gamma(T)\int_0^t|q_\epsilon^{u_\epsilon}(s)-g^u(s)|^2\,ds.\end{aligned}$$

By the Gronwall lemma, this allows to conclude that

$$\sup_{t\in[0,T]} |q_\epsilon^{u_\epsilon}(t) - g^u(t)| \leq c_\gamma(T) \sup_{t\in[0,T]} \left| \int_0^t \frac{\sigma(g^u(s))}{\alpha(g^u(s))} [u_\epsilon(s) - u(s)]\, ds \right| + c_\gamma(T) \sup_{t\in[0,T]} |R_\epsilon(t)|. \qquad (3.16)$$

Now, for any $\epsilon > 0$, we define

$$\Gamma_\epsilon(t) = \int_0^t \frac{\sigma(g^u(s))}{\alpha(g^u(s))} [u_\epsilon(s) - u(s)]\, ds.$$

For any $0 < s < t$ we have

$$\Gamma_\epsilon(t) - \Gamma_\epsilon(s) = \int_s^t \frac{\sigma(g^u(r))}{\alpha(g^u(r))} [u_\epsilon(r) - u(r)]\, dr,$$

so that, as u_ϵ and u are both in S_T^γ,

$$|\Gamma_\epsilon(t) - \Gamma_\epsilon(s)| \leq c_\gamma \sqrt{t-s}, \quad \epsilon > 0.$$

As $\Gamma_\epsilon(0) = 0$, this implies that the family of continuous functions $\{\Gamma_\epsilon\}_{\epsilon>0}$ is equibounded and equicontinuous, so that, by the Ascoli-Arzelà theorem, there exists $\epsilon_n \downarrow 0$ and $v \in C([0,T];\mathbb{R}^d)$ such that

$$\lim_{n\to 0} \sup_{t\in[0,T]} |\Gamma_{\epsilon_n}(t) - v(t)| = 0, \quad \mathbb{P}-\text{a.s.}$$

On the other hand, as (3.1) holds, for any $h \in \mathbb{R}^d$ we have

$$\lim_{\epsilon\to 0} \langle \Gamma_\epsilon(t), h\rangle = \lim_{\epsilon\to 0} \left\langle u_\epsilon - u, \frac{\sigma(g^u(\cdot))}{\alpha(g^u(\cdot))} h \right\rangle_{L^2(0,T;\mathbb{R}^d)} = 0,$$

so that we can conclude that $v = 0$ and

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} |\Gamma_\epsilon(t)| = 0.$$

Thanks to (3.16), this implies that

$$\limsup_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} |q_\epsilon^{u_\epsilon}(t) - g^u(t)| \leq c \limsup_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} |R_\epsilon(t)|,$$

so that (3.14) follows if we show that

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} |R_\epsilon(t)| = 0. \qquad (3.17)$$

Thus, we have

$$|I_\epsilon^1(t)| = \frac{|p|}{\epsilon} \left| \int_0^t e^{-A_\epsilon(s)}\, ds \right| \leq c\,|p|\,\epsilon^{-1} \int_0^t e^{-\frac{\alpha_0 s}{\epsilon^2}}\, ds \leq c\,|p|\,\epsilon. \qquad (3.18)$$

Moreover

$$|I_\epsilon^2(t)| = \frac{1}{|\alpha(q_\epsilon^{u_\epsilon}(t))|} \left| \int_0^t e^{-A_\epsilon(t,s)} b(q_\epsilon^{u_\epsilon}(s))\, ds \right| \leq c \int_0^t e^{-\frac{\alpha_0(t-s)}{\epsilon^2}} (1 + |q_\epsilon^{u_\epsilon}(s)|)\, ds \leq c\,\epsilon^2 \left(1 + \sup_{t\in[0,T]} |q_\epsilon^{u_\epsilon}(t)|\right).$$

