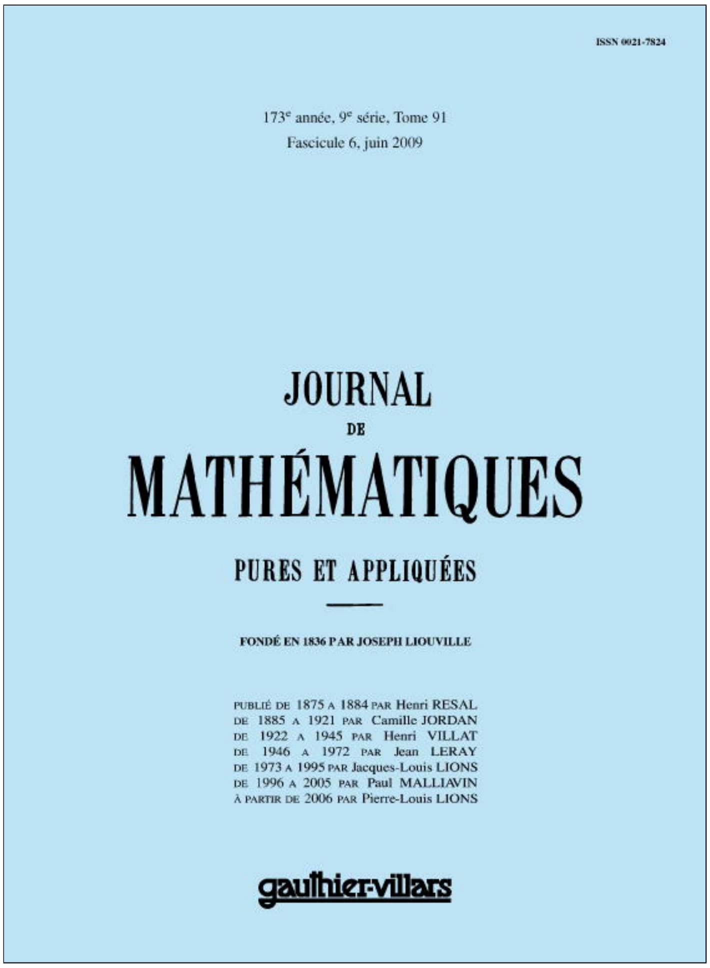
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Normal deviations from the averaged motion for some reaction–diffusion equations with fast oscillating perturbation

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Abstract

We study the normalized difference between the solution u_ϵ of a reaction–diffusion equation in a bounded interval $[0, L]$, perturbed by a fast oscillating term arising as the solution of a stochastic reaction–diffusion equation with a strong mixing behavior, and the solution $\bar{u}$ of the corresponding averaged equation. We assume the smoothness of the reaction coefficient and we prove that a central limit type theorem holds. Namely, we show that the normalized difference $(u_\epsilon - \bar{u})/\sqrt{\epsilon}$ converges weakly in $C([0, T]; L^2(0, L))$ to the solution of the linearized equation, where an extra Gaussian term appears. Such a term is explicitly given.

Résumé

Nous étudions la différence normalisée entre la solution u_ϵ d'une équation de réaction–diffusion sur un intervalle borné $[0, L]$, perturbée par un terme rapidement oscillant qui apparaît comme solution d'une équation stochastique de réaction–diffusion avec un comportement fortement mélangeant, et la solution $\bar{u}$ de l'équation correspondante moyennée. Nous supposons que le coefficient de réaction–diffusion est régulier et qu'un théoréme du type de la limite centrale s'applique. Nous montrons que la différence normalisée $(u_\epsilon - \bar{u})/\sqrt{\epsilon}$ converge faiblement dans $C([0, T]; L^2(0, L))$ vers la solution de l'équation linéarisée, où un terme gaussien supplémentaire, donné explicitement, apparaît.

MSC: 60H15; 60F05; 70K65; 37C40

Keywords: Stochastic reaction–diffusion equations; Invariant measures; Ergodic and strongly mixing processes; Averaging principle

1. Introduction

Let $\epsilon > 0$ be a small parameter. In the present paper we are dealing with the following class of reaction–diffusion equations in the bounded interval $[0, L]$:

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t}(t, \xi) = \mathcal{A} u_\epsilon(t, \xi) + f\big(\xi, u_\epsilon(t, \xi), v(t/\epsilon, \xi)\big), \quad t \geqslant 0,\ \xi \in [0, L], \\ \mathcal{N}_1 u_\epsilon(t, 0) = \mathcal{N}_1 u_\epsilon(t, L) = 0, \quad t \geqslant 0,\ u_\epsilon(0, \xi) = x(\xi),\ \xi \in [0, L], \end{cases} \tag{1.1}$$

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where v is the solution of the stochastic reaction–diffusion equation:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,\xi) = \mathcal{B}v(t,\xi) + g\big(\xi, v(t,\xi)\big) + \sigma\big(\xi, v(t,\xi)\big)\dfrac{\partial w}{\partial t}(t,\xi), \quad t \geqslant 0,\ \xi \in [0,L], \\ \mathcal{N}_2 v(t,0) = \mathcal{N}_2 v(t,L) = 0, \quad t \geqslant 0,\ v(0,\xi) = y,\ \xi \in [0,L]. \end{cases} \tag{1.2}$$

Here $\mathcal{A}$ and $\mathcal{B}$ are second order uniformly elliptic operators and $\mathcal{N}_1$ and $\mathcal{N}_2$ are some operators acting on the boundary.

The reaction coefficients $f : [0,L] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : [0,L] \times \mathbb{R} \to \mathbb{R}$ and the diffusion coefficient $\sigma : [0,L] \times \mathbb{R} \to \mathbb{R}$ are measurable and satisfy usual Lipschitz-continuity assumptions.

The stochastic perturbation in the fast motion equation is given by a cylindrical Wiener process $\partial w/\partial t$ which is white both in time and in space and is defined on a complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

What is of interest in applications is the study of the limiting behavior of the motion $u_\epsilon(t)$, for time t in intervals of order ϵ^{-1}, as it is indeed on such time scales that the most significant changes happen, like for example the exit from the neighborhood of a periodic trajectory or of an equilibrium point.

Recently, in the paper [5] and in the paper [6] written in collaboration with Mark Freidlin, we have studied this aspect, that is the occurrence of an averaging principle for a general class of systems of coupled stochastic reaction–diffusion equations, describing respectively a fast and a slow motion. We have shown that, under the main assumption that the fast motion equation, with frozen slow component x in L^2, admits a unique invariant measure μ^x which is strongly mixing, then the slow motion $u_\epsilon^{x,y}$ converges (either weakly or in probability, depending on the structure of the system) to the solution $\bar{u}^x$ of a suitable stochastic evolution equation in the Hilbert space L^2, obtained by taking the average of the coefficients of the slow motion equation with respect to the invariant measure μ^x of the fast motion equation.

In the present paper we are going one step further. Namely, we are interested in the study of the normal deviations of the slow motion $u_\epsilon^{x,y}(t)$ from the averaged motion $\bar{u}^x$. As far as we know, up to now this problem has been treated only in the case of systems with a finite number of degrees of freedom (to this purpose we refer to the fundamental paper by Khasminskii [10] appeared in 1966). In the infinite-dimensional setting the problem was completely open, not only concerning the type of results which can be obtained but also concerning the techniques which can be used in their proofs.

The situation we are considering here is much more simple than the general one considered in [5]. Actually, here we do not have a system of two fully coupled stochastic equations in any space dimension, but a deterministic reaction–diffusion equation in dimension $d = 1$ describing the slow motion, perturbed by a fast motion, obtained as the solution of a stochastic equation and independent of the slow motion.

Nevertheless, the analysis of the present situation is completely new. First of all, it was not even clear what sort of limiting motion one could have for the normalized difference $(u_\epsilon^{x,y} - \bar{u}^x)/\sqrt{\epsilon}$. Actually, it was reasonable to expect to obtain a Gaussian motion, but the structure of the covariance, which we describe explicitly, was something a priori not at all intuitive. Secondly, for the techniques which we have used in the proof of our results we are greatly indebted to the fundamental paper [10], but the passage from a finite-dimensional to an infinite-dimensional setting has required a substantial introduction of new techniques. Moreover, at present such techniques do not allow to treat the more general case of fully coupled stochastic systems in any space dimension for which an averaging phenomenon occurs.

In the situation we are considering, we assume that Eq. (1.2) admits a unique invariant measure μ and there exists a constant $\delta > 0$ such that for any $y_1, y_2 \in H := L^2(0,L)$,

$$\mathbb{E}\big|v^{y_1}(t) - v^{y_2}(t)\big|_H^2 \leqslant c e^{-2\delta t}|y_1 - y_2|_H^2, \quad t \geqslant 0.$$

This in particular implies that a spectral gap occurs for the transition semigroup P_t associated with Eq. (1.2), which means that for any Lipschitz-continuous function $\varphi : H \to \mathbb{R}$, and for any $y \in H$:

$$\big|P_t\varphi(y) - \langle \varphi, \mu\rangle\big| \leqslant e^{-\delta t}\big(|y|_H + 1\big)[\varphi]_{\mathrm{Lip}}, \quad t \geqslant 0. \tag{1.3}$$

Now, we denote by $\bar{u}^x$ the solution of the problem:

$$\begin{cases} \dfrac{\partial \bar{u}^x}{\partial t}(t,\xi) = \mathcal{A}\bar{u}^x(t,\xi) + \bar{F}\big(\bar{u}^x(t)\big)(\xi), \quad t \geqslant 0,\ \xi \in [0,L], \\ \mathcal{N}_1 \bar{u}^x(t,0) = \mathcal{N}_1 \bar{u}^x(t,L) = 0, \quad t \geqslant 0,\ \bar{u}^x(0,\xi) = x(\xi),\ \xi \in [0,L], \end{cases}$$

where for any $x, y \in H$,

$$\bar{F}(x) := \int_H F(x, y)\,\mu(dy),$$

and

$$F(x, y)(\xi) = f\big(\xi, x(\xi), y(\xi)\big), \quad \xi \in [0, L].$$

As proved both in [5] and in [6], for any $T > 0$ the family $\{u_\epsilon^{x,y}\} \subset C([0, T]; H)$ converges weakly to $\bar{u}^x$, as $\epsilon \to 0$. Moreover, if the diffusion coefficient in the slow motion equation does not depend on the fast motion, the convergence is in probability. All this means that the system of the two Eqs. (1.1) and (1.2) satisfies an *averaging principle*.

In the present paper we are interested in the analysis of the deviation of $u_\epsilon^{x,y}$ from the averaged motion $\bar{u}^x$. Namely, we want to prove that, under a smoothness assumption for the reaction coefficient f in the slow motion equation, a central limit type result holds, in the sense that

$$z_\epsilon^{x,y} := \frac{1}{\sqrt{\epsilon}}\big(u_\epsilon^{x,y} - \bar{u}^x\big) \rightharpoonup z^x, \quad \text{as } \epsilon \downarrow 0, \tag{1.4}$$

in $C([0, T]; H)$. Moreover, we want to identify the weak limit z^x as the solution of the linear problem:

$$z^x(t) = \Gamma^x(t) + \int_0^t e^{(t-s)A} D\bar{F}\big(\bar{u}^x(s)\big) z^x(s)\,ds,^1 \quad t \in [0, T], \tag{1.5}$$

where Γ^x is a Gaussian process taking values in H, having continuous trajectories and independent increments. We will characterize Γ^x by showing that it has zero mean and covariance operator given by:

$$\mathbb{E}\big\langle \Gamma^x(t), h\big\rangle_H \big\langle \Gamma^x(t), k\big\rangle_H = \int_0^t \big\langle \Phi\big(\bar{u}^x(s)\big) e^{(t-s)A} h, e^{(t-s)A} k\big\rangle_H\,ds, \tag{1.6}$$

for any $h, k \in H$ and $t \geqslant 0$, where $\Phi : H \to \mathcal{L}(H)$ is defined by,

$$\Phi(x)(k, h) := \int_0^\infty \big[\big\langle \big(F_k(x, \cdot) P_r F_h(x, \cdot) + F_h(x, \cdot) P_r F_k(x, \cdot)\big), \mu\big\rangle - 2\bar{F}_k(x)\bar{F}_h(x)\big]\,dr,$$

and where $F_h(x, y) := \langle F(x, y), h\rangle_H$ and $\bar{F}_h(x) := \langle F_h(x, \cdot), \mu\rangle_H$, for any $x, h, k \in H$.

In particular, when it is possible to factorize $\Phi(x)$ as $\Psi^\star(x)\Psi(x)$, for some $\Psi : H \to \mathcal{L}(H)$, then z^x turns out to be the mild solution of the linear stochastic partial differential equation with non-local coefficients:

$$\begin{cases} \dfrac{\partial z^x}{\partial t}(t, \xi) = \mathcal{A} z^x(t, \xi) + \big[D\bar{F}\big(\bar{u}^x(t)\big) z^x(t)\big](\xi) + \Psi\big(\bar{u}^x(t)\big)\dfrac{\partial w}{\partial t}(t, \xi), \quad t \geqslant 0,\ \xi \in [0, L], \\ \mathcal{N}_1 z^x(t, 0) = \mathcal{N}_1 z^x(t, L) = 0, \quad t \geqslant 0,\ z^x(0, \xi) = 0,\ \xi \in [0, L], \end{cases}$$

for some space–time white noise $w(t, \xi)$.

