

On the small noise limit in the Smoluchowski-Kramers approximation of nonlinear wave equations with variable friction

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Abstract

We study the validity of a large deviation principle for a class of stochastic nonlinear damped wave equations, of Klein-Gordon type, in the joint small mass and small noise limit. The friction term is assumed to be state dependent.

Key words: Smoluchowski-Kramers approximation, Large deviations, stochastic nonlinear damped wave equations.

1 Introduction

In this article we deal with this class of stochastic wave equations with state-dependent damping on a bounded smooth domain $\mathcal{O} \subset \mathbb{R}^d$

$$\begin{cases} \mu \partial_t^2 u_\mu(t, x) = \Delta u_\mu(t, x) - \gamma(u_\mu(t, x)) \partial_t u_\mu(t, x) + f(x, u_\mu(t, x)) + \sigma(u_\mu(t, \cdot)) \partial_t w^\mathcal{Q}(t, x), \\ u_\mu(0, x) = u_0(x), \quad \partial_t u_\mu(0, x) = v_0(x), \quad u_\mu(t, x) = 0, \quad x \in \partial \mathcal{O}, \end{cases} \quad (1.1)$$

depending on a parameter $0 < \mu \ll 1$. Here the friction coefficient γ is strictly positive and bounded and the nonlinearity f is either a Lipschitz-continuous function (in this case we can consider any $d \geq 1$) or a locally Lipschitz-continuous function of the Klein-Gordon type (in this case we can only take $d = 1$). The noise $w^\mathcal{Q}(t)$ is a cylindrical Q -Wiener process and σ is a suitable Lipschitz-continuous operator-valued function.

The solution u_μ of equation (1.1) can be seen as the displacement field of some particles in a domain \mathcal{O} , subject to interaction forces represented by the Laplacian and to nonlinear reactions represented by f , in the presence of a random external forcing $\sigma(u_\mu(t, \cdot)) \partial_t w^\mathcal{Q}(t)$ and a state-dependent friction $\gamma(u_\mu(t)) \partial_t u_\mu(t)$. A series of papers has investigated the validity of the so-called Smoluchowski-Kramers approximation, that describes the limiting behavior of the solution u_μ , as the density μ of the particles vanishes. For the finite dimensional case, the existing literature is quite broad and we refer in particular to [15], [16], [19], [20] and [32] (see also [6], [13] and [23] for systems subject to a magnetic field and [21] and [29] for some related multiscale problems).

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In recent years there has been an intense activity dealing with the Smoluchowski-Kramers approximation of infinite dimensional systems. To this purpose, we refer to [4], [5], [30] and [25] for the case of constant damping term (see also [12] where systems subject to a magnetic field are studied), and to [14] for the case of state-dependent damping. As a matter of fact, these two situations are quite different. When γ is constant, u_μ converges to the solution of the stochastic parabolic problem

$$\begin{cases} \gamma \partial_t u(t, x) = \Delta u(t, x) + f(x, u(t, x)) + \sigma(u(t, \cdot)) \partial_t w^\mathcal{Q}(t, x), \\ u(0, x) = u_0(x), \quad \partial_t u(0, x) = v_0(x), \quad u_\mu(t, x) = 0, \quad x \in \partial \mathcal{O}. \end{cases} \quad (1.2)$$

However, when γ is not constant, because of the interplay between the state-dependent friction and the noise, an extra drift is created and in [14] it has been proven that the limiting equation becomes

$$\begin{cases} \gamma(u(t, x)) \partial_t u(t, x) = \Delta u(t, x) + f(u(t, x)) - \frac{\gamma'(u(t, x))}{2\gamma^2(u(t, x))} \sum_{i=1}^{\infty} |[\sigma(u(t, \cdot)) Q e_i](x)|^2 + \sigma(u(t, \cdot)) \partial_t w^\mathcal{Q}(t, x), \\ u(0, x) = u_0(x), \quad u(t)|_{\partial \mathcal{O}} = 0, \end{cases} \quad (1.3)$$

where $\{Q e_i\}_{i \in \mathbb{N}}$ is a complete orthonormal basis of the reproducing kernel of the noise.

Once proved the validity of the small mass limit, it is important to understand how stable such an approximation is with respect to other important asymptotic features of the two systems, such as for example the long time behavior. To this purpose, in [9] and [4] it is shown that the statistically invariant states of equation (1.1) (in case of constant friction) converge in a suitable sense to the invariant measure of equation (1.2). In the same spirit, the two papers [10] and [11] are devoted to an analysis of the convergence of the quasi-potential, that describes, as known, the asymptotics of the exit times and the large deviation principle for the invariant measure.

In the present paper we are interested in studying the validity of a large deviation principle for the following equation

$$\begin{cases} \mu \partial_t^2 u_\mu(t, x) = \Delta u_\mu(t, x) - \gamma(u_\mu(t, x)) \partial_t u_\mu(t, x) + f(x, u_\mu(t, x)) + \sqrt{\mu} \sigma(u_\mu(t, \cdot)) \partial_t w^\mathcal{Q}(t, x), \\ u_\mu(0, x) = u_0(x), \quad \partial_t u_\mu(0, x) = v_0(x), \quad u_\mu(t, x) = 0, \quad x \in \partial \mathcal{O}, \end{cases} \quad (1.4)$$

where, together with the mass, we are also assuming that the intensity of the noise vanishes. Our aim is proving that in the joint small mass and small noise limit the family of random variables $\{u_\mu\}_{\mu > 0}$ satisfy a large deviation principle in the space $C([0, T]; L^p(\mathcal{O}))$, (for some $p > 2$ depending on the dimension d), with respect to the action functional

$$I_T(u) = \frac{1}{2} \left\{ \int_0^T \|\varphi(t)\|_H^2 dt : u(t) = u^\varphi(t), t \in [0, T] \right\},$$

where $u^\varphi(t)$ denotes the solution of the controlled quasi-linear parabolic equation

$$\begin{cases} \gamma(u(t, x)) \partial_t u(t, x) = \Delta u(t, x) + f(x, u(t, x)) + \sigma(u(t, \cdot)) \varphi(t, x), \\ u(0, x) = u_0(x), \quad u(t, x) = 0, \quad x \in \partial \mathcal{O}. \end{cases} \quad (1.5)$$

This means in particular that, in spite of the fact that in the presence of a non-constant friction coefficient the Smoluchowski-Kramers approximation of equation (1.1) leads to equation (1.3), the large deviation principle is consistent with equation (1.2).

Due to the nature of our problem, the weak-convergence approach to large deviation, as developed in [3] for SPDEs, is the ideal tool for our proof. As known, such an approach requires a thorough analysis of the following controlled version of equation (1.4)

$$\begin{cases} \mu \partial_t^2 u_\mu(t, x) = \Delta u_\mu(t, x) - \gamma(u_\mu(t, x)) \partial_t u_\mu(t, x) + f(x, u_\mu(t, x)) + \sigma(u_\mu(t, \cdot)) \mathcal{Q}\varphi(t, x) \\ \quad + \sqrt{\mu} \sigma(u_\mu(t, \cdot)) \partial_t w^\mathcal{Q}(t, x), \quad t > 0, \quad x \in \mathcal{O}, \\ u_\mu(0, x) = u_0(x), \quad \partial_t u_\mu(0, x) = v_0(x), \quad u_\mu(t, x) = 0, \quad x \in \partial \mathcal{O}, \end{cases} \quad (1.6)$$

In [14], the case of Lipschitz f and bounded σ is considered, and, as a matter of fact, the Smoluchowski-Kramers approximation for systems with variable friction is only known under these assumptions. However, in relevant models it is important to consider non-Lipschitz nonlinearities. For this reason, in this paper we are considering also nonlinearities f having polynomial growth and satisfying some dissipativity conditions. We would like to stress that this fact, together with the fact that we allow the diffusion coefficients σ to have linear growth and we have to add a control in equation (1.6), has required the introduction of new arguments, compared with [14], already in the proof of the well-posedness result.

After we have shown that equation (1.6) admits a unique solution u_μ^φ , for every fixed $\mu > 0$ and for every predictable control, we have proven suitable a priori bounds for such solution and its time derivative. Then, we have introduced $\rho_\mu := g(u_\mu^\varphi)$, where $g' = \gamma$ and $\{\varphi_\mu\}_{\mu>0}$ is a family of controls all contained \mathbb{P} -a.s. in a ball of $L^2(0, T; L^2(\mathcal{O}))$, and we have shown that these estimates imply the tightness of the family $\{\rho_\mu\}_{\mu \in (0, \mu_T)}$ in $C([0, T]; H^\delta)$, for some $\mu_T > 0$ and for every $\delta < 1$.

Next, we have shown how, for every sequence $\{\mu_k\}_{k \in \mathbb{N}}$ converging to zero, every limit point ρ of $\{\rho_{\mu_k}\}_{k \in \mathbb{N}}$ is a weak solution of the deterministic controlled problem

$$\begin{cases} \partial_t \rho(t, x) = \operatorname{div}[b(\rho(t, x))] + f_g(x, \rho(t, x)) + \sigma_g(\rho(t, \cdot)) \varphi(t, x), \quad t > 0, \quad x \in \mathcal{O}, \\ \rho(0, x) = g(u_0(x)), \quad \rho(t, x) = 0, \quad x \in \partial \mathcal{O}, \end{cases}$$

where $b = 1/\gamma \circ g^{-1}$, $f_g = f \circ g^{-1}$, and $\sigma_g = \sigma \circ g^{-1}$. In order to identify uniquely the limit point and prove that $\{\rho_{\mu_k}\}_{k \in \mathbb{N}}$ converges to ρ , we had to extend some results proved in [24] about nonlinear evolution equations with locally monotone coefficients to prove that the equation above has a unique solution. Finally, by defining $u := g^{-1}(\rho)$, we have obtained the convergence of $u_{\mu_k}^\varphi$ to the solution of the controlled equation (1.6) and this has allowed us to conclude our proof.

2 Notations and assumptions

Throughout the present paper \mathcal{O} is a bounded domain in \mathbb{R}^d , with smooth boundary. We denote by H the Hilbert space $L^2(\mathcal{O})$ and by $\langle \cdot, \cdot \rangle_H$ the corresponding inner product. H^1 is the completion of $C_0^\infty(\mathcal{O})$ with respect to norm

$$\|u\|_{H^1}^2 := \|\nabla u\|_H^2 = \int_{\mathcal{O}} |\nabla u(x)|^2 dx,$$

and H^{-1} is the dual space to H^1 . Then H^1 , H and H^{-1} are all complete separable metric spaces, and $H^1 \subset H \subset H^{-1}$, with compact embeddings. In what follows, we shall denote

$$\mathcal{H} = H \times H^{-1}, \quad \mathcal{H}_1 = H^1 \times H.$$

Given the domain \mathcal{O} , we denote by $\{e_i\}_{i \in \mathbb{N}} \subset H^1$ the complete orthonormal basis of H which diagonalizes the Laplacian Δ , endowed with Dirichlet boundary conditions on $\partial \mathcal{O}$. Moreover, we denote by $\{-\alpha_i\}_{i \in \mathbb{N}}$ the corresponding sequence of eigenvalues, i.e.

$$\Delta e_i = -\alpha_i e_i, \quad i \in \mathbb{N}.$$

2.1 The stochastic term

We assume that $w^{\mathcal{Q}}(t)$ is a cylindrical Q -Wiener process, defined on a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. This means that $w^{\mathcal{Q}}(t)$ can be formally written as

$$w^{\mathcal{Q}}(t) = \sum_{i=1}^{\infty} Q e_i \beta_i(t),$$

where $\{\beta_i\}_{i \in \mathbb{N}}$ is a sequence of independent standard Brownian motions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, $\{e_i\}_{i \in \mathbb{N}}$ is the complete orthonormal system introduced above that diagonalizes the Laplace operator, endowed with Dirichlet boundary conditions, and $Q : H \rightarrow H$ is a bounded linear operator,. When $Q = I$, $w^{\mathcal{Q}}(t)$ will be denoted by $w(t)$. In particular, we have $w^{\mathcal{Q}}(t) = Q w(t)$.