Thanks to (3.13), this implies

$$\mathbb{E} \sup_{t\in[0,T]} |I_\epsilon^2(t)| \leq c_\gamma(T)(|p|+|q|+1)\epsilon^{\frac{3}{2}}, \quad \epsilon \in (0,1]. \tag{3.19}$$

Next

$$\mathbb{E} \sup_{t\in[0,T]} |I_\epsilon^3(t)| = \sqrt{\epsilon}\,\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t \frac{\sigma(q_\epsilon^{u_\epsilon}(s))}{\alpha(q_\epsilon^{u_\epsilon}(s))}\, dw(s) \right| \leq c(T)\sqrt{\epsilon}. \tag{3.20}$$

Concerning $I^4(t)$, we have

$$\begin{aligned} |I_\epsilon^4(t)| &= \left| \int_0^t \left(\int_0^s e^{-A_\epsilon(s,r)} b(q_\epsilon^{u_\epsilon}(r))\, dr \right) \frac{1}{\alpha^2(q_\epsilon^{u_\epsilon}(s))} \left\langle \nabla\alpha(q_\epsilon^{u_\epsilon}(s)), p_\epsilon^{u_\epsilon}(s) \right\rangle ds \right| \\ &\leq \epsilon^2 c \left(1 + \sup_{t\in[0,T]} |q_\epsilon^{u_\epsilon}(t)| \right) \int_0^t |p_\epsilon^{u_\epsilon}(s)|\, ds, \end{aligned}$$

so that, due to (3.13) we obtain

$$\mathbb{E} \sup_{t\in[0,T]} |I_\epsilon^4(t)| \leq \epsilon^2 c_\gamma(T)(|q|+|p|+1)\,\epsilon^{-\frac{1}{2}} \left(\mathbb{E} \int_0^t |p_\epsilon^{u_\epsilon}(s)|^2\, ds \right)^{\frac{1}{2}}.$$

As a consequence of (3.4) and (3.12), this yields

$$\mathbb{E} \sup_{t\in[0,T]} |I_\epsilon^4(t)| \leq \epsilon\, c_\gamma(T)(|q|^2+|p|^2+1), \quad \epsilon \in (0,\epsilon_0]. \tag{3.21}$$

Concerning $I_\epsilon^5(t)$, according to (3.5) we have

$$\mathbb{E} \sup_{t\in[0,T]} |I_\epsilon^5(t)| \leq c\,\mathbb{E} \sup_{t\in[0,T]} |H_\epsilon(t)| \leq \sqrt{\epsilon}\, c_\gamma(T)(1+|q|+|p|). \tag{3.22}$$

Finally, it remains to estimate $I_\epsilon^6(t)$. We have

$$|I_\epsilon^6(t)| = \left| \int_0^t \frac{1}{\alpha^2(q_\epsilon^{u_\epsilon}(s))} H_\epsilon(s) \left\langle \nabla\alpha(q_\epsilon^{u_\epsilon}(s)), p_\epsilon^{u_\epsilon}(s) \right\rangle ds \right| \leq c \int_0^t |H_\epsilon(s)||p_\epsilon^{u_\epsilon}(s)|\, ds,$$

so that

$$\mathbb{E} \sup_{t\in[0,T]} |I_\epsilon^6(t)| \leq c \left(\int_0^T \mathbb{E}|H_\epsilon(s)|^2\, ds \int_0^T \mathbb{E}|p_\epsilon^{u_\epsilon}(s)|^2\, ds \right)^{\frac{1}{2}}.$$

By using (3.12), this gives

$$\mathbb{E} \sup_{t\in[0,T]} |I_\epsilon^6(t)| \leq c_\gamma(T)(1+|q|+|p|) \left(\int_0^T \mathbb{E}|H_\epsilon(s)|^2\, ds \right)^{\frac{1}{2}} + \frac{c(T)}{\epsilon^2} \int_0^T \mathbb{E}|H_\epsilon(s)|^2\, ds,$$

so that, from (3.4) we get

$$\mathbb{E} \sup_{t\in[0,T]} |I_\epsilon^6(t)| \leq \epsilon\, c_\gamma(T)(1+|q|+|p|), \quad \epsilon \in (0,\epsilon_0].$$