In order to prove the validity of limit (1.4), we introduce for each $\epsilon > 0$ the linear problem:

$$\zeta_\epsilon(t) = \Gamma_\epsilon^{x,y}(t) + \int_0^t e^{(t-s)A} D\bar{F}\big(\bar{u}^x(s)\big)\zeta_\epsilon(s)\,ds, \tag{1.7}$$

where

$$\Gamma_\epsilon^{x,y}(t) = \frac{1}{\sqrt{\epsilon}} \int_0^t e^{(t-s)A}\big[F\big(\bar{u}^x(s), v^y(s/\epsilon)\big) - \bar{F}\big(\bar{u}^x(s)\big)\big]\,ds. \tag{1.8}$$

[1] Here and in what follows we shall denote by e^{tA} the semigroup generated by the realization in H of the operator $\mathcal{A}$, endowed with the boundary condition $\mathcal{N}_1$.

The two key steps in the proof of (1.4) consist in showing that

1. the solution $\zeta_\epsilon^{x,y}$ of problem (1.7) weakly converges in $C([0,T];H)$ to the solution of problem (1.5), as $\epsilon \downarrow 0$,
2. the second moment of the $C([0,T];H)$-norm of the difference $z_\epsilon^{x,y} - \zeta_\epsilon^{x,y}$ converges to zero, as $\epsilon \downarrow 0$.

The first step follows once we prove that the process $\Gamma_\epsilon^{x,y}$ weakly converges in $C([0,T];H)$ to the Gaussian process Γ^x described above. This is the major task of the paper. Actually, we have to prove that, as a consequence of (1.3), the sequence $\{\mathcal{L}(\Gamma_\epsilon^{x,y})\}_{\epsilon\in(0,1]}$ is tight. Tightness is proved in Theorem 4.1 and is a consequence of Lemma 4.2, whose proof (postponed to Appendix A) is rather technical even if the only two basis ingredients are the spectral gap (1.3) and the strong Feller property of P_t.

In order to identify the weak limit with the process Γ^x, we have to show that the weak limit of any subsequence $\{\Gamma_{\epsilon_n}^{x,y}\}_n$, with $\epsilon_n \downarrow 0$, has independent increments, continuous trajectories, zero mean and covariance given by (1.6). This means that such a weak limit has to coincide with the Gaussian process Γ^x in $C([0,T];H)$. We would like to stress that the proof of identity (1.6) is quite involved and is based again on (1.3) and on a priori estimates for the processes $u_\epsilon^{x,y}(t)$ and $v^y(t/\epsilon)$. The proof of the independence of increments is based on estimate (3.11), which is on its turn a consequence of spectral gap and strong Feller property.

The second step is obtained by looking at the equation satisfied by the difference $\rho_\epsilon^{x,y} := z_\epsilon^{x,y} - \zeta_\epsilon^{x,y}$ and by proceeding with a priori bounds. Hence, the limit

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} \left|\rho_\epsilon^{x,y}(t)\right|_H^2 = 0$$

follows from Lemma 5.3, whose proof is postponed to Appendix B. For the theory of averaging of systems with a finite number of degrees of freedom we refer to the monographies [1,2,8,19] and to the papers [7,9–15,18,21,22]. For averaging of infinite dimensional systems, we refer to [16,17,20].

2. Setup

We denote by H the Hilbert space $L^2(0,L)$ of square integrable functions, endowed with the scalar product $\langle\cdot,\cdot\rangle_H$ and the corresponding norm $|\cdot|_H$. Moreover, we denote by $|\cdot|_\infty$ the usual sup-norm in $L^\infty(0,L)$.

We denote by $B_b(H)$ the Banach space of Borel bounded functions $\varphi : H \to \mathbb{R}$, endowed with the sup-norm,

$$\|\varphi\|_0 := \sup_{x\in H}\left|\varphi(x)\right|.$$

$C_b(H)$ is the subspace of uniformly continuous functions and $\mathrm{Lip}_b(H)$ is the subspace of Lipschitz-continuous functions. Moreover, we denote by $\mathrm{Lip}(H)$ the space of Lipschitz-continuous functions $\varphi : H \to \mathbb{R}$ (not necessarily bounded). $\mathrm{Lip}(H)$ is a Banach space, endowed with the norm,

$$\|\varphi\|_{\mathrm{Lip}} := \left|\varphi(0)\right| + \sup_{\substack{x,y\in H\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|_H} =: \left|\varphi(0)\right| + [\varphi]_{\mathrm{Lip}}.$$

In particular, for any $\varphi \in \mathrm{Lip}(H)$ we have:

$$\left|\varphi(x)\right| \leqslant \|\varphi\|_{\mathrm{Lip}}\big(1+|x|_H\big), \quad x\in H. \tag{2.1}$$

Finally $C_b^1(H)$ is the space of continuously Fréchet differentiable functions with bounded derivative, endowed with the norm,

$$\|\varphi\|_1 := \|\varphi\|_0 + [\varphi]_1 =: \|\varphi\|_0 + \sup_{x\in H}\left|D\varphi(x)\right|_H.$$

Next, $\mathcal{L}(H)$ is the Banach space of bounded linear operators $A : H \to H$, endowed with the sup-norm,

$$\|A\|_0 := \sup_{|x|_H \leqslant 1} |Ax|_H,$$

and $\mathcal{L}_2(H)$ is the subspace of Hilbert–Schmidt operators, endowed with the norm,

$$\|A\|_2 := \sqrt{\mathrm{Tr}[A^\star A]}.$$

The operators $\mathcal{A}$ and $\mathcal{B}$ appearing in Eq. (1.1) are second order uniformly elliptic operators with uniformly continuous coefficients and the boundary operators $\mathcal{N}_1$ and $\mathcal{N}_2$ can be either the identity operator (Dirichlet boundary conditions) or a first order operator satisfying the non-tangentiality condition (Neumann, or even general Robin boundary conditions).

As known, the realizations A and B in H of the second order operators $\mathcal{A}$ and $\mathcal{B}$, endowed respectively with the boundary conditions $\mathcal{N}_1$ and $\mathcal{N}_2$, generate two analytic semigroups with dense domain, which will be denoted by e^{tA} and e^{tB}, $t \geqslant 0$, respectively. Their domains $D(A)$ and $D(B)$ are given by:

$$W^{2,2}_{\mathcal{N}_i}(0,L) := \big\{x \in W^{2,2}(0,L)\colon \mathcal{N}_i x(0) = \mathcal{N}_i x(L) = 0\big\}, \quad i = 1,2.$$

By interpolation we have that for any $0 \leqslant r \leqslant s \leqslant 1/2$ and $t > 0$ the semigroups e^{tA} and e^{tB} map $W^{r,2}(0,L)$ into $W^{s,2}(0,L)$,[2] and

$$\big|e^{tA}h\big|_{s,2} + \big|e^{tB}h\big|_{s,2} \leqslant c_{r,s}(t\wedge 1)^{-\frac{s-r}{2}} e^{\gamma_{r,s}t}|h|_{r,2}, \tag{2.2}$$

for some constants $c_{r,s} \geqslant 1$ and $\gamma_{r,s} \in \mathbb{R}$.

In what follows, we shall assume that the semigroup e^{tA} satisfies the following conditions.

Hypothesis 1. *There exists an orthonormal basis $\{e_k\}_{k\in\mathsf{N}}$ in H and a non-negative sequence $\{\alpha_k\}_{k\in\mathsf{N}}$ such that*

$$Ae_k = -\alpha_k e_k, \quad k \in \mathbb{N}.$$

Moreover,

$$\alpha_k \sim k^2.$$

In view of Hypothesis 1, for any $s \in [0,1]$ we denote:

$$|h|_s^2 := |h|^2_{D((-A)^s)} = \sum_{k\in\mathsf{N}} \big(1 + \alpha_k^{2s}\big) h_k^2.$$

It is immediate to check that for any $s \in [0,1]$ and $h \in D((-A)^s)$,

$$\big|\big(e^{tA} - I\big)h\big|_H \leqslant c_s(t\wedge 1)^s |h|_s. \tag{2.3}$$

Together with (2.2), this implies that for any $\theta \in (0,1)$ and $p > 1/(1-\theta)$ and for any $f \in L^p(0,T;H)$ and $0 \leqslant t \leqslant t+h \leqslant T$:

$$\left|\int_0^{t+h} e^{(t+h-s)A} f(s)\,ds - \int_0^t e^{(t-s)A} f(s)\,ds\right|_H \leqslant c_{T,p} h^\theta |f|_{L^p(0,T;H)}. \tag{2.4}$$

Moreover, it is possible to show that for any $s \in [0,1/4)$ there exists $c_1, c_2 > 0$ such that

$$c_1|h|_{2s,2} \leqslant |h|_s \leqslant c_2|h|_{2s,2}, \tag{2.5}$$

so that $D((-A)^s) = W^{2s,2}(0,L)$, with equivalence of norms.

The stochastic perturbation in the fast motion equation is given by a Gaussian noise $\partial w/\partial t(t,\xi)$, for $(t,\xi) \in [0,\infty) \times [0,L]$, which is white both in time and in space. Formally, the cylindrical Wiener process $w(t,\xi)$ is given by the series,

$$w(t,\xi) = \sum_{j\in\mathsf{N}} e_j(\xi)\beta_j(t),$$

[2] For any $s > 0$, $W^{s,2}(0,L)$ denotes the set of functions $h \in H$ such that

$$[h]_{s,2} := \int_{[0,L]^2} \frac{|h(\xi) - h(\eta)|^2}{|\xi - \eta|^{2s+1}}\,d\xi\,d\eta < \infty.$$

$W^{s,2}(0,L)$ is a Banach space, endowed with the norm $|h|_{s,2} := |h|_H + [h]_{s,2}$.

for some orthonormal basis $\{e_j\}_{j\in\mathrm{N}}$ of H and some sequence of mutually independent standard Brownian motions $\{\beta_j\}_{j\in\mathrm{N}}$, defined on the same complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

The semigroup e^{tB} generated by the diffusion operator $\mathcal{B}$, with the boundary condition $\mathcal{N}_2$, appearing in Eq. (1.2) is assumed to satisfy the following conditions.

Hypothesis 2.

1. *There exists* $\lambda > 0$ *such that*

$$\left\| e^{tB} \right\|_0 \leqslant e^{-\lambda t}, \quad t \geqslant 0. \tag{2.6}$$

2. *There exist two operators* $C : D(C) \subset H \to H$ *and* $L : D(L) \subset H \to H$ *such that* $B = C + L$ *and* $|L^\star e^{tC} x|_H \leqslant c(t \wedge 1)^{-\frac{1}{2}} |x|_H$, *for any* $t > 0$ *and* $x \in H$.
3. *There exist a complete orthonormal system* $\{f_k\}_k$ *in* H *and a non-negative sequence* $\{\gamma_k\}_k$ *such that* $C f_k = -\gamma_k f_k$, *with* $\sup_{k\in\mathrm{N}} |f_k|_\infty < \infty$ *and* $\gamma_k \sim k^2$.

Remark 2.1. Assume that

$$\mathcal{B}x = ax'' + bx',$$

with $a \in C^1[0, L]$ and $b \in C[0, L]$. If set,

$$\mathcal{C}x = [ax']', \qquad \mathcal{L}x = [b - a']x',$$

and denote by C and L the realizations of $\mathcal{C}$ and $\mathcal{L}$ in H, we have that $B = C + L$ and conditions 2 and 3 introduced in Hypothesis 2 are satisfied.

Concerning the coefficients f, g and σ we assume the following conditions:

Hypothesis 3.

1. *The mapping* $f : [0, L] \times \mathbb{R}^2 \to \mathbb{R}$ *is measurable and the mapping* $f(\xi, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ *is Lipschitz-continuous, uniformly with respect to* $\xi \in [0, L]$. *Moreover, the mapping* $f(\xi, \cdot, \rho_2) : \mathbb{R} \to \mathbb{R}$ *is twice continuously differentiable, for any* $\xi \in [0, L]$ *and* $\rho_2 \in \mathbb{R}$, *with uniformly bounded derivatives.*
2. *The mapping* $g : [0, L] \times \mathbb{R} \to \mathbb{R}$ *is measurable and the mapping* $g(\xi, \cdot) : \mathbb{R} \to \mathbb{R}$ *is differentiable with,*

$$\sup_{(\xi,\rho)\in[0,L]\times\mathrm{R}} \left| \frac{\partial g(\xi, \rho)}{\partial \rho} \right| =: L_g < \lambda, \tag{2.7}$$

where λ *is the constant introduced in* (2.6).