In what follows we shall denote by H_Q the set $Q(H)$. H_Q is the reproducing kernel of the noise $w^{\mathcal{Q}}$ and is a Hilbert space, endowed with the inner product

$$\langle Qh, Qk \rangle_{H_Q} = \langle h, k \rangle_H, \quad h, k \in H.$$

Notice that the sequence $\{Qe_i\}_{i \in \mathbb{N}}$ is a complete orthonormal system in H_Q . Moreover, if U is any Hilbert space containing H_Q such that the embedding of H_Q into U is Hilbert-Schmidt, we have that

$$w^{\mathcal{Q}} \in C([0, T]; U). \quad (2.1)$$

Next, we recall that for every two separable Hilbert spaces E and F , $\mathcal{L}_2(E, F)$ denotes the space of Hilbert-Schmidt operators from E into F . $\mathcal{L}_2(E, F)$ is a Hilbert space, endowed with the inner product

$$\langle A, B \rangle_{\mathcal{L}_2(E, F)} = \text{Tr}_E [A^* B] = \text{Tr}_F [B A^*].$$

As well known, $\mathcal{L}_2(E, F) \subset \mathcal{L}(E, F)$ and

$$\|A\|_{\mathcal{L}(E, F)} \leq \|A\|_{\mathcal{L}_2(E, F)}. \quad (2.2)$$

Hypothesis 1. *The mapping $\sigma : H \rightarrow \mathcal{L}_2(H_Q, H)$ is defined by*

$$[\sigma(h) Q e_i](x) = \sigma_i(x, h(x)), \quad x \in \mathcal{O} \quad h \in H, \quad i \in \mathbb{N},$$

for some mapping $\sigma_i : \mathcal{O} \times \mathbb{R} \rightarrow \mathbb{R}$. We assume that there exists $L > 0$ such that

$$\sup_{x \in \mathcal{O}} \sum_{i=1}^{\infty} |\sigma_i(x, y_1) - \sigma_i(x, y_2)|^2 \leq L |y_1 - y_2|^2, \quad y_1, y_2 \in \mathbb{R}. \quad (2.3)$$

Moreover,

$$\sup_{x \in \mathcal{O}} \sum_{i=1}^{\infty} |\sigma_i(x, 0)|^2 =: \sigma_0^2 < \infty. \quad (2.4)$$

Remark 2.1. 1. Condition (2.3) implies that σ is Lipschitz continuous. Namely for any $h_1, h_2 \in H$

$$\|\sigma(h_1) - \sigma(h_2)\|_{\mathcal{L}_2(H_Q, H)} \leq \sqrt{L} \|h_1 - h_2\|_H. \quad (2.5)$$

This, together with condition (2.4), implies also that σ has linear growth, that is

$$\|\sigma(h)\|_{\mathcal{L}_2(H_Q, H)} \leq \sqrt{L} \|h\|_H + |\mathcal{O}|^{1/2} \sigma_0. \quad (2.6)$$

2. If σ is constant, then Hypothesis 1 means that σQ is a Hilbert-Schmidt operator in H .
3. If σ is not constant, Hypothesis 1 is satisfied for example when

$$[\sigma(h)Qk](x) = s(x, h(x))Qk(x), \quad x \in \mathcal{O}, \quad h, k \in H,$$

for some measurable function $s : \mathcal{O} \times \mathbb{R} \rightarrow \mathbb{R}$ such that $s(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous, uniformly with respect to $x \in \mathcal{O}$, and for some $Q \in \mathcal{L}(H)$ such that

$$\sum_{i=1}^{\infty} \|Qe_i\|_{L^\infty(\mathcal{O})}^2 < \infty. \quad (2.7)$$

In case Q is diagonalizable with respect the basis $(e_i)_{i \in \mathbb{N}}$, with $Qe_i = \lambda_i e_i$, condition (2.7) reads

$$\sum_{i=1}^{\infty} \lambda_i^2 \|e_i\|_{L^\infty(\mathcal{O})}^2 < \infty. \quad (2.8)$$

In general (see [18]), we have

$$\|e_i\|_{L^\infty(\mathcal{O})} \leq c i^\alpha,$$

for some $\alpha > 0$, and (2.8) becomes

$$\sum_{i=1}^{\infty} \lambda_i^2 i^{2\alpha} < \infty.$$

In particular, when $d = 1$ or the domain is a hyperrectangle when $d > 1$ the eigenfunctions $(e_i)_{i \in \mathbb{N}}$ are equi-bounded and (2.8) becomes $Q \in \mathcal{L}_2(H)$.

2.2 The coefficients γ and f

Throughout the paper, we shall assume that the friction coefficient satisfies the following condition

Hypothesis 2. *The mapping γ belongs to $C_b^1(\mathbb{R})$ and there exist γ_0 and γ_1 such that*

$$0 < \gamma_0 \leq \gamma(r) \leq \gamma_1, \quad r \in \mathbb{R}. \quad (2.9)$$

In what follows, we shall define

$$g(r) = \int_0^r \gamma(\sigma) d\sigma, \quad r \in \mathbb{R}.$$

Remark 2.2. 1. Clearly $g(0) = 0$ and $g'(r) = \gamma(r)$. In particular, due to (2.9), g is uniformly Lipschitz continuous on \mathbb{R} .

2. The function g is strictly increasing and

$$(g(r_1) - g(r_2))(r_1 - r_2) \geq \gamma_0 |r_1 - r_2|^2, \quad r_1, r_2 \in \mathbb{R}.$$

As far as the nonlinearity f is concerned, in this paper we shall consider two situations: f is Lipschitz continuous and \mathcal{O} is a bounded smooth domain in \mathbb{R}^d , for any arbitrary $d \geq 1$, or f is only locally Lipschitz continuous with polynomial growth and \mathcal{O} is a bounded interval in \mathbb{R} .

Hypothesis 3. *The mapping $f : \mathcal{O} \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable and there exists $c > 0$ such that*

$$\sup_{x \in \mathcal{O}} |f(x, r) - f(x, s)| \leq c |r - s|, \quad r, s \in \mathcal{O}.$$

Moreover

$$\sup_{x \in \mathcal{O}} |f(x, 0)| < \infty.$$

In what follows, for every function $u : \mathcal{O} \rightarrow \mathbb{R}$, we shall denote

$$F(u)(x) = f(x, u(x)), \quad x \in \mathcal{O}.$$

Hypothesis 4. *We have $\mathcal{O} = [0, L]$ and the mapping $f : [0, L] \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable and satisfies the following conditions.*

1. *There exist $\theta > 1$ and $c_1 > 0$ such that for every $r \in \mathbb{R}$*

$$\sup_{x \in [0, L]} |f(x, r)| \leq c_1 (1 + |r|^\theta), \quad \sup_{x \in [0, L]} |\partial_r f(x, r)| \leq c_1 (1 + |r|^{\theta-1}). \quad (2.10)$$

Moreover, there exists $c_2 > 0$ such that for every $r \in \mathbb{R}$ and $x \in [0, L]$

$$f(x, r) := \int_0^r f(x, s) ds \leq c_2 (1 - |r|^{\theta+1}). \quad (2.11)$$

2. *For every $x \in [0, L]$, the function $f(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable and*

$$\sup_{(x, r) \in [0, L] \times \mathbb{R}} \partial_r f(x, r) \leq 0. \quad (2.12)$$

3. *For every $r \in \mathbb{R}$, the function $f(\cdot, r) : [0, L] \rightarrow \mathbb{R}$ is differentiable and*

$$\sup_{x \in [0, L]} |\partial_x f(x, r)| \leq c(1 + |r|), \quad r \in \mathbb{R}.$$

Remark 2.3. 1. A typical example of a function f satisfying Hypothesis 4 is

$$f(r) = -a|r|^{\theta-1}r.$$

2. When $d = 1$, we have that $H^1 \hookrightarrow L^\infty(\mathcal{O})$, and then $f(u) \in H$, for every $u \in H^1$.

3. We are assuming (2.12) just for the sake of simplicity. In fact, our results remain true under the condition

$$\sup_{(x,r) \in [0,L] \times \mathbb{R}} \partial_r f(x,r) < \infty.$$

4. Thanks to (2.12) we have

$$\langle F(u) - F(v), u - v \rangle_H \leq 0, \quad u, v \in H^1.$$

In particular, there exists some $c > 0$ such that

$$\langle F(u), u \rangle_H \leq c \|u\|_H, \quad u \in H^1. \quad (2.13)$$

5. Due to (2.10), for every $u, v \in H^1$, we have

$$\|F(u) - F(v)\|_H^2 \leq c \int_{\mathcal{O}} \left(1 + |u(x)|^{2(\theta-1)} + |v(x)|^{2(\theta-1)}\right) |u(x) - v(x)|^2 dx,$$

so that

$$\|F(u) - F(v)\|_H \leq c \left(1 + \|u\|_{H^1}^{\theta-1} + \|v\|_{H^1}^{\theta-1}\right) \|u - v\|_H. \quad (2.14)$$

In particular, we have

$$\|F(u)\|_H \leq c \left(1 + \|u\|_{H^1}^{\theta}\right), \quad u \in H^1.$$

6. In the same way

$$\begin{aligned} \|F(u) - F(v)\|_{L^1} &\leq c \int_{\mathcal{O}} \left(1 + |u(x)|^{\theta-1} + |v(x)|^{\theta-1}\right) |u(x) - v(x)| dx \\ &\leq c \left(1 + \|u\|_{L^{2(\theta-1)}}^{\theta-1} + \|v\|_{L^{2(\theta-1)}}^{\theta-1}\right) \|u - v\|_H. \end{aligned} \quad (2.15)$$

Now, by proceeding as in [28] (see also [5]), for every $n \in \mathbb{N}$ and $x \in \mathcal{O}$ we define

$$f_n(x,r) := \begin{cases} f(x,n) + (r-n)\partial_r f(x,n), & \text{if } r \geq n, \\ f(x,r), & \text{if } r \in [-n,n], \\ f(x,-n) + (r+n)\partial_r f(x,-n), & \text{if } r \leq -n. \end{cases} \quad (2.16)$$

Clearly, for every $n \in \mathbb{N}$, the mapping $f_n(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous, uniformly with respect to $x \in [0, L]$, and

$$f_n(x,r) = f(x,r), \quad x \in [0, L], \quad r \in [-n, n].$$

Moreover,

$$\sup_{x \in [0, L]} |f_n(x,r)| \leq c \left(1 + |r|^{\theta}\right), \quad \sup_{x \in [0, L]} |\partial_r f_n(x,r)| \leq c \left(1 + |r|^{\theta-1}\right), \quad (2.17)$$

for some constant c independent of n , and there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$

$$f_n(x,r) := \int_0^r f_n(x,s) ds \leq c \left(1 - n^{\theta+1}\right), \quad r \in \mathbb{R}, \quad x \in [0, L]. \quad (2.18)$$

3 The problem and the method

As we mentioned in the introduction, we are interested in the study of the validity of a large deviation principle, as $\mu \downarrow 0$ for the family $\{\mathcal{L}(u_\mu)\}_{\mu>0}$, where u_μ is the solution of equation (1.1). Our final goal is proving the following result.

Theorem 3.1. *Assume Hypotheses 1 and 2 and either Hypothesis 3 or Hypothesis 4 and fix $p < \infty$, if $d = 1, 2$, and $p < 2d/(d-2)$, if $d > 2$. Then, for every $(u_0, v_0) \in \mathcal{H}_1$ and $T > 0$, the family $\{\mathcal{L}(u_\mu)\}_{\mu>0}$ satisfy a large deviation principle in $C([0, T]; L^p(\mathcal{O}))$, as $\mu \downarrow 0$, with action functional*

$$I_T(u) = \frac{1}{2} \left\{ \int_0^T \|\varphi(t)\|_H^2 dt : u(t) = u^\varphi(t), t \in [0, T] \right\}, \quad (3.1)$$

where $u^\varphi(t)$ denotes the unique weak solution to the quasi-linear parabolic equation

$$\begin{cases} \partial_t u(t, x) = \gamma^{-1}(u(t, x)) [\Delta u(t, x) + f(x, u(t, x)) + \sigma(u(t, \cdot))\varphi(t, x)], \\ u(0, x) = u_0(x), \quad u(t, x) = 0, \quad x \in \partial \mathcal{O}. \end{cases} \quad (3.2)$$

Theorem 3.1 is proved by following the classical weak convergence approach to large deviations, as developed for SPDEs in [3]. To this purpose, we need first to introduce some notations. For every $T > 0$, we denote by \mathcal{P}_T the set of predictable processes in $L^2(\Omega \times [0, T]; H)$, and for every $M > 0$ we introduce the sets

$$\mathcal{S}_{T, M} := \left\{ \varphi \in L^2_w(0, T; H) : \|\varphi\|_{L^2(0, T; H)} \leq M \right\},$$

and

$$\Lambda_{T, M} := \{ \varphi \in \mathcal{P}_T : \varphi \in \mathcal{S}_{T, M}, \mathbb{P} - \text{a.s.} \}.$$

Next, for every $\varphi \in \mathcal{P}_T$ we consider the controlled version of equation (1.1)

$$\begin{cases} \mu \partial_t^2 u_\mu(t, x) = \Delta u_\mu(t, x) - \gamma(u_\mu(t, x)) \partial_t u_\mu(t, x) + f(x, u_\mu(t, x)) + \sigma(u_\mu(t, \cdot)) \varphi(t, x) \\ \quad + \sqrt{\mu} \sigma(u_\mu(t, \cdot)) \partial_t w^\varphi(t, x), \quad t > 0, \quad x \in \mathcal{O}, \\ u_\mu(0, x) = u_0(x), \quad \partial_t u_\mu(0, x) = v_0(x), \quad u_\mu(t, x) = 0, \quad x \in \partial \mathcal{O}, \end{cases} \quad (3.3)$$

The well-posedness of the equation above has been proven in [14] in the case the nonlinearity f is Lipschitz-continuous, the diffusion coefficient σ is bounded and the control $\varphi = 0$. In what follows, we will prove that also under the more general conditions we are assuming for f and σ , the following results holds.

