March 26, 2014

Borrower: RAPID:UMC

Lending String:

Patron:

Journal Title: Communications in applied analysis.
ISSN: 1083-2564
Volume: 2 **Issue:** 2
Month/Year: 1998 **Pages:** 271-297

Article Author: Cerrai, Sandra

Article Title: Kolmogorov equations in Hilbert spaces with nonsmooth coefficients

Imprint:

ILL Number: -7763623

Call #: ARC QA299.6 .C65 v. 2 1998

Location:

Charge
Maxcost:

Shipping Address:
NEW: Main Library

Odyssey: 128.8.66.142

Communications in Applied Analysis **2** (1998), no. 2, 271–297

KOLMOGOROV EQUATIONS IN HILBERT SPACES WITH NON SMOOTH COEFFICIENTS

Sandra Cerrai
Università degli Studi di Firenze, Dimadefas
Via Lombroso 6/17, 50134 Firenze, Italy

Communicated by G. Da Prato

ABSTRACT: We prove existence, uniqueness and optimal regularity of solutions to a class of linear parabolic and elliptic equations in Hilbert spaces. The coefficients of the first order derivatives are not necessarily differentiable and may be unbounded. The drift term is the Nemytskii operator associated to a nonlinear real function. We give a probabilistic representation of solutions and we use some preliminary result proved for the associated stochastic partial differential equation.

AMS (MOS) Subject Classification. 47D07, 60H15, 35K22, 35B45

1. INTRODUCTION

Let H denote the separable Hilbert space $L^2(0,1)$ (with norm $|\cdot|_H$ and inner product $<\cdot,\cdot>$) and let $A : D(A) \subset H \to H$ be the infinitesimal generator of a strongly continuous semigroup e^{tA}, $t \geq 0$,on H. Moreover, let F be the Nemytskii operator associated to a real function f having linear growth

$$F(x)(\xi) = f(x(\xi)), \quad \xi \in [0,1]. \tag{1.1}$$

We are here concerned with the second order linear differential operator

$$\mathcal{A}_0(x) = \frac{1}{2}\mathrm{Tr}\,[QD^2] + < Ax + F(x), D >_H, \quad x \in D(A), \tag{1.2}$$

where $Q = Q^\star$ is a nonnegative and non degenerate bounded linear operator on H. D denotes the gradient.

Our aim is to study existence and uniqueness of the solution and Schauder estimates for the parabolic and the elliptic infinite dimensional problems associated to the operator $\mathcal{A}_0(\cdot)$. This kind of problem has just been studied by Cannarsa and Da Prato [3] and Da Prato [8], in the case $F = 0$ and in the case of bounded linear perturbations ($|F| \leq K$) satisfying some regularity properties. In the present work we

Received March 20, 1997

consider the case of unbounded nonlinearities F, which are not Fréchet differentiable. Actually, we assume that $f \in C^3(\mathbb{R})$ and

$$\begin{cases} |f(t)| \leq C\,(1+|t|) \\ \sup\limits_{t\in\mathbb{R}} |f^{(j)}(t)| < +\infty, \quad j=1,2,3. \end{cases} \tag{1.3}$$

As well known, the operator $\mathcal{A}_0(\cdot)$ is related in a natural way to the stochastic partial differential equation

$$\begin{cases} dX_t = & (AX_t + F(X_t))\,dt + \sqrt{Q}dW_t \\ X_0 = & x. \end{cases} \tag{1.4}$$

Indeed, the solution process $X_t(x)$ to (1.4) is a diffusion on H with generator $\mathcal{A}_0(\cdot)$.

As (1.3) holds, then F is Lipschitz continuous and equation (1.4) admits an unique mild solution $X_t(x)$, $t \geq 0$, for any initial datum $x \in H$ (for a proof see e.g. Da Prato and Zabczyk [10]). If $\varphi \in B_b(H)$, the space of all borelian and bounded functions from H to $\mathbb{R}$, we define

$$P_t\varphi(x) = \mathbb{E}\left(\varphi\left(X_t(x)\right)\right), \quad t \geq 0, \quad x \in H.$$

P_t is the *transition semigroup* associated to equation (1.4). By means of P_t we can give a probabilistic representation for the solutions of the following infinite dimensional parabolic and elliptic problems

$$\begin{cases} u_t(t,x) = \mathcal{A}_0(x)u(t,x), & t>0, \quad x \in D(A) \\ u(0,x) = \varphi(x), & x \in H \end{cases} \tag{1.5}$$

and

$$\lambda\psi(x) - \mathcal{A}_0(x)\psi(x) = \varphi(x), \quad x \in D(A), \tag{1.6}$$

with $\lambda > 0$ and $\varphi \in C_b(H)$, the space of all uniformly continuous and bounded functions from H to $\mathbb{R}$. Actually, if we define

$$u(t,x) = P_t\varphi(x), \quad t \geq 0, \quad x \in H, \tag{1.7}$$

we are able to show that u is the unique classical solution of problem (1.5). In the same way, if we define

$$\psi(x) = \int_0^{+\infty} e^{-\lambda t} P_t\varphi(x)\,dt, \quad x \in H \tag{1.8}$$

(and this is meaningful, due to the fact that P_t is weakly continuous, see Cerrai [6] and [4] for the definition and main properties), we can verify that for any $\varphi \in C_b(H)$, ψ is the unique strong solution of problem (1.6). Moreover, as proved in Cerrai [6], for any $\varphi \in B_b(H)$ and $t > 0$

$$\|D^j\left(P_t\varphi\right)\|_\infty \leq C\,(t\wedge 1)^{-j/2}\,\|\varphi\|_\infty, \quad j=1,2,3. \tag{1.9}$$

Then, by using interpolation in spaces of functions of infinite variables (see Cannarsa and Da Prato [2]) and a general method due to Lunardi [18], we get Schauder estimates.

The function f is not assumed to be linear in general, so that F is not Fréchet differentiable. Then, in order to get differentiable dependence on initial data for the solution $X_t(x)$ of the problem (1.4), and hence smoothing property for P_t, we cannot proceed as in the literature, where F is assumed to be differentiable. Nevertheless in Cerrai [6], by assuming *ultracontractivity* for the semigroup e^{tA}, we proved that $X_t(x)$ is mean-square differentiable with respect to $x \in H$, up to the k-th order, for some k depending on the ultracontractivity constant of e^{tA}. Then we have been able to prove that the semigroup P_t has a regularizing effect, that is

$$\varphi \in B_b(H) \Rightarrow P_t\varphi \in C_b^k(H), \quad t > 0.$$

Therefore, by using the results proved in Cerrai [6], we first show that the function u defined by (1.7) is the unique classical solution of (1.5). To this purpose we first show that for any $t > 0$ and $x \in H$ the operator $u_{xx}(t,x)$ is trace-class and

$$\sup_{x \in H} |\mathrm{Tr}\,(u_{xx}(t,x))| \leq \rho(t)\|\varphi\|_\infty,$$

for a suitable $\rho :]0,+\infty[\to [0,+\infty[$ whose behaviour near 0 is studied. This follows if we assume that there exists a basis $\{e_k\}$ for H, such that $e_k \in L^4(0,1)$ and

$$\sum_{k=1}^{+\infty} |e^{tA}e_k|_4 \leq Ct^{-\beta},$$

for a certain $\beta < 3/4$. We remark that such a hypothesis is satisfied by the realization of the second order derivative with Dirichlet boundary condition. Existence and uniqueness of the solution of (1.5) are first proved for regular initial data ($\varphi \in C_b^2(H)$) and then, by an approximating procedure, for any $\varphi \in C_b(H)$ ($C_b^2(H)$ is not dense in $C_b(H)$, but it is possible to construct a bounded sequence in $C_b^2(H)$ pointwise convergent to φ, see Peszat and Zabczyk [20]).

Concerning the elliptic problem, we prove that for any $\varphi \in C_b(H)$ there exists a unique strong solution. If $\varphi \in C_b^1(H)$, we show that the function ψ which is given by (1.8) is the unique strict solution. To this purpose we have first to verify that ψ belongs to $C_b^2(H)$ and $D^2\psi(x)$ is a trace-class operator for any $x \in H$. Then we have to prove that the function

$$t \mapsto \mathrm{Tr}[QD^2(P_t\varphi)(x)], \quad x \in H$$

is well defined and integrable near 0. The general case of $\varphi \in C_b(H)$ follows, since $C_b^1(H)$ is dense in $C_b(H)$.

We conclude the paper by giving the Schauder estimates, both for the elliptic and the parabolic problem. They follow from (1.9) and a general method based on interpolation due to Lunardi (see Lunardi [18]).

Remark 1.1. We will denote by C (without any index) any positive constant appearing in inequalities, whose dependence on some parameters is not important. Such constants may change even in the same chain of inequalities. We will denote with C_p any positive constant whose dependence on a parameter p we want to emphasize.

2. NOTATIONS, HYPOTHESES AND PRELIMINARY RESULTS

Throughout the paper H will be the separable Hilbert space $L^2(0,1)$, with norm $|\cdot|_H$ and scalar product $\langle\cdot,\cdot\rangle$. We will denote the usual norm in $L^p(0,1)$ by $|\cdot|_p$, for any $p \geq 2$.

$\mathcal{L}(H)$ will denote the Banach algebra of all linear bounded operators from H to H endowed with the norm

$$\|T\|_{\mathcal{L}(H)} = \sup_{|x|_H \leq 1} |Tx|_H.$$

$\mathcal{L}^+(H)$ is the subspace of nonnegative bounded operators. Other subspaces of $\mathcal{L}(H)$ are $\mathcal{L}_1(H)$, the set of all trace-class operators, which is a Banach space endowed with the norm

$$\|T\|_{\mathcal{L}_1(H)} = \text{Tr}\,\sqrt{T^*T}$$

and $\mathcal{L}_2(H)$, the set of all Hilbert-Schmidt operators, which is a Banach space endowed with the norm

$$\|T\|_{\mathcal{L}_2(H)} = \sqrt{\text{Tr}\,[T^*T]}.$$

2.1. Functional spaces and interpolation theory

$B_b(H)$ is the Banach space of all Borelian and bounded functions from H to $\mathbb{R}$ whereas $C_b(H)$ is the space of all mappings from H to $\mathbb{R}$ which are uniformly continuous and bounded. $C_b(H)$, endowed with the norm

$$\|\varphi\|_\infty = \sup_{x\in H} |\varphi(x)|,$$

is a Banach space. We shall also consider several subspaces of $C_b(H)$.