This, together with (3.18), (3.19), (3.20), (3.21) and (3.22), implies (3.17) and (3.14) follows. □

4 Some Applications and Remarks

Let G be a bounded domain in $\mathbb{R}^d$, with a smooth boundary ∂G. We consider here the exit problem for the process $q^\epsilon(t)$ defined as the solution of Eq. (1.1). For every $\epsilon > 0$, we define

$$\tau^\epsilon := \min\{t \geq 0 \,:\, q^\epsilon(t) \notin G\}, \qquad \tau_\epsilon := \min\{t \geq 0 \,:\, q_\epsilon(t) \notin G\},$$

where $q_\epsilon(t) = q^\epsilon(t/\epsilon)$ is the solution of Eq. (2.6). It is clear that

$$\tau^\epsilon = \frac{1}{\epsilon}\tau_\epsilon, \qquad q^\epsilon(\tau^\epsilon) = q_\epsilon(\tau_\epsilon).$$

In what follows, we shall assume that the dynamical system

$$\dot{q}(t) = b(q(t)), \qquad t \geq 0, \tag{4.1}$$

satisfies the following conditions.

Hypothesis 2 The point $O \in G$ is asymptotically stable for the dynamical system (4.1) and for any initial condition $q \in \mathbb{R}^d$

$$\lim_{t\to\infty} q(t) = O.$$

Moreover, we have

$$\langle b(q), \nu(q)\rangle > 0, \qquad q \in \partial G,$$

where $\nu(q)$ is the inward normal vector at $q \in \partial G$.

Now, we introduce the quasi-potential associated with the action functional I defined in (2.4)

$$\begin{aligned} V(q) &= \inf \left\{ I(f),\ f \in C([0,T];\mathbb{R}^d),\ f(0) = O,\ f(T) = q,\ T > 0 \right\} \\ &= \frac{1}{2}\inf \left\{ \int_0^T \left| \alpha(f(s))\sigma^{-1}(f(s)) \left(\dot{f}(s) - \frac{b(f(s))}{\alpha(f(s))} \right) \right|^2 ds, \right. \\ &\qquad \left. f(0) = O,\ f(T) = q,\ T > 0 \right\}. \end{aligned}$$

It is easy to check that, under our assumptions on $\alpha(q)$, the quasi-potential V coincides with

$$\frac{1}{2}\inf \left\{ \int_0^T \left| \sigma^{-1}(f(s)) \left(\dot{f}(s) - \alpha(f(s))b(f(s)) \right) \right|^2 ds,\ f(0) = O,\ f(T) = q,\ T > 0 \right\}. \tag{4.2}$$

Theorem 4.1 *Under Hypotheses 1 and 2, for each $q \in \{q \in G \,:\, V(q) \leq V_0\}$ and $p \in \mathbb{R}^d$, we have*

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}_{(q,p)}\tau^\epsilon = \lim_{\epsilon\to 0} \epsilon \log \mathbb{E}_{(q,p)}\tau_\epsilon = V_0, \tag{4.3}$$

and

$$\lim_{\epsilon\to 0} \epsilon \log \tau^\epsilon = \lim_{\epsilon\to 0} \epsilon \log \tau_\epsilon = V_0, \quad \textit{in probability}, \tag{4.4}$$

where

$$V_0 := \min_{q\in\partial G} V(q).$$

Moreover, if the minimum of V on ∂G is achieved at a unique point $q^\star \in \partial G$, then

$$\lim_{\epsilon\to 0} q^\epsilon(\tau^\epsilon) = \lim_{\epsilon\to 0} q_\epsilon(\tau_\epsilon) = q^\star. \tag{4.5}$$