Theorem 3.2. *Under Hypotheses 1 and 2 and either Hypothesis 3 or Hypothesis 4, for every $T, M > 0$ and $\varphi \in \Lambda_{T, M}$ and for every initial conditions $(u_0, v_0) \in \mathcal{H}_1$, there exists a unique adapted process*

$(u_\mu, v_\mu) \in L^2(\Omega, C([0, T]; \mathcal{H}_1))$ which solves the systems of equations

$$\begin{cases} u_\mu(t, x) = u_0(x) + \int_0^t v_\mu(s, x) ds, \\ \mu v_\mu(t, x) = \mu v_0(x) + \int_0^t [\Delta u_\mu(s, x) - \gamma(u_\mu(s, x))v_\mu(s, x) + f(x, u_\mu(s, x)) + \sigma(u_\mu(s))Q\varphi(s, x)] ds \\ \quad + \sqrt{\mu} \int_0^t \sigma(u_\mu(s)) dw^\mathcal{Q}(s). \end{cases} \quad (3.4)$$

Once proved Theorem 3.2, we introduce the following two conditions.

C1. Let $\{\varphi_\mu\}_{\mu>0}$ be an arbitrary family of processes in $\Lambda_{T,M}$ such that

$$\lim_{\mu \rightarrow 0} \varphi_\mu = \varphi, \quad \text{in distribution in } L_w^2(0, T; H),$$

where $L_w^2(0, T; H)$ is the space $L^2(0, T; H)$ endowed with the weak topology and $\varphi \in \Lambda_{T,M}$. Then, for every $p < \infty$ we have

$$\lim_{\mu \rightarrow 0} u_\mu^{\varphi_\mu} = u^\varphi, \quad \text{weakly in } C([0, T], L^p(\mathcal{O})),$$

where $u_\mu^{\varphi_\mu}$ is the solution to (3.3), corresponding to the control φ_μ , and u^φ is the solution to (3.2), corresponding to the control φ .

C2. For every $T, R > 0$ and $p < \infty$, the level sets $\Phi_{T,R} = \{I_T \leq R\}$ are compact in the space $C([0, T]; L^p(\mathcal{O}))$.

As shown in [3], Conditions C1. and C2. imply the validity of a Laplace principle with action functional I_T in the space $C([0, T]; L^p(\mathcal{O}))$ for the family $\{u_\mu\}_{\mu>0}$. Due to the compactness of the level sets $\Phi_{T,R}$ stated in C1. this is equivalent to the validity of Theorem 3.1.

Thus, in what follows our strategy will be first proving Theorem 3.2, for every fixed $\mu > 0$, and then proving conditions C1. and C2.

4 Well-posedness of equation (3.4)

In Theorem 3.2 the parameter $\mu > 0$ is fixed. This means that in this section, without any loss of generality, we can assume $\mu = 1$. If we denote

$$\eta := v + g(u), \quad z = (u, \eta),$$

then system (3.4) can be rewritten as the following abstract stochastic evolution equation

$$dz(t) = [A(z(t)) + B_\varphi(t, z(t))] dt + \Sigma(z(t)) dw^\mathcal{Q}(t), \quad z(0) = (u_0, v_0 + g(u_0)), \quad (4.1)$$

where

$$A(u, \eta) = (-g(u) + \eta, \Delta u + F(u)), \quad (u, \eta) \in D(A) = \mathcal{H}_1,$$

$$B_\varphi(t, (u, \eta)) = (0, \sigma(u)Q\varphi(t)), \quad (u, \eta) \in \mathcal{H}, \quad t \in [0, T],$$

and

$$\Sigma(u, \eta) = (0, \sigma(u)), \quad (u, \eta) \in \mathcal{H}.$$

This means that the adapted \mathcal{H}_1 -valued process $z(t) = (u(t), \eta(t))$ is the unique solution to the equation

$$z(t) = (u_0, v_0 + g(u_0)) + \int_0^t [A(z(s)) + B_\varphi(s, z(s))] ds + \int_0^t \Sigma(z(s)) dw^\mathcal{Q}(s), \quad (4.2)$$

if and only if the adapted \mathcal{H}_1 -valued process $(u(t), v(t)) = (u(t), -g(u(t)) + \eta(t))$ is the unique solution of

$$\begin{cases} u(t, x) = u_0(x) + \int_0^t v(s, x) ds, \\ v(t, x) = \mu v_0(x) + \int_0^t [\Delta u(s, x) - \gamma(u(s, x))v(s, x) + f(x, u(s, x)) + \sigma(u(s, \cdot))\mathcal{Q}\varphi(s, x)] ds \\ \quad + \int_0^t \sigma(u(s, \cdot)) dw^\mathcal{Q}(s, xx). \end{cases} \quad (4.3)$$

In our proof of Theorem 3.2 we first assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz-continuous and then we consider the case Hypothesis 4 holds.

4.1 The case when f satisfies Hypothesis 3

In [14, Section 3] an analogous result has been proved, in the case $\varphi = 0$ (and hence $B_\varphi = 0$) and σ . Here, we extend the arguments used in [14] to consider the case of an arbitrary $\varphi \in \Lambda_{T, M}$, σ having linear growth. As in [14, Section 3], the arguments we are using here are based on classical tools from the theory of monotone non-linear operators and we refer to the monograph [1] for all details.

Since f is assumed to be Lipschitz continuous, we have

$$\|A(z)\|_{\mathcal{H}} \leq c(1 + \|z\|_{\mathcal{H}_1}), \quad z \in D(A), \quad (4.4)$$

and, as shown in [14, Lemma 3.1], there exists $\kappa \in \mathbb{R}$ such that

$$\langle A(z_1) - A(z_2), z_1 - z_2 \rangle_{\mathcal{H}} \leq \kappa \|z_1 - z_2\|_{\mathcal{H}}^2.$$

Moreover, for every $\lambda > 0$ small enough

$$\text{Range}(I - \lambda A) = \mathcal{H}.$$

This means that the operator $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ is *quasi m -dissipative*. In particular, this implies that there exists $\lambda_0 > 0$ such that

$$J_\lambda := (I - \lambda A)^{-1}, \quad \lambda \in (0, \lambda_0),$$

is a well-defined Lipschitz-continuous mapping in \mathcal{H} and we can introduce the *Yosida approximation* of A , defined as

$$A_\lambda(z) := A(J_\lambda(z)) = \frac{1}{\lambda} (J_\lambda(z) - z), \quad z \in \mathcal{H}.$$

Notice that

$$\langle A_\lambda(z_1) - A_\lambda(z_2), z_1 - z_2 \rangle_{\mathcal{H}} \leq \frac{\kappa}{1 - \lambda \kappa} \|z_1 - z_2\|_{\mathcal{H}}^2, \quad (4.5)$$

and

$$\|A_\lambda(z_1) - A_\lambda(z_2)\|_{\mathcal{H}} \leq \frac{2}{\lambda(1 - \lambda \kappa)} \|z_1 - z_2\|_{\mathcal{H}}. \quad (4.6)$$

Moreover, for every $z \in D(A)$

$$\|A^\lambda(z)\|_{\mathcal{H}} \leq \frac{1}{1 - \lambda \kappa} \|A(z)\|_{\mathcal{H}},$$

so that for every $z \in D(A)$

$$\|J_\lambda(z) - z\|_{\mathcal{H}} = \lambda \|A^\lambda(z)\| \leq \frac{\lambda}{1 - \lambda \kappa} \|A(z)\|_{\mathcal{H}},$$

and

$$\lim_{\lambda \rightarrow 0} \|A_\lambda(z) - A(z)\|_{\mathcal{H}} = 0.$$

In [14, Proof of Theorem 3.2], it has been shown that there exists some $\lambda_1 \in (0, \lambda_0)$ such that for every $\lambda \in (0, \lambda_1)$

$$\langle A_\lambda(z), z \rangle_{\mathcal{H}_1} \leq -\frac{\gamma_0}{2} \|J_\lambda(z)_1\|_{H^1}^2 + c \|J_\lambda(z)_2\|_{H^{-1}}^2, \quad z \in \mathcal{H}_1. \quad (4.7)$$

Furthermore, for every $\lambda, \nu \in (0, \lambda_0)$ and $z_1, z_2 \in \mathcal{H}_1$ it holds

$$\langle A_\lambda(z_1) - A_\nu(z_2), z_1 - z_2 \rangle_{\mathcal{H}} \leq \|z_1 - z_2\|_{\mathcal{H}}^2 + c(\lambda + \nu) (\|z_1\|_{\mathcal{H}_1}^2 + \|z_2\|_{\mathcal{H}_1}^2 + 1). \quad (4.8)$$

Concerning the random operator B_φ , according to Hypothesis 1 for every $t \in [0, T]$ we have that $B_\varphi(t, \cdot) : \mathcal{H} \rightarrow \mathcal{H}_1$ is well defined and, in view of (2.5), for any $z_1, z_2 \in \mathcal{H}$

$$\|B_\varphi(t, z_1) - B_\varphi(t, z_2)\|_{\mathcal{H}_1} = \|(\sigma(u_1) - \sigma(u_2)) Q\varphi(t)\|_H \leq \sqrt{L} \|u_1 - u_2\|_H \|\varphi(t)\|_H. \quad (4.9)$$

Finally, Hypothesis 1 implies that the mapping $\Sigma : \mathcal{H} \rightarrow \mathcal{L}_2(H_Q, \mathcal{H}_1)$ is well-defined and due to (2.5) for any $z_1, z_2 \in \mathcal{H}$

$$\|\Sigma(z_1) - \Sigma(z_2)\|_{\mathcal{L}_2(H_Q, \mathcal{H}_1)} = \|\sigma(u_1) - \sigma(u_2)\|_{\mathcal{L}_2(H_Q, H)} \leq \sqrt{L} \|u_1 - u_2\|_H. \quad (4.10)$$

Step 1. For every $\lambda \in (0, \lambda_0)$ the approximating problem

$$dz(t) = [A_\lambda(z(t)) + B_\varphi(t, z(t))] dt + \Sigma(z(t)) dw^Q(t), \quad z(0) = (u_0, v_0 + g(u_0)) \quad (4.11)$$

admits a unique solution $z_\lambda \in L^2(\Omega; C([0, T]; \mathcal{H}))$.

Proof of Step 1. According to (4.6), we have

$$\int_0^T \|A_\lambda(z_1(s)) - A_\lambda(z_2(s))\|_{\mathcal{H}}^2 ds \leq \frac{c}{\lambda^2} \int_0^T \|z_1(s) - z_2(s)\|_{\mathcal{H}}^2 ds \leq \frac{cT}{\lambda^2} \sup_{t \in [0, T]} \|z_1(t) - z_2(t)\|_{\mathcal{H}}^2. \quad (4.12)$$

According to (4.9), if $\varphi \in \Lambda_{T,M}$, we have

$$\begin{aligned} \int_0^T \|B_\varphi(t, z_1) - B_\varphi(t, z_2)\|_{\mathcal{H}_1}^2 dt &\leq L \int_0^T \|u_1(t) - u_2(t)\|_H^2 \|\varphi(t)\|_H^2 dt \\ &\leq LM^2 \sup_{t \in [0, T]} \|z_1(t) - z_2(t)\|_{\mathcal{H}}^2, \quad \mathbb{P} - \text{a.s.} \end{aligned} \quad (4.13)$$

Finally, according to (4.10), we have

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} \left\| \int_0^t (\Sigma(z_1(s)) - \Sigma(z_2(s))) dw^\mathcal{Q}(s) \right\|_{\mathcal{H}_1}^2 &\leq c \int_0^T \mathbb{E} \|\Sigma(z_1(s)) - \Sigma(z_2(s))\|_{\mathcal{L}_2(H_\mathcal{Q}, \mathcal{H}_1)}^2 ds \\ &\leq TL \mathbb{E} \sup_{t \in [0, T]} \|z_1(t) - z_2(t)\|_{\mathcal{H}}^2. \end{aligned} \quad (4.14)$$

Therefore, in view of (4.12), (4.13) and (4.14), for every $\lambda \in (0, \lambda_0)$ the mapping

$$\Phi_\lambda(z)(t) = (u_0, v_0 + g(u_0)) + \int_0^t [A_\lambda(z(s)) + B_\varphi(s, z(s))] ds + \int_0^t \Sigma(z(s)) dw^\mathcal{Q}(s)$$

is Lipschitz continuous from $L^2(\Omega; C([0, T]; \mathcal{H}))$ into itself, and then for every $\lambda \in (0, \lambda_0)$ equation (4.11) admits a unique solution $z_\lambda \in L^2(\Omega; C([0, T]; \mathcal{H}))$.