3. *The mapping* $\sigma : [0, L] \times \mathbb{R} \to \mathbb{R}$ *is measurable and the mapping* $\sigma(\xi, \cdot) : \mathbb{R} \to \mathbb{R}$ *is differentiable, with uniformly bounded derivative. Moreover,*

$$\inf_{(\xi,\rho)\in[0,L]\times\mathrm{R}} \left| \sigma(\xi, \rho) \right| =: c_\sigma > 0, \tag{2.8}$$

and there exists $\beta \in [0, 1)$ *such that*

$$\sup_{(\xi,\rho)\in[0,L]\times\mathrm{R}} \frac{|\sigma(\xi, \rho)|}{1 + |\rho|^\beta} =: M_\sigma < \infty. \tag{2.9}$$

In what follows, for any $x, y, z \in H$ and $\xi \in [0, L]$, we shall set:

$$F(x, y)(\xi) := f\big(\xi, x(\xi), y(\xi)\big), \qquad G(y)(\xi) := g\big(\xi, y(\xi)\big), \qquad \big[\Sigma(y)z\big](\xi) := \sigma\big(\xi, y(\xi)\big) z(\xi).$$

Due to the conditions in Hypothesis 3, the mappings,

$$F : H \times H \to H, \qquad G : H \to H,$$

are both Lipschitz-continuous and, due to (2.9), the mappings,

$$\Sigma : H \to \mathcal{L}\big(H; L^{\frac{2}{\beta+1}}(0,L)\big), \qquad \Sigma : H \to \mathcal{L}\big(L^{\frac{2}{1-\beta}}(0,L); H\big),$$

are both Lipschitz-continuous. Moreover, from (2.8) we have that for any $y, z \in H$,

$$\big[\Sigma^{-1}(y)z\big](\xi) := \frac{z(\xi)}{\sigma(\xi, y(\xi))}, \quad \xi \in [0, L],$$

is well defined, and

$$\Sigma^{-1}(y)\Sigma(y) = \Sigma(y)\Sigma^{-1}(y) = I, \quad y \in H. \tag{2.10}$$

According to Hypotheses 2 and 3, both Eqs. (1.1) and (1.2) admit unique mild solutions in $L^p(\Omega, C([0,T]; H))$, for any $p \geqslant 1$ and $T > 0$. Namely, as proved for example in [3], for any $y \in H$ there exists a unique adapted process $v^y \in L^p(\Omega, C([0,T]; H))$ such that

$$v^y(t) = e^{tB}y + \int_0^t e^{(t-s)B} G\big(v^y(s)\big)\, ds + \int_0^t e^{(t-s)B} \Sigma\big(v^y(s)\big)\, dw(s),$$

and in correspondence of such v^y, for any $\epsilon > 0$ and $x \in H$ there exists a unique adapted process $u_\epsilon^{x,y} \in L^p(\Omega, C([0,T]; H))$ such that

$$u_\epsilon^{x,y}(t) = e^{tA}x + \int_0^t e^{(t-s)A} F\big(u_\epsilon^{x,y}(s), v^y(s/\epsilon)\big)\, ds.$$

2.1. The fast transition semigroup

Now, we introduce the transition semigroup associated with Eq. (1.2), by setting for any $\varphi \in B_b(H)$ and $t \geqslant 0$,

$$P_t\varphi(y) = \mathbb{E}\varphi\big(v^y(t)\big), \quad y \in H.$$

Due to the differentiability assumptions on g and σ and to (2.8) which implies (2.10), we have that P_t is a strong Feller semigroup. More precisely, it maps $B_b(H)$ into $C_b^1(H)$ and for any $\varphi \in B_b(H)$

$$[P_t\varphi]_1 \leqslant \frac{c}{\sqrt{t}} \|\varphi\|_0, \quad t > 0. \tag{2.11}$$

Moreover, as proved in [4, Theorem 7.3], due to (2.6) and (2.7) and to the growth condition (2.9), for any $p \geqslant 1$,

$$\mathbb{E}\big|v^y(t)\big|_H^p \leqslant c\big(1 + e^{-\gamma p t}|y|_H^p\big), \tag{2.12}$$

with $\gamma = (\lambda - L_g)/2$, and there exists some $\bar\theta > 0$ such that for any $t_0 > 0$,

$$\sup_{t \geqslant t_0} \mathbb{E}\big|v^y(t)\big|_{D((-B)^{\bar\theta})} \leqslant c_{t_0}\big(1 + |y|_H\big). \tag{2.13}$$

This in particular implies that the family $\{\mathcal{L}(v^y(t))\}_{t \geqslant t_0}$ is tight in $\mathcal{P}(H)$, so that the semigroup P_t admits an invariant measure μ.

Next, we assume that the fast equation (1.2) satisfies the following condition.

Hypothesis 4. *There exists $\delta > 0$ such that for any $y_1, y_2 \in H$:*

$$\mathbb{E}\big|v^{y_1}(t) - v^{y_2}(t)\big|_H^2 \leqslant c e^{-4\delta t}|y_1 - y_2|_H^2, \quad t \geqslant 0. \tag{2.14}$$

This implies that for any $\varphi \in \mathrm{Lip}(H)$,

$$\big|P_t\varphi(y_1) - P_t\varphi(y_2)\big| \leqslant c e^{-2\delta t}[\varphi]_{\mathrm{Lip}}|y_1 - y_2|_H, \tag{2.15}$$

so that the invariant measure μ is unique and strongly mixing and

$$\big|P_t\varphi(y) - \langle\varphi,\mu\rangle\big| \leqslant ce^{-2\delta t}\big(|y|_H + 1\big)[\varphi]_{\mathrm{Lip}}, \quad t > 0, \tag{2.16}$$

where

$$\langle\varphi,\mu\rangle := \int_H \varphi(z)\,\mu(dz).$$

Moreover, according to (2.11) and to the semigroup law, this implies that for any $\varphi \in B_b(H)$,

$$\big|P_t\varphi(y) - \langle\varphi,\mu\rangle\big| \leqslant \frac{ce^{-\delta t}}{\sqrt{t}}\big(|y|_H + 1\big)\|\varphi\|_0, \quad t > 0. \tag{2.17}$$

Notice that, as proved for example in [6, Lemma 3.4], from (2.12) we have:

$$\int_H |z|_H^p\,\mu(dz) =: c_p < \infty,$$

for any $p \geqslant 1$. Then, by using (2.12) and (2.14), we easily obtain that for any $\varphi, \psi \in \mathrm{Lip}(H)$:

$$\big|P_t(\varphi\psi)(y) - \langle\varphi\psi,\mu\rangle\big| \leqslant ce^{-2\delta t}\big(1 + |y|_H^2\big)\|\varphi\|_{\mathrm{Lip}}\|\psi\|_{\mathrm{Lip}}. \tag{2.18}$$

Remark 2.2.

1. As proved in [4, Theorem 7.3], (2.12) and (2.13) are still valid if in (2.9) we take $\beta = 1$ and we assume also the condition,
$$k_{1,p}\left(\frac{L_g}{\lambda}\right)^p + k_{2,p}\frac{M_\sigma^p}{\lambda^{c_p}} < 1,$$
for suitable constants $k_{1,p}, k_{2,p}, c_p > 0$.
2. In [4, Theorem 7.4] it is proved that for any p large enough there exist some constants $h_{1,p}, h_{2,p}, c_p > 0$ such that the condition,
$$h_{1,p}\left(\frac{L_g}{\lambda}\right)^p + h_{2,p}\frac{M_\sigma^p}{\lambda^{c_p}} < 1,$$
implies that there exists $\delta_p > 0$ such that
$$\mathbb{E}\big|v^{y_1}(t) - v^{y_2}(t)\big|_H^p \leqslant e^{-\delta_p t}|y_1 - y_2|_H^p,$$
for any $y_1, y_2 \in H$ and $t \geqslant 0$, and hence (2.14) holds.
3. When the stochastic perturbation in Eq. (1.2) is of additive type, that is $\sigma \equiv 1$, thanks to (2.6) and (2.7) condition (2.14) is always satisfied with $\delta = (\lambda - L_g)/2$.
4. In the case $\sigma \equiv 1$, in Eq. (1.2) we do not need to have a noise which is white in space but we can also consider a cylindrical Wiener process of the following type,
$$w^Q(t,\xi) = \sum_{j=1}^\infty Qe_j(\xi)\beta_j(t),$$
for some $Q \in \mathcal{L}^+(H)$. In this case, in Hypothesis 2 we have to add to (2.6) the following two conditions:
(a) for any $t > 0$ the operator $e^{tB}Q$ belongs to $\mathcal{L}_2(H)$ and there exists $\gamma \in (0, 1/2)$ such that
$$\int_0^\infty t^{-\gamma}\big\|e^{tB}Q\big\|_2^2\,dt < \infty, \tag{2.19}$$
(b) there exists $\eta < 1$ such that
$$\mathrm{Im}(-B)^{-\frac{\eta}{2}} \subset \mathrm{Im}\,Q. \tag{2.20}$$

Due to the closed graph theorem, condition (2.20) means that there exists some $\Gamma_\eta \in \mathcal{L}(H)$ such that

$$Q^{-1} = \Gamma_\eta(-B)^{\frac{\eta}{2}}.$$

Such a condition assures that the semigroup associated with the fast equation has a smoothing effect.

Moreover, if we assume that there exists a orthonormal basis $\{e_j\}_{j\in\mathrm{N}}$ in H and two sequences of non-negative real numbers $\{\theta_j\}_{j\in\mathrm{N}}$ and $\{\lambda_j\}_{j\in\mathrm{N}}$ such that

$$Be_j = -\theta_j e_j, \qquad Qe_j = \lambda_j e_j,$$

condition (2.19) becomes,

$$\sum_{j\in\mathrm{N}} \frac{\lambda_j^2}{\theta_j^{1-\gamma}} < \infty,$$

and condition (2.20) becomes,

$$\inf_{j\in\mathrm{N}} \lambda_j \theta_j^{\frac{\eta}{2}} > 0.$$

In the interval $[0, L]$ we have $\theta_j \sim j^2$, so that it is not difficult to check that the two conditions above can be both satisfied, by a suitable choice of the sequence $\{\lambda_j\}$.

2.2. *The averaged equation*

In what follows, for any fixed $x, y, h \in H$ we define:

$$F_h(x, y) := \langle F(x, y), h\rangle_H.$$

According to Hypothesis 3-1, the mapping,

$$y \in H \mapsto F_h(x, y) := \langle F(x, y), h\rangle_H \in \mathbb{R},$$

is Lipschitz-continuous and

$$[F_h(x, \cdot)]_{\mathrm{Lip}} \leqslant L_f |h|_H, \tag{2.21}$$

where L_f is the Lipschitz constant of the mapping $f(\xi, \cdot) : \mathbb{R}^2 \to \mathbb{R}$. Then, if we define,

$$\bar{F}(x) := \int_H F(x, y)\,\mu(dy), \quad x \in H,$$

due to (2.16) for any fixed $t \geqslant 0$ and $x, y, h \in H$ we have:

$$\left|\mathbb{E}F_h\big(x, v^y(t)\big) - \bar{F}_h(x)\right| \leqslant cL_f e^{-2\delta t}\big(1 + |y|_H\big)|h|_H, \tag{2.22}$$

so that in particular for any $\tau \geqslant 0$ and $T > 0$,

$$\left|\frac{1}{T}\int_\tau^{T+\tau} \mathbb{E}F_h\big(x, v^y(t)\big)\,dt - \bar{F}_h(x)\right| \leqslant \frac{c}{T}\big(1 + |y|_H\big)|h|_H. \tag{2.23}$$

In [5] (and also in [6] in the case of additive noise), we have proved that under Hypotheses 2, 3 and 4 an averaging principle holds for the process u_ϵ (in fact in [6] and [5] much more general situations are treated). Namely, we have proved that in the setting we are considering here for any $x \in D((-A)^\alpha)$, with $\alpha > 0$, and $y \in H$ and for any $T, \eta > 0$ it holds,

$$\lim_{\epsilon\to 0} \mathbb{P}\big(\big|u_\epsilon^{x,y} - \bar{u}^x\big|_{C([0,T];H)} > \eta\big) = 0, \tag{2.24}$$

where $\bar{u}^x$ is the solution in $C([0, +\infty); H)$ of the averaged equation:

$$\frac{du}{dt}(t) = Au(t) + \bar{F}\big(u(t)\big), \quad u(0) = x. \tag{2.25}$$

Notice that, due to our assumptions on A and F (and hence on $\bar{F}$), for any $T > 0$, $\alpha \in [0,1)$ and $x \in D((-A)^\alpha)$, we have:

$$\sup_{t\in[0,T]} \left|\bar{u}^x(t)\right|_\alpha = c_{T,\alpha}\left(1+|x|_\alpha\right) < \infty. \tag{2.26}$$

Moreover, from (2.3) and (2.26) for any $\alpha \in [0,1)$ and $x \in D((-A)^\alpha)$ we obtain:

$$\left|\bar{u}^x(t) - \bar{u}^x(s)\right|_H \leqslant c_{T,\alpha}(t-s)^\alpha\left(1+|x|_\alpha\right), \quad 0 \leqslant s < t \leqslant T. \tag{2.27}$$

3. Some consequences of the spectral gap

In the previous section we have seen that, as an immediate consequence of the spectral gap (2.16), estimate (2.23) holds. The following result is maybe less immediate, but it is again a consequence of (2.16).

Lemma 3.1. *Under Hypotheses* 2, 3 *and* 4, *for any* $\varphi, \psi \in \mathrm{Lip}(H)$ *and* $y \in H$ *and for any* $\tau \geqslant 0$ *we have*:

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \int_\tau^{\tau+T}\int_\tau^{\tau+T} \mathbb{E}\left[\varphi\left(v^y(t)\right) - \mathbb{E}\varphi\left(v^y(t)\right)\right]\left[\psi\left(v^y(s)\right) - \mathbb{E}\psi\left(v^y(s)\right)\right] dt\, ds \\
&\quad = \int_H \left(\varphi(z)\int_0^\infty \left(P_r\psi(z) - \langle\psi,\mu\rangle\right) dr + \psi(z)\int_0^\infty \left(P_r\varphi(z) - \langle\varphi,\mu\rangle\right) dr\right)\mu(dz) \\
&\quad = \int_0^\infty \left(\left\langle[\varphi P_r\psi + \psi P_r\varphi],\mu\right\rangle - 2\langle\varphi,\mu\rangle\langle\psi,\mu\rangle\right) dr,
\end{aligned} \tag{3.1}$$

and the limit is uniform with respect to τ.