Proof First, note that $q_\epsilon(t)$ is the first component of the $2d$-dimensional Markov process $z_\epsilon(t) = (q_\epsilon(t), p_\epsilon(t))$. Because of the structure of the p-component of the drift of this process and our assumptions on the vector field b, starting from $(q, p) \in \mathbb{R}^{2d}$, the trajectory of $z_\epsilon(t)$ spends most of the time in a small neighborhood of the point $q = O$ and $p = 0$, with probability close to 1, as $0 < \epsilon << 1$. From time to time, the process $z_\epsilon(t)$ deviates from this point and, as proven in Theorem 2.1, the deviations of $q_\epsilon(t)$ are governed by the large deviation principle with action functional I, defined in (2.4). This allows to prove the validity of (4.3), (4.4) and (4.5) in the same way as Theorems 4.41, 4.42 and 4.2.1 from [11] are proven. We omit the details. □

As an immediate consequence of (4.2) and [11, Theorem 4.3.1], we have the following result.

Theorem 4.2 *Assume $a(q) := \sigma(q)\sigma^\star(q) = I$ and $\alpha(q)b(q) = -\nabla U(q) + l(q)$, for any $q \in \mathbb{R}^d$, for some smooth function $U : \mathbb{R}^d \to \mathbb{R}$ having a unique critical point (a minimum) at $O \in \mathbb{R}^d$ and such that*

$$\langle \nabla U(q), l(q)\rangle = 0, \quad q \in \mathbb{R}^d.$$

Then

$$V(q) = 2U(q), \quad q \in \mathbb{R}^d.$$

From Theorems 4.1 and 4.2, it is possible to get a number of results concerning the asymptotic behavior, as $\epsilon \downarrow 0$, of the solutions of the degenerate parabolic and the elliptic problems associated with the differential operator $\mathcal{L}^\epsilon$ defined by

$$\mathcal{L}^\epsilon u(q, p) = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(q)\frac{\partial^2 u}{\partial p_i \partial p_j}(q, p) + \left(b(q) - \frac{1}{\epsilon}\alpha(q)p\right)\cdot \nabla_p u(q, p) \\ + p \cdot \nabla_q u(q, p).$$

Assume now that the dynamical system (4.1) has several asymptotically stable attractors. Assume, for the sake of brevity, that all attractors are just stable equilibriums $O_1, O_2,\dots,O_l$. Denote by $\mathcal{E}$ the set of separatrices separating the basins of these attractors, and assume the set $\mathcal{E}$ to have dimension strictly less than d. Moreover, let each trajectory $q(t)$, starting at $q_0 \in \mathbb{R}^d \setminus \mathcal{E}$, be attracted to one of the stable equilibriums O_i, $i = 1,\dots,l$, as $t \to \infty$. Finally, assume that the projection of $b(q)$ on the radius connecting the origin in $\mathbb{R}^d$ and the point $q \in \mathbb{R}^d$ is directed to the origin and its length is bounded from below by some uniform constant $\theta > 0$ (this condition provides the positive recurrence of the process $z_\epsilon(t) = (q_\epsilon(t), p_\epsilon(t)), t \geq 0$).

In what follows, we shall denote

$$V(q_1, q_2) = \frac{1}{2}\inf \int_0^T \left|\alpha(f(s))\sigma^{-1}(f(s))\left(\dot f(s) - \frac{b(f(s))}{\alpha(f(s))}\right)\right|^2 ds, \\ f(0) = q_1,\ f(T) = q_2,\ T > 0\}$$

and

$$V_{ij} = V(O_i, O_j), \quad i, j \in \{1, \ldots, l\}.$$

In a generic case, the behavior of the process $(q^\epsilon(t), p^\epsilon(t))$, on time intervals of order $\exp(\lambda\epsilon^{-1})$, $\lambda > 0$ and $0 < \epsilon << 1$, can be described by a hierarchy of cycles as in [6,11]. The cycles are defined by the numbers V_{ij}. For (almost) each initial point q and a time scale λ, these numbers define also the metastable state $O_{i^\star}$, $i^\star = i^\star(q, \lambda)$, where $q^\epsilon(t)$ spends most of the time during the time interval $[0, \exp(\lambda\epsilon^{-1})]$. Slow changes of the field $b(q)$ and/or of the damping coefficient $\alpha(q)$ can lead to stochastic resonance (compare with [7]).