Step 2. There exists $c_T > 0$ such that

$$\mathbb{E} \sup_{t \in [0, T]} \|z_\lambda(t)\|_{\mathcal{H}_1}^2 \leq c_T (1 + \|z_0\|_{\mathcal{H}_1}^2), \quad \lambda \in (0, \lambda_1). \quad (4.15)$$

Proof of Step 2. As a consequence of Itô's formula, we have

$$\begin{aligned} \|z_\lambda(t)\|_{\mathcal{H}_1}^2 &= \|z_0\|_{\mathcal{H}_1}^2 + 2 \int_0^t \langle A_\lambda(z_\lambda(s)), z_\lambda(s) \rangle_{\mathcal{H}_1} ds + 2 \int_0^t \langle B_\varphi(s, z_\lambda(s)), z_\lambda(s) \rangle_{\mathcal{H}_1} ds \\ &\quad + \int_0^t \|\Sigma(z_\lambda(s))\|_{\mathcal{L}_2(H_\mathcal{Q}, \mathcal{H}_1)}^2 ds + 2 \int_0^t \langle z_\lambda(s), \Sigma(z_\lambda(s)) dw^\mathcal{Q}(s) \rangle_{\mathcal{H}_1}. \end{aligned}$$

Due to (4.7), we have

$$\begin{aligned} \int_0^t \langle A_\lambda(z_\lambda(s)), z_\lambda(s) \rangle_{H^{-1}} ds &\leq -\frac{\gamma_0}{2} \int_0^t \|J_\lambda(z_\lambda(s))_1\|_{H^1}^2 ds + c \int_0^t \|J_\lambda(z_\lambda(s))\|_{\mathcal{H}}^2 ds \\ &\leq -\frac{\gamma_0}{2} \int_0^t \|J_\lambda(z_\lambda(s))_1\|_{H^1}^2 ds + \int_0^t \|z_\lambda(s)\|_{\mathcal{H}}^2 ds. \end{aligned} \quad (4.16)$$

Next, recalling that $\varphi \in \Lambda_{T,M}$, due to (4.9) for every $\delta > 0$ we have

$$\begin{aligned} \left| \int_0^t \langle B_\varphi(s, z_\lambda(s)), z_\lambda(s) \rangle_{\mathcal{H}_1} ds \right| &\leq \delta \int_0^t \|B_\varphi(s, z_\lambda(s))\|_{\mathcal{H}_1}^2 ds + \frac{c}{\delta} \int_0^t \|z_\lambda(s)\|_{\mathcal{H}_1}^2 ds \\ &\leq \delta c_M \left(1 + \sup_{r \in [0, t]} \|z_\lambda(r)\|_{\mathcal{H}}^2 \right) + \frac{c}{\delta} \int_0^t \|z_\lambda(s)\|_{\mathcal{H}_1}^2 ds, \quad \mathbb{P} - \text{a.s.} \end{aligned} \quad (4.17)$$

In the same way, thanks to (4.10) for every $\delta > 0$

$$\begin{aligned} \mathbb{E} \sup_{r \in [0, t]} \left| \int_0^r \langle z_\lambda(s), \Sigma(z_\lambda(s)) dw^{\mathcal{Q}}(s) \rangle_{\mathcal{H}_1} \right| &\leq c \mathbb{E} \left(\sup_{r \in [0, t]} \|z_\lambda(r)\|_{\mathcal{H}_1}^2 \int_0^t \|\Sigma(z_\lambda(s))\|_{\mathcal{L}_2(H_{\mathcal{Q}}, \mathcal{H}_1)}^2 ds \right)^{\frac{1}{2}} \\ &\leq \delta \mathbb{E} \sup_{r \in [0, t]} \|z_\lambda(r)\|_{\mathcal{H}_1}^2 + \frac{c}{\delta} \int_0^t \mathbb{E} \|z_\lambda(s)\|_{\mathcal{H}_1}^2 ds + \frac{cT}{\delta}. \end{aligned} \quad (4.18)$$

Finally, due to (4.10) we get

$$\int_0^t \|\Sigma(z_\lambda(s))\|_{\mathcal{L}_2(H_{\mathcal{Q}}, \mathcal{H}_1)}^2 ds \leq cT \left(1 + \int_0^t \mathbb{E} \|z_\lambda(s)\|_{\mathcal{H}}^2 ds \right) \quad (4.19)$$

Therefore, if we choose $\delta > 0$ small enough, from (4.16), (4.17), (4.18) and (4.19) we get

$$\mathbb{E} \sup_{r \in [0, t]} \|z_\lambda(r)\|_{\mathcal{H}_1}^2 + \frac{\gamma_0}{2} \int_0^t \mathbb{E} \|J_\lambda(z_\lambda(s))_1\|_{H^1}^2 ds \leq cT \int_0^t \mathbb{E} \sup_{r \in [0, s]} \|z_\lambda(r)\|_{\mathcal{H}_1}^2 ds + cT.$$

Now, Gronwall's lemma allows to obtain (4.15).

Step 3. There exists $z \in L^2(\Omega; C([0, T]; \mathcal{H}))$ such that

$$\lim_{\lambda \rightarrow 0} \mathbb{E} \sup_{t \in [0, T]} \|z_\lambda(t) - z(t)\|_{\mathcal{H}}^2 = 0. \quad (4.20)$$

Proof of Step 3. For every $\lambda, \nu \in (0, \lambda_1)$, we denote $\rho_{\lambda, \nu}(t) := z_\lambda(t) - z_\nu(t)$. We have

$$\begin{aligned} d\rho_{\lambda, \nu}(t) &= [A_\lambda(z_\lambda(t)) - A_\nu(z_\nu(t))] dt \\ &\quad + [B_\varphi(t, z_\lambda(t)) - B_\varphi(t, z_\nu(t))] dt + [\Sigma(z_\lambda(t)) - \Sigma(z_\nu(t))] dw^{\mathcal{Q}}(t). \end{aligned}$$

We have

$$\begin{aligned} \|\rho_{\lambda, \nu}(t)\|_{\mathcal{H}}^2 &= 2 \int_0^t \langle A_\lambda(z_\lambda(s)) - A_\nu(z_\nu(s)), \rho_{\lambda, \nu}(s) \rangle_{\mathcal{H}} ds \\ &\quad + 2 \int_0^t \langle B_\varphi(s, z_\lambda(s)) - B_\varphi(s, z_\nu(s)), \rho_{\lambda, \nu}(s) \rangle_{\mathcal{H}} ds + \int_0^t \|\Sigma(z_\lambda(s)) - \Sigma(z_\nu(s))\|_{\mathcal{L}_2(H_{\mathcal{Q}}, \mathcal{H})}^2 ds \\ &\quad + 2 \int_0^t \langle \rho_{\lambda, \nu}(s), (\Sigma(z_\lambda(s)) - \Sigma(z_\nu(s))) dw^{\mathcal{Q}}(s) \rangle_{\mathcal{H}} =: \sum_{k=1}^4 I_k(t). \end{aligned}$$

Due to (4.8), we have

$$|I_1(t)| \leq c \int_0^t \|\rho_{\lambda, \nu}(s)\|_{\mathcal{H}}^2 ds + c(\lambda + \nu) \int_0^t (\|z_\lambda(s)\|_{\mathcal{H}_1}^2 + \|z_\nu(s)\|_{\mathcal{H}_1}^2 + 1) ds, \quad (4.21)$$

and due to (4.10) we have

$$|I_3(t)| \leq c \int_0^t \|\rho_{\lambda, \nu}(s)\|_{\mathcal{H}}^2 ds. \quad (4.22)$$

Moreover, by proceeding as in the proof of (4.17) and (4.18), for every $\delta > 0$ we have

$$\mathbb{E} \sup_{r \in [0, t]} |I_2(r)| + \mathbb{E} \sup_{r \in [0, t]} |I_4(r)| \leq \delta \mathbb{E} \sup_{r \in [0, t]} \|\rho_{\lambda, \nu}(r)\|_{\mathcal{H}}^2 + \frac{c}{\delta} \int_0^t \mathbb{E} \|\rho_{\lambda, \nu}(r)\|_{\mathcal{H}}^2 dr. \quad (4.23)$$

Therefore, if we choose $\delta > 0$ sufficiently small, from (4.21), (4.22) and (4.23), we obtain

$$\begin{aligned} \mathbb{E} \sup_{r \in [0, t]} \|\rho_{\lambda, \nu}(r)\|_{\mathcal{H}}^2 &\leq c \int_0^t \mathbb{E} \sup_{r \in [0, s]} \|\rho_{\lambda, \nu}(r)\|_{\mathcal{H}}^2 ds \\ &\quad + c(\lambda + \nu) \int_0^t (\|z_\lambda(s)\|_{\mathcal{H}_1}^2 + \|z_\nu(s)\|_{\mathcal{H}_1}^2 + 1) ds, \end{aligned}$$

and the Gronwall lemma, together with (4.15), gives

$$\begin{aligned} \mathbb{E} \sup_{r \in [0, T]} \|\rho_{\lambda, \nu}(r)\|_{\mathcal{H}}^2 &\leq c_T(\lambda + \nu) \int_0^T (\|z_\lambda(s)\|_{\mathcal{H}_1}^2 + \|z_\nu(s)\|_{\mathcal{H}_1}^2 + 1) ds \\ &\leq c_T(\lambda + \nu) (1 + \|z_0\|_{\mathcal{H}}^2). \end{aligned}$$

This implies that

$$\lim_{\lambda, \nu \rightarrow 0} \mathbb{E} \sup_{r \in [0, T]} \|\rho_{\lambda, \nu}(r)\|_{\mathcal{H}}^2 = 0,$$

so that the family $\{z_\lambda\}_{\lambda \in (0, \lambda_1)}$ is Cauchy and (4.20) follows.

Step 4. There exists a unique solution $z \in L^2(\Omega; C([0, T]; \mathcal{H}_1))$ for equation (4.1).

Proof of step 4. For every $\lambda \in (0, \lambda_0)$ we have that z_λ satisfies equation (4.11). Then, by proceeding as in [14, Proof of Theorem 3.2], we take the limit, as λ goes to zero, of both sides of (4.11) in $L^2(\Omega; C([0, T]; \mathcal{H}_{-1}))$ and thanks to (4.20) we obtain that z satisfies the equation

$$z(t) = (u_0, v_0 + g(u_0)) + \int_0^t [A(z(s)) + B_\varphi(s, z(s))] ds + \int_0^t \Sigma(z(s)) dw^\mathcal{Q}(s),$$

and $z \in L^2(\Omega; L^\infty(0, T; \mathcal{H}_1))$.

Next, by using again arguments analogous to those used in [14, Proof of Theorem 3.2], we can show that z has continuous trajectories and is the unique solution of equation (4.2).

4.2 The case when f satisfies Hypothesis 4

In view of what we have seen in Subsection 4.1, for every $n \in \mathbb{N}$ and for every $\varphi \in \Lambda_{T, M}$ and $(u_0, v_0) \in \mathcal{H}_1$ there exists a unique solution $(u_n, v_n) \in L^2(\Omega; C([0, T]; \mathcal{H}_1))$ for the equation

$$\begin{cases} u_n(t, x) = u_0(x) + \int_0^t v_n(s, x) ds, \\ v_n(t, x) = \mu v_0(x) + \int_0^t [\Delta u_n(s, x) - \gamma(u_n(s, x))v_n(s, x) + f_n(x, u(s, x)) + \sigma(u_n(s, \cdot))\mathcal{Q}\varphi(s, x)] ds \\ \quad + \int_0^t \sigma(u(s, \cdot)) dw^\mathcal{Q}(s, x), \end{cases} \quad (4.24)$$

where f_n is the function defined in (2.16).