Proof. We have:

$$\int_\tau^{\tau+T}\int_\tau^{\tau+T} \mathbb{E}\left[\varphi\left(v^y(t)\right) - \mathbb{E}\varphi\left(v^y(t)\right)\right]\left[\psi\left(v^y(s)\right) - \mathbb{E}\psi\left(v^y(s)\right)\right] dt\, ds = \int_\tau^{\tau+T} \left(I^y_{\varphi,\psi}(t) + I^y_{\psi,\varphi}(t)\right) dt, \tag{3.2}$$

where

$$\begin{aligned}
I^y_{\varphi,\psi}(t) &:= \int_t^{\tau+T} \mathbb{E}\left[\varphi\left(v^y(t)\right) - \mathbb{E}\varphi\left(v^y(t)\right)\right]\left[\psi\left(v^y(s)\right) - \mathbb{E}\psi\left(v^y(s)\right)\right] ds \\
&= \int_t^{\tau+T} \left[\mathbb{E}\varphi\left(v^y(t)\right)\psi\left(v^y(s)\right) - P_t\varphi(y)P_s\psi(y)\right] ds.
\end{aligned}$$

Then, from the Markov property, we obtain:

$$\begin{aligned}
I^y_{\varphi,\psi}(t) &= \int_t^{\tau+T} \left[\mathbb{E}\varphi\left(v^y(t)\right)P_{s-t}\psi\left(v^y(t)\right) - P_t\varphi(y)P_s\psi(y)\right] ds \\
&= \int_t^{\tau+T} \mathbb{E}\varphi\left(v^y(t)\right)\left[P_{s-t}\psi\left(v^y(t)\right) - P_s\psi(y)\right] ds \\
&= \mathbb{E}\varphi\left(v^y(t)\right) \int_0^{\tau+T-t} \left[P_r\psi\left(v^y(t)\right) - P_t(P_r\psi)(y)\right] dr.
\end{aligned}$$

This implies:

$$\begin{aligned}
&\int_\tau^{\tau+T}\left[\mathbb{E}\varphi\big(v^y(t)\big)\int_0^\infty\big(P_r\psi\big(v^y(t)\big)-\langle\psi,\mu\rangle\big)\,dr-I^y_{\varphi,\psi}(t)\right]dt\\
&\quad=\int_\tau^{\tau+T}\mathbb{E}\varphi\big(v^y(t)\big)\int_{\tau+T-t}^\infty\big(P_r\psi\big(v^y(t)\big)-\langle\psi,\mu\rangle\big)\,dr\,dt+\int_\tau^{\tau+T}\mathbb{E}\varphi\big(v^y(t)\big)\int_0^{\tau+T-t}\big(P_t(P_r\psi)(y)-\langle\psi,\mu\rangle\big)\,dr\,dt\\
&\quad=:J_1(\tau,T)+J_2(\tau,T).
\end{aligned}$$

Due to (2.1) and (2.16) we have:

$$\big|J_1(\tau,T)\big|\leqslant c[\psi]_{\mathrm{Lip}}\|\varphi\|_{\mathrm{Lip}}\int_\tau^{\tau+T}\big(1+\mathbb{E}\big|v^y(t)\big|_H^2\big)\int_{\tau+T-t}^\infty e^{-2\delta r}\,dr\,dt,$$

and then, thanks to (2.12), we obtain:

$$\big|J_1(\tau,T)\big|\leqslant c[\psi]_{\mathrm{Lip}}\|\varphi\|_{\mathrm{Lip}}\big(1+|y|_H^2\big)\big(1-e^{-2\delta T}\big). \tag{3.3}$$

Analogously, we have:

$$\begin{aligned}
\big|J_2(\tau,T)\big|&\leqslant c[\psi]_{\mathrm{Lip}}\|\varphi\|_{\mathrm{Lip}}\int_\tau^{\tau+T}\big(1+\mathbb{E}\big|v^y(t)\big|_H^2\big)\int_t^{\tau+T} e^{-2\delta r}\,dr\,dt\\
&\leqslant c[\psi]_{\mathrm{Lip}}\|\varphi\|_{\mathrm{Lip}}\big(1+|y|_H^2\big)\big(1-e^{-2\delta T}\big).
\end{aligned}$$

Together with (3.3) this yields:

$$\lim_{T\to\infty}\frac{1}{T}\int_\tau^{\tau+T}\left[\mathbb{E}\varphi\big(v^y(t)\big)\int_0^\infty\big(P_r\psi\big(v^y(t)\big)-\langle\psi,\mu\rangle\big)\,dr-I^y_{\varphi,\psi}(t)\right]dt=0, \tag{3.4}$$

and the limit is uniform with respect to τ. Due to (2.15), for any $\psi\in\mathrm{Lip}(H)$ the mapping,

$$x\in H\to\int_0^\infty\big(P_r\psi(x)-\langle\psi,\mu\rangle\big)\,dr\in\mathbb{R},$$

is Lipschitz-continuous and then, thanks to (2.18), we have:

$$\lim_{T\to\infty}\frac{1}{T}\int_\tau^{\tau+T}\mathbb{E}\varphi\big(v^y(t)\big)\int_0^\infty\big(P_r\psi\big(v^y(t)\big)-\langle\psi,\mu\rangle\big)\,dr\,dt=\int_H\varphi(z)\int_0^\infty\big(P_r\psi(z)-\langle\psi,\mu\rangle\big)\,dr\,\mu(dz),$$

uniformly with respect to τ. From (3.4), this yields,

$$\lim_{T\to\infty}\frac{1}{T}\int_\tau^{\tau+T}I^y_{\varphi,\psi}(t)\,dt=\int_H\varphi(z)\int_0^\infty\big(P_r\psi(z)-\langle\psi,\mu\rangle\big)\,dr\,\mu(dz),$$

uniformly with respect to $\tau\geqslant 0$. The same is true for $I^y_{\psi,\varphi}(t)$ (clearly with φ and ψ exchanged) and then we get (3.1). □

In what follows, for any $x,h,k\in H$ we shall define:

$$\Phi(x)(k,h):=\int_0^\infty\big[\big\langle\big[F_k(x,\cdot)P_rF_h(x,\cdot)+F_h(x,\cdot)P_rF_k(x,\cdot)\big],\mu\big\rangle-2\bar F_k(x)\bar F_h(x)\big]\,dr. \tag{3.5}$$

Notice that $\Phi(x): H \times H \to \mathbb{R}$ is a symmetric bi-linear map, and

$$\left|\Phi(x)(k,h)\right| \leqslant c\left(1+|x|_H^2\right)|h|_H|k|_H. \tag{3.6}$$

This means that $\Phi(x) \in \mathcal{L}^+(H)$, for any $x \in H$, and

$$\Phi(x)(k,h) = \left\langle \Phi(x)k, h\right\rangle_H, \quad (k,h) \in H \times H.$$

With these notations, in view of (3.1), we have:

$$\lim_{T\to\infty} \frac{1}{T} \int_\tau^{\tau+T}\int_\tau^{\tau+T} \mathbb{E}\left[F_k\left(x, v^y(t)\right) - \mathbb{E}F_k\left(x, v^y(t)\right)\right]\left[F_h\left(x, v^y(s)\right) - \mathbb{E}F_h\left(x, v^y(s)\right)\right] dt\, ds$$
$$= \left\langle \Phi(x)k, h\right\rangle_H. \tag{3.7}$$

Next, for any $0 \leqslant s \leqslant t \leqslant \infty$ and $y \in H$ we denote:

$$\mathcal{H}_s^t(y) := \sigma\left(v^y(r),\ s \leqslant r \leqslant t\right).$$

The σ-algebra $\mathcal{H}_s^t(y)$ is clearly generated by the family $\mathcal{C}_s^t(y)$ of cylindrical sets, that is the family of all sets of the following type $\{v^y(r_1) \in A_1, \dots, v^y(r_k) \in A_k\}$, for $k \in \mathbb{N}$, $s \leqslant r_1 < r_2 < \cdots < r_k \leqslant t$ and $A_1, \dots, A_k \in \mathcal{B}(H)$.

Lemma 3.2. *Under Hypotheses* 2, 3 *and* 4, *for any $y \in H$ and $s, t > 0$ it holds*:

$$\sup\left\{\left|\mathbb{P}(B_1 \cap B_2) - \mathbb{P}(B_1)\mathbb{P}(B_2)\right|:\ B_1 \in \mathcal{H}_0^t(y),\ B_2 \in \mathcal{H}_{t+s}^\infty(y)\right\} \leqslant \frac{ce^{-\delta s}}{\sqrt{s}}\left(1+|y|_H\right). \tag{3.8}$$

Proof. Let $B_1 \in \mathcal{C}_0^t(y)$ and $B_2 \in \mathcal{C}_{s+t}^\infty(y)$. We have:

$$B_1 = \bigcap_{i=1}^{k_1}\left\{v^y(r_{1,i}) \in A_{1,i}\right\}, \qquad B_2 = \bigcap_{i=1}^{k_2}\left\{v^y(r_{2,i}) \in A_{2,i}\right\},$$

where $0 \leqslant r_{1,1} < \cdots < r_{1,k_1} \leqslant t$ and $s+t \leqslant r_{2,1} < \cdots < r_{2,k_2} < \infty$ and $A_{j,i} \in \mathcal{B}(H)$, for $j = 1,2$ and $i = 1, \dots, k_j$.

We have:

$$\mathbb{P}(B_1 \cap B_2) = \mathbb{E}\left(\prod_{i=1}^{k_1} \mathbb{I}_{A_{1,i}}\left(v^y(r_{1,i})\right)\prod_{i=1}^{k_2} \mathbb{I}_{A_{2,i}}\left(v^y(r_{2,i})\right)\right)$$
$$= \mathbb{E}\left(\prod_{i=1}^{k_1} \mathbb{I}_{A_{1,i}}\left(v^y(r_{1,i})\right)\mathbb{E}\left(\prod_{i=1}^{k_2} \mathbb{I}_{A_{2,i}}\left(v^y(r_{2,i})\right)|\mathcal{F}_t\right)\right).$$

Now, as $t+s \leqslant r_{2,1} < \cdots < r_{2,k_2}$, we have:

$$\mathbb{E}\left(\prod_{i=1}^{k_2} \mathbb{I}_{A_{2,i}}\left(v^y(r_{2,i})\right)|\mathcal{F}_t\right) = \mathbb{E}\left(\mathbb{I}_{A_{2,1}}\left(v^y(r_{2,1})\right)\mathbb{E}\left(\prod_{i=2}^{k_2} \mathbb{I}_{A_{2,i}}\left(v^y(r_{2,i})\right)|\mathcal{F}_{r_{2,1}}\right)|\mathcal{F}_t\right)$$
$$= \mathbb{E}\left(\mathbb{I}_{A_{2,1}}\left(v^y(r_{2,1})\right)\mathbb{E}\left(\mathbb{I}_{A_{2,2}}\left(v^y(r_{2,2})\right)\mathbb{E}\left(\prod_{i=3}^{k_2} \mathbb{I}_{A_{2,i}}\left(v^y(r_{2,i})\right)|\mathcal{F}_{r_{2,2}}\right)|\mathcal{F}_{r_{2,1}}\right)|\mathcal{F}_t\right),$$

and then, by iterating this procedure, we obtain,

$$\mathbb{E}\left(\prod_{i=1}^{k_2} \mathbb{I}_{A_{2,i}}\left(v^y(r_{2,i})\right)|\mathcal{F}_t\right) = \mathbb{E}\left(\mathbb{I}_{A_{2,1}}\left(v^y(r_{2,1})\right)\mathbb{E}\left(\mathbb{I}_{A_{2,2}}\left(v^y(r_{2,2})\right)\cdots\mathbb{E}\left(\mathbb{I}_{A_{2,k_2}}\left(v^y(r_{2,k_2})\right)|\mathcal{F}_{r_{2,k_2-1}}\right)|\cdots|\mathcal{F}_{r_{2,1}}\right)|\mathcal{F}_t\right).$$

This implies:

$$\mathbb{E}\left(\prod_{i=1}^{k_2} \mathbb{I}_{A_{2,i}}\left(v^y(r_{2,i})\right)|\mathcal{F}_t\right) = P_{r_{2,1}-t}\left[\mathbb{I}_{A_{2,1}}P_{r_{2,2}-r_{2,1}}\left(\mathbb{I}_{A_{2,2}}P_{r_{2,3}-r_{2,2}}(\mathbb{I}_{A_{2,3}}\cdots)\right)\right]\left(v^y(t)\right).$$

Analogously, we have:

$$\mathbb{E}\prod_{i=1}^{k_2}\mathbb{I}_{A_{2,i}}\big(v^y(r_{2,i})\big)=P_{r_{2,1}-t}\big[\mathbb{I}_{A_{2,1}}P_{r_{2,2}-r_{2,1}}\big(\mathbb{I}_{A_{2,2}}P_{r_{2,3}-r_{2,2}}(\mathbb{I}_{A_{2,3}}\cdots)\big)\big](y).$$

Therefore,

$$\mathbb{P}(B_1\cap B_2)-\mathbb{P}(B_1)\mathbb{P}(B_2)=\mathbb{E}\prod_{i=1}^{k_1}\mathbb{I}_{A_{1,i}}\big(v^y(r_{1,i})\big)\big(P_{r_{2,1}-t}\big[\mathbb{I}_{A_{2,1}}P_{r_{2,2}-r_{2,1}}\big(\mathbb{I}_{A_{2,2}}P_{r_{2,3}-r_{2,2}}(\mathbb{I}_{A_{2,3}}\cdots)\big)\big]\big(v^y(t)\big)$$
$$-P_{r_{2,1}-t}\big[\mathbb{I}_{A_{2,1}}P_{r_{2,2}-r_{2,1}}\big(\mathbb{I}_{A_{2,2}}P_{r_{2,3}-r_{2,2}}(\mathbb{I}_{A_{2,3}}\cdots)\big)\big](y)\big).$$

Thanks to (2.17) and (2.12), this yields:

$$\big|\mathbb{P}(B_1\cap B_2)-\mathbb{P}(B_1)\mathbb{P}(B_2)\big|\leqslant\frac{ce^{-\delta(r_{2,1}-t)}}{\sqrt{r_{2,1}-t}}\big(\mathbb{E}\big|v^y(t)\big|_H+|y|_H+1\big)\leqslant\frac{ce^{-\delta s}}{\sqrt{s}}\big(1+|y|_H\big),$$

so that (3.8) holds for $B_1\in\mathcal{C}_0^t(y)$ and $B_2\in\mathcal{C}_{t+s}^\infty(y)$.