1. $C_b^k(H)$, $k \geq 1$, is the subset of $C_b(H)$ of all functions φ which are k times Fréchet differentiable, with bounded and uniformly continuous derivatives up to the k-th order. If we define

   $$[\varphi]_h = \sup_{x\in H} |D^h\varphi(x)|, \quad h = 1,\dots,k,$$

   then $C_b^k(H)$ endowed with the norm

   $$\|\varphi\|_k = \|\varphi\|_\infty + \sum_{h=1}^{k} [\varphi]_h$$

   is a Banach space.

2. $C_b^{0,1}(H)$ is the subspace of $C_b(H)$ of all fuctions φ such that

$$[\varphi]_{0,1} = \sup_{\substack{x,y\in H\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|} < +\infty.$$

$C_b^{0,1}(H)$ is a Banach space endowed with the norm

$$\|\varphi\|_{0,1} = \|\varphi\|_\infty + [\varphi]_{0,1}.$$

3. $C_b^{k,1}(H)$, $k \geq 1$, is the subspace of $C_b^k(H)$ of all functions φ such that

$$[\varphi]_{k,1} = \sup_{\substack{x,y\in H\\ x\neq y}} \frac{|D^k\varphi(x)-D^k\varphi(y)|}{|x-y|} < +\infty.$$

$C_b^{k,1}$ is a Banach space with the norm

$$\|\varphi\|_{k,1} = \|\varphi\|_k + [\varphi]_{k,1}.$$

4. $C_b^\alpha(H)$, $\alpha \in]0,1[$, is the subspace of $C_b(H)$ of all functions φ such that

$$[\varphi]_\alpha = \sup_{\substack{x,y\in H\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha} < +\infty.$$

$C_b^\alpha(H)$ is a Banach space endowed with the norm

$$\|\varphi\|_\alpha = \|\varphi\|_\infty + [\varphi]_\alpha.$$

5. $C_b^{k+\alpha}(H)$, $k \geq 1$ and $\alpha \in]0,1[$, is the subspace of $C_b^k(H)$ of all functions φ such that

$$[\varphi]_{k+\alpha} = \sup_{\substack{x,y\in H\\ x\neq y}} \frac{|D^k\varphi(x)-D^k\varphi(y)|}{|x-y|^\alpha} < +\infty.$$

$C_b^{k+\alpha}(H)$ with the norm

$$\|\varphi\|_{k+\alpha} = \|\varphi\|_k + [\varphi]_{k+\alpha}$$

is a Banach space.

As well known, if $\dim H < \infty$ then the space $C_b^\infty(H)$ is dense in $C_b(H)$. In 1954, J. Kurtzweil [16] proved that if $\dim H = \infty$, then for any $\varphi \in C_b(H)$ there exists a sequence $\{\varphi_n\} \subset C^\infty(H) \cap C_b(H)$ converging to φ in $C_b(H)$. However, in 1973 A.S. Nemirovski and S.M. Semenov [19] proved that $C_b^2(H)$ is not dense in $C_b(H)$, whereas $C_b^{1,1}(H)$ is.

Now, we briefly recall the K-definition of interpolation spaces (for more details see Triebel [21]). Let X and Y be Banach spaces such that $Y \subset X$ with continuous embedding. For any $x \in X$ and $t \geq 0$ we define

$$K(t,x) = \inf\{\|a\|_X + t\|b\|_Y \ : \ x = a + b, \ a \in X, \ b \in Y\}.$$

For arbitrary $\theta \in [0,1]$ we set

$$[x]_{(X,Y)_{\theta,\infty}} = \sup_{t \in]0,1]} t^{-\theta} K(t,x)$$

and

$$(X,Y)_{\theta,\infty} = \{x \in X \ : \ [x]_{(X,Y)_{\theta,\infty}} < +\infty\}.$$

It is possible to show that $(X,Y)_{\theta,\infty}$, endowed with the norm

$$\|x\|_{(X,Y)_{\theta,\infty}} = \|x\|_X + [x]_{(X,Y)_{\theta,\infty}},$$

is a Banach space. Moreover, we can check that $x \in (X,Y)_{\theta,\infty}$ and $[x]_{(X,Y)_{\theta,\infty}} \leq L$ iff for all $t \in]0,1]$ there exist $a_t \in X$ and $b_t \in Y$ such that

$$t^{-\theta}\|a_t\|_X + t^{1-\theta}\|b_t\|_Y \leq L.$$

As an application of Reiteration Theorem (see e.g. Triebel [21]), we have the following useful interpolatory inclusions (for a proof see Da Prato [8] or Cannarsa and Da Prato [3]).

Proposition 2.1. *Let $\theta \in]0,1[$. We have*

$$\begin{aligned} &\left(C_b^\alpha(H), C_b^\beta(H)\right)_{\theta,\infty} = C_b^{\alpha+\theta(\beta-\alpha)}(H), && 0 \leq \alpha < \beta \leq 1 \\ &\left(C_b^\beta(H), C_b^{2+\beta}(H)\right)_{1-\frac{\beta}{2}(1-\theta),\infty} \subset C_b^{2+\theta\beta}(H), && 0 \leq \beta \leq 1. \end{aligned} \tag{2.1}$$

2.2. Main assumptions

In the sequel we shall assume

Hypothesis 2.2. $f : \mathbb{R} \to \mathbb{R}$ *is a C^3 function and*

$$\left\{ \begin{array}{ll} |f(t)| \leq C(1+|t|), & t \in \mathbb{R}, \\ \sup_{t\in\mathbb{R}} \left|f^{(j)}(t)\right| < +\infty, & j = 1,2,3. \end{array} \right.$$

Hypothesis 2.3. *1. The operator $Q \in \mathcal{L}^+(H)$ and there exists $\nu > 0$ such that*

$$\frac{1}{\nu} I \leq Q \leq \nu I. \tag{2.2}$$

2. $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous contraction semigroup e^{tA}, $t \geq 0$.

3. $e^{tA} \in \mathcal{L}_2(H)$, *for any* $t > 0$, *and we have*

$$\int_0^t Tr\left[e^{sA}Qe^{sA^*}\right] ds < +\infty, \quad t > 0. \tag{2.3}$$

Hypothesis 2.4. *1. For any* $p \geq 2$, e^{tA} *is a semigroup of contractions in* $L^p(0,1)$.

2. For any $t > 0$, e^{tA} *is bounded from* H *to* $L^\infty(0,1)$ *and there exists* $\alpha < \frac{1}{2}$ *such that*

$$|e^{tA}x|_\infty \leq Ct^{-\alpha}|x|_H, \quad x \in H. \tag{2.4}$$

In particular, from Hypothesis 2.4-2. it follows that for any $p \geq 2$ and $t > 0$

$$\left|e^{tA}x\right|_p \leq \left|e^{tA}x\right|_\infty^{\frac{p-2}{p}} \left|e^{tA}x\right|_H^{\frac{2}{p}} \leq C_p t^{-\frac{\alpha(p-2)}{p}}|x|_H. \tag{2.5}$$

Hypothesis 2.5. *There exists a complete orthonormal basis* $\{e_k\}$ *in* H *such that* $e_k \in L^4(0,1)$, *for any* $k \in \mathbb{N}$, *and*

$$\sum_{k=1}^{+\infty} |e^{tA}e_k|_4 \leq Ct^{-\beta}, \quad t > 0, \tag{2.6}$$

for a certain $\beta \in]0, 3/4[$.

Hypothesis 2.6. *There exists* $\epsilon > 0$ *such that*

$$\int_0^t s^{-2\epsilon} Tr[e^{sA}Qe^{sA^*}]\, ds < +\infty, \quad t > 0.$$

Remark 2.7. As shown in Cerrai [6], Hypotheses 2.2-2.4 are satisfied by the second derivative operator with Dirichlet boundary conditions

$$\begin{cases} D(A) = \{x \in H^2(0,1) : x(0) = x(1) = 0\}, \\ Ax(\xi) = x''(\xi), \quad \xi \in [0,1]. \end{cases} \tag{2.7}$$

Moreover, it is possible to show that the operator (2.7), enjoys also Hypotheses 2.5 and 2.6, For example, concerning Hypothesis 2.5, for any k we have $e_k \in C^\infty(0,1)$ and

$$\sup_{k \in \mathbb{N}} |e_k|_4 < +\infty.$$

Then

$$\sum_{k=1}^{+\infty} \left|e^{tA}e_k\right|_4 = \sum_{k=1}^{+\infty} e^{-\pi^2 k^2 t}|e_k|_4 \leq C \sum_{k=1}^{+\infty} e^{-\pi^2 k^2 t}.$$

In this case it is possible to verify that if we choose $\beta = 1/2$, then (2.6) holds true.

2.3. Preliminary results

We recall that the cylindrical Wiener process W_t is defined as

$$W_t = \sum_{k=1}^{+\infty} \beta_k(t) e_k, \tag{2.8}$$

where $\{e_k\}$ is a complete orthonormal system in H and $\{\beta_k\}$ is a sequence of mutually independent real Brownian motions defined on a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and adapted to the nonanticipative filtration $\mathcal{F}_t$. The series (2.8) defining W_t does not converge in H, but it is convergent in any Hilbert space U such that the embedding

$$H \subset U$$

is Hilbert-Schmidt (see Da Prato and Zabczyk [10], Chapter 4).

Definition 2.8. *A H-valued predictable process X is said to be a* mild *solution of problem (1.4) if*

$$X_t(x) = e^{tA}x + \int_0^t e^{(t-s)A} F(X_s(x))\, ds + W_t^A,$$

where W_t^A is the unique mild solution of the linear problem

$$\begin{cases} dX_t = & AX_t\, dt + \sqrt{Q} dW_t, \\ X_0 = & 0. \end{cases}$$

Due to Hypothesis 2.3, it can be shown that W_t^A, which is the so-called *stochastic convolution*, is a Gaussian process and is mean-square continuous with values in H (see Da Prato and Zabczyk [10]).