Step 1. There exists $c_T > 0$ independent of $n \in \mathbb{N}$ such that

$$\mathbb{E} \sup_{t \in [0, T]} \|(u_n(t), v_n(t))\|_{\mathcal{H}_1}^2 + \gamma_0 \int_0^T \mathbb{E} \|v_n(s)\|_H^2 ds \leq c_T \left(1 + \|u_0\|_{H^1}^{\theta+1} + \|v_0\|_H^2\right), \quad n \geq n_0. \quad (4.25)$$

Proof of Step 1. From the Itô formula we have

$$\begin{aligned} \|u_n(t)\|_{H^1}^2 + \|v_n(t)\|_H^2 &= \|u_0\|_{H^1}^2 + \|v_0\|_H^2 - \int_0^t \langle \gamma(u_n(s))v_n(s), v_n(s) \rangle_H ds \\ &+ \int_{\mathcal{O}} \mathfrak{f}_n(u_n(t, x)) dx - \int_{\mathcal{O}} \mathfrak{f}_n(u_0(x)) dx + \int_0^t \langle \sigma(u_n(s))\mathcal{Q}\varphi(s), v_n(s) \rangle_H ds \\ &+ \int_0^t \langle \sigma(u_n(s))dw^{\mathcal{Q}}(s), v_n(s) \rangle_H. \end{aligned}$$

Thanks to (2.17) and (2.18) we have

$$\int_{\mathcal{O}} \mathfrak{f}_n(u_n(s, x)) dx \leq c \left(1 - n^{\theta+1}\right) \leq c, \quad n \geq n_0,$$

and

$$\left| \int_{\mathcal{O}} \mathfrak{f}_n(u_0(x)) dx \right| \leq c \left(1 + \int_{\mathcal{O}} |u_0(x)|^{\theta+1} dx\right) = c \left(1 + \|u_0\|_{L^{\theta+1}}^{\theta+1}\right) \leq c \left(1 + \|u_0\|_{H^1}^{\theta+1}\right).$$

Therefore,

$$\begin{aligned} &\sup_{r \in [0, t]} \|u_n(r)\|_{H^1}^2 + \sup_{r \in [0, t]} \|v_n(r)\|_H^2 + \gamma_0 \int_0^t \|v_n(s)\|_H^2 ds \\ &\leq c \left(1 + \|u_0\|_{H^1}^{\theta+1} + \|v_0\|_H^2\right) + \sup_{r \in [0, t]} \left| \int_0^t \langle \sigma(u_n(s))\mathcal{Q}\varphi(s), v_n(s) \rangle_H ds \right| \\ &+ \sup_{r \in [0, t]} \left| \int_0^t \langle \sigma(u_n(s))dw^{\mathcal{Q}}(s), v_n(s) \rangle_H \right|. \end{aligned}$$

Since $\varphi \in \Lambda_{T, M}$, by proceeding as in (4.17) and (4.18), this implies that for every $\delta > 0$

$$\begin{aligned} &\mathbb{E} \sup_{r \in [0, t]} \|u_n(r)\|_{H^1}^2 + \mathbb{E} \sup_{r \in [0, t]} \|v_n(r)\|_H^2 + \gamma_0 \int_0^t \mathbb{E} \|v_n(s)\|_H^2 ds \\ &\leq c \left(1 + \|u_0\|_{H^1}^{\theta+1} + \|v_0\|_H^2\right) + \delta \mathbb{E} \sup_{r \in [0, t]} \|u_n(r)\|_H^2 + \frac{c}{\delta} \int_0^t \mathbb{E} \|v_n(s)\|_H^2 ds. \end{aligned}$$

In particular, if we choose δ small enough we have

$$\begin{aligned} &\mathbb{E} \sup_{r \in [0, t]} \|u_n(r)\|_{H^1}^2 + \mathbb{E} \sup_{r \in [0, t]} \|v_n(r)\|_H^2 + \gamma_0 \int_0^t \mathbb{E} \|v_n(s)\|_H^2 ds \\ &\leq c \left(1 + \|u_0\|_{H^1}^{\theta+1} + \|v_0\|_H^2\right) + c \int_0^t \mathbb{E} \|v_n(s)\|_H^2 ds, \end{aligned}$$

and Gronwall's lemma allows to obtain (4.25).

Step 2. There exists $z = (u, v) \in L^2(\Omega; C([0, T]; \mathcal{H}_1))$ solution to problem (4.3) such that

$$\mathbb{E} \sup_{t \in [0, T]} \|(u(t), v(t))\|_{\mathcal{H}_1}^2 + \gamma_0 \int_0^T \mathbb{E} \|v(s)\|_H^2 ds \leq c_T \left(1 + \|u_0\|_{H^1}^{\theta+1} + \|v_0\|_H^2 \right). \quad (4.26)$$

Proof of Step 2. For every $n \geq n_0$, we define

$$\tau_n := \inf \{ t \geq 0 : \|u_n(t)\|_{H^1} \geq n \},$$

with $\inf \emptyset = +\infty$. Since $\{\tau_n\}_{n \in \mathbb{N}}$ is an increasing sequence of stopping times, if we define $\tau := \sup_{n \in \mathbb{N}} \tau_n$, we have that

$$\mathbb{P}(\tau < \infty) = \lim_{T \rightarrow \infty} \mathbb{P}(\tau \leq T) = \lim_{T \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P}(\tau_n \leq T).$$

Now, according to (4.25), for every $T > 0$ we have

$$\mathbb{P}(\tau_n \leq T) \leq \mathbb{P} \left(\sup_{t \in [0, T]} \|u_n(t)\|_{H^1} \geq n \right) \leq \frac{1}{n^2} \mathbb{E} \sup_{t \in [0, T]} \|u_n(t)\|_{H^1}^2 \leq \frac{c}{n^2},$$

so that

$$\lim_{n \rightarrow \infty} \mathbb{P}(\tau_n \leq T) = 0,$$

and hence $\mathbb{P}(\tau = \infty) = 1$.

Now, for every $t \in [0, T]$ and $\omega \in \{\tau = +\infty\}$, there exists $n \in \mathbb{N}$ such that $t \leq \tau_n(\omega)$. Thus, we define

$$z(t)(\omega) = z_n(t)(\omega).$$

Notice that this is a good definition, as $f_n(r) = f_m(r)$, for every $n \leq m$ and $|r| \leq n$. Moreover, since for $\omega \in \{\tau = \infty\}$ and $t \leq \tau_n(\omega)$

$$\|u_n(t)(\omega)\|_{L^\infty([0, t])} \leq \|u_n(t)(\omega)\|_{H^1} \leq n,$$

we have that $f_n(u(s)(\omega)) = f(u(s)(\omega))$, for every $s \in [0, t]$, so that z belongs to $L^2(\Omega; C([0, T]; \mathcal{H}_1))$ and solves equation (4.3). By taking the limit as $n \rightarrow \infty$ in (4.25), we get (4.26).

Step 3. The solution z is unique in $L^2(\Omega; C([0, T]; \mathcal{H}_1))$.

Proof of Step 3. Let z_1 and z_2 be two solutions of equation (4.1) in $L^2(\Omega; C([0, T]; \mathcal{H}_1))$. For every $R > 0$, we define

$$\tau_R := \tau_{1,R} \wedge \tau_{2,R},$$

where

$$\tau_{i,R} := \inf \{ t \geq 0 : \|u_i(t)\|_{H^1} \geq R \}, \quad i = 1, 2.$$

Since z_1 and z_2 belong to $L^2(\Omega; C([0, T]; \mathcal{H}_1))$, we have

$$\lim_{R \rightarrow \infty} \mathbb{P}(\tau_R < T) = 0. \quad (4.27)$$

Now, if we define $\rho = z_1 - z_2$, from the Itô formula we have

$$\begin{aligned} \|\rho(t \wedge \tau_R)\|_{\mathcal{H}}^2 &= 2 \int_0^{t \wedge \tau_R} \langle A(z_1(s)) - A(z_2(s)), \rho(s) \rangle_{\mathcal{H}} ds \\ &+ 2 \int_0^{t \wedge \tau_R} \langle B_\Phi(s, z_1(s)) - B_\Phi(s, z_2(s)), \rho(s) \rangle_{\mathcal{H}} ds + \int_0^{t \wedge \tau_R} \|\Sigma(z_1(s)) - \Sigma(z_2(s))\|_{\mathcal{L}_2(H_0, \mathcal{H})}^2 ds \\ &+ 2 \int_0^{t \wedge \tau_R} \langle \rho(s), (\Sigma(z_1(s)) - \Sigma(z_2(s))) dw^Q(s) \rangle_{\mathcal{H}} =: \sum_{k=1}^4 I_k(t \wedge \tau_R). \end{aligned}$$

We have

$$\begin{aligned} \langle A(z_1(s)) - A(z_2(s)), \rho(s) \rangle_{\mathcal{H}} &= -\langle (g(u_1(s)) - g(u_2(s))), u_1(s) - u_2(s) \rangle_H \\ &+ \langle F(u_1(s)) - F(u_2(s)), \eta_1(s) - \eta_2(s) \rangle_{H^{-1}}, \end{aligned}$$

so that, according to (2.14)

$$\begin{aligned} |\langle A(z_1(s)) - A(z_2(s)), \rho(s) \rangle_{\mathcal{H}}| &\leq c \|\rho(s)\|_{\mathcal{H}}^2 + c \|F(u_1(s)) - F(u_2(s))\|_H^2 \\ &\leq c \|\rho(s)\|_{\mathcal{H}}^2 + c \left(1 + \|u_1(s)\|_{H^1}^{2(\theta-1)} + \|u_2(s)\|_{H^1}^{2(\theta-1)}\right) \|u_1(s) - u_2(s)\|_H^2. \end{aligned}$$

This implies that

$$\sup_{r \in [0, t]} |I_1(r \wedge \tau_R)| \leq c(R) \int_0^{t \wedge \tau_R} \|\rho(s \wedge \tau_R)\|_{\mathcal{H}}^2 ds. \quad (4.28)$$

Moreover, by proceeding as in (4.17) and (4.18), for every $\delta > 0$ we have

$$\mathbb{E} \sup_{r \in [0, t]} (|I_2(r \wedge \tau_R)| + |I_4(r \wedge \tau_R)|) \leq \delta \mathbb{E} \sup_{r \in [0, t]} \|\rho(r \wedge \tau_R)\|_{\mathcal{H}}^2 + \frac{c}{\delta} \int_0^{t \wedge \tau_R} \mathbb{E} \|\rho(s \wedge \tau_R)\|_{\mathcal{H}}^2 ds. \quad (4.29)$$

Therefore, since

$$\sup_{r \in [0, t]} |I_3(r \wedge \tau_R)| \leq c \int_0^{t \wedge \tau_R} \|\rho(s \wedge \tau_R)\|_{\mathcal{H}}^2 ds,$$

thanks to (4.28) and (4.29), if we fix some $\delta > 0$ small enough, we get

$$\mathbb{E} \sup_{r \in [0, t]} \|\rho(r \wedge \tau_R)\|_{\mathcal{H}}^2 \leq c(R) \int_0^{t \wedge \tau_R} \sup_{r \in [0, s]} \|\rho(r \wedge \tau_R)\|_{\mathcal{H}}^2 dr.$$

This implies that for every $R > 0$

$$\mathbb{E} \sup_{r \in [0, T]} \|\rho(r \wedge \tau_R)\|_{\mathcal{H}}^2 = 0.$$

In view of (4.27), by taking the limit as $R \uparrow \infty$ this gives $\mathbb{E} \sup_{r \in [0, T]} \|\rho(r)\|_{\mathcal{H}}^2 = 0$ and uniqueness follows.

5 A-priori bounds and tightness

In the previous section we have proved that for any $\mu > 0$ and any $T > 0$ there exists a unique solution $(u_\mu, \partial_t u_\mu) \in L^2(\Omega; C([0, T]; \mathcal{H}_1))$ to system (4.3). Our purpose here is proving a bound for $(u_\mu, \partial_t u_\mu)$, which is uniform with respect to μ .

Lemma 5.1. *Under Hypotheses 1 and 2 and either Hypothesis 3 or Hypothesis 4, for every $T, M > 0$ and for every initial condition $(u_0, v_0) \in \mathcal{H}_1$, there exist $c_T > 0$ and $\mu_T > 0$ such that for every $\varphi \in \Lambda_{T, M}$ and $\mu \in (0, \mu_T)$*

$$\mathbb{E} \sup_{t \in [0, T]} \|u_\mu(t)\|_{H^1}^2 + \mathbb{E} \sup_{t \in [0, T]} \|u_\mu(t)\|_{L^{\theta+1}}^{\theta+1} + \mu \mathbb{E} \sup_{t \in [0, T]} \|\partial_t u_\mu(t)\|_H^2 + \int_0^T \mathbb{E} \|\partial_t u_\mu(t)\|_H^2 dt \leq c_T. \quad (5.1)$$

Proof. We give our proof in case Hypothesis 4 holds and we leave to the reader the proof in case Hypothesis 3 holds.