Consider next the reaction diffusion equation in $\mathbb{R}^d$

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, q) = \mathcal{L}u(t, q) + c(q, u(t, q))u(t, q), \\ u(0, q) = g(q), \quad q \in \mathbb{R}^d, \quad t > 0. \end{cases} \tag{4.6}$$

Here $\mathcal{L}$ is a linear second order uniformly elliptic operator, with regular enough coefficients. Let $q(t)$ be the diffusion process in $\mathbb{R}^d$ associated with the operator $\mathcal{L}$. The Feynman–Kac formula says that u can be seen as the solution of the problem

$$u(t, q) = \mathbb{E}_q\, g(q(t)) \exp \int_0^t c(q(s), u(t-s, q(s)))\, ds. \tag{4.7}$$

Reaction–diffusion equations describe the interaction between particle transport defined by $q(t)$ and reaction which consists of multiplication (if $c(q, u) > 0$) and annihilation (if $c(q, u) < 0$) of particles. In classical reaction–diffusion equations, the Langevin dynamics which describes a diffusion with inertia is replaced by its vanishing mass approximation. If the transport is described by the Langevin dynamics itself, Eq. (4.6) should be replaced by an equation in $\mathbb{R}^{2d}$. Assuming that the drift is equal to zero ($b(q) = 0$), and the damping is of order ϵ^{-1}, as $\epsilon \downarrow 0$, this equation has the form

$$\begin{cases} \dfrac{\partial u^\epsilon}{\partial t}(t, q, p) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^d a_{i,j}(q)\frac{\partial^2 u^\epsilon}{\partial p_i \partial p_j}(q, p) - \frac{1}{\epsilon}\alpha(q)p \cdot \nabla_p u^\epsilon(q, p) + p \cdot \nabla_q u^\epsilon(q, p) \\ \qquad\qquad + c(q, u^\epsilon(t, q, p))u^\epsilon(t, q, p), \quad t > 0, \quad (q, p) \in \mathbb{R}^{2d}, \\ u^\epsilon(0, q, p) = g(q) \geq 0, \quad (q, p) \in \mathbb{R}^{2d}. \end{cases} \tag{4.8}$$

Now, we define

$$R(t, q) = \sup\left\{\int_0^t c(f(s), 0)\, ds - I_t(f) \;:\; f(0) = q,\; f(t) \in G_0\right\},$$

where

$$I_t(f) = \frac{1}{2}\int_0^t \alpha^2(f(s))a^{-1}(f(s))\dot{f}(s) \cdot \dot{f}(s)\, ds,$$

and $G_0 = \text{supp}\{g(q),\; q \in \mathbb{R}^d\}$.

Definition 4.3 1. We say that Condition (N) is satisfied if $R(t, x)$ can be characterized, for any $t > 0$ and $x \in \Sigma_t = \{q \in \mathbb{R}^d,\; R(t, q) = 0\}$, as

$$\sup\left\{\int_0^t c(f(s), 0)\, ds - I_t(f),\; f(0) = q,\; f(t) \in G_0,\; R(t-s, f(s)) \leq 0,\; 0 \leq s \leq t\right\}.$$

2. We say that the non-linear term $f(q,u) = c(q,u)u$ in Eq. (4.8) is of Kolmogorov–Petrovskii–Piskunov (KPP) type if $c(q,u)$ is Lipschitz-continuous, $c(q,0) \geq c(q,u) > 0$, for any $0 < u < 1$, $c(q,1) = 0$ and $c(q,u) < 0$, for any $u > 1$.