For any $\varphi \in B_b(H)$, we set

$$P_t\varphi(x) = \mathbb{E}\left(\varphi(X_t(x))\right), \quad x \in H \quad t \geq 0.$$

As known P_t is the *transition semigroup* associated to the problem (1.4). In Cerrai [6] we studied regularity properties of P_t. We proved that P_t is a *weakly continuous* semigroup on $C_b(H)$, and, as usual, we introduced its infinitesimal generator $\mathcal{A} : D(\mathcal{A}) \subset C_b(H) \to C_b(H)$ as the unique closed linear operator such that

$$R(\lambda, \mathcal{A})\varphi(x) = \int_0^{+\infty} e^{\lambda t} P_t\varphi(x)\, dt, \quad x \in H.$$

We also proved that P_t has a smoothing effect, that is

$$\varphi \in B_b(H) \Rightarrow P_t\varphi \in C_b^k(H), \quad t > 0, \tag{2.9}$$

for some $k \geq 3$. Moreover, the following estimates hold

$$\|D^j(P_t\varphi)\|_\infty \leq C(t \wedge 1)^{-j/2} \|\varphi\|_\infty, \quad t > 0. \tag{2.10}$$

In the present paper we want to show that from P_t we can get existence, uniqueness and optimal regularity for the parabolic and the elliptic infinite dimensional problem associated to the equation (1.4).

3. THE PARABOLIC PROBLEM

We shall consider the parabolic problem associated to the operator $\mathcal{A}_0(\cdot)$

$$\begin{cases} u_t(t,x) = \frac{1}{2}\mathrm{Tr}\,[Q u_{xx}(t,x)] + \langle Ax + F(x), u_x(t,x)\rangle\,, & x \in D(A), \quad t>0 \\ u(0,x) = \varphi(x), & x \in H. \end{cases} \tag{3.1}$$

Definition 3.1. *A continuous function $u : [0,+\infty[\to \mathbb{R}$ is said to be a* classical *solution to problem (3.1) if*

1. *for any $t > 0$, $u(t,\cdot) \in C_b^2(H)$ and for any $h,k \in H$ the function*

   $$]0,+\infty[\times H \to \mathbb{R}, \quad (t,x) \mapsto \langle Q u_{xx}(t,x)h, k\rangle$$

   is continuous;

2. *for any $t > 0$ and $x \in H$, the operator $u_{xx}(t,x) \in \mathcal{L}_1(H)$ and the function*

   $$]0,+\infty[\times H \to \mathbb{R}, \quad (t,x) \mapsto Tr\,[Q u_{xx}(t,x)]$$

   is continuous;

3. *for any $x \in D(A)$, the function $u(\cdot,x)$ is differentiable on $]0,+\infty[$;*

4. *function u satisfies equation (3.1).*

Definition 3.2. *A continuous function $u : [0,+\infty[\times H \to \mathbb{R}$ is said to be a* strict *solution to problem (3.1) if it satisfies conditions 1-4 of Definition 3.1 with $t > 0$ and $]0,+\infty[$ replaced respectively by $t \geq 0$ and $[0,+\infty[$.*

If $\varphi \in C_b^2(H)$ and $D^2\varphi(x) \in \mathcal{L}_1(H)$, for any $x \in H$, by using Itô formula and Propositions 3.1 and 3.4 of Cerrai [6], where differentiability of $X_t(x)$ with respect to $X \in H$ is proved, it is possible to show that the function

$$u : [0,+\infty[\times H \to \mathbb{R}, \quad (t,x) \mapsto P_t\varphi(x) \tag{3.2}$$

is a strict solution to problem (3.1) (for a proof see Appendix B). In this section we want to prove that for any $\varphi \in C_b(H)$ the same function u is the unique classical solution.

In Cerrai [6] we have proved that if $\varphi \in C_b(H)$, then $P_t\varphi \in C_b^2(H)$, for any $t > 0$. Our next step is to prove that operator $D^2(P_t\varphi)(x)$ belongs to $\mathcal{L}_1(H)$, for any $t > 0$ and $x \in H$.

3.1. Trace-class property of $D^2(P_t\varphi)$

Before proving that $D^2(P_t\varphi)(x) \in \mathcal{L}_1(H)$, for all $\varphi \in C_b(H)$ and $t > 0$, we need a preliminary result.

Lemma 3.3. *If $\{e_k\}$ is the orthonormal basis of H introduced in Hypothesis 2.5, we have*

$$\sup_{x\in H} \sum_{k=1}^{+\infty} |D_x X_t(x) e_k|_4 \leq \alpha(t), \quad t > 0, \tag{3.3}$$

where $\alpha :]0,+\infty[\to [0,+\infty[$ is a locally integrable function such that for any $T > 0$

$$\sup_{t\in (0,T]} t^{\beta}\alpha(t) < \infty. \tag{3.4}$$

Furthermore, the convergence is uniform for $x \in H$.

Proof. As proved in Cerrai [6], we have

$$D_x X_t(x) e_k = e^{tA} e_k + \int_0^t e^{(t-s)A} \left(DF(X_s(x)) D_x X_s(x) e_k \right) ds,$$

and as $e_k \in L^4(0,1)$, then $D_x X_t(x) e_k \in L^4(0,1)$, for any $t \geq 0$ and $x \in H$. Moreover,

$$|DF(X_t(x)) D_x X_t(x) e_k|_4 \leq C\, |D_x X_t(x) e_k|_4 ,$$

so that

$$|D_x X_t(x) e_k|_4 \leq |e^{tA} e_k|_4 + C \int_0^t |D_x X_s(x) e_k|_4 \, ds.$$

Now, if we define

$$L_t^n(x) = \sum_{k=1}^{n} |D_x X_t(x) e_k|_4 , \quad n \in \mathbb{N},$$

according to (2.6), we have

$$L_t^n(x) \ \leq C t^{-\beta} + C \int_0^t L_s^n(x)\, ds.$$

We recall that we are assuming $\beta < 3/4 < 1$, then the integral equation

$$\alpha(t) = C t^{-\beta} + C \int_0^t \alpha(s)\, ds, \quad t \in [0,T] \tag{3.5}$$

admits an unique locally integrable continuous solution $\alpha(t)$, and by comparison

$$\sum_{k=1}^{n} |D_x X_t(x) e_k|_4 \leq \alpha(t), \quad n \in \mathbb{N}.$$

As $\alpha(t)$ is independent of $n \in \mathbb{N}$, (3.3) follows. An easy generalization of Gronwall's Lemma implies (3.4). Finally, uniformity of convergence with respect to $x \in H$ holds, as integral equation (3.5) is indipendent of x. □

Another preliminary result that we will use in next proposition is the following (for a proof see Dunford and Schwartz [12]).

Lemma 3.4. *Let $T \in \mathcal{L}(H)$. Assume that there exists $K > 0$ such that, for all finite rank linear bounded operators N, one has*

$$|Tr(NT)| \leq K\|N\|_{\mathcal{L}(H)}.$$

Then $T \in \mathcal{L}_1(H)$ and

$$|TrT| \leq K.$$

Proposition 3.5. *Under Hypotheses 2.2-2.5, $D^2(P_t\varphi)(x) \in \mathcal{L}_1(H)$, $\forall \varphi \in C_b(H)$ and $t > 0$, and there exists a function $\rho :]0, +\infty[\rightarrow [0, +\infty[$ such that*

$$\sup_{x \in H} \left| Tr[D^2(P_t\varphi)(x)] \right| \leq \rho(t)\|\varphi\|_\infty, \quad t > 0. \tag{3.6}$$

Moreover, it holds

$$\sup_{t \in (0,1]} \rho(t)t^{1+\beta} < +\infty. \tag{3.7}$$

Proof. Let $\varphi \in C_b^2(H)$ and let $\{e_k\}$ be the basis introduced in Hypothesis 2.5. We have $P_t\varphi(x) = \mathbb{E}(\varphi(X_t(x)))$ and then, by deriving under the sign of integral, for any $h, k \in H$ we get

$$\begin{aligned} \left\langle D^2(P_t\varphi)(x)h, k \right\rangle = \; & \mathbb{E}\left\langle D^2\varphi(X_t(x))D_xX_t(x)h, D_xX_t(x)k \right\rangle \\ & + \mathbb{E}\left\langle D\varphi(X_t(x)), D_x^2X_t(x)(h,k) \right\rangle. \end{aligned} \tag{3.8}$$

Now, let $N \in \mathcal{L}(H)$ be a finite rank operator. In Cerrai [6] we proved that for any $x, h_1, \ldots, h_3 \in H$ and $j = 1, 2, 3$

$$\sup_{x \in H} \left| D_x^j X_t(x)(h_1, \ldots, h_j) \right|_H \leq \nu_{j,2} \prod_{i=1}^{j} |h_i|_H, \tag{3.9}$$

for some continuous increasing functions $\nu_{j,2} : [0, +\infty[\rightarrow [0, +\infty[$. Then, for any $k \in \mathbb{N}$ we have

$$\begin{aligned} & \left| \left\langle ND^2(P_t\varphi)(x)e_k, e_k \right\rangle \right| \\ & \leq \|\varphi\|_2 \left(\mathbb{E}\left(|D_xX_t(x)e_k|_H \, |D_xX_t(x)N^\star e_k|_H \right) + \mathbb{E}\left| D_x^2X_t(x)(e_k, N^\star e_k) \right|_H \right) \\ & \leq \|\varphi\|_2 \left(\|N\|\nu_1(t)\mathbb{E}\, |D_xX_t(x)e_k|_H + \mathbb{E}\left| D_x^2X_t(x)(e_k, N^\star e_k) \right|_H \right). \end{aligned}$$

As proved in Cerrai [6]

$$D^2X_t(x)(e_k,N^\star e_k)=\int_0^t e^{(t-s)A}\left(DF(X_s(x))D^2X_s(x)(e_k,N^\star e_k)\right)\,ds$$
$$+\int_0^t e^{(t-s)A}\left(D^2F(X_s(x))\,(D_xX_s(x)e_k,D_xX_s(x)N^\star e_k)\right)\,ds,$$

and then, as

$$\left|D^2F(X_t(x))\,(D_xX_t(x)e_k,D_xX_t(x)N^\star e_k)\right|_H \leq C\,|D_xX_s(x)e_k|_4\,|D_xX_s(x)N^\star e_k|_4\,,$$

we have

$$\left|D^2X_t(x)(e_k,N^\star e_k)\right|_H \leq C\int_0^t \left|D^2X_s(x)(e_k,N^\star e_k)\right|_H\,ds+C\int_0^t |D_xX_s(x)e_k|_4\,|D_xX_s(x)N^\star e_k|_4\,ds.$$