Step 1. There exists $c_T > 0$ such that for all $\mu \in (0, 1)$

$$\begin{aligned} & \mathbb{E} \sup_{r \in [0, t]} \|u_\mu(r)\|_H^2 + \int_0^t \mathbb{E} \|u_\mu(s)\|_{H^1}^2 ds \\ & \leq c_T \left(1 + \int_0^t \mathbb{E} \|u_\mu(s)\|_H^2 ds + \mu \int_0^t \mathbb{E} \|\partial_t u_\mu(s)\|_H^2 ds + \mu^2 \mathbb{E} \sup_{r \in [0, t]} \|\partial_t u_\mu(r)\|_H^2 \right). \end{aligned} \quad (5.2)$$

Proof of Step 1. As shown in [14, Proof of Lemma 4.1], for every $\mu \in (0, 1)$ we have

$$\begin{aligned} & \frac{\gamma_0}{4} \|u_\mu(t)\|_H^2 + \int_0^t \|u_\mu(s)\|_{H^1}^2 ds \leq c + c\mu^2 \|\partial_t u_\mu(t)\|_H^2 + \mu \int_0^t \|\partial_t u_\mu(s)\|_H^2 ds \\ & + \int_0^t \langle F(u_\mu(s)), u_\mu(s) \rangle_H ds + \int_0^t \langle u_\mu(s), \sigma(u_\mu(s)) Q \varphi_\mu(s) \rangle_H ds + \sqrt{\mu} \int_0^t \langle u_\mu(s), \sigma(u_\mu(s)) dw^\mathcal{Q}(s) \rangle_H. \end{aligned} \quad (5.3)$$

Due to (2.13), we have

$$\int_0^t \langle F(u_\mu(s)), u_\mu(s) \rangle_H ds \leq c_T \sup_{r \in [0, t]} (1 + \|u_\mu(r)\|_H^2). \quad (5.4)$$

Moreover, by proceeding as in the proof of (4.17) and (4.18), for every $\delta > 0$ we have

$$\begin{aligned} & \mathbb{E} \sup_{r \in [0, t]} \left| \int_0^r \langle u_\mu(s), \sigma(u_\mu(s)) Q \varphi_\mu(s) \rangle_H ds \right| + \sqrt{\mu} \mathbb{E} \sup_{r \in [0, t]} \left| \int_0^r \langle u_\mu(s), \sigma(u_\mu(s)) dw^\mathcal{Q}(s) \rangle_H \right| \\ & \leq \delta \mathbb{E} \sup_{r \in [0, t]} \|u_\mu(r)\|_H^2 + \frac{c}{\delta} \int_0^t \mathbb{E} \|u_\mu(s)\|_H^2 ds. \end{aligned}$$

Therefore, if we choose $\delta > 0$ sufficiently small above, this, together with (5.4) and (5.3) allows to conclude that (5.2) holds true.

Step 2. For every $T > 0$ there exist $c_T > 0$ and $\mu_T > 0$ such that

$$\sup_{\mu \in (0, \mu_T)} \mathbb{E} \sup_{t \in [0, T]} \|u_\mu(t)\|_H^2 \leq c_T, \quad (5.5)$$

and (5.1) holds.

Proof of Step 2. As in the previous section, from Itô's formula, we have

$$\begin{aligned} \|u_\mu(t)\|_{H^1}^2 + \mu \|\partial_t u_\mu(t)\|_H^2 &= \|u_0\|_{H^1}^2 + \mu \|v_0\|_H^2 - \int_0^t \langle \gamma(u_\mu(s)) \partial_t u_\mu(s), \partial_t u_\mu(s) \rangle_H ds \\ &+ \int_{\mathcal{O}} \mathfrak{f}(u_\mu(t, x)) dx - \int_{\mathcal{O}} \mathfrak{f}(u_0(x)) dx + \int_0^t \langle \sigma(u_\mu(s)) \mathcal{Q}\varphi(s), \partial_t u_\mu(s) \rangle_H ds \\ &+ \sqrt{\mu} \int_0^t \langle \sigma(u_\mu(s)) dw^{\mathcal{Q}}(s), \partial_t u_\mu(s) \rangle_H + \int_0^t \|\sigma(u_\mu(s))\|_{\mathcal{L}_2(H_{\mathcal{Q}}, H)}^2 ds. \end{aligned}$$

Due to (2.6), (2.9) and (2.11), this gives

$$\begin{aligned} &\|u_\mu(t)\|_{H^1}^2 + c \|u_\mu(s)\|_{L^{\theta+1}}^{\theta+1} + \mu \|\partial_t u_\mu(t)\|_H^2 + \gamma_0 \int_0^t \|\partial_t u_\mu(s)\|_H^2 ds \\ &\leq c \left(1 + \|u_0\|_{H^1}^{\theta+1} + \mu \|v_0\|_H^2 \right) + \int_0^t \|\sigma(u_\mu(s))\|_{\mathcal{L}_2(H_{\mathcal{Q}}, H)}^2 ds \\ &+ \int_0^t \langle \sigma(u_\mu(s)) \mathcal{Q}\varphi(s), \partial_t u_\mu(s) \rangle_H ds + \sqrt{\mu} \int_0^t \langle \sigma(u_\mu(s)) dw^{\mathcal{Q}}(s), \partial_t u_\mu(s) \rangle_H. \end{aligned}$$

Therefore, by proceeding as in Step 1, for every $\delta > 0$ we have

$$\begin{aligned} &\mathbb{E} \sup_{r \in [0, t]} \|u_\mu(r)\|_{H^1}^2 + \mathbb{E} \sup_{r \in [0, T]} \|u_\mu(r)\|_{L^{\theta+1}}^{\theta+1} + \mu \mathbb{E} \sup_{r \in [0, t]} \|\partial_t u_\mu(r)\|_H^2 + \int_0^t \mathbb{E} \|\partial_t u_\mu(s)\|_H^2 ds \\ &\leq c \left(1 + \|u_0\|_{H^1}^{\theta+1} + \mu \|v_0\|_H^2 \right) + \delta \mathbb{E} \sup_{r \in [0, t]} \|u_\mu(r)\|_H^2 + \frac{c}{\delta} \int_0^t \mathbb{E} \|\partial_t u_\mu(s)\|_H^2 ds + c \int_0^t \|u_\mu(s)\|_H^2 ds. \end{aligned} \quad (5.6)$$

If we take $\delta > 0$ in (5.6), we get

$$\begin{aligned} &\mathbb{E} \sup_{r \in [0, t]} \|u_\mu(r)\|_{H^1}^2 + \mathbb{E} \sup_{r \in [0, T]} \|u_\mu(r)\|_{L^{\theta+1}}^{\theta+1} + \mu \mathbb{E} \sup_{r \in [0, t]} \|\partial_t u_\mu(r)\|_H^2 \\ &\leq c \left(1 + \int_0^t \mathbb{E} \|\partial_t u_\mu(s)\|_H^2 ds \right) + c \int_0^t \|u_\mu(s)\|_H^2 ds, \end{aligned} \quad (5.7)$$

and if we take $\delta > 0$ large enough we get

$$\mathbb{E} \sup_{r \in [0, T]} \|u_\mu(r)\|_{L^{\theta+1}}^{\theta+1} + \mu \mathbb{E} \sup_{r \in [0, t]} \|\partial_t u_\mu(r)\|_H^2 + \int_0^t \mathbb{E} \|\partial_t u_\mu(s)\|_H^2 ds \leq c \left(1 + \mathbb{E} \sup_{r \in [0, t]} \|u_\mu(r)\|_H^2 \right). \quad (5.8)$$

By combining together (5.2) and (5.8), we can fix $\mu_T > 0$ such that for every $\mu \in (0, \mu_T)$

$$\mathbb{E} \sup_{r \in [0, t]} \|u_\mu(r)\|_H^2 \leq c_T \left(1 + \int_0^t \mathbb{E} \|u_\mu(s)\|_H^2 ds \right),$$

which implies (5.5). Thus, from (5.5), (5.7) and (5.8), we obtain (5.1). □

Now, for every $T > 0$ and $\mu > 0$ we define

$$\rho_\mu(t, x) = g(u_\mu(t, x)), \quad (t, x) \in [0, T] \times \mathcal{O}.$$

According to Hypothesis 2, we know that

$$|g(r)| \leq \gamma_1 |r|, \quad |g'(r)| \leq \gamma_1, \quad r \in \mathbb{R}$$

so that for every $\mu > 0$ and $t \in [0, T]$,

$$\|\rho_\mu(t)\|_H \leq \gamma_1 \|u_\mu(t)\|_H, \quad \|\rho_\mu(t)\|_{H^1} \leq \gamma_1 \|u_\mu(t)\|_{H^1}, \quad \|\partial_t \rho_\mu(t)\|_H \leq \gamma_1 \|\partial_t u_\mu(t)\|_H. \quad (5.9)$$

Since the function g is strictly increasing, it is invertible and we have

$$u_\mu(t, x) = g^{-1}(\rho_\mu(t, x)), \quad (t, x) \in [0, T] \times \mathcal{O},$$

which implies that

$$\Delta u_\mu(t, x) = \operatorname{div} [\nabla g^{-1}(\rho_\mu(t))] = \operatorname{div} \left[\frac{1}{\gamma(g^{-1}(\rho_\mu(t)))} \nabla \rho_\mu(t) \right].$$

Moreover, by the definition of ρ_μ ,

$$\nabla \rho_\mu(t) = \gamma(u_\mu(t)) \nabla u_\mu(t), \quad \partial_t \rho_\mu(t) = \gamma(u_\mu(t)) \partial_t u_\mu(t).$$

This means that if we integrate equation (3.3) (with φ_μ) with respect to t we have

$$\begin{aligned} \rho_\mu(t) + \mu \partial_t u_\mu(t) &= g(u_0) + \mu v_0 + \int_0^t \operatorname{div} [b(\rho_\mu(s)) \nabla \rho_\mu(s)] ds + \int_0^t F_g(\rho_\mu(s)) ds \\ &\quad + \int_0^t \sigma_g(\rho_\mu(s)) \mathcal{Q} \varphi_\mu(s) ds + \sqrt{\mu} \int_0^t \sigma_g(\rho_\mu(s)) dw^\mathcal{Q}(s), \end{aligned} \quad (5.10)$$

where for every $r \in \mathbb{R}$ and $x \in \mathcal{O}$

$$b(r) := \frac{1}{\gamma(g^{-1}(r))}, \quad f_g(x, r) := f(x, g^{-1}(r)),$$

and for every $u \in H^1$

$$F_g(u) = f_g \circ u, \quad \sigma_g(u) := \sigma(g^{-1} \circ u).$$

Theorem 5.2. *Assume Hypotheses 1 and 2 and either Hypothesis 3 or Hypothesis 4 and fix an arbitrary $T > 0$ and $(u_0, v_0) \in \mathcal{H}_1$. Then, for any family of predictable controls $\{\varphi_\mu\}_{\mu \in (0, \mu_T)} \subset \Lambda_{T, M}$, the family of probabilities $\{\mathcal{L}(\rho_\mu)\}_{\mu \in (0, \mu_T)}$ is tight in $C([0, T]; H^\delta)$, for every $\delta < 1$.*

Proof. According to (5.1) and (5.9), we have that

$$\mathbb{E} \sup_{r \in [0, t]} \|\rho_\mu(r)\|_H^2 + \int_0^t \mathbb{E} \|\partial_t \rho_\mu(s)\|_{H^1}^2 ds \leq c_T, \quad \mu \in (0, \mu_T).$$

This means that for every $\varepsilon > 0$ there exists $L_\varepsilon > 0$ such that if we denote by K_ε the ball of radius L_ε in $C([0, T]; H^1) \cap W^{1,2}([0, T]; H)$, then

$$\inf_{\mu \in (0, \mu_T)} \mathbb{P}(\rho_\mu \in K_\varepsilon) \geq 1 - \varepsilon.$$

This allows to conclude as, due to the Aubin-Lions lemma, the set K_ε is compact in $C([0, T]; H^\delta)$, for every $\delta < 1$. \square

6 The limit controlled problem

In order to prove conditions C1 and C2, we need first to understand better the controlled quasi-linear parabolic problem

$$\begin{cases} \partial_t \rho(t, x) = \operatorname{div}[b(\rho(t, x))] + f_g(x, \rho(t, x)) + \sigma_g(\rho(t, \cdot))\varphi(t, x), & t > 0, \quad x \in \mathcal{O}, \\ \rho(0, x) = g(u_0(x)), \quad \rho(t, x) = 0, & x \in \partial \mathcal{O}. \end{cases} \quad (6.1)$$

In view of Hypothesis 2, we have

$$\frac{1}{\gamma_1} |r| \leq |g^{-1}(r)| \leq \frac{1}{\gamma_0}, \quad r \in \mathbb{R}.$$

When Hypothesis 4 holds, thanks to (2.10), this means that for every $u \in L^{\theta+1}([0, L])$

$$\|F_g(u)\|_{H^{-1}} \leq c \int_0^L (1 + |g^{-1}(u(x))|^\theta) dx \leq c \left(1 + \|u\|_{H^{\frac{2}{\theta-1}}}^2 \|u\|_{L^{\theta+1}}^{\frac{(\theta+1)(\theta-2)}{\theta-1}} \right). \quad (6.2)$$