Theorem 4.4 *Let the non-linear term in* (4.8) *be of KPP type. Assume that Condition (N) is satisfied and assume that the closure of* $G_0 = \mathrm{supp}\{g(q),\ q \in \mathbb{R}^d\}$ *coincides with the closure of the interior of* G_0*. Then,*

$$\lim_{\epsilon\to 0} u^\epsilon(t/\epsilon, q, p) = 0, \quad \textit{if } R(t,q) < 0, \tag{4.9}$$

and

$$\lim_{\epsilon\to 0} u^\epsilon(t/\epsilon, q, p) = 1, \quad \textit{if } R(t,q) > 0, \tag{4.10}$$

so that equation $R(t,q) = 0$ *in* $\mathbb{R}^{2d}$ *defines the interface separating the area where* u^ϵ*, the solution of* (4.8)*, is close to* 1 *and to* 0*, as* $\epsilon \downarrow 0$.

Proof If we define $u_\epsilon(t,q,p) = u^\epsilon(t/\epsilon, q, p)$, the analog of (4.7) yields

$$u_\epsilon(t,q,p) = \mathbb{E}_{(q,p)} g(q_\epsilon(t)) \exp\left(\frac{1}{\epsilon}\int_0^t c(q_\epsilon(s), u_\epsilon(t-s, q_\epsilon(s), p_\epsilon(s))\, ds\right), \tag{4.11}$$

where $z_\epsilon(t) = (q_\epsilon(s), p_\epsilon(s))$ is the solution to Eq. (2.6). By taking into account our assumptions on $c(q,u)$, we derive from (4.11)

$$u_\epsilon(t,q,p) \leq \mathbb{E}_{(q,p)}\, g(q_\epsilon(t)) \exp\left(\frac{1}{\epsilon}\int_0^t c(q_\epsilon(s), 0)\, ds\right).$$

Theorem 2.1 and the Laplace formula imply that the right hand side of the above inequality is logarithmically equivalent , as $\epsilon \downarrow 0$, to $\exp\left(\frac{1}{\epsilon} R(t,q)\right)$ and this implies (4.9).

In order to prove (4.10), first of all one should check that if $R(t,q) = 0$, then for each $\delta > 0$

$$u_\epsilon(t,q,p) \geq \exp\left(-\frac{1}{\epsilon}\delta\right), \tag{4.12}$$

when $\epsilon > 0$ is small enough. This follows from (4.11) and Condition (N), if one takes into account the continuity of $c(q,u)$. The strong Markov property of the process $(q_\epsilon(t), p_\epsilon(t))$ and bound (4.12) imply (4.10) (compare with [5]). □

Consider, as an example, the case $c(q,0) = c = \text{const}$. Then

$$R(t,q) = ct - \inf\{I_t(f),\ f(0) = q,\ f(t) \in G_0\}.$$

The infimum in the equality above coincides with

$$\frac{1}{2t}\rho^2(q, G_0),$$

(see, for instance, [5] for a proof), where $\rho(q_1,q_2)$, $q_1, q_2 \in \mathbb{R}^d$, is the distance in the Riemaniann metric

$$ds = \alpha(q)\sqrt{\sum_{i,j=1}^d a_{i,j}(q)dq_i\, dq_j}.$$

This implies that the interface moves according to the Huygens principle with the constant speed $\sqrt{2c}$, if calculated in the Riemannian metric ds.

If $\alpha(q) = 0$ in a domain $G_1 \subset \mathbb{R}^d$, the points of G_1 should be identified. The Riemaniann metric in $\mathbb{R}^d$ induces now, in a natural way, a new metric $\tilde{\rho}$ in this space with identified points. The motion of the interface, in this case, can be described by the Huygens principle with constant velocity $\sqrt{2c}$ in the metric $\tilde{\rho}$.