The general case of $B_1\in\mathcal{H}_0^t(y)$ and $B_2\in\mathcal{H}_{s+t}^\infty(y)$ follows from a monotone class argument, as $\mathcal{C}_s^t(y)$ is an algebra which generates $\mathcal{H}_s^t(y)$, for any $0\leqslant s\leqslant t\leqslant\infty$. □

An important consequence of the previous lemma is given by the following result.

Proposition 3.3. *Assume Hypotheses* 2, 3 *and* 4. *Let* $y\in H$ *and let* $\xi_1,\dots,\xi_n$ *be complex-valued random variables such that* ξ_i *is* $\mathcal{H}_{s_i}^{t_i}(y)$*-measurable, for any* $i=1,\dots,n$ *and* $0\leqslant s_1\leqslant t_1<s_2\leqslant\cdots<s_n\leqslant t_n$.

1. *If* $|\xi_i|\leqslant 1$, $\mathbb{P}$*-a.s., then*
$$\left|\mathbb{E}\prod_{i=1}^n\xi_i-\prod_{i=1}^n\mathbb{E}\xi_i\right|\leqslant c(n-1)\frac{e^{-\delta\Delta}}{\sqrt{\Delta}},\tag{3.9}$$
where $\Delta:=\min\{s_2-t_1,\dots,s_n-t_{n-1}\}$.
2. *If there exists* $\rho\in(0,1)$ *such that*
$$\sup_{i=1,\dots,n}|\xi_i|_{L^{\frac{2(n-1)}{1-\rho}}(\Omega;\mathsf{C})}:=\kappa_{n,\rho}<\infty,\tag{3.10}$$
then
$$\left|\mathbb{E}\prod_{i=1}^n\xi_i-\prod_{i=1}^n\mathbb{E}\xi_i\right|\leqslant c_{n,\rho}\kappa_{n,\rho}^n\left(\frac{e^{-\delta\Delta}}{\sqrt{\Delta}}\right)^{\frac{\rho}{2+\rho}},\tag{3.11}$$
for some positive constant $c_{n,\rho}$.

Proof. The proof of (3.9) is as in [19, Lemmas IV.11.1 and IV.11.2] by induction on n. We recall it for the reader's convenience.

Let $n=2$. We have:

$$|\mathbb{E}\xi_1\xi_2-\mathbb{E}\xi_1\mathbb{E}\xi_2|=\big|\mathbb{E}\big(\xi_1\big[\mathbb{E}\big(\xi_2|\mathcal{H}_{s_1}^{t_1}(y)\big)-\mathbb{E}\xi_2\big]\big)\big|\leqslant\big|\mathbb{E}\big(\xi_1'\big[\mathbb{E}\big(\xi_2|\mathcal{H}_{s_1}^{t_1}(y)\big)-\mathbb{E}\xi_2\big]\big)\big|=\big|\mathbb{E}\xi_1'\xi_2-\mathbb{E}\xi_1'\mathbb{E}\xi_2\big|,$$

where

$$\xi_1':=2\mathbb{I}_{\{\mathsf{E}(\xi_2|\mathcal{H}_{s_1}^{t_1}(y))-\mathsf{E}\xi_2>0\}}-1.$$

Analogously, we have:

$$\big|\mathbb{E}\xi_1'\xi_2-\mathbb{E}\xi_1'\mathbb{E}\xi_2\big|\leqslant\big|\mathbb{E}\xi_1'\xi_2'-\mathbb{E}\xi_1'\mathbb{E}\xi_2'\big|,$$

where

$$\xi_2':=2\mathbb{I}_{\{\mathsf{E}(\xi_1'|\mathcal{H}_{s_2}^{t_2}(y))-\mathsf{E}\xi_1'>0\}}-1.$$

Therefore, we have:

$$|\mathbb{E}\xi_1\xi_2 - \mathbb{E}\xi_1\mathbb{E}\xi_2| \leqslant \big|\mathbb{E}\xi_1'\xi_2' - \mathbb{E}\xi_1'\mathbb{E}\xi_2'\big| = 4\big|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)\big|,$$

where

$$A := \big\{\mathbb{E}\big(\xi_2|\mathcal{H}_{s_1}^{t_1}(y)\big) - \mathbb{E}\xi_2 > 0\big\} \in \mathcal{H}_{s_1}^{t_1}(y),$$

and

$$B := \big\{\mathbb{E}\big(\xi_1'|\mathcal{H}_{s_2}^{t_2}(y)\big) - \mathbb{E}\xi_1' > 0\big\} \in \mathcal{H}_{s_2}^{t_2}(y).$$

According to (3.8), this implies:

$$|\mathbb{E}\xi_1\xi_2 - \mathbb{E}\xi_1\mathbb{E}\xi_2| \leqslant c\frac{e^{-\delta(s_2-t_1)}}{\sqrt{s_2-t_1}}\big(1+|y|_H\big),$$

so that (3.9) holds for $n=2$.

Next, assume that (3.9) is true for $n-1$. We have:

$$\left|\mathbb{E}\prod_{i=1}^n\xi_i - \prod_{i=1}^n\mathbb{E}\xi_i\right| \leqslant \left|\mathbb{E}\prod_{i=1}^{n-1}\xi_i\xi_n - \mathbb{E}\prod_{i=1}^{n-1}\xi_i\mathbb{E}\xi_n\right| + \left|\left(\mathbb{E}\prod_{i=1}^{n-1}\xi_i - \prod_{i=1}^{n-1}\mathbb{E}\xi_i\right)\mathbb{E}\xi_n\right|, \tag{3.12}$$

and then, from the inductive hypothesis,

$$\left|\mathbb{E}\prod_{i=1}^n\xi_i - \prod_{i=1}^n\mathbb{E}\xi_i\right| \leqslant \frac{ce^{-\delta(s_n-t_{n-1})}}{\sqrt{s_n-t_{n-1}}} + c(n-2)\frac{ce^{-\delta\Delta}}{\sqrt{\Delta}} \leqslant c(n-1)\frac{ce^{-\delta\Delta}}{\sqrt{\Delta}}.$$

Now we prove (3.11), in the case the moduli of the random variables ξ_i are not pointwise bounded by 1, but their momenta satisfy condition (3.10). As before, we proceed by induction on n and we first verify (3.11) for $n=2$.

For any $R>0$ let us define:

$$A_{1,R} := \big\{|\xi_1| \leqslant R\big\}, \qquad A_{2,R} := \big\{|\xi_2| \leqslant R\big\}.$$

We have:

$$\begin{aligned}\mathbb{E}\xi_1\xi_2 - \mathbb{E}\xi_1\mathbb{E}\xi_2 &= \mathbb{E}(\xi_1\xi_2; A_{1,R}\cap A_{2,R}) + \mathbb{E}\big(\xi_1\xi_2; A_{1,R}^c\cup A_{2,R}^c\big) - \mathbb{E}\xi_1\mathbb{E}\xi_2\\ &= (\mathbb{E}\xi_1\mathbb{I}_{A_{1,R}}\xi_2\mathbb{I}_{A_{2,R}} - \mathbb{E}\xi_1\mathbb{I}_{A_{1,R}}\mathbb{E}\xi_2\mathbb{I}_{A_{2,R}}) + \mathbb{E}\xi_1\xi_2\mathbb{I}_{A_{1,R}^c\cup A_{2,R}^c}\\ &\quad - (\mathbb{E}\xi_1\mathbb{I}_{A_{1,R}}\mathbb{E}\xi_2\mathbb{I}_{A_{2,R}^c} + \mathbb{E}\xi_1\mathbb{I}_{A_{1,R}^c}\mathbb{E}\xi_2\mathbb{I}_{A_{2,R}} + \mathbb{E}\xi_1\mathbb{I}_{A_{1,R}^c}\mathbb{E}\xi_2\mathbb{I}_{A_{2,R}^c})\\ &=: J_{1,R} + J_{2,R} + J_{3,R}.\end{aligned}$$

For $J_{1,R}$ we have:

$$J_{1,R} = R^2\left(\mathbb{E}\frac{\xi_1}{R}\mathbb{I}_{A_{1,R}}\frac{\xi_2}{R}\mathbb{I}_{A_{2,R}} - \mathbb{E}\frac{\xi_1}{R}\mathbb{I}_{A_{1,R}}\mathbb{E}\frac{\xi_2}{R}\mathbb{I}_{A_{2,R}}\right),$$

and then, as for $i=1,2$

$$\frac{\xi_i}{R}\mathbb{I}_{A_{i,R}} \in \mathcal{H}_{s_i}^{t_i}(y), \qquad \left|\frac{\xi_i}{R}\mathbb{I}_{A_{i,R}}\right| \leqslant 1, \quad \mathbb{P}\text{-a.s.},$$

according to (3.9) we have:

$$|J_{1,R}| \leqslant c\frac{e^{-\delta\Delta}}{\sqrt{\Delta}}R^2. \tag{3.13}$$

For $J_{2,R}$, if $\rho\in(0,1)$ is a constant fulfilling (3.10), we have:

$$|J_{2,R}|^{\frac{2}{1-\rho}} \leqslant \mathbb{E}|\xi_1|^{\frac{2}{1-\rho}}\mathbb{E}|\xi_2|^{\frac{2}{1-\rho}}\big(\mathbb{P}\big(A_{1,R}^c\big)+\mathbb{P}\big(A_{2,R}^c\big)\big)^{\frac{2\rho}{1-\rho}} \leqslant \mathbb{E}|\xi_1|^{\frac{2}{1-\rho}}\mathbb{E}|\xi_2|^{\frac{2}{1-\rho}}R^{-\frac{2\rho}{1-\rho}}\big(\mathbb{E}|\xi_1|+\mathbb{E}|\xi_2|\big)^{\frac{2\rho}{1-\rho}},$$

so that from (3.10) we obtain,

$$|J_{2,R}| \leqslant c_\rho\kappa_{2,\rho}^{2+\rho}R^{-\rho}. \tag{3.14}$$

Analogously, for $J_{3,R}$ we have:

$$|J_{3,R}| \leqslant c_\rho \kappa_{2,\rho}^{2+\rho} R^{-\rho}. \tag{3.15}$$

Therefore, collecting together (3.13), (3.14) and (3.15), we conclude that for any $R > 0$,

$$|\mathbb{E}\xi_1\xi_2 - \mathbb{E}\xi_1\mathbb{E}\xi_2| \leqslant c\frac{e^{-\delta\Delta}}{\sqrt{\Delta}} R^2 + c_\rho \kappa_{2,\rho}^{2+\rho} R^{-\rho}.$$

By taking the minimum on $R > 0$, this yields:

$$|\mathbb{E}\xi_1\xi_2 - \mathbb{E}\xi_1\mathbb{E}\xi_2| \leqslant c_\rho \kappa_{2,\rho}^2 \left(\frac{e^{-\delta\Delta}}{\sqrt{\Delta}}\right)^{\frac{\rho}{2+\rho}}. \tag{3.16}$$

Next, if we assume that (3.11) is true for $n-1$, we conclude that it is true also for n. Actually, as

$$\left(\mathbb{E}\prod_{i=1}^{n-1} |\xi_i|^{\frac{2}{1-\rho}}\right)^{\frac{1-\rho}{2}} \leqslant \left(\prod_{i=1}^{n-1} \mathbb{E}|\xi_i|^{\frac{2(n-1)}{1-\rho}}\right)^{\frac{1-\rho}{2(n-1)}} = \kappa_{n,\rho},$$

due to (3.12), (3.16) and (3.11) for $n-1$, we obtain:

$$\left|\mathbb{E}\prod_{i=1}^{n}\xi_i - \prod_{i=1}^{n}\mathbb{E}\xi_i\right| \leqslant c_\rho \kappa_{n,\rho}^2 \left(\frac{e^{-\delta\Delta}}{\sqrt{\Delta}}\right)^{\frac{\rho}{2+\rho}} + c_{n-1,\rho}\kappa_{n-1,\rho}^{n-1}\kappa_{n,\rho}\left(\frac{e^{-\delta\Delta}}{\sqrt{\Delta}}\right)^{\frac{\rho}{2+\rho}},$$

which implies (3.11) for n, as $\kappa_{n-1,\rho} \leqslant \kappa_{n,\rho}$. □

4. Construction of the limiting diffusion

For any $\epsilon > 0$ and $x, y \in H$, let us consider the problem:

$$\frac{d\Gamma}{dt}(t) = A\Gamma(t) + \frac{1}{\sqrt{\epsilon}}\left[F\left(\bar{u}^x(t), v^y(t/\epsilon)\right) - \bar{F}\left(\bar{u}^x(t)\right)\right], \quad \Gamma(0) = 0, \tag{4.1}$$

where $\bar{u}^x(t)$ is the solution of the averaged equation (2.25). By the variation of constants formula, the solution $\Gamma_\epsilon^{x,y}(t)$ is given by:

$$\Gamma_\epsilon^{x,y}(t) = \frac{1}{\sqrt{\epsilon}}\int_0^t e^{(t-s)A}\left[F\left(\bar{u}^x(s), v^y(s/\epsilon)\right) - \bar{F}\left(\bar{u}^x(s)\right)\right] ds.$$

In this section we are interested in studying the weak limit, as $\epsilon \downarrow 0$, of the sequence $\{\Gamma_\epsilon^{x,y}\}_{\epsilon>0}$ in $C([0,T];H)$, for any $T > 0$. To this purpose, we first prove that such a sequence is tight and then we identify uniquely the weak limit of any subsequence and hence of the whole sequence.