Next, again due to Hypothesis 2 we have

$$\frac{1}{\gamma_1} \leq \frac{g^{-1}(r)}{r} \leq \frac{1}{\gamma_0}, \quad r \in \mathbb{R},$$

so that, thanks to Hypothesis 4, we get,

$$f_g(r)r = f(g^{-1}(r))g^{-1}(r) \cdot \frac{r}{g^{-1}(r)} \leq c \left(1 - |r|^{\theta+1} \right), \quad r \in \mathbb{R},$$

(here we define $g^{-1}(0)/0 = 1/\gamma(0)$). In particular, we have

$$\langle F_g(u), u \rangle_H \leq c \left(1 - \|u\|_{L^{\theta+1}}^{\theta+1} \right), \quad u \in H^1. \quad (6.3)$$

Moreover, since g^{-1} is increasing and f is decreasing, we have that f_g is decreasing, so that

$$\langle F_g(u_1) - F_g(u_2), u_1 - u_2 \rangle_H \leq 0. \quad (6.4)$$

Finally, thanks to Hypotheses 2 and 1,

$$|b(r)| \leq \frac{1}{\gamma_0}, \quad b(r) \geq \frac{1}{\gamma_1}, \quad |b(r) - b(s)| \leq c|r - s|, \quad r, s \in \mathbb{R}, \quad (6.5)$$

and

$$\|\sigma_g(u_1) - \sigma_g(u_2)\|_{\mathcal{L}(H_0, H)} \leq c \|u_1 - u_2\|_H, \quad u_1, u_2 \in H. \quad (6.6)$$

Definition 6.1. A function $\rho \in L^2(0, T; H^1)$ is a weak solution to equation (6.1) if for every test function $\psi \in C_0^\infty(\mathcal{O})$ and $t \in [0, T]$

$$\langle \rho(t), \psi \rangle_H = \langle g(u_0), \psi \rangle_H - \int_0^t \langle b(\rho(s)) \nabla \rho(s), \nabla \psi \rangle_H ds + \int_0^t \langle F_g(\rho(s)) + \sigma_g(\rho(s)) \mathcal{Q}\varphi(s), \psi \rangle_H ds. \quad (6.7)$$

Theorem 6.2. Assume Hypotheses 1 and 2 and either Hypothesis 3 or Hypothesis 4 and fix any $T > 0$ and $\varphi \in L^2(0, T; H)$. Then, for every $u_0 \in H^1$ there exists a unique weak solution ρ to equation (6.1) such that

$$\rho \in C([0, T]; H) \cap L^2(0, T; H^1), \quad \partial_t \rho \in L^2(0, T; H^{-1}). \quad (6.8)$$

Proof. We prove the result above in case Hypothesis 4 is satisfied. We leave to the reader the proof in the case Hypothesis 3 is satisfied. For every $\varphi \in L^2(0, T; H)$ and $\psi \in H^1$, equation (6.7) can be written as

$$\langle \rho(t), \psi \rangle_H = \langle g(u_0), \psi \rangle_H + \int_0^t \langle S_\varphi(s, \rho(s)), \psi \rangle_H ds,$$

where we have defined

$$S_\varphi(t, \rho) := \operatorname{div}[b(\rho) \nabla \rho] + F_g(\rho) + \sigma_g(\rho) \mathcal{Q}\varphi(t), \quad t \in [0, T], \quad \rho \in H^1.$$

Due to the presence of φ , the operator S_φ does not satisfy the assumptions of [24, Theorem 1.1] and for this reason we cannot apply directly the well-posedness result proved therein. However, as we are going to show below, the operator S_φ satisfies some generalized conditions that still allow to use techniques introduced in [24] to prove our result.

First of all, we notice that for every $t \in [0, T]$ and $\rho_1, \rho_2, \rho \in H^1$ the mapping $s \in \mathbb{R} \mapsto \langle S_\varphi(t, \rho_1 + s\rho_2), \rho \rangle_H$, is continuous, so that the operator S_φ is *hemi-continuous*.

Thanks to (6.4), (6.5) and (6.6), for every $t \in [0, T]$ and $\rho_1, \rho_2 \in H^1$

$$\begin{aligned} & \langle \mathcal{S}_\varphi(t, \rho_1) - \mathcal{S}_\varphi(t, \rho_2), \rho_1 - \rho_2 \rangle_H = \langle \operatorname{div}[b(\rho_1) \nabla \rho_1 - b(\rho_2) \nabla \rho_2], \rho_1 - \rho_2 \rangle_H \\ & + \langle F_g(\rho_1) - F_g(\rho_2), \rho_1 - \rho_2 \rangle_H + \langle (\sigma_g(\rho_1) - \sigma_g(\rho_2)) \mathcal{Q}\varphi(t), \rho_1 - \rho_2 \rangle_H \\ & = -\langle b(\rho_1) \nabla(\rho_1 - \rho_2), \nabla(\rho_1 - \rho_2) \rangle_H - \langle (b(\rho_1) - b(\rho_2)) \nabla \rho_2, \nabla(\rho_1 - \rho_2) \rangle_H \\ & + \langle F_g(\rho_1) - F_g(\rho_2), \rho_1 - \rho_2 \rangle_H + \langle (\sigma_g(\rho_1) - \sigma_g(\rho_2)) \mathcal{Q}\varphi(t), \rho_1 - \rho_2 \rangle_H \\ & \leq -\frac{1}{\gamma_1} \|\rho_1 - \rho_2\|_{H^1}^2 + c \|\rho_1 - \rho_2\|_H^2 \|\rho_2\|_{H^1}^2 + \frac{1}{2\gamma_1} \|\rho_1 - \rho_2\|_{H^1}^2 + c(1 + \|\varphi(t)\|_H^2) \|\rho_1 - \rho_2\|_H^2 \end{aligned}$$

so that

$$\langle \mathcal{S}_\varphi(t, \rho_1) - \mathcal{S}_\varphi(t, \rho_2), \rho_1 - \rho_2 \rangle_H \leq c(1 + \|\varphi(t)\|_H^2 + \|\rho_2\|_{H^1}^2) \|\rho_1 - \rho_2\|_H^2. \quad (6.9)$$

Moreover, thanks to (6.3), for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for every $t \in [0, T]$ and $\rho \in H^1$

$$\begin{aligned} \langle \mathcal{S}_\varphi(t, \rho), \rho \rangle_H &= \langle \operatorname{div}[b(\rho)\nabla\rho], \rho \rangle_H + \langle F_g(\rho), \rho \rangle_H + \langle \sigma_g(\rho)Q\varphi(t), \rho \rangle_H \\ &\leq -\frac{1}{\gamma_1} \|\rho\|_{H^1}^2 - c\|\rho\|_{L^{\theta+1}}^{\theta+1} + c_\varepsilon \|\rho\|_H^2 + c + \varepsilon \|\varphi(t)\|_H^2 \|\rho\|_H^2. \end{aligned} \quad (6.10)$$

Finally, according to (6.2) and (6.5), for every $t \in [0, T]$ and $\rho \in H^1$

$$\|\mathcal{S}(t, \rho)\|_{H^{-1}} \leq c \left(\|\rho\|_{H^1} + \|\rho\|_{L^{\theta+1}}^{\theta+1} + \|\rho\|_H^2 + \|\varphi(t)\|_H^2 \right). \quad (6.11)$$

For every n , we denote $H_n := \operatorname{span}\{e_1, \dots, e_n\}$ and we denote by P_n the projection of H onto H_n . Next, we introduce the finite dimensional problem

$$u'_n(t) = P_n \mathcal{S}_\varphi(t, \rho_n(t)), \quad u_n(0) = P_n \rho(0) \in H_n.$$

Since $P_n \mathcal{S}_\varphi$ is quasi-monotone and coercive in H_n , there exists a unique solution $\rho_n \in C([0, T]; H_n)$, such that $\rho'_n \in L^2(0, T; H_n)$.

Now, we are showing that there exists $c_T > 0$ such that for every $n \in \mathbb{N}$

$$\sup_{t \in [0, T]} \|\rho_n(t)\|_H^2 + \int_0^T \|\rho_n(t)\|_{L^{\theta+1}}^{\theta+1} dt + \int_0^T \|\mathcal{S}_\varphi(t, \rho_n(t))\|_{H^{-1}}^2 dt \leq c_T. \quad (6.12)$$

Due to (6.10), we have

$$\begin{aligned} \|\rho_n(t)\|_H^2 &= \|\rho_n(0)\|_H^2 + \int_0^t \langle \mathcal{S}_\varphi(s, \rho_n(s)), \rho_n(s) \rangle_H ds \leq \|\rho_n(0)\|_H^2 + cT \\ &\quad - \frac{1}{\gamma_1} \int_0^t \|\rho_n(s)\|_{H^1}^2 ds - c \int_0^t \|\rho_n(s)\|_{L^{\theta+1}}^{\theta+1} ds + c_\varepsilon \int_0^t \|\rho_n(s)\|_H^2 ds + \varepsilon \int_0^t \|\varphi(s)\|_H^2 \|\rho_n(s)\|_H^2 ds. \end{aligned}$$

Hence, if we take $\bar{\varepsilon} = (2\|\varphi\|_{L^2(0, T; H)})^{-1}$, we get

$$\begin{aligned} &\frac{1}{2} \sup_{r \in [0, t]} \|\rho_n(r)\|_H^2 + \frac{1}{\gamma_1} \int_0^t \|\rho_n(s)\|_{H^1}^2 ds + c \int_0^t \|\rho_n(s)\|_{L^{\theta+1}}^{\theta+1} ds \\ &\leq \|\rho_n(0)\|_H^2 + cT + c_{\bar{\varepsilon}} \int_0^t \|\rho_n(s)\|_H^2 ds, \end{aligned}$$

and the Gronwall lemma implies

$$\sup_{r \in [0, T]} \|\rho_n(r)\|_H^2 + \int_0^T \|\rho_n(s)\|_{H^1}^2 ds + \int_0^T \|\rho_n(s)\|_{L^{\theta+1}}^{\theta+1} ds \leq c_T. \quad (6.13)$$

Finally, according to (6.11) and (6.13), we have

$$\int_0^T \|\mathcal{S}_\varphi(t, \rho_n(t))\|_{H^{-1}}^2 dt \leq c \int_0^T \left(\|\rho_n(t)\|_{H^1} + \|\rho_n(t)\|_{L^{\theta+1}}^{\theta+1} + \|\rho_n(t)\|_H^2 \right) dt + c \|\varphi\|_{L^2(0, T; H)}^2,$$

and this, together with (6.13), implies (6.12).

As a consequence of (6.12), we have that there exists a subsequence, still denoted by ρ_n , and there exists $\rho \in L^2(0, T; H^1)$, with $\partial_t \rho \in L^2(0, T; H^{-1})$, and $\eta \in L^2(0, T; H^{-1})$ such that, as $n \rightarrow \infty$,

$$\rho_n \rightharpoonup \rho, \quad \text{in } L^2(0, T; H^1), \quad \partial_t \rho_n \rightharpoonup \partial_t \rho, \quad \text{in } L^2(0, T; H^{-1}),$$

and

$$S_\varphi(\cdot, \rho_n) \rightharpoonup \eta \quad \text{in } L^2(0, T; H^{-1}).$$

Thanks to (6.9), we can use the same arguments used in [24, Lemma 2.4] to show that $\eta = S_\varphi(\cdot, u)$, as elements of $L^2(0, T; H^{-1})$. Moreover we can prove that $\partial_t \rho(t) = \eta(t)$. Therefore, we conclude that ρ is a weak solution of equation (6.1).

As far as uniqueness is concerned, as in [24] it is again a consequence of (6.9). □

Corollary 6.3. *Assume Hypotheses 1 and 2 and either Hypothesis 3 or Hypothesis 4 and fix any $T > 0$, $u_0 \in H^1$ and $\varphi \in L^2(0, T; H)$. Then, if we define $u := g^{-1}(\rho)$, where ρ is the unique weak solution of equation (6.1), we have that u is the unique weak solution to equation (3.2) and*

$$u \in C([0, T]; H) \cap L^2(0, T; H^1), \quad \partial_t u \in L^2(0, T; H^{-1}). \quad (6.14)$$

Proof. We define $u = g^{-1}(\rho)$, where ρ is the unique solution to equation (6.1). Due to the fact that ρ satisfies (6.8) and g^{-1} is differentiable with bounded derivative, we have that

$$u \in C([0, T]; H) \cap L^2(0, T; H^1),$$

and

$$\nabla(g(u(t))) = g'(u(t))\nabla u(t) = \gamma(u(t))\nabla u(t).$$

Recalling how b , F_g and σ_g were defined, this implies

$$\begin{aligned} S_\varphi(t, \rho(t)) &= \operatorname{div} [b(\rho(t))\nabla \rho(t)] + F_g(\rho(t)) + \sigma_g(\rho(t))Q\varphi(t) \\ &= \operatorname{div} \left[\frac{1}{\gamma(u(t))} \nabla(g(u(t))) \right] + f(u(t)) + \sigma(u(t))Q\varphi(t) = \Delta u(t) + f(u(t)) + \sigma(u(t))Q\varphi(t) \end{aligned}$$

in H^{-1} sense. Moreover, by mollfying ρ with respect to t and x and then by taking the limit, we have that

$$\partial_t u \in L^2(0, T; H^{-1})$$

and

$$\partial_t \rho(t) = g'(u(t))\partial_t u(t) = \gamma(u(t))\partial_t u(t),$$

in H^{-1} sense. We can now conclude, as we know that $\partial_t \rho(t) = S_\varphi(t, \rho(t))$, in H^{-1} sense. □

7 Proof of Theorem 3.1

In order to prove Theorem 3.1, we will show that the conditions C1 and C2 that we introduced in Section 3 are both satisfied. We will consider here the case the nonlinearity f satisfies Hypothesis 4 and we leave to the reader to adapt our proof to the case f satisfies Hypothesis 3.