In Cerrai [6] we proved that for any $p\geq 2$

$$\sup_{x\in H}\sup_{s\in(0,t]} s^{\frac{\alpha(p-2)}{p}}|D_xX_s(x)h|_p\leq \mu_p(t)|h|_H,$$

for an increasing function $\mu_p:[0,+\infty[\to[0,+\infty[$. Then we have

$$|D_xX_s(x)N^\star e_k|_4\leq s^{-\alpha/2}\mu_4(t)|N^\star e_k|_H\leq s^{-\alpha/2}\mu_4(t)\|N\|,$$

so that

$$\left|D^2X_t(x)(e_k,N^\star e_k)\right|_H \leq C\int_0^t \left|D^2X_s(x)(e_k,N^\star e_k)\right|_H\,ds+\mu_4(t)\|N\|\int_0^t s^{-\alpha/2}\,|D_xX_s(x)e_k|_4\,ds.$$

Now, for any $n\in\mathbb{N}$, let us define

$$M_t^n(x)=\sum_{k=1}^n\left|D_x^2X_t(x)(e_k,N^\star e_k)\right|_H.$$

From (3.3), we have

$$M_t^n(x)\leq C\int_0^t M_s^n(x)\,ds+C\|N\|\mu_4(t)\int_0^t s^{-\alpha/2}\alpha(s)\,ds,$$

and then, as $\alpha<1/2$ and $\beta<3/4$, the function $s\mapsto s^{-\alpha/2}\alpha(s)$ is integrable near 0 and by Gronwall lemma there exists a continuous increasing function $\gamma:[0,+\infty[\to[0,+\infty[$, which is independent of $x\in H$, such that

$$M_t^n(x)\leq\|N\|\gamma(t),\quad t\geq 0. \tag{3.10}$$

Therefore, for any $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} \left|\left\langle N^* D^2(P_t\varphi)(x)e_k, e_k\right\rangle\right|_H \leq \|\varphi\|_2 \|N\| \left(\nu_1(t)\mathbb{E}\sum_{k=1}^{n} |DX_t(x)e_k|_H + \gamma(t)\right),$$

so that, as

$$|D_x X_t(x)e_k|_H \leq |D_x X_t(x)e_k|_4,$$

for any $t > 0$ we get

$$\left|\mathrm{Tr}\left[ND^2(P_t\varphi)(x)\right]\right| \leq C\|N\|\|\varphi\|_2 \left(\nu_1(t)\alpha(t) + \gamma(t)\right).$$

Moreover, as γ is independent of $x \in H$, the convergence is uniform with respect to $x \in H$.

Finally, let $\varphi \in C_b(H)$. By semigroup law, it is easy to check that

$$\mathrm{Tr}\left[ND^2(P_t\varphi)(x)\right] \leq C\|N\|\|P_{t/2}\varphi\|_2 \left(\nu_1(t/2)\alpha(t/2) + \gamma(t/2)\right) \tag{3.11}$$

and then, by using (2.10), our statement follows from Lemma 3.4 with

$$\rho(t) = C\,(t/2 \wedge 1)^{-1} \left(\nu_1(t/2)\alpha(t/2) + \gamma(t/2)\right). \tag{3.12}$$

□

3.2. Existence and uniqueness

For each $n \in \mathbb{N}$ we set

$$\begin{cases} \Pi_n : H \to \mathbb{R}^n, & x \mapsto (< x, e_k >)_{k=1,\dots,n} \\ T_n : \mathbb{R}^n \to H, & \xi \mapsto \sum_{k=1}^{n} \xi_k e_k. \end{cases}$$

Theorem 3.6. *Assume that Hypotheses 2.2-2.6 hold. Then $\forall\,\varphi \in C_b(H)$ there exists an unique classical solution $u : [0, +\infty[\times H \to \mathbb{R}$ to problem (3.1). It is given by*

$$u(t, x) = P_t\varphi(x) = \mathbb{E}\left(\varphi(X_t(x))\right), \quad t \geq 0, \quad x \in H. \tag{3.13}$$

Proof. As in Peszat and Zabczyk [20], for any $n \in \mathbb{N}$ we define

$$\varphi_n(x) = \int_{\mathbb{R}^n} \varphi(T_n\xi)\rho_n(\xi - \Pi_n x)\, d\xi, \quad x \in H,$$

where $\rho_n : \mathbb{R}^n \to \mathbb{R}^n$ are nonnegative smooth functions such that

$$\mathrm{supp}\,(\rho_n) \subset \{\xi \in \mathbb{R}^n : |\xi|_{\mathbb{R}^n} \leq 1/n\}, \quad \int_{\mathbb{R}^n} \rho_n(\xi)\, d\xi = 1.$$

It is easy to show that $\varphi_n \in C_b^2(H)$ and

$$\begin{cases} \|\varphi_n\|_\infty \leq \|\varphi\|_\infty, & n \in \mathbb{N}, \\ \lim_{n\to+\infty} \varphi_n(x) = \varphi(x), & x \in H. \end{cases} \tag{3.14}$$

Moreover, $D^2\varphi_n(x) \in \mathcal{L}_1(H)$ and the mapping $x \mapsto \mathrm{Tr}\,[D^2\varphi_n(x)]$ belongs to $C_b(H)$.

Then if we define

$$u^n(t,x) = P_t\varphi_n(x),$$

we have that u^n is a strict solution of

$$\begin{cases} u^n_t(t,x) = \dfrac{1}{2}\mathrm{Tr}\,[Qu^n_{xx}(t,x)] + \langle Ax + F(x), u^n_x(t,x)\rangle, & x \in D(A), \quad t \geq 0, \\ u^n(0,x) = \varphi_n(x), \quad x \in H \end{cases}$$

(for a proof see Appendix B, Proposition A.1).

Now, we want to show that for any $t > 0$ and $x \in H$

$$\mathrm{Tr}[Qu^n_{xx}(t,x)] \to \mathrm{Tr}\,[Qu_{xx}(t,x)], \quad \text{as } n \to +\infty. \tag{3.15}$$

By using semigroup law and trace-class property of u^n and u, we have

$$\begin{aligned} &\mathrm{Tr}[u^n_{xx}(t,x)] - \mathrm{Tr}[u_{xx}(t,x)] \\ &= \lim_{h\to\infty}\sum_{k=1}^{h}\mathbb{E}\left\langle D^2\psi_{n,t}(X_{t/2}(x))D_xX_{t/2}(x)e_k, D_xX_{t/2}(x)e_k\right\rangle \\ &+ \lim_{h\to\infty}\mathbb{E}\left\langle D\psi_{n,t}(X_{t/2}(x)), \sum_{k=1}^{h} D^2_xX_{t/2}(x)(e_k,e_k)\right\rangle, \end{aligned}$$

where $\psi_{n,t} = P_{t/2}(\varphi_n - \varphi)$. For any $h \in \mathbb{N}$ we have

$$\begin{aligned} &\left|\sum_{k=1}^{h}\mathbb{E}\left\langle D^2\psi_{n,t}(X_{t/2}(x))D_xX_{t/2}(x)e_k, D_xX_{t/2}(x)e_k\right\rangle\right| \\ &\leq \mathbb{E}\left(\left|D^2\psi_{n,t}(X_{t/2}(x))\right|_{\mathcal{L}(H)}\sum_{k=1}^{h}\left|D_xX_{t/2}(x)e_k\right|^2_H\right) \\ &\leq \mathbb{E}\left|D^2\psi_{n,t}(X_{t/2}(x))\right|_{\mathcal{L}(H)} C_t t^{-\beta}, \end{aligned}$$

last inequality following from Lemma 3.3. In the same way, by using (3.10), we have

$$\left|\mathbb{E}\left\langle D\psi_n(X_{t/2}(x)), \sum_{k=1}^{h} D^2_xX_{t/2}(x)(e_k,e_k)\right\rangle\right| \leq \mathbb{E}\left|D\psi_{n,t}(X_{t/2}(x))\right|_H \gamma(t/2).$$

We recall that for any $\varphi \in B_b(H)$ and $x, h \in H$ we have

$$\langle D(P_t\varphi)(x), h\rangle = \frac{1}{t}\mathbb{E}\left(\varphi(X_t(x))\int_0^t\left\langle D_xX_s(x)h, Q^{-1/2}dW_s\right\rangle\right), \quad t > 0. \tag{3.16}$$

Then, as $\varphi_n(x) \to \varphi(x)$ and $\|\varphi_n\|_\infty \leq \|\varphi\|_\infty$, we get

$$D\psi_{n,t}(x) \to 0, \quad \forall\, x \in H \quad \text{and} \quad \|D\psi_{n,t}\|_\infty \leq C(t\wedge 1)^{-1/2}\|\varphi\|_\infty.$$

Thanks to Proposition 5.3 of Cerrai [6], this easily implies that

$$D^2\psi_{n,t}(x) \to 0, \quad \forall\, x \in H \quad \text{and} \quad \|D^2\psi_{n,t}\|_\infty \leq C(t\wedge 1)^{-1}\|\varphi\|_\infty,$$

and then we can conclude that (3.15) holds true. Moreover, by using again (3.16) and (2.10), for any $x \in H$ and $t > 0$ we have

$$\langle Ax + F(x), u^n_x(t,x)\rangle \to \langle Ax + F(x), u_x(t,x)\rangle\,, \quad \text{as } n \to +\infty.$$

Hence, since

$$\lim_{n\to+\infty} u^n(t,x) = u(t,x), \quad (t,x) \in [0,+\infty[\times H$$

and

$$\begin{cases} |\langle Ax + F(x), u^n_x(t,x)\rangle| \leq (|Ax|_H + |F(x)|_H)\,(t\wedge 1)^{-1/2}\|\varphi\|_\infty \\ |\mathrm{Tr}[Qu^n_{xx}(t,x)]| \leq \|Q\|\rho(t)\|\varphi\|_\infty, \end{cases}$$

it follows that there exists $u_t(t,x)$ for any $t > 0$ and $x \in D(A)$ and u satisfies the Kolmogorov equation (3.1).

Uniqueness of the classical solution follows from a standard method based upon Itô formula (see e.g. Da Prato and Zabczyk [10] and Appendix B). □

4. THE ELLIPTIC PROBLEM

We are here concerned with the elliptic problem

$$\lambda\psi(x) - \frac{1}{2}\mathrm{Tr}\,[QD^2\psi(x)] - < Ax + F(x), D\psi(x) >= \varphi(x), \quad x \in D(A), \quad (4.1)$$

for $\lambda > 0$ and $\varphi \in C_b(H)$.