Theorem 4.1. *Assume Hypotheses* 1–4 *and fix $x, y \in H$ and $T > 0$. Then*

1. *for any $\rho \in (0,1)$ there exist $p_\rho \geqslant 1$ such that for any $0 \leqslant t \leqslant t+h \leqslant T$ and $p \geqslant p_\rho$,*
$$\sup_{\epsilon>0}\mathbb{E}\left|\Gamma_\epsilon^{x,y}(t+h) - \Gamma_\epsilon^{x,y}(t)\right|_H^p \leqslant c_{T,p}\left(1 + |x|_H^p + |y|_H^p\right)|h|^{1+\rho}, \tag{4.2}$$
2. *there exists $\theta \in (0,1/2)$ such that for any $p \geqslant 1$,*
$$\sup_{\epsilon>0}\mathbb{E}\sup_{t\in[0,T]}\left|\Gamma_\epsilon^{x,y}(t)\right|_\theta^p \leqslant c_{T,p}\left(1 + |x|_H^p + |y|_H^p\right). \tag{4.3}$$

In particular, the sequence $\{\mathcal{L}(\Gamma_\epsilon^{x,y})\}_{\epsilon>0}$ is tight in $\mathcal{P}(C([0,T];H))$.

Proof. Step 1, proof of (4.2). For any $T > 0$ and $0 \leqslant t \leqslant t+h \leqslant T$ we have:

$$\Gamma_\epsilon^{x,y}(t+h) - \Gamma_\epsilon^{x,y}(t) = \frac{1}{\sqrt{\epsilon}}\big(e^{hA} - I\big)\int_0^t e^{(t-r)A}\big[F\big(\bar{u}^x(r), v^y(r/\epsilon)\big) - \bar{F}\big(\bar{u}^x(r)\big)\big]\,dr$$
$$+ \frac{1}{\sqrt{\epsilon}}\int_t^{t+h} e^{(t+h-r)A}\big[F\big(\bar{u}^x(r), v^y(r/\epsilon)\big) - \bar{F}\big(\bar{u}^x(r)\big)\big]\,dr.$$

This means that

$$\big|\Gamma_\epsilon^{x,y}(t+h) - \Gamma_\epsilon^{x,y}(t)\big|_H^2 \leqslant \frac{2}{\epsilon}\left|\big(e^{hA} - I\big)\int_0^t e^{(t-r)A}\big[F\big(\bar{u}^x(r), v^y(r/\epsilon)\big) - \bar{F}\big(\bar{u}^x(r)\big)\big]\,dr\right|_H^2$$
$$+ \frac{2}{\epsilon}\left|\int_t^{t+h} e^{(t+h-r)A}\big[F\big(\bar{u}^x(r), v^y(r/\epsilon)\big) - \bar{F}\big(\bar{u}^x(r)\big)\big]\,dr\right|_H^2$$
$$=: \frac{2}{\epsilon}(I_{\epsilon,1} + I_{\epsilon,2}).$$

Now, we estimate the two terms $I_{\epsilon,1}$ and $I_{\epsilon,2}$. We have:

$$I_{\epsilon,1} = \sum_{k=1}^\infty \big(e^{-h\alpha_k} - 1\big)^2\left|\int_0^t e^{-(t-r)\alpha_k}\big\langle F\big(\bar{u}^x(r), v^y(r/\epsilon)\big) - \bar{F}\big(\bar{u}^x(r)\big), e_k\big\rangle_H\,dr\right|^2.$$

Then, as $\alpha_k \sim k^2$, for any $\rho \in (0,1)$ and $n \geqslant 2$,

$$I_{\epsilon,1}^n = h^{1+\rho}\left[\sum_{k=1}^\infty \frac{(e^{-h\alpha_k} - 1)^2}{(\alpha_k h)^{\frac{1+\rho}{n}}}\alpha_k^{-\frac{(1+\rho)(n-1)}{2n}}\alpha_k^{\frac{(1+\rho)(n+1)}{2n}}\left|\int_0^t e^{-(t-r)\alpha_k}\big\langle F\big(\bar{u}^x(r), v^y(r/\epsilon)\big) - \bar{F}\big(\bar{u}^x(r)\big), e_k\big\rangle_H\,dr\right|^2\right]^n$$
$$\leqslant c h^{1+\rho}\left(\sum_{k=1}^\infty \alpha_k^{-\frac{(1+\rho)}{2}}\right)^{n-1}\sum_{k=1}^\infty \alpha_k^{\frac{(1+\rho)(n+1)}{2}}$$
$$\times \int_{[0,t]^{2n}}\prod_{i=1}^{2n} e^{-(t-r_i)\alpha_k}\big\langle F\big(\bar{u}^x(r_i), v^y(r_i/\epsilon)\big) - \bar{F}\big(\bar{u}^x(r_i)\big), e_k\big\rangle_H\,dr_1\cdots dr_{2n}.$$

Therefore, we have:

$$\mathbb{E}I_{\epsilon,1}^n \leqslant c h^{1+\rho}\sum_{k=1}^\infty \alpha_k^{\frac{(1+\rho)(n+1)}{2}}\int_{[0,t]^{2n}}\mathbb{E}\prod_{i=1}^{2n} e^{-(t-r_i)\alpha_k}\big\langle F\big(\bar{u}^x(r_i), v^y(r_i/\epsilon)\big) - \bar{F}\big(\bar{u}^x(r_i)\big), e_k\big\rangle_H\,dr_1\cdots dr_{2n}.$$

In order to estimate the term above we need the following crucial lemma, whose proof is postponed to Appendix A.

Lemma 4.2. *Assume Hypotheses* 2, 3 *and* 4 *and fix any* $\alpha > 0$ *and* $\beta \in [0, 1/3)$. *For any* $x, y, h \in H$, $r \geqslant 0$ *and* $\epsilon > 0$, *define*:

$$\Psi_{\epsilon,h}(r) := \big\langle F\big(\bar{u}^x(\epsilon r), v^y(r)\big) - \bar{F}\big(\bar{u}^x(\epsilon r)\big), h\big\rangle_H, \qquad \vartheta_{\alpha,\beta}(r) := e^{-r\alpha}r^{-\beta}.$$

Then, for any $j \in \mathbb{N}$ *and* $0 \leqslant s < t \leqslant T$ *we have*:

$$\left|\int_{[s,t]^j}\mathbb{E}\prod_{i=1}^j \vartheta_{\alpha,\beta}(t-r_i)\Psi_{\epsilon,h}(r_i/\epsilon)\,dr_1\cdots dr_j\right| \leqslant c_{T,j}\big(1 + |x|_H^j + |y|_H^j\big)\epsilon^{\frac{j}{2}}\left(\frac{(t-s)\alpha \wedge 1}{\alpha}\right)^{\frac{(1-2\beta)j}{2}}|h|_H^j. \tag{4.4}$$

If we apply Lemma 4.2 in the time interval $[0,t]$, with $\alpha=\alpha_k$, $\beta=0$, $j=2n$ and $h=e_k$, we obtain:

$$\mathbb{E}I^n_{\epsilon,1}\leqslant c\,c_{T,2n}h^{1+\rho}\big(1+|x|^{2n}_H+|y|^{2n}_H\big)\epsilon^n\sum_{k=1}^\infty \alpha_k^{\frac{(1+\rho)(n+1)}{2}-n}.$$

Hence, if we take $\rho\in(0,1)$, we can find $n_\rho\in\mathbb{N}$ such that

$$n-\frac{(1+\rho)(n+1)}{2}>\frac{1}{2},\quad n\geqslant n_\rho,$$

so that

$$\mathbb{E}I^n_{\epsilon,1}\leqslant c_{T,2n}h^{1+\rho}\big(1+|x|^{2n}_H+|y|^{2n}_H\big)\epsilon^n,\quad n\geqslant n_\rho. \tag{4.5}$$

For the term $I_{\epsilon,2}$ we have:

$$I_{\epsilon,2}=\sum_{k=1}^\infty\left|\int_t^{t+h}e^{-2(t+h-r)\alpha_k}\big\langle F\big(\bar u^x(r),v^y(r/\epsilon)\big)-\bar F\big(\bar u^x(r)\big),e_k\big\rangle_H\,dr\right|^2,$$

so that, by proceeding as for $I_{\epsilon,1}$, for any $n\in\mathbb{N}$ we have:

$$I^n_{\epsilon,2}\leqslant\left(\sum_{k=1}^\infty\alpha_k^{-\frac{(1+\rho)}{2}}\right)^{n-1}\sum_{k=1}^\infty\alpha_k^{\frac{(1+\rho)(n-1)}{2}}$$
$$\times\int_{[t,t+h]^{2n}}\prod_{i=1}^{2n}e^{-(t+h-r_i)\alpha_k}\big\langle F\big(\bar u^x(r_i),v^y(r_i/\epsilon)\big)-\bar F\big(\bar u^x(r_i)\big),e_k\big\rangle_H\,dr_1\cdots dr_{2n}.$$

Therefore, by applying again Lemma 4.2 in the time interval $[t,t+h]$, with $\alpha=\alpha_k$, $\beta=0$, $j=2n$ and $h=e_k$, we obtain:

$$\mathbb{E}I^n_{\epsilon,2}\leqslant c_{T,2n}\big(1+|x|^{2n}_H+|y|^{2n}_H\big)\epsilon^n\sum_{k=1}^\infty\alpha_k^{\frac{(1+\rho)(n-1)}{2}-n}(h\alpha_k\wedge 1)^n$$
$$\leqslant c_{T,2n}h^{1+\rho}\big(1+|x|^{2n}_H+|y|^{2n}_H\big)\epsilon^n\sum_{k=1}^\infty\alpha_k^{\frac{(1+\rho)(n+1)}{2}-n},$$

so that, as above for $I_{\epsilon,1}$,

$$\mathbb{E}I^n_{\epsilon,2}\leqslant c_{T,2n}h^{1+\rho}\big(1+|x|^{2n}_H+|y|^{2n}_H\big)\epsilon^n,\quad n\geqslant n_\rho.$$

Together with (4.5), this implies (4.2) with $p_\rho=2n_\rho$.

Step 2, proof of (4.3). By stochastic factorization, for any $\theta\geqslant 0$ and $\beta\in(0,1/2)$ we have:

$$(-A)^\theta\Gamma^{x,y}_\epsilon(t)=\frac{1}{\sqrt{\epsilon}}\frac{\sin\pi\beta}{\beta}\int_0^t(t-s)^{\beta-1}e^{(t-s)A}Y_\epsilon(s)\,ds,$$

where

$$Y_\epsilon(s)=\int_0^s(s-r)^{-\beta}(-A)^\theta e^{(s-r)A}\big[F\big(\bar u^x(r),v^y(r/\epsilon)\big)-\bar F\big(\bar u^x(r)\big)\big]\,dr.$$

Hence, for any $n\in\mathbb{N}$ and $\beta>1/2n$ we have:

$$\big|(-A)^\theta\Gamma^{x,y}_\epsilon(t)\big|^{2n}_H\leqslant\frac{c_{n,\beta}}{\epsilon^n}\left(\int_0^t s^{\frac{(\beta-1)2n}{2n-1}}\,ds\right)^{2n-1}\int_0^t\big|Y_\epsilon(s)\big|^{2n}_H\,ds.$$

Now, if $\theta > 0$ we have:

$$\begin{aligned}\left|Y_\epsilon(s)\right|_H^{2n} &= \left(\sum_{k=1}^\infty \alpha_k^{2\theta}\left|\int_0^s (s-r)^{-\beta} e^{-(s-r)\alpha_k}\left\langle F\left(\bar u^x(r), v^y(r/\epsilon)\right) - \bar F\left(\bar u^x(r)\right), e_k\right\rangle_H dr\right|^2\right)^n \\ &\leqslant \left(\sum_{k=1}^\infty \alpha_k^{-\frac{1+\theta}{2}}\right)^{n-1} \sum_{k=1}^\infty \alpha_k^{2n\theta+\frac{(n-1)(1+\theta)}{2}} \\ &\quad\times \int_{[0,s]^{2n}} \prod_{i=1}^{2n}(s-r_i)^{-\beta} e^{-(s-r_i)\alpha_k}\left\langle F\left(\bar u^x(r_i), v^y(r_i/\epsilon)\right) - \bar F\left(\bar u^x(r_i)\right), e_k\right\rangle_H dr_1\cdots dr_{2n}.\end{aligned}$$

Therefore, if we assume $n \geqslant 2$, we can apply Lemma 4.2 in the time interval $[0,s]$, with $\beta \in (1/2n, 1/3)$, $\alpha = \alpha_k$, $j = 2n$ and $h = e_k$, and we obtain:

$$\mathbb{E}\left|Y_\epsilon(s)\right|_H^{2n} \leqslant c_{T,2n}\epsilon^n\left(1 + |x|_H^{2n} + |y|_H^{2n}\right)\sum_{k=1}^\infty \alpha_k^{2n\theta+\frac{(n-1)(1+\theta)}{2}-n(1-2\beta)}.$$

Hence, if we assume $n > 2$ and $\theta \in (0, 1/14)$ we can find $\beta \in (1/2n, 1/4)$ such that

$$n(1-2\beta) - 2n\theta - \frac{(n-1)(1+\theta)}{2} > \frac{1}{2}.$$

This implies

$$\mathbb{E}\left|Y_\epsilon(s)\right|_H^{2n} \leqslant c_{T,2n}\epsilon^n\left(1 + |x|_H^{2n} + |y|_H^{2n}\right),$$

and then

$$\mathbb{E}\sup_{t\in[0,T]}\left|(-A)^\theta \Gamma_\epsilon^{x,y}(t)\right|_H^{2n} \leqslant c_{T,2n}\left(1 + |x|_H^{2n} + |y|_H^{2n}\right). \qquad \Box$$

As the family $\{\mathcal{L}(\Gamma_\epsilon^{x,y})\}_{\epsilon>0}$ is tight in $C([0,T];H)$, for any sequence $\{\epsilon_n\} \downarrow 0$ there exists a subsequence $\{\epsilon_{n_k}\}_{k\in\mathrm{N}}$ such that $\Gamma_{\epsilon_{n_k}}^{x,y}$ weakly converges to some random element Γ^x taking values in $C([0,T];H)$. In what remains of the present section, we characterize Γ^x and in particular we obtain the weak convergence in $C([0,T];H)$ of $\{\Gamma_\epsilon^{x,y}\}_{\epsilon>0}$ to Γ^x, as $\epsilon \downarrow 0$.