Theorem 7.1. *Under Hypotheses 1 and 2 and either Hypothesis 3 or Hypothesis 4, condition C1 holds.*

Proof. Let us fix $T, M > 0$ and let $\{\varphi_\mu\}$ be a family of processes in $\Lambda_{T,M}$ such that

$$\lim_{\mu \rightarrow 0} \varphi_\mu = \varphi, \quad \text{in distribution in } L_w^2(0, T; H),$$

where $L_w^2(0, T; H)$ is the space $L^2(0, T; H)$ endowed with the weak topology and $\varphi \in \Lambda_{T,M}$.

For every sequence $\{\mu_k\}_{k \in \mathbb{N}}$ converging to 0, as $k \uparrow \infty$, we denote $\rho_k = g(u_k)$, where $u_k := u_{\mu_k}^{\varphi_{\mu_k}}$ is the solution of equation (3.3), corresponding to the control φ_{μ_k} . Thanks to Theorem 5.2 and Lemma 5.1, we have that the family

$$\{\mathcal{L}(\rho_k, \mu_k \partial_t u_k, \varphi_{\mu_k})\}_{k \in \mathbb{N}} \subset \mathcal{P}\left(C([0, T]; H^\delta) \times C([0, T]; H) \times \mathcal{S}_{T,M}\right)$$

is tight. We denote by ρ a weak limit point for the sequence $\{\rho_k\}_{k \in \mathbb{N}}$ and we denote

$$\Lambda := C([0, T]; H^\delta) \times C([0, T]; H) \times \mathcal{S}_{T,M} \times C([0, T], U),$$

where U is the Hilbert space such that (2.1) holds. By the Skorokhod Theorem there exist random variables

$$\mathcal{Y} = (\hat{\rho}, 0, \hat{\varphi}, \hat{w}^Q), \quad \mathcal{Y}_k = (\hat{\rho}_k, \hat{\theta}_k, \hat{\varphi}_k, \hat{w}_k^Q), \quad k \in \mathbb{N},$$

defined on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0, T]}, \hat{\mathbb{P}})$, such that

$$\mathcal{L}(\mathcal{Y}) = \mathcal{L}(\rho, 0, \varphi, w^Q), \quad \mathcal{L}(\mathcal{Y}_k) = \mathcal{L}(\rho_k, \mu_k \partial_t u_k, \varphi_{\mu_k}, w^Q), \quad k \in \mathbb{N},$$

and such that

$$\lim_{k \rightarrow \infty} \mathcal{Y}_k = \mathcal{Y} \quad \text{in } \Lambda, \quad \hat{\mathbb{P}} - \text{a.s.} \quad (7.1)$$

For every $k \in \mathbb{N}$ and $\psi \in H^2$, we have

$$\begin{aligned} \langle \hat{\rho}_k(t) + \hat{\theta}_k(t), \psi \rangle_H &= \langle g(u_0) + \mu_k v_0, \psi \rangle_H + \int_0^t \langle \operatorname{div} [b(\hat{\rho}_k(s)) \nabla \hat{\rho}_k(s)], \psi \rangle_H ds \\ &+ \int_0^t \langle F_g(\hat{\rho}_k(s)), \psi \rangle_H ds + \int_0^t \langle \sigma_g(\hat{\rho}_k(s)) Q \hat{\varphi}_k(s), \psi \rangle_H ds + \sqrt{\mu_k} \int_0^t \langle \sigma_g(\hat{\rho}_{\mu_k}(s)) d\hat{w}_k^Q(s), \psi \rangle_H. \end{aligned}$$

Thanks to (7.1), for every $t \in [0, T]$ we have

$$\lim_{k \rightarrow \infty} \langle \hat{\rho}_k(t) + \hat{\theta}_k(t), \psi \rangle_H = \langle \hat{\rho}(t), \psi \rangle_H, \quad \hat{\mathbb{P}} - \text{a.s.} \quad (7.2)$$

Next, if we define $\hat{u}_k := g^{-1}(\hat{r}_k)$ and $\hat{u} := g^{-1}(\hat{r})$, we have

$$\begin{aligned} & \int_0^t \langle b(\hat{\rho}_k(s)) \nabla \hat{\rho}_k(s), \nabla \Psi \rangle_H ds - \int_0^t \langle b(\hat{\rho}(s)) \nabla \hat{\rho}(s), \nabla \Psi \rangle_H ds \\ &= \int_0^t \langle \nabla \hat{u}_k(s), \nabla \Psi \rangle_H ds - \int_0^t \langle \nabla \hat{u}(s), \nabla \Psi \rangle_H ds = - \int_0^t \langle (\hat{u}_k(s) - \hat{u}(s)), \Delta \Psi \rangle_H ds. \end{aligned}$$

In particular, since (7.1) implies the \mathbb{P} -a.s. convergence of \hat{u}_k to \hat{u} in $C([0, T]; H)$, we get that

$$\lim_{k \rightarrow \infty} \int_0^t \langle b(\hat{\rho}_k(s)) \nabla \hat{\rho}_k(s), \nabla \Psi \rangle_H ds = \int_0^t \langle b(\hat{\rho}(s)) \nabla \hat{\rho}(s), \nabla \Psi \rangle_H ds, \quad \hat{\mathbb{P}} - \text{a.s.} \quad (7.3)$$

It is immediate to check that estimate (2.15) extends to F_g . Thus we have

$$\begin{aligned} & \left| \int_0^t \langle F_g(\hat{\rho}_k(s)), \Psi \rangle_H ds - \int_0^t \langle F_g(\hat{\rho}(s)), \Psi \rangle_H ds \right| \leq \int_0^t \|F_g(\hat{\rho}_k(s)) - F_g(\hat{\rho}(s))\|_{L^1} ds \|\Psi\|_{H^1} \\ & \leq c \int_0^t \left(1 + \|\hat{\rho}_k(s)\|_{L^{2(\theta-1)}}^{\theta-1} + \|\hat{\rho}(s)\|_{L^{2(\theta-1)}}^{\theta-1} \right) \|\hat{\rho}_k(s) - \hat{\rho}(s)\|_H ds \|\Psi\|_{H^1}, \end{aligned}$$

so that, thanks to (7.1),

$$\lim_{k \rightarrow \infty} \int_0^t \langle F_g(\hat{\rho}_k(s)), \Psi \rangle_H ds = \int_0^t \langle F_g(\hat{\rho}(s)), \Psi \rangle_H ds, \quad \hat{\mathbb{P}} - \text{a.s.} \quad (7.4)$$

Now, for any $h \in L^2(0, T; H)$,

$$\left| \int_0^t \langle \sigma_g(\hat{\rho}(s)) Qh(s), \Psi \rangle_H ds \right| \leq c \|\Psi\|_H \left(\int_0^T (1 + \|\hat{\rho}(s)\|_H^2) ds \right)^{\frac{1}{2}} \left(\int_0^T \|h(s)\|_H^2 ds \right)^{\frac{1}{2}}$$

and this implies that the mapping

$$h \in L^2(0, t; H) \mapsto \int_0^t \langle \sigma_g(\hat{\rho}(s)) Qh(s), \Psi \rangle_H ds \in \mathbb{R}$$

is a linear functional so that, thanks to (7.1)

$$\lim_{k \rightarrow \infty} \int_0^t \langle \sigma_g(\hat{\rho}(s)) Q\hat{\phi}_k(s), \Psi \rangle_H ds = \int_0^t \langle \sigma_g(\hat{\rho}(s)) Q\hat{\phi}(s), \Psi \rangle_H ds, \quad \hat{\mathbb{P}} - \text{a.s.} \quad (7.5)$$

Moreover, we have

$$\begin{aligned} & \left| \int_0^t \langle (\sigma_g(\hat{\rho}_k(s)) - \sigma_g(\hat{\rho}(s))) Q\hat{\phi}_k(s), \Psi \rangle_H ds \right| \\ & \leq \|\Psi\|_H \int_0^t \|\sigma_g(\hat{\rho}_k(s)) - \sigma_g(\hat{\rho}(s))\|_{\mathcal{L}(H_Q, H)} \|\hat{\phi}_k(s)\|_H ds \leq c \|\Psi\|_H \|\hat{\rho}_k - \hat{\rho}\|_{L^2(0, T; H)} \|\hat{\phi}_k\|_{L^2(0, T; H)}, \end{aligned}$$

and by using again (7.1) we get

$$\lim_{k \rightarrow \infty} \int_0^t \langle (\sigma_g(\hat{\rho}_k(s)) - \sigma_g(\hat{\rho}(s))) Q\hat{\phi}_k(s), \Psi \rangle_H ds = 0, \quad \hat{\mathbb{P}} - \text{a.s.}$$

This, together with (7.5), implies

$$\lim_{k \rightarrow \infty} \int_0^t \langle \sigma_g(\hat{\rho}_k(s)) Q \hat{\phi}_k(s), \Psi \rangle_H ds = \int_0^t \langle \sigma_g(\hat{\rho}(s)) Q \hat{\phi}(s), \Psi \rangle_H ds, \quad \hat{\mathbb{P}} - \text{a.s.} \quad (7.6)$$

Finally, since

$$\sup_{k \in \mathbb{N}} \hat{\mathbb{E}} \sup_{t \in [0, T]} \left| \int_0^t \langle \sigma_g(\hat{\rho}_{\mu_k}(s)) d\hat{w}_k^Q(s), \Psi \rangle_H \right|^2 \leq c \sup_{k \in \mathbb{N}} \|\Psi\|_H^2 \int_0^T (1 + \hat{\mathbb{E}} \|\hat{\rho}_{\mu_k}(s)\|_H^2) ds < \infty,$$

we conclude that

$$\lim_{k \rightarrow \infty} \sqrt{\mu_k} \sup_{t \in [0, T]} \hat{\mathbb{E}} \left| \int_0^t \langle \sigma_g(\hat{\rho}_{\mu_k}(s)) d\hat{w}_k^Q(s), \Psi \rangle_H \right|^2 = 0.$$

This, together with (7.2), (7.3), (7.4) and (7.6) implies that

$$\begin{aligned} \langle \hat{\rho}(t), \Psi \rangle_H &= \langle g(u_0), \Psi \rangle_H - \int_0^t \langle b(\hat{\rho}(s)) \nabla \hat{\rho}(s), \nabla \Psi \rangle_H ds \\ &\quad - \int_0^t [\langle F_g(\hat{\rho}(s)), \Psi \rangle_H + \langle \sigma_g(\hat{\rho}(s)) Q \hat{\phi}(s), \Psi \rangle_H] ds. \end{aligned}$$

As proven in Theorem 6.2, the equation above has a unique solution. Thus, for every sequence $\{\mu_k\}_{k \in \mathbb{N}} \downarrow 0$ the sequence $\{\rho_k\}_{k \in \mathbb{N}}$ converges in distribution to the solution ρ of equation (6.1) with respect to the strong topology of $C([0, T]; H^\delta)$, for any arbitrary $\delta < 1$, and hence with respect to the strong topology of $C([0, T]; L^p(\mathcal{O}))$, for every $p < \infty$, if $d = 1, 2$ and $p < 2d/(d-2)$, if $d > 2$. In particular, this implies that the sequence $\{u_k\}_{k \in \mathbb{N}}$ converges in distribution to the solution u^φ of equation (3.2) with respect to the strong topology of $C([0, T]; L^p(\mathcal{O}))$ and condition C1 holds.

As a consequence of the arguments used above to prove condition C1, we have that the mapping

$$\varphi \in L_w^2(0, T; H) \mapsto u^\varphi \in C([0, T]; L^p(\mathcal{O}))$$

is continuous. Therefore, since $\Lambda_{T, M}$ is compact in $L_w^2(0, T; H)$, for every $M > 0$, we have that

$$\Phi_{T, R} = \{I_T \leq R\} = \{u^\varphi : \varphi \in \Lambda_{T, 2R^2}\}$$

is compact, and condition C2 follows. \square

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