Definition 4.1. *A function ψ is called a* strict solution *of (4.1) if*

1. $\psi \in C^2_b(H)$,
2. $D^2\psi(x) \in \mathcal{L}_1(H)$, *for all $x \in H$ and $Tr[D^2\psi(x)] \in C_b(H)$,*
3. *ψ satisfies equation (4.1).*

Definition 4.2. *A function ψ is called a* strong solution *of (4.1) if there exist two sequences $\{\psi_n\}$ and $\{\varphi_n\}$ in $C_b(H)$ such that*

1. *for any $n \in \mathbb{N}$, ψ_n is a strict solution to problem*

$$\lambda\psi_n(x) - \frac{1}{2}Tr[QD^2\psi_n(x)] - < Ax + F(x), D\psi_n(x) >= \varphi_n(x);$$

2. *$\psi_n \to \psi$ and $\varphi_n \to \varphi$ in $C_b(H)$, as $n \to +\infty$.*

Theorem 4.3. *Under Hypotheses 2.2-2.6, for any $\lambda > 0$ and $\varphi \in C_b(H)$ there exists a unique strong solution ψ to equation (4.1) which is given by*

$$\psi(x) = R(\lambda, \mathcal{A})\varphi(x) = \int_0^{+\infty} e^{-\lambda t}P_t\varphi(x)\,dt, \quad x \in H. \quad (4.2)$$

Proof. Step 1: If $\varphi \in C_b^1(H)$, then $R(\lambda,\mathcal{A})\varphi$ is a strict solution to (4.1).

We first show that $R(\lambda,\mathcal{A})\varphi \in C_b^2(H)$. For any $x, h \in H$, we have

$$\begin{aligned}
&R(\lambda,\mathcal{A})\varphi(x+h) - R(\lambda,\mathcal{A})\varphi(x)\\
&= \int_0^{+\infty} e^{-\lambda t}\left(P_t\varphi(x+h) - P_t\varphi(x)\right)\,dt = \int_0^{+\infty} e^{-\lambda t} < D(P_t\varphi)(x), h > \,dt\\
&+ \int_0^{+\infty} e^{-\lambda t}\int_0^1 < D(P_t\varphi)(x+\theta h) - D(P_t\varphi)(x), h > \,d\theta\,dt,
\end{aligned}$$

last two integrals being meaningful, as the mappings

$$t \mapsto < D(P_t\varphi)(x), h >, \quad \text{and} \quad t \mapsto \int_0^1 < D(P_t\varphi)(x+\theta h), h > \,d\theta$$

are measurable. Then, as $\|P_t\varphi\|_1 \le C\|\varphi\|_1$, $\forall\, t \ge 0$, it follows that

$$\lim_{h\to 0} \frac{\left|\int_0^{+\infty} e^{-\lambda t}\int_0^1 < D(P_t\varphi)(x+\theta h) - D(P_t\varphi)(x), h > \,d\theta\,dt\right|}{|h|_H} = 0.$$

Moreover, if we define

$$< D\left(R(\lambda,\mathcal{A})\varphi\right)(x), h > = \int_0^{+\infty} e^{-\lambda t}\left\langle D(P_t\varphi)(x), h\right\rangle\,dt,$$

we have $D\left(R(\lambda,\mathcal{A})\varphi\right)(x) \in \mathcal{L}(H;\mathbb{R})$ and $|D\left(R(\lambda,\mathcal{A})\varphi\right)(x)|_H \le C\|\varphi\|_1/\lambda$, so that $R(\lambda,\mathcal{A})\varphi$ is differentiable. Now, in order to prove existence and continuity of second derivative, let us fix $k \in H$. We have

$$\begin{aligned}
&\left\langle D\left(R(\lambda,\mathcal{A})\varphi\right)(x+k) - D\left(R(\lambda,\mathcal{A})\varphi\right)(x), h\right\rangle\\
&= \int_0^{+\infty} e^{-\lambda t}\left\langle D(P_t\varphi)(x+k) - D(P_t\varphi)(x), h\right\rangle\,dt\\
&= \int_0^{+\infty} e^{-\lambda t} < D^2(P_t\varphi)(x)k, h > \,dt\\
&+ \int_0^{+\infty} e^{-\lambda t}\int_0^1 < D^2(P_t\varphi)(x+\theta k)k - D^2(P_t\varphi)(x)k, h > \,d\theta\,dt
\end{aligned}$$

and the last two integrals are well defined by the same arguments used for the first derivative. As $\|P_t\varphi\|_2 \le C(t\wedge 1)^{-1/2}\|\varphi\|_1$ it follows that

$$\lim_{h,k\to 0} \frac{\left|\int_0^{+\infty} e^{-\lambda t}\int_0^1 < D^2(P_t\varphi)(x+\theta k)k - D^2(P_t\varphi)(x)k, h > \,d\theta\,dt\right|}{|h|_H|k|_H} = 0.$$

Then, by setting

$$< D^2\left(R(\lambda,\mathcal{A})\varphi\right)(x)k, h > = \int_0^{+\infty} e^{-\lambda t} < D^2(P_t\varphi)(x)k, h > \,dt,$$

we have that $D^2(R(\lambda,\mathcal{A})\varphi) \in \mathcal{L}(H)$ and $\|D^2(R(\lambda,\mathcal{A})\varphi)\| \leq C\lambda^{-1/2}\|\varphi\|_1$, so that $R(\lambda,\mathcal{A})\varphi$ is twice differentiable. Finally let us prove continuity of the second derivative. For any $x,y \in H$ we have

$$\begin{aligned}&\left|\left\langle\left(D^2(R(\lambda,\mathcal{A})\varphi)(x) - D^2(R(\lambda,\mathcal{A})\varphi)(y)\right)k,h\right\rangle\right|\\&\leq \int_0^{+\infty} e^{-\lambda t}\left|\left\langle\left(D^2(P_t\varphi)(x) - D^2(P_t\varphi)(x)\right)k,h\right\rangle\right|dt.\end{aligned}$$

Besides, since for any $t>0$ it holds

$$\begin{cases} P_t : C_b^1(H) \to C_b^1(H), & \|P_t\varphi\|_1 \leq C\|\varphi\|_1 \\ P_t : C_b^1(H) \to C_b^3(H), & \|P_t\varphi\|_3 \leq C(t\wedge 1)^{-1}\|\varphi\|_1, \end{cases}$$

by interpolation, for any $\epsilon \in]0,1[$ we get

$$P_t : C_b^1(H) \to C_b^{2+\epsilon}, \quad \|P_t\varphi\|_{2+\epsilon} \leq C(t\wedge 1)^{-\frac{1+\epsilon}{2}}\|\varphi\|_1.$$

This implies that

$$\begin{aligned}&\left|\left\langle\left(D^2(R(\lambda,\mathcal{A})\varphi)(x) - D^2(R(\lambda,\mathcal{A})\varphi)(y)\right)k,h\right\rangle\right|\\&\leq \int_0^{+\infty} e^{-\lambda t}(t\wedge 1)^{-\frac{1+\epsilon}{2}}\,dt|h||k||x-y|^\epsilon\|\varphi\|_1,\end{aligned}$$

so that $R(\lambda,\mathcal{A})\varphi \in C_b^{2+\epsilon}(H)$.

Now, we prove that $D^2(R(\lambda,\mathcal{A})\varphi)(x) \in \mathcal{L}_1(H)$, for any $x \in H$. Let $\{e_k\}$ be the basis of H introduced in Hypothesis 2.5 and let $N \in \mathcal{L}(H)$ be a finite rank operator. By differentiating each side of (3.16), it easily follows

$$\begin{aligned}&\left\langle ND^2(P_t\varphi)(x)e_k,e_k\right\rangle\\&=\frac{1}{t}\mathbb{E}\left(\langle D\varphi(X_t(x)),D_xX_t(x)N^\star e_k\rangle\int_0^t\left\langle D_xX_s(x)e_k,Q^{-1/2}dW_s\right\rangle\right)\\&+\frac{1}{t}\mathbb{E}\left(\varphi(X_t(x))\int_0^t\left\langle D_x^2X_s(x)(e_k,N^\star e_k),Q^{-1/2}dW_s\right\rangle\right).\end{aligned}$$

Then, for any $h \in \mathbb{N}$ we get

$$\begin{aligned}&\sum_{k=1}^h\left\langle ND^2(P_t\varphi)(x)e_k,e_k\right\rangle\\&=\frac{1}{t}\mathbb{E}\left\langle D\varphi(X_t(x)),D_xX_t(x)N^\star\left(\sum_{k=1}^h\int_0^t\left\langle D_xX_s(x)e_k,Q^{-1/2}dW_s\right\rangle e_k\right)\right\rangle\\&+\frac{1}{t}\mathbb{E}\left(\varphi(X_t(x))\int_0^t\left\langle\sum_{k=1}^h D_x^2X_s(x)(e_k,N^\star e_k),Q^{-1/2}dW_s\right\rangle\right)\\&=I_1^h(t,x)+I_2^h(t,x).\end{aligned}$$

Concerning I_1^h, as (3.9) holds, we have

$$\begin{aligned}\left|I_1^h(t,x)\right| &\leq \frac{\|N\|}{t}\|\varphi\|_1\nu_{1,2}(t)\mathbb{E}\left|\sum_{k=1}^h\int_0^t\left\langle D_xX_s(x)e_k,Q^{-1/2}dW_s\right\rangle e_k\right|_H\\ &\leq \frac{\|N\|}{t}\|\varphi\|_1\nu_{1,2}(t)\left(\mathbb{E}\sum_{k=1}^h\left|\int_0^t\left\langle D_xX_s(x)e_k,Q^{-1/2}dW_s\right\rangle\right|^2\right)^{1/2}\\ &\leq \|Q\|^{-1/2}\frac{\|N\|}{t}\|\varphi\|_1\nu_{1,2}(t)\left(\mathbb{E}\int_0^t\sum_{k=1}^h|D_xX_s(x)e_k|_H^2\,ds\right)^{1/2}.\end{aligned}$$

Hence, from Lemma 3.3 and (3.9) it follows

$$\begin{aligned}\left|I_1^h(t,x)\right| &\leq \frac{C\|N\|}{t}\|\varphi\|_1\nu_{1,2}(t)\left(\int_0^t\nu_{1,2}(s)\alpha(s)\,ds\right)^{1/2}\\ &\leq C\|N\|\|\varphi\|_1\nu_{1,2}^{3/2}(t)t^{-\frac{1+\beta}{2}}.\end{aligned} \tag{4.3}$$