Lemma 4.3. *Under Hypotheses* 1–4*, for any $x, y, h \in H$ and for any $T > 0$ we have*:

$$\sup_{t\in[0,T]}\left|\mathbb{E}\left\langle \Gamma_\epsilon^{x,y}(t), h\right\rangle_H\right| \leqslant c_T\left(1 + |y|_H\right)|h|_H\sqrt{\epsilon}, \quad \epsilon > 0. \tag{4.6}$$

In particular, for any $t \in [0,T]$ and $h \in H$,

$$\mathbb{E}\left\langle \Gamma^x(t), h\right\rangle_H = 0.$$

Proof. With the notation introduced in Section 2, for any $h \in H$ we have:

$$\begin{aligned}\left\langle \Gamma_\epsilon^{x,y}(t), h\right\rangle_H &= \frac{1}{\sqrt{\epsilon}}\int_0^t \left\langle F\left(\bar u^x(s), v^y(s/\epsilon)\right) - \bar F\left(\bar u^x(s)\right), e^{(t-s)A}h\right\rangle_H ds \\ &= \frac{1}{\sqrt{\epsilon}}\int_0^t \left(F_{e^{(t-s)A}h}\left(\bar u^x(s), v^y(s/\epsilon)\right) - \bar F_{e^{(t-s)A}h}\left(\bar u^x(s)\right)\right) ds.\end{aligned}$$

In view of (2.22), with a change of variables, this yields for any $T > 0$,

$$\left|\mathbb{E}\langle \Gamma_\epsilon^{x,y}(t),h\rangle_H\right| \leqslant \frac{cL_f}{\sqrt{\epsilon}}\big(1+|y|_H\big)|h|_H\int_0^t \big\|e^{(t-s)A}\big\|_0 e^{-\frac{2\delta}{\epsilon}s}\,ds$$
$$\leqslant c_T\sqrt{\epsilon}\big(1+|y|_H\big)|h|_H\int_0^{\frac{t}{\epsilon}} e^{-2\delta s}\,ds \leqslant c_T\sqrt{\epsilon}\big(1+|y|_H\big)|h|_H,\quad t\in[0,T],$$

so that (4.6) follows. □

Lemma 4.4. *Under Hypotheses* 1–4*, for any* $t>0$*,* $x,h,k\in D((-A)^\alpha)$*, with* $\alpha>0$*, and* $y\in H$ *we have*:

$$\lim_{\epsilon\to 0}\mathbb{E}\langle \Gamma_\epsilon^{x,y}(t),h\rangle_H\langle \Gamma_\epsilon^{x,y}(t),k\rangle_H=\int_0^t\big\langle \Phi\big(\bar u^x(s)\big)e^{(t-s)A}h,e^{(t-s)A}k\big\rangle_H\,ds, \tag{4.7}$$

where $\Phi: H\to\mathcal{L}(H)$ *is the mapping defined in* (3.5).

In particular, for any $t\in[0,T]$ *and* $h,k\in H$,

$$\mathbb{E}\langle \Gamma^{x}(t),h\rangle_H\langle \Gamma^{x}(t),k\rangle_H=\int_0^t\big\langle \Phi\big(\bar u^x(s)\big)e^{(t-s)A}h,e^{(t-s)A}k\big\rangle_H\,ds. \tag{4.8}$$

Proof. For any $y,h\in H$, $0\leqslant s<t$ and $\epsilon>0$ we set $v_\epsilon(t):=v^y(t/\epsilon)$ and $h_t(s):=e^{(t-s)A}h$. We have:

$$\begin{aligned}
&\mathbb{E}\langle \Gamma_\epsilon^{x,y}(t),h\rangle_H\langle \Gamma_\epsilon^{x,y}(t),k\rangle_H\\
&\quad=\frac{1}{\epsilon}\int_0^t\int_0^t \mathbb{E}\big[F_{h_t(s)}\big(\bar u^x(s),v_\epsilon(s)\big)-\bar F_{h_t(s)}\big(\bar u^x(s)\big)\big]\big[F_{k_t(r)}\big(\bar u^x(r),v_\epsilon(r)\big)-\bar F_{k_t(r)}\big(\bar u^x(r)\big)\big]\,dr\,ds\\
&\quad=\frac{1}{\epsilon}\int_0^t\int_0^t \mathbb{E}\big[F_{h_t(s)}\big(\bar u^x(s),v_\epsilon(s)\big)-\mathbb{E}F_{h_t(s)}\big(\bar u^x(s),v_\epsilon(s)\big)\big]\big[F_{k_t(r)}\big(\bar u^x(r),v_\epsilon(r)\big)-\mathbb{E}F_{k_t(r)}\big(\bar u^x(r),v_\epsilon(r)\big)\big]\,dr\,ds\\
&\qquad+\frac{1}{\epsilon}\int_0^t\int_0^t \big[\mathbb{E}F_{h_t(s)}\big(\bar u^x(s),v_\epsilon(s)\big)-\bar F_{h_t(s)}\big(\bar u^x(s)\big)\big]\big[\mathbb{E}F_{k_t(r)}\big(\bar u^x(r),v_\epsilon(r)\big)-\bar F_{k_t(r)}\big(\bar u^x(r)\big)\big]\,dr\,ds\\
&\quad=: J_{1,\epsilon}(t)+J_{2,\epsilon}(t).
\end{aligned}$$

Since

$$J_{2,\epsilon}(t)=\frac{1}{\epsilon}\int_0^t\big[\mathbb{E}F_{h_t(s)}\big(\bar u^x(s),v_\epsilon(s)\big)-\bar F_{h_t(s)}\big(\bar u^x(s)\big)\big]\,ds\int_0^t\big[\mathbb{E}F_{k_t(s)}\big(\bar u^x(s),v_\epsilon(s)\big)-\bar F_{k_t(s)}\big(\bar u^x(s)\big)\big]\,ds,$$

according to (2.22) we obtain:

$$\big|J_{2,\epsilon}(t)\big|\leqslant\frac{c_T}{\epsilon}\big(1+|y|_H^2\big)|h|_H|k|_H\bigg(\int_0^t e^{-\frac{2\delta s}{\epsilon}}\,ds\bigg)^2\leqslant c_T\epsilon\big(1+|y|_H^2\big)|h|_H|k|_H,$$

so that

$$\lim_{\epsilon\to 0}\mathbb{E}\langle \Gamma_\epsilon^{x,y}(t),h\rangle_H\langle \Gamma_\epsilon^{x,y}(t),k\rangle_H=\lim_{\epsilon\to 0}J_{1,\epsilon}(t). \tag{4.9}$$

In order to compute the limit of $J_{1,\epsilon}(t)$, we divide $[0,t]$ into n intervals of size $t/n:=\eta_n$ and we define:

$$C_i:=\big[i\eta_n,(i+1)\eta_n\big]\times\big[i\eta_n,(i+1)\eta_n\big],\quad i=0,\dots,n-1.$$

Moreover we set:

$$C := \bigcup_{i=0}^{n-1} C_i, \qquad D := \big([0,t]\times[0,t]\big)\setminus C.$$

This means that

$$D = \left(\bigcup_{i=1}^{n-1}[0, i\eta_n]\times\big[i\eta_n,(i+1)\eta_n\big]\right)\cup\left(\bigcup_{i=1}^{n-1}\big[i\eta_n,(i+1)\eta_n\big]\times[0,i\eta_n]\right).$$

Next, we define:

$$\begin{aligned}\Lambda_{\epsilon,t}(s,r) := \mathbb{E}\big[&F_{h_t(s)}\big(\bar u^x(s),v_\epsilon(s)\big)-\mathbb{E}F_{h_t(s)}\big(\bar u^x(s),v_\epsilon(s)\big)\big]\\ &\times\big[F_{k_t(r)}\big(\bar u^x(r),v_\epsilon(r)\big)-\mathbb{E}F_{k_t(r)}\big(\bar u^x(r),v_\epsilon(r)\big)\big],\quad 0\leqslant s,r\leqslant t.\end{aligned}$$

If $s>r$ we have:

$$\begin{aligned}\Lambda_{\epsilon,t}(s,r) &= \mathbb{E}\big[\big(F_{k_t(r)}\big(\bar u^x(r),v_\epsilon(r)\big)-\mathbb{E}F_{k_t(r)}\big(\bar u^x(r),v_\epsilon(r)\big)\big)\\ &\quad\times\mathbb{E}\big(F_{h_t(s)}\big(\bar u^x(s),v_\epsilon(s)\big)-\mathbb{E}F_{h_t(s)}\big(\bar u^x(s),v_\epsilon(s)\big)\big|\mathcal{F}_{\frac{r}{\epsilon}}\big)\big]\\ &= \mathbb{E}\big[\big(F_{k_t(r)}\big(\bar u^x(r),v_\epsilon(r)\big)-\mathbb{E}F_{k_t(r)}\big(\bar u^x(r),v_\epsilon(r)\big)\big)\\ &\quad\times\big(P_{\frac{s-r}{\epsilon}}F_{h_t(s)}\big(\bar u^x(s),\cdot\big)\big(v_\epsilon(r)\big)-P_{\frac{s}{\epsilon}}F_{h_t(s)}\big(\bar u^x(s),\cdot\big)(y)\big)\big]\\ &= \mathbb{E}F_{k_t(r)}\big(\bar u^x(r),v_\epsilon(r)\big)\big(P_{\frac{s-r}{\epsilon}}F_{h_t(s)}\big(\bar u^x(s),\cdot\big)\big(v_\epsilon(r)\big)-P_{\frac{s}{\epsilon}}F_{h_t(s)}\big(\bar u^x(s),\cdot\big)(y)\big).\end{aligned}$$

From this it follows:

$$\big|\Lambda_{\epsilon,t}(s,r)\big|\leqslant c_T|k|_H\mathbb{E}\big(\big(1+\big|\bar u^x(r)\big|_H+\big|v_\epsilon(r)\big|_H\big)\big|P_{\frac{s-r}{\epsilon}}F_{h_t(s)}\big(\bar u^x(s),\cdot\big)\big(v_\epsilon(r)\big)-P_{\frac{s}{\epsilon}}F_{h_t(s)}\big(\bar u^x(s),\cdot\big)(y)\big|\big).$$

Due to (2.16) we have:

$$\begin{aligned}&\big|P_{\frac{s-r}{\epsilon}}F_{h_t(s)}\big(\bar u^x(s),\cdot\big)\big(v_\epsilon(r)\big)-P_{\frac{s}{\epsilon}}F_{h_t(s)}\big(\bar u^x(s),\cdot\big)(y)\big|\\ &\quad\leqslant\big|P_{\frac{s-r}{\epsilon}}F_{h_t(s)}\big(\bar u^x(s),\cdot\big)\big(v_\epsilon(r)\big)-\bar F_{h_t(s)}\big(\bar u^x(s)\big)\big|+\big|P_{\frac{s}{\epsilon}}F_{h_t(s)}\big(\bar u^x(s),\cdot\big)(y)-\bar F_{h_t(s)}\big(\bar u^x(s)\big)\big|\\ &\quad\leqslant c_T|h|_H\big(1+\big|v_\epsilon(r)\big|_H\big)e^{-\frac{2\delta(s-r)}{\epsilon}},\end{aligned}$$

and then

$$\big|\Lambda_{\epsilon,t}(s,r)\big|\leqslant c_T|h|_H|k|_H\big(1+\big|\bar u^x(s)\big|_H^2+\mathbb{E}\big|v_\epsilon(r)\big|_H^2\big)e^{-\frac{2\delta(s-r)}{\epsilon}}.$$

Thanks to (2.12) and (2.26), this implies that for any $i=0,\dots,n-1$,

$$\begin{aligned}\left|\int_0^{i\eta_n}\int_{i\eta_n}^{(i+1)\eta_n}\Lambda_{\epsilon,t}(s,r)\,ds\,dr\right| &\leqslant c_T|h|_H|k|_H\big(1+|x|_H^2+|y|_H^2\big)\int_0^{i\eta_n}e^{\frac{2\delta r}{\epsilon}}\,dr\int_{i\eta_n}^{(i+1)\eta_n}e^{-\frac{2\delta s}{\epsilon}}\,ds\\ &\leqslant c_T|h|_H|k|_H\big(1+|x|_H^2+|y|_H^2\big)\epsilon^2.\end{aligned}$$

In particular,

$$\left|\iint_D\Lambda_{\epsilon,t}(s,r)\,ds\,dr\right|\leqslant 2nc_T|h|_H|k|_H\big(1+|x|_H^2+|y|_H^2\big)\epsilon^2. \tag{4.10}$$