If $c(q, 0)$ is not constant, the motion of the interface, in general, cannot be described by a Huygens principle. Actually, the motion can have jumps and other specific features (compare with [5]).

Finally, if the Condition (N) is not satisfied, the function $R(t, q)$ should be replaced by another one. Define

$$\tilde{R}(t, q) = \sup \left\{ \min_{0 \leq a \leq t} \left(\int_0^a c(f(s), 0)\, ds - I_a(f) \right) \, : \, f(0) = q, \ f(t) \in G_0 \right\}.$$

The function $\tilde{R}(t, q)$ is Lipschitz continuous and non-positive and if Condition (N) is satisfied, then

$$\tilde{R}(t, q) = \min \{R(t, q), 0\}.$$

By proceeding as in [6], it is possible to prove that

$$\lim_{\epsilon \to 0} u^\epsilon(t/\epsilon, q, p) = 0, \quad \text{if } R(t, q) < 0,$$

and

$$\lim_{\epsilon \to 0} u^\epsilon(t/\epsilon, q, p) = 1,$$

if (t, q) is in the interior of the set $\{(t, q) \, : \, t > 0, \ q \in \mathbb{R}^d, \ \tilde{R}(t, q) = 0\}$.

Finally, we would like to mention a few generalizations.

1. The arguments that we we have used in the proof of Theorem 2.1, can be used to prove the same result for the equation

$$\begin{cases} \ddot{q}^{\,\epsilon}(t) = b(q^\epsilon(t)) - \dfrac{\alpha(q^\epsilon(t))}{\epsilon} \dot{q}^{\,\epsilon}(t) + \dfrac{1}{\epsilon^\beta} \sigma(q^\epsilon(t)) \dot{B}(t), \\ q^\epsilon(0) = q \in \mathbb{R}^d, \quad \dot{q}^\epsilon(0) = p \in \mathbb{R}^d, \end{cases}$$

   for any $\beta < 1/2$. As a matter of fact, with the very same method we can show that also in this case the family $\{q_\epsilon\}_{\epsilon > 0}$ satisfies a large deviation principle in $C([0, T]; \mathbb{R}^d)$ with action functional I and with normalizing factor $\epsilon^{1-2\beta}$.
2. The damping can be assumed to be anisotropic. This means that the coefficient $\alpha(q)$ can be replaced by a matrix $\alpha(q)$, with all eigenvalues having negative real part.
3. Systems with strong *non-linear damping* can be considered. Namely, let (q^ϵ, p^ϵ) be the time-inhomogeneous Markov process corresponding to the following initial-boundary value problem for a degenerate quasi-linear equation on a bounded regular domain $G \subset \mathbb{R}^d$

$$\begin{cases} \dfrac{\partial u^\epsilon(t, p, q)}{\partial t} = \dfrac{1}{2} \displaystyle\sum_{i,j=1}^d a_{i,j}(q) \dfrac{\partial^2 u^\epsilon(t, q, p)}{\partial p_i \partial p_j} + b(q) \cdot \nabla_p u^\epsilon(t, q, p) \\ \qquad\qquad - \dfrac{\alpha(q, u^\epsilon(t, q, p))}{\epsilon} p \cdot \nabla p u^\epsilon(t, q, p) + p \cdot \nabla_q u^\epsilon(t, q, p). \\ u^\epsilon(0, q, p) = g(q), \quad u^\epsilon(t, q, p)_{|_{q \in \partial G}} = \psi(q), \end{cases}$$

Existence and uniqueness of such degenerate problem, under some mild conditions, follows from [4, Chap. 5]. The non-linearity of the damping leads to some pecularities in the exit problem and in metastability. In particular, in the generic case, metastable distributions can be distributions among several asymptotic attractors and the limiting exit distributions may have a density (see [10]).

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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