Concerning I_2^h, we have

$$\begin{aligned}\left|I_2^h(t,x)\right| &\leq \frac{1}{t}\|\varphi\|_\infty\mathbb{E}\left|\int_0^t\left\langle\sum_{k=1}^h D_x^2X_s(x)(e_k,N^\star e_k),Q^{-1/2}dW_s\right\rangle\right|\\ &\leq \|Q\|^{-1/2}\frac{1}{t}\|\varphi\|_\infty\left(\mathbb{E}\int_0^t\left|\sum_{k=1}^h D_x^2X_s(x)(e_k,N^\star e_k)\right|_H^2ds\right)^{1/2}\end{aligned}$$

and thanks to (3.10) we get

$$\left|I_2^h(t,x)\right| \leq \frac{\|N\|C}{t}\|\varphi\|_\infty\left(\int_0^t\gamma^2(s)\,ds\right)^{1/2} \leq C\|N\|t^{-1/2}\|\varphi\|_\infty\gamma(t). \tag{4.4}$$

As h is arbitrary, from (3.4) we conclude that

$$\left|\mathrm{Tr}\left[ND^2\left(P_t\varphi\right)(x)\right]\right| \leq C\|N\|t^{-\frac{1+\beta}{2}}\left(t^{\beta/2}\gamma(t)+\nu_1(t)\right)\|\varphi\|_1,\quad t\in(0,1]$$

and the convergence is uniform with respect to $x \in H$. Now, if $t > 1$, by using (3.11) and the semigroup law, we have

$$\begin{aligned}&\mathrm{Tr}\left[ND^2(P_t\varphi)(x)\right] = \mathrm{Tr}\left[ND^2(P_1(P_{t-1}\varphi))(x)\right]\\ &\leq C\|N\|\|P_{1/2}(P_{t-1}\varphi)\|_2\left(\nu_1(1/2)\alpha(1/2)+\gamma(1/2)\right) \leq C\|\varphi\|_\infty,\end{aligned}$$

and then

$$\left|\mathrm{Tr}\left[ND^2(P_t\varphi)(x)\right]\right| \leq C(t\wedge 1)^{-\frac{\beta+1}{2}}\|\varphi\|_1,\quad t>0. \tag{4.5}$$

By Lemma 3.4 this implies that $D^2(R(\lambda,\mathcal{A})\varphi)(x) \in \mathcal{L}_1(H)$, for any $x \in H$ and

$$\begin{aligned}&\left|\mathrm{Tr}\left[D^2(R(\lambda,\mathcal{A})\varphi)(x)\right]\right|\\ &\leq C\int_0^{+\infty}e^{-\lambda t}(t\wedge 1)^{-\frac{\beta+1}{2}}\,dt\|\varphi\|_1 \leq C\lambda^{\frac{\beta-1}{2}}\|\varphi\|_1.\end{aligned} \tag{4.6}$$

Besides, for $x \in D(A)$ and $t \geq 0$ we have

$$|\langle D(P_t\varphi)(x), Ax + F(x)\rangle| \leq C\,(|Ax|_H + |F(x)|_H)\,\|\varphi\|_1. \tag{4.7}$$

Therefore

$$|\langle D(R(\lambda, \mathcal{A})\varphi)(x), Ax + F(x)\rangle| \leq \frac{C}{\lambda}\|\varphi\|_\infty. \tag{4.8}$$

We are now able to show that $R(\lambda, \mathcal{A})\varphi$ is a strict solution to equation (4.1). If we define $u(t,x) = P_t\varphi(x)$, $(t,x) \in [0, +\infty[\times H$, according to Theorem 3.6, we have

$$u_t(t,x) = \frac{1}{2}\mathrm{Tr}\,[QD^2(P_t\varphi)(x)] + \langle D(P_t\varphi)(x), Ax + F(x)\rangle\,, \quad t > 0, \quad x \in D(A)$$

so that, by (4.5) and (4.7) it holds

$$|u_t(t,x)| \leq C\left((t\wedge 1)^{-\frac{\theta+1}{2}} + (|Ax|_H + F(x))\right)\|\varphi\|_1.$$

Then, if $\psi = R(\lambda, \mathcal{A})\varphi$ from (4.6) and (4.8) we get

$$\begin{aligned}&\frac{1}{2}\mathrm{Tr}\,[QD^2\psi(x)] + \langle D\psi(x), Ax + F(x)\rangle\\ &= \int_0^{+\infty} e^{-\lambda t}u_t(t,x)\,dt = -\varphi(x) + \lambda\psi(x).\end{aligned}$$

Step 2: If $\varphi \in C_b(H)$, then $R(\lambda, \mathcal{A})\varphi$ is the unique strong solution to (4.1).

Indeed, let $\{\varphi_n\} \subset C_b^1(H)$ be a sequence convergent to φ in $C_b(H)$. If we define $\psi_n = R(\lambda, \mathcal{A})\varphi_n$, by previous step we have that ψ_n is a strict solution to problem

$$\lambda\psi_n(x) - \frac{1}{2}\mathrm{Tr}\,[QD^2\psi_n(x)] - \langle D\psi_n(x), Ax + F(x)\rangle = \varphi_n(x), \quad x \in D(A).$$

Then, as $\lim_{n\to+\infty}\psi_n = \psi$ in $C_b(H)$, it follows that ψ is a strong solution.

Finally, let us prove uniqueness. If φ is a strict solution of the equation $\lambda\psi - \mathcal{A}_0(\cdot)\psi = 0$, then the function

$$u(t,x) = e^{\lambda t}\psi(x)$$

is a classical solution to problem (3.1). By Theorem 3.6, u is unique, so that ψ is unique, as well. Uniqueness for general φ follows immediately by linearity. If ψ is a strong solution to (4.1), let $\{\psi_n\}$ and $\{\varphi_n\}$ be as in Definition 4.2. By uniqueness of strict solutions we have

$$\psi_n = R(\lambda, \mathcal{A})\varphi_n, \quad n \in \mathbb{N},$$

so that, letting $n \to +\infty$, we have $\psi = R(\lambda, \mathcal{A})\varphi$. □

5. SCHAUDER ESTIMATES

In Cerrai [6] we showed that for any $\varphi \in C_b(H)$

$$\|P_t\varphi\|_j \leq C(t\wedge 1)^{-j/2}\|\varphi\|_\infty, \quad j=1,2,3.$$

In the following proposition we prove estimates in the Hölder norms.

Proposition 5.1. *Let $\theta \in]0,1[$ and $\alpha \in]\theta,3]$. Then for any $\varphi \in C_b^\theta(H)$ we have*

$$\|P_t\varphi\|_\alpha \leq C(t\wedge 1)^{-\frac{\alpha-\theta}{2}}\|\varphi\|_\theta, \quad t>0. \tag{5.1}$$

Moreover, it holds

$$\sup_{x\in H}\left|Tr[D^2(P_t\varphi)(x)]\right| \leq C(t\wedge 1)^{-1+\frac{\theta-\beta}{2}}\|\varphi\|_\theta, \quad t>0. \tag{5.2}$$

Proof. If $\varphi \in C_b^1(H)$ then for any $x,h\in H$

$$\langle D(P_t\varphi)(x),h\rangle = \mathbb{E}\left(\langle D\varphi(X_t(x)), D_xX_t(x)h\rangle\right), \quad t\geq 0,$$

so that, by using (3.9) and (2.10), we have

$$\|P_t\varphi\|_1 \leq C\|\varphi\|_1, \quad t\geq 0. \tag{5.3}$$

By the Markov property, for any $k\in H$ we have

$$\begin{aligned}&\left\langle D(P_{t/2}\varphi)(X_{t/2}(x)), D_xX_{t/2}(x)k\right\rangle\\ &= \frac{2}{t}\mathbb{E}\left(\varphi(X_t(x))\int_{t/2}^t\left\langle D_xX_s(x)k, Q^{-1/2}\,dW_s\right\rangle\Big|\ X_s(x)\ :\ 0\leq s\leq t/2\right).\end{aligned}$$

Moreover, as proved in Cerrai [6], if we define $\psi = P_{t/2}\varphi$, for any $x,h,k\in H$ we have

$$\left\langle D^2(P_t\varphi)(x)h,k\right\rangle = I_t^1(x)(h,k)+I_t^2(x)(h,k), \tag{5.4}$$

where

$$\begin{aligned}I_t^1(x)(h,k) &= \frac{2}{t}\mathbb{E}\left(\left\langle D\psi\left(X_{t/2}(x)\right), D_xX_{t/2}(x)k\right\rangle\int_0^{t/2}\left\langle D_xX_s(x)h, Q^{-1/2}dW_s\right\rangle\right)\\ I_t^2(x)(h,k) &= \frac{2}{t}\mathbb{E}\left(\psi\left(X_{t/2}(x)\right)\int_0^{t/2}\left\langle D_x^2X_s(x)(h,k), Q^{-1/2}dW_s\right\rangle\right).\end{aligned}$$

Then we get

$$\begin{aligned}&\left\langle D^2(P_t\varphi)(x)h,k\right\rangle\\ &= \frac{4}{t^2}\mathbb{E}\left(\varphi(X_t(x))\int_0^{t/2}\left\langle D_xX_s(x)h, Q^{-1/2}\,dW_s\right\rangle\int_{t/2}^t\left\langle D_xX_s(x)k, Q^{-1/2}\,dW_s\right\rangle\right)\\ &+\frac{2}{t}\mathbb{E}\left(P_{t/2}\varphi(X_{t/2}(x))\int_0^{t/2}\left\langle D_x^2X_s(x)(h,k), Q^{-1/2}\,dW_s\right\rangle\right).\end{aligned} \tag{5.5}$$

Now, if $\varphi \in C_b^1(H)$ we can differentiate each side of (5.5) along any direction $l \in H$ and it follows

$$\begin{aligned} D^3(P_t\varphi)(x)(h,k,l) &= \frac{4}{t^2}\mathbb{E}\bigg(\langle D\varphi(X_t(x)), D_xX_t(x)l\rangle \\ &\int_0^{t/2} \left\langle D_xX_s(x)h, Q^{-1/2}\,dW_s\right\rangle \int_{t/2}^t \left\langle D_xX_s(x)k, Q^{-1/2}\,dW_s\right\rangle \bigg) \\ &+\frac{4}{t^2}\mathbb{E}\left(\varphi(X_t(x)) \int_0^{t/2} \left\langle D_x^2X_s(x)(h,l), Q^{-1/2}\,dW_s\right\rangle \int_{t/2}^t \left\langle D_xX_s(x)k, Q^{-1/2}\,dW_s\right\rangle\right) \\ &+\frac{4}{t^2}\mathbb{E}\left(\varphi(X_t(x)) \int_0^{t/2} \left\langle D_xX_s(x)h, Q^{-1/2}\,dW_s\right\rangle \int_{t/2}^t \left\langle D_x^2X_s(x)(k,l), Q^{-1/2}\,dW_s\right\rangle\right) \\ &\frac{2}{t}\mathbb{E}\left(\left\langle D(P_{t/2}\varphi)(X_{t/2}(x)), D_xX_t(x)l\right\rangle \int_0^{t/2} \left\langle D_x^2X_s(x)(h,k), Q^{-1/2}\,dW_s\right\rangle\right) \\ &+\frac{2}{t}\mathbb{E}\left(P_{t/2}\varphi(X_{t/2}(x)) \int_0^{t/2} \left\langle D_x^3X_s(x)(h,k,l), Q^{-1/2}\,dW_s\right\rangle\right). \end{aligned}$$

Then, by using (3.9), this implies that

$$\|P_t\varphi\|_3 \leq C(t\wedge 1)^{-1}\|\varphi\|_1. \tag{5.6}$$

Estimate (5.1) follows from Proposition 2.1 and from (5.3) and (5.6), by using interpolation (for a similar proof see Cerrai [5], Proposition 8.1). In order to prove (5.2), we recall that for any $\varphi \in C_b^1(H)$

$$\left|\mathrm{Tr}\,[D^2(P_t\varphi)(x)]\right| \leq C(t\wedge 1)^{-\frac{\beta+1}{2}}\|\varphi\|_1.$$

Besides, from the semigroup law we get

$$\left|\mathrm{Tr}\,[D^2(P_t\varphi)(x)]\right| \leq C(t/2\wedge 1)^{-\frac{\beta+1}{2}}\|P_{t/2}\varphi\|_1 \leq C(t\wedge 1)^{-1+\frac{\beta}{2}}\|\varphi\|_\infty$$

and then by interpolation (5.2) follows. □

By using previous Proposition and proceeding as in the proof of the first step of Theorem 4.3, we get

Corollary 5.2. *Let $\lambda > 0$ and set $\psi = R(\lambda, \mathcal{A})\varphi$, with $\varphi \in C_b(H)$. The following statements hold.*

1. *If $\theta \in]0,1[$, then $\psi \in C_b^{1+\theta}(H)$.*
2. *If $\varphi \in C_b^\theta(H)$, with $\theta \in]0,1[$, then $\psi \in C_b^2(H)$ and it holds*

$$\|\psi\|_2 \leq \Gamma(\theta/2)\lambda^{-\theta/2}\|\varphi\|_\theta. \tag{5.7}$$

3. *If $\varphi \in C_b^\theta(H)$, with $\theta \in]\beta,1[$, then ψ is a strict solution to equation (4.1).*

We are now able to prove Schauder estimates for the elliptic problem (4.1) associated to the operator $\mathcal{A}_0(\cdot)$.

Theorem 5.3. *Assume that $\lambda > 0$, $\theta \in]0,1[$ and $\varphi \in C_b^\theta(H)$. Then the function $\psi = R(\lambda, \mathcal{A})\varphi \in C^{2+\theta}(H)$ and it holds*

$$\|\psi\|_{2+\theta} \leq C_\lambda \|\varphi\|_\theta. \tag{5.8}$$

Proof. We apply a general method due to Lunardi [18]. Since

$$\left(C_b^\alpha(H), C_b^{2+\alpha}(H)\right)_{1-\frac{\alpha-\theta}{2},\infty} \subset C_b^{2+\theta}(H),$$

it suffices to show that

$$R(\lambda, \mathcal{A})\varphi \in \left(C_b^\alpha(H), C_b^{2+\alpha}(H)\right)_{1-\frac{\alpha-\theta}{2},\infty}.$$

This follows from the very first definition of interpolation spaces. Indeed for any $t \in [0,1]$ we have

$$R(\lambda, \mathcal{A})\varphi(x) = a_t(x) + b_t(x), \quad x \in H,$$

where

$$a_t(x) = \int_0^t e^{-\lambda s} P_s\varphi(x)\, ds$$
$$b_t(x) = \int_t^{+\infty} e^{-\lambda s} P_s\varphi(x)\, ds.$$

By using (5.1) and proceeding as in the proof of the first step of Theorem 4.3, we get that $a_t \in C_b^\alpha(H)$ and $b_t \in C_b^{2+\alpha}(H)$, for $t > 0$, and the good estimates hold (for more details see e.g. Cerrai [5], Theorem 8.2). □

ACKNOWLEDGEMENT

The author wants to thank Professors G. Da Prato and K.D. Elworthy for the useful suggestions they gave to her.

REFERENCES

[1] J.M. Bismut, *Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions*, Z. Wahrscheinlichkeitstheorie verw. Gebiete, **56** (1981), 469–505.

[2] P. Cannarsa and G. Da Prato, On a functional analysis approach to parabolic equations in infinite dimensions, *J. Funct. Anal.*, **118** (1993), no. 1, 22–42.

[3] P. Cannarsa and G. Da Prato, Infinite-dimensional elliptic equations with Hölder continuous coefficients, *Advances in Differential Equations*, **1** (1996), no. 3, 425–452.

[4] S. Cerrai, A Hille Yosida theorem for weakly continuous semigroups, *Semigroup Forum,* **49** (1994), 349–367.

[5] S. Cerrai, Elliptic and parabolic equations in $\mathbb{R}^n$ with coefficients having polynomial growth, *Comm. in Partial Differential Equations,* **21** (1996), 281–317.

[6] S. Cerrai, *Differentiability with respect to initial datum for solutions of SPDE's with non Fréchet differentiable coefficients*, Preprint 36, Scuola Normale Superiore Pisa, 1996.

[7] J.L. Daleckij, Differential equations with functional derivatives and stochastic equations for generalized random processes, *Dokl. Akad. Nauk. SSSR,* **166** (1965), 1035–1038.

[8] G. Da Prato, *Elliptic and parabolic equations in Hilbert spaces*, Preprint 13, Scuola Normale Superiore Pisa, 1996.

[9] G. Da Prato, K. D. Elworthy and J. Zabczyk, Strong Feller property for stochastic semilinear equations, *Stochastic Analysis and Applications,* **13** (1995), no. 1, 35–45.

[10] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[11] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press, Cambridge, 1996.

[12] N. Dunford and J.T. Schwartz, *Linear Operators,* Vol. II, 1956.

[13] K.D. Elworthy and X.M. Li, Formulae for the derivatives of heat semigroups, *J. Functional Analysis,* **125** (1994), 252–286.

[14] M. Fuhrman, *On a class of stochastic equations in Hilbert spaces, solvability and smoothing properties*, Preprint, 1996.

[15] L. Gross, Potential theory in Hilbert spaces, *J. Functional Analysis,* **1** (1967), 123–189.

[16] J. Kurtzweil, On approximation in real Banach spaces, *Studia Math.,* **14** (1954), 213–231.

[17] J.M. Lasry and P.L. Lions, A remark on regularization in Hilbert spaces, *Israel J. Math.,* **55** (1986), 257–266.

[18] A. Lunardi, *An interpolation method to characterize domains of generators of semigroups*, Preprint, Scuola Normale Superiore Pisa, 1995, *Semigroup Forum,* (to appear).

[19] A.S. Nemirovski and S.M. Semenov, The polynomial approximation of functions in Hilbert spaces, *Mat. Sb. (N.S.),* **92** (1973), 257–281.

[20] S. Peszat and J. Zabczyk, *Strong Feller property and irriducibility for diffusion processes on Hilbert spaces*, Preprint, Institute of Mathematics, Polish Academy of Sciences, 1993.

[21] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators,* North-Holland, Amsterdam, 1986.

A. APPENDIX

Our aim is to prove that if φ is regular enough then the function

$$u : [0, +\infty) \times H \to \mathbb{R}, \quad (t, x) \mapsto P_t\varphi(x)$$

is a strict solution to problem (3.1). If $A \in \mathcal{L}(H)$ and $\operatorname{Tr} Q < +\infty$, the unique mild solution $X_t(x)$ of equation (1.4) is a *strong* solution, that is

$$X_t(x) = x + \int_0^t (AX_s(x) + F(X_s(x))\, ds + \sqrt{Q}\, dW_t.$$

Then $X_t(x)$ is an Itô process and we can apply Itô formula for Hilbert space processes. Hence, if $\varphi \in C_b^2(H)$ we have

$$\begin{aligned} &P_t\varphi(x) = \varphi(x) \\ &+\mathbb{E}\int_0^t \Big(\langle AX_s(x) + F(X_s(x)), D\varphi(X_s(x))\rangle + \frac{1}{2}\operatorname{Tr}\left[D^2\varphi(X_s(x))Q\right]\Big)\, ds. \end{aligned} \tag{A.1}$$

In the present paper we do not assume A to be bounded and Q to be trace-class, so that we can not apply Itô formula for Hilbert space processes. To this purpose we have to proceed by approximation, assuming further hypotheses for φ.

Proposition A.1. *Assume that $\varphi \in C_b^2(H)$ and that $D^2\varphi(x) \in \mathcal{L}_1(H)$ for any $x \in H$. Moreover, assume that the mapping $x \mapsto Tr[D^2\varphi(x)]$ is continuous and*

$$\sup_{x\in H}\left|Tr\left[D^2\varphi(x)\right]\right| < +\infty. \tag{A.2}$$

Then, under Hypotheses 2.2-2.6, φ is a strict solution to problem (3.1).

Proof. We split up the proof into two steps.

Step 1: Assume that $A \in \mathcal{L}(H)$. For any $n \in \mathbb{N}$ define $Q_n = \sqrt{Q}P_n\sqrt{Q}$. Then $Q_n \in \mathcal{L}^+(H)$ and for each $x \in H$ we have $Q_n x \to Qx$ and $\sqrt{Q_n}x \to \sqrt{Q}x$, as $n \to +\infty$. Moreover, it is possible to check that $\operatorname{Tr} Q_n < +\infty$ and then $Q_n \in \mathcal{L}_1(H)$. For each $n \in \mathbb{N}$, let us consider the problem

$$\left\{\begin{aligned} dX_t^n &= (AX_t^n + F(X_t^n))\, dt + \sqrt{Q_n}\, dW_t \\ X_0^n &= x. \end{aligned}\right. \tag{A.3}$$

Let $X_t^n(x)$ be its strong (and then mild) solution. If we introduce the transition semigroup relative to $X_t^n(x)$, $P_t^n\varphi(x) = \mathbb{E}(\varphi(X_t^n(x)))$, we have

$$\begin{aligned} &P_t^n\varphi(x) = \varphi(x) \\ &+\mathbb{E}\int_0^t \Big(\langle AX_s^n(x) + F(X_s^n(x)), D\varphi(X_s^n(x))\rangle + \frac{1}{2}\operatorname{Tr}\left[D^2\varphi(X_s^n(x))Q_n\right]\Big)\, ds. \end{aligned} \tag{A.4}$$

We remark that $X^n(x) \to X(x)$, as $n \to +\infty$, in $C([0,T];H)$, $\mathbb{P}$-a.s. Indeed we have

$$X_t^n(x) - X_t(x) = \int_0^t e^{(t-s)A}\left(F(X_s^n(x)) - F(X_s(x))\right) ds + Z_t^n(x) - Z_t(x),$$

where processes $Z_t^n(x)$ and $Z_t(x)$ are introduced in Fuhrman [14], in the proof of Lemma 6.2. This implies that for any integer $m > 1/(2\epsilon) > 1$ (ϵ is the constant introduced in Hypothesis 2.6)

$$|X_t^n(x) - X_t(x)|^{2m} \le C_{T,m} \sup_{t\in[0,T]} |Z_t^n(x) - Z_t(x)|^{2m}, \quad t \in [0,T]$$

and by using again the proof of Lemma 6.2, it follows that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t^n(x) - X_t(x)|^{2m}\right) \to 0, \quad \text{as } n \to +\infty. \tag{A.5}$$

Now, let us prove that for each $x \in H$ and $t > 0$

$$\lim_{n\to+\infty} \mathbb{E}\int_0^t \left(\mathrm{Tr}\,[D^2\varphi(X_s^n(x))Q_n] - \mathrm{Tr}\,[D^2\varphi(X_s(x))Q]\right) ds = 0. \tag{A.6}$$

We have

$$\begin{aligned}&\mathrm{Tr}\,[D^2\varphi(X_t^n(x))Q_n] - \mathrm{Tr}\,[D^2\varphi(X_t(x))Q]\\ &= \mathrm{Tr}\,\left[\left(D^2\varphi(X_t^n(x)) - D^2\varphi(X_t(x))\right)Q_n\right] + \mathrm{Tr}\,[D^2\varphi(X_t(x))(Q_n - Q)].\end{aligned}$$

Concerning the first term,

$$\begin{aligned}&\left|\mathrm{Tr}\,\left[\left(D^2\varphi(X_t^n(x)) - D^2\varphi(X_t(x))\right)Q_n\right]\right|\\ &\le \|Q_n\| \left|\mathrm{Tr}\,\left[D^2\varphi(X_t^n(x)) - D^2\varphi(X_t(x))\right]\right|.\end{aligned}$$

We recall that the mapping $x \mapsto \mathrm{Tr}\,[D^2\varphi(x)]$ is continuous and (A.2) holds, then, as $\|Q_n\| \le \|Q\|$, by dominated convergence theorem, from (A.5) we get

$$\mathbb{E}\int_0^t \left(\mathrm{Tr}\,\left[\left(D^2\varphi(X_s^n(x)) - D^2\varphi(X_s(x))\right)Q_n\right]\right) ds \to 0, \quad \text{as } n \to +\infty.$$

As far as the second term is concerned, we have that $(Q_n - Q)x \to 0$ as $n \to +\infty$, for any $x \in H$, and $\|Q_n - Q\| \le 2\|Q\|$. Then, as $D^2\varphi(x) \in \mathcal{L}_1(H)$, it follows that

$$\mathrm{Tr}\,\left[D^2\varphi(X_t(x))(Q_n - Q)\right] \to 0, \quad \text{as } n \to +\infty.$$

Moreover, for any $t \ge 0$ we have

$$\mathrm{Tr}\,\left[D^2\varphi(X_t(x))(Q_n - Q)\right] \le 2\|Q\| \sup_{x\in H}\left|\mathrm{Tr}\,\left[D^2\varphi(x)\right]\right|,$$

so that, by dominated convergence theorem we get

$$\mathbb{E}\int_0^t \mathrm{Tr}\,\left[(Q_n - Q)D^2\varphi(X_s(x))\right] ds \to 0, \quad \text{as } n \to +\infty.$$

Finally

$$\langle AX_s^n(x) + F(X_s^n(x)), D\varphi(X_s^n(x))\rangle \to \langle AX_s(x) + F(X_s(x)), D\varphi(X_s(x))\rangle,$$

uniformly in $[0,T]$, $\mathbb{P}$-a.s., as $n \to +\infty$. Then, as $P_t^n\varphi(x) \to P_t\varphi(x)$, by taking the limit in each side of (A.4), (A.1) follows. If φ satisfies the conditions of the proposition, it is easy to check that $P_t\varphi$ satisfies the same conditions, for any $t \geq 0$ and then, as proved in Da Prato and Zabczyk [10], by using the semigroup law, we can conclude that the function $u(t,x) = P_t\varphi(x)$ is differentiable with respect to $t \geq 0$ and

$$\begin{cases} u_t(t,x) = \frac{1}{2}\mathrm{Tr}\,[u_{xx}(t,x)Q] + \langle Ax + F(x), u_x(t,x)\rangle, & x \in D(A),\ t > 0, \\ u(0,x) = \varphi(x), & x \in H. \end{cases}$$

Step 2: Let A be the infinitesimal generator of a strongly continuous semigroup. We introduce the Yosida approximations A_n of A and for each n we consider the approximating problems

$$\begin{cases} dX_t^n & = (A_nX_t^n + F(X_t^n))\,dt + \sqrt{Q}\,dW_t \\ X_0^n & = x. \end{cases} \tag{A.7}$$

By the previous step we have that the function $u^n(t,x) = \mathbb{E}(\varphi(X_t^n(x)))$ is differentiable with respect to $t > 0$ and

$$\begin{cases} u_t^n(t,x) = \frac{1}{2}\mathrm{Tr}\,[u_{xx}^n(t,x)Q] + \langle A_nx + F(x), u_x^n(t,x)\rangle, & x \in D(A),\ t > 0, \\ u^n(0,x) = \varphi(x), & x \in H. \end{cases}$$

Let $X_t^n(x)$, $D_xX_t^n(x)$ and $D_x^2X_t^n(x)$ denote respectively the solution of equation (A.7) and its first two derivatives with respect to initial datum. As shown in Da Prato and Zabczyk [10], for any $x, h, k \in H$, we have

$$\begin{cases} X_t^n(x) \to X_t(x) \\ D_xX_t^n(x)h \to D_xX_t(x)h \\ D_x^2X_t^n(x)(h,k) \to D_x^2X_t(x)(h,k), \end{cases} \tag{A.8}$$

as $n \to +\infty$, uniformly in $[0,T]$, $\mathbb{P}$-a.s. Moreover, since e^{tA_n} is of contraction for any $n \in \mathbb{N}$, it is clear that

$$\begin{cases} \sup_{x\in H} |D_xX_t^n(x)h|_H \leq \nu_{1,1}(t)|h|_H \\ \sup_{x\in H} \left|D_x^2X_t^n(x)(h,k)\right|_H \leq \nu_{2,2}(t)|h|_H|k|_H, \end{cases} \tag{A.9}$$

where ν_1 and $\nu_{2,2}$ are the same functions defined in (3.9).

It holds

$$< u_x^n(t,x), h >= \mathbb{E}\left(< D\varphi(X_t^n(x)), D_xX_t^n(x)h >\right),$$

then, according to (A.9), as $\varphi \in C_b^2(H)$ it follows

$$< u_x^n(t,x), h > \to < u_x(t,x), h >, \quad \text{as } n \to +\infty.$$

This implies that for any $x \in D(A)$

$$< u_x^n(t,x), A_n x + F(x) > \to < u_x(t,x), Ax + F(x) >, \quad \text{as } n \to +\infty.$$

If we prove that

$$\mathrm{Tr}\,[u_{xx}^n(t,x)] \to \mathrm{Tr}\,[u_{xx}(t,x)], \quad \text{as } n \to +\infty,$$

we conclude that $u(t,x)$ is differentiable with respect to $t \geq 0$ and is a strict solution of (3.1). We have

$$\begin{aligned} < u_{xx}^n(t,x)h, k > &= \mathbb{E}\left(\left\langle D^2\varphi(X_t^n(x))D_x X_t^n(x)h, D_x X_t^n(x)k\right\rangle\right) \\ &\quad + \mathbb{E}\left(\left\langle D\varphi(X_t^n(x)), D_x^2 X_t^n(x)(h,k)\right\rangle\right). \end{aligned}$$

By using (A.8), (A.9) and assumptions made for φ, it holds

$$\begin{aligned} &\lim_{n\to+\infty} \sum_{k=1}^{+\infty} \mathbb{E}\left(\left\langle D^2\varphi(X_t^n(x))D_x X_t^n(x)e_k, D_x X_t^n(x)e_k\right\rangle\right) \\ &= \sum_{k=1}^{+\infty} \mathbb{E}\left(\left\langle D^2\varphi(X_t(x))D_x X_t(x)e_k, D_x X_t(x)e_k\right\rangle\right). \end{aligned}$$

For the second term we have

$$\begin{aligned} &\left\langle D\varphi(X_t^n(x)), D_x^2 X_t^n(x)(e_k,e_k)\right\rangle - \left\langle D\varphi(X_t(x)), D_x^2 X_t(x)(e_k,e_k)\right\rangle \\ &= \left\langle D\varphi(X_t^n(x)), D_x^2 X_t^n(x)(e_k,e_k) - D_x^2 X_t(x)(e_k,e_k)\right\rangle \\ &+ \left\langle D\varphi(X_t^n(x)) - D\varphi(X_t(x)), D_x^2 X_t(x)(e_k,e_k)\right\rangle \end{aligned}$$

and then, by (3.10) we get

$$\begin{aligned} &\sum_{k=1}^{\infty} \left|\left\langle D\varphi(X_t^n(x)), D_x^2 X_t^n(x)(e_k,e_k)\right\rangle - \left\langle D\varphi(X_t(x)), D_x^2 X_t(x)(e_k,e_k)\right\rangle\right| \\ &\leq \|D\varphi\|_\infty \sum_{k=1}^{+\infty} \left|D_x^2 X_t^n(x)(e_k,e_k) - D_x^2 X_t(x)(e_k,e_k)\right|_H \\ &+ |D\varphi(X_t^n(x)) - D\varphi(X_t(x))|_H\, \gamma(t) \to 0. \end{aligned}$$

□