Next, for any $i=0,\dots,n-1$ we have:

$$\begin{aligned}
\int_{i\eta_n}^{(i+1)\eta_n}\int_{i\eta_n}^{(i+1)\eta_n} \Lambda_{\epsilon,t}(s,r)\,ds\,dr &= \int_{i\eta_n}^{(i+1)\eta_n}\int_{i\eta_n}^{(i+1)\eta_n} \mathbb{E}\big[F_{h_t(i\eta_n)}\big(\bar u^x(i\eta_n), v_\epsilon(s)\big) - \mathbb{E}F_{h_t(i\eta_n)}\big(\bar u^x(i\eta_n), v_\epsilon(s)\big)\big] \\
&\quad \times \big[F_{k_t(i\eta_n)}\big(\bar u^x(i\eta_n), v_\epsilon(r)\big) - \mathbb{E}F_{k_t(i\eta_n)}\big(\bar u^x(i\eta_n), v_\epsilon(r)\big)\big]\,ds\,dr \\
&\quad + \int_{i\eta_n}^{(i+1)\eta_n}\int_{i\eta_n}^{(i+1)\eta_n} \mathbb{E}\big[F_{h_t(s)}\big(\bar u^x(s), v_\epsilon(s)\big)F_{k_t(r)}\big(\bar u^x(r), v_\epsilon(r)\big) \\
&\quad - F_{h_t(i\eta_n)}\big(\bar u^x(i\eta_n), v_\epsilon(s)\big)F_{k_t(i\eta_n)}\big(\bar u^x(i\eta_n), v_\epsilon(r)\big)\big]\,ds\,dr \\
&\quad - \int_{i\eta_n}^{(i+1)\eta_n}\int_{i\eta_n}^{(i+1)\eta_n} \big[\mathbb{E}F_{h_t(s)}\big(\bar u^x(s), v_\epsilon(s)\big)\mathbb{E}F_{k_t(r)}\big(\bar u^x(r), v_\epsilon(r)\big) \\
&\quad - \mathbb{E}F_{h_t(i\eta_n)}\big(\bar u^x(i\eta_n), v_\epsilon(s)\big)\mathbb{E}F_{k_t(i\eta_n)}\big(\bar u^x(i\eta_n), v_\epsilon(r)\big)\big]\,ds\,dr \\
&=: I_{1,i} + I_{2,i} + I_{3,i}.
\end{aligned}$$

Due to the Lipschitz-continuity of F and to (2.3), from estimates (2.12) and (2.26) we get:

$$\begin{aligned}
|I_{2,i}| &\leqslant c_T|h|_H|k|_H \int_{i\eta_n}^{(i+1)\eta_n}\int_{i\eta_n}^{(i+1)\eta_n} \big(\big|\bar u^x(s) - \bar u^x(i\eta_n)\big|_H + \big|\bar u^x(r) - \bar u^x(i\eta_n)\big|_H\big) \\
&\quad \times \big(1 + \big|\bar u^x(s)\big|_H + \big|\bar u^x(r)\big|_H + \mathbb{E}\big|v_\epsilon(s)\big|_H + \mathbb{E}\big|v_\epsilon(r)\big|_H\big)\,ds\,dr \\
&\quad + c_T \int_{i\eta_n}^{(i+1)\eta_n}\int_{i\eta_n}^{(i+1)\eta_n} \big(|k|_H\big|\big(e^{(t-s)A} - e^{(t-i\eta_n)A}\big)h\big|_H + |h|_H\big|\big(e^{(t-r)A} - e^{(t-i\eta_n)A}\big)k\big|_H\big) \\
&\quad \times \big(1 + \big|\bar u^x(s)\big|_H^2 + \big|\bar u^x(r)\big|_H^2 + \mathbb{E}\big|v_\epsilon(s)\big|_H^2 + \mathbb{E}\big|v_\epsilon(r)\big|_H^2\big)\,ds\,dr \\
&\leqslant 2\eta_n c_T|h|_\alpha|k|_\alpha\big(1 + |x|_H^2 + |y|_H^2\big) \int_{i\eta_n}^{(i+1)\eta_n} \big(\big|\bar u^x(s) - \bar u^x(i\eta_n)\big|_H + (s - i\eta_n)^\alpha\big)\,ds.
\end{aligned}$$

Then, thanks to (2.27), we conclude that

$$|I_{2,i}| \leqslant 2\eta_n^{2+\alpha}c_T|h|_\alpha|k|_\alpha\big(1 + |x|_\alpha^3 + |y|_H^3\big).$$

The same arguments can be used for $I_{3,i}$ and we get:

$$\sum_{i=0}^{n-1}\big(|I_{2,i}| + |I_{3,i}|\big) \leqslant c_T n^{-(1+\alpha)}|h|_\alpha|k|_\alpha\big(1 + |x|_\alpha^3 + |y|_H^3\big). \tag{4.11}$$

Concerning the terms $I_{1,i}$, with a change of variables we have:

$$I_{1,i} = \epsilon\left[\frac{\epsilon}{\eta_n}\int_{\frac{i\eta_n}{\epsilon}}^{\frac{(i+1)\eta_n}{\epsilon}}\int_{\frac{i\eta_n}{\epsilon}}^{\frac{(i+1)\eta_n}{\epsilon}} \hat I_{1,i}(s,r)\,ds\,dr\right]\eta_n,$$

where

$$\begin{aligned}
\hat I_{1,i}(s,r) &:= \mathbb{E}\big[F_{h_t(i\eta_n)}\big(\bar u^x(i\eta_n), v^y(s)\big) - \mathbb{E}F_{h_t(i\eta_n)}\big(\bar u^x(i\eta_n), v^y(s)\big)\big] \\
&\quad \times \big[F_{k_t(i\eta_n)}\big(\bar u^x(i\eta_n), v^y(r)\big) - \mathbb{E}F_{k_t(i\eta_n)}\big(\bar u^x(i\eta_n), v^y(r)\big)\big].
\end{aligned}$$

Then,

$$\begin{aligned}I_{1,i} &= \epsilon\big\langle \Phi\big(\bar u^x(i\eta_n)\big)e^{(t-i\eta_n)A}h, e^{(t-i\eta_n)A}k\big\rangle_H \eta_n\\ &\quad + \epsilon\left[\frac{\epsilon}{\eta_n}\int_{\frac{i\eta_n}{\epsilon}}^{\frac{(i+1)\eta_n}{\epsilon}}\int_{\frac{i\eta_n}{\epsilon}}^{\frac{(i+1)\eta_n}{\epsilon}} \hat I_{1,i}(s,r)\,ds\,dr - \big\langle \Phi\big(\bar u^x(i\eta_n)\big)e^{(t-i\eta_n)A}h, e^{(t-i\eta_n)A}k\big\rangle_H\right]\eta_n\\ &=: \epsilon\big\langle \Phi\big(\bar u^x(i\eta_n)\big)e^{(t-i\eta_n)A}h, e^{(t-i\eta_n)A}k\big\rangle_H \eta_n + \epsilon \hat J_{\epsilon,i}\eta_n.\end{aligned}$$

Therefore, collecting all terms, we have:

$$\begin{aligned}J_{1,\epsilon}(t) &= \frac{1}{\epsilon}\int_0^t\int_0^t \Lambda_{\epsilon,t}(s,r)\,ds\,dr\\ &= \frac{1}{\epsilon}\iint_D \Lambda_{\epsilon,t}(s,r)\,ds\,dr + \frac{1}{\epsilon}\sum_{i=0}^{n-1}(I_{2,i}+I_{3,i})\\ &\quad + \sum_{i=0}^{n-1}\big\langle \Phi\big(\bar u^x(i\eta_n)\big)e^{(t-i\eta_n)A}h, e^{(t-i\eta_n)A}k\big\rangle_H\eta_n + \sum_{i=0}^{n-1}\hat J_{\epsilon,i}\eta_n.\end{aligned}$$

If we take $n_\epsilon \sim \epsilon^{-\gamma}$, with $(1+\alpha)^{-1} < \gamma < 1$, we have:

$$\lim_{\epsilon\to 0}\epsilon n_\epsilon = 0, \qquad \lim_{\epsilon\to 0}\epsilon n_\epsilon^{1+\alpha} = \infty,$$

and hence, in view of (4.10), (4.11) and (3.1), we obtain:

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\iint_D \Lambda_{\epsilon,t}(s,r)\,ds\,dr + \frac{1}{\epsilon}\sum_{i=0}^{n_\epsilon-1}(I_{2,i}+I_{3,i}) + \sum_{i=0}^{n_\epsilon-1}\hat J_{\epsilon,i}\eta_{n_\epsilon} = 0.$$

Moreover, we have:

$$\lim_{\epsilon\to 0}\sum_{i=0}^{n_\epsilon-1}\big\langle \Phi\big(\bar u^x(i\eta_{n_\epsilon})\big)e^{(t-i\eta_{n_\epsilon})A}h, e^{(t-i\eta_{n_\epsilon})A}k\big\rangle_H\eta_{n_\epsilon} = \int_0^t \big\langle \Phi\big(\bar u^x(s)\big)e^{(t-s)A}h, e^{(t-s)A}k\big\rangle_H\,ds,$$

so that

$$\lim_{\epsilon\to 0}J_{1,\epsilon}(t) = \int_0^t \big\langle \Phi\big(\bar u^x(s)\big)e^{(t-s)A}h, e^{(t-s)A}k\big\rangle_H\,ds.$$

Together with (4.9), this yields (4.7).

From (4.7) we get (4.8) for $h,k \in D((-A)^\alpha)$. Now, due to (3.6), the mapping,

$$(h,k)\in H\times H \mapsto \int_0^t \big\langle e^{(t-s)A}\Phi\big(\bar u^x(s)\big)e^{(t-s)A}h, k\big\rangle_H\,ds \in \mathbb{R},$$

is continuous. Then, as the mapping,

$$(h,k)\in H\times H\mapsto \mathbb{E}\big\langle \Gamma^x(t)h, \Gamma^x(t)k\big\rangle_H \in \mathbb{R},$$

is continuous and $D((-A)^\alpha)$ is dense in H, we obtain (4.8) for any $h,k\in H$. □

Lemma 4.5. *Under Hypotheses* 1–4*, the process $\Gamma^x(t)$, $t\in[0,T]$, has independent increments.*

Proof. Let $n\in\mathbb{N}$ and $0\leqslant s_1 < t_1 < s_2 < \cdots < s_n < t_n \leqslant T$. With the notations introduced in Section 3, we have:

$$\exp i\big\langle \Gamma_\epsilon^{x,y}(t_j) - \Gamma_\epsilon^{x,y}(s_j), h\big\rangle_H \in \mathcal{H}_{s_j/\epsilon}^{t_j/\epsilon}(y),$$

for any $h \in H$ and $j = 1, \dots, n$. Then, according to (3.9) we have:

$$\left| \mathbb{E} \exp\left(i \sum_{j=1}^n \left\langle \Gamma_\epsilon^{x,y}(t_j) - \Gamma_\epsilon^{x,y}(s_j), h \right\rangle_H \right) - \prod_{j=1}^n \mathbb{E} \exp\left(i \left\langle \Gamma_\epsilon^{x,y}(t_j) - \Gamma_\epsilon^{x,y}(s_j), h \right\rangle_H \right) \right| \leqslant c(n-1) \frac{e^{-\frac{\delta}{\epsilon}\Delta}}{\sqrt{\frac{\Delta}{\epsilon}}},$$

where $\Delta := \min\{s_2 - t_1, \dots, s_n - t_{n-1}\}$. As

$$\lim_{\epsilon \to 0} \mathbb{E} \exp\left(i \sum_{j=1}^n \left\langle \Gamma_\epsilon^{x,y}(t_j) - \Gamma_\epsilon^{x,y}(s_j), h \right\rangle_H \right) = \mathbb{E} \exp\left(i \sum_{j=1}^n \left\langle \Gamma^x(t_j) - \Gamma^x(s_j), h \right\rangle_H \right),$$

and

$$\lim_{\epsilon \to 0} \prod_{j=1}^n \mathbb{E} \exp\left(i \left\langle \Gamma_\epsilon^{x,y}(t_j) - \Gamma_\epsilon^{x,y}(s_j), h \right\rangle_H \right) = \prod_{j=1}^n \mathbb{E} \exp\left(i \left\langle \Gamma^x(t_j) - \Gamma^x(s_j), h \right\rangle_H \right),$$

this implies

$$\mathbb{E} \exp\left(i \sum_{j=1}^n \left\langle \Gamma^x(t_j) - \Gamma^x(s_j), h \right\rangle_H \right) = \prod_{j=1}^n \mathbb{E} \exp\left(i \left\langle \Gamma^x(t_j) - \Gamma^x(s_j), h \right\rangle_H \right),$$

so that independence of increments follows. □

As the process $\Gamma^x(t)$ has continuous trajectories and independent increments, we have that $\Gamma^x(t)$ is a Gaussian process. This means that it is characterized by its mean and covariance. Thanks to Lemmas 4.3 and 4.4, this allows to obtain the following result.

Theorem 4.6. *Assume Hypotheses* 1–4 *and fix* $x \in D((-A)^\alpha)$, *with* $\alpha > 0$ *and* $y \in H$. *Then* $\{\Gamma_\epsilon^{x,y}\}_{\epsilon > 0}$ *weakly converges in* $C([0,T]; H)$, *as* $\epsilon \downarrow 0$, *to the Gaussian process* Γ^x, *with independent increments, zero mean and covariance operator given by* (4.8).

5. The limiting result

Since we are assuming that the mapping $f(\xi, \cdot, \rho_2) : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with bounded derivative, uniformly with respect to $\xi \in [0, L]$ and $\rho \in \mathbb{R}^2$, it is immediate to check that the mapping $F(\cdot, y) : H \to H$ is Gâteaux differentiable, and

$$\left[D_x F(x,y) z \right](\xi) = \frac{\partial f}{\partial \rho_1}\left(\xi, x(\xi), y(\xi) \right) z(\xi), \quad \xi \in [0, L].$$

In particular, the averaged coefficient $\bar{F} : H \to H$ is Gâteaux differentiable, and

$$D\bar{F}(x) z = \int_H \left[D_x F(x,y) z \right] \mu(dy), \quad x, z \in H.$$

Now, for each $\epsilon > 0$ and $x, y \in H$, we consider the problem:

$$\frac{\partial \zeta}{\partial t}(t) = A\zeta(t) + D\bar{F}\left(\bar{u}^x(t) \right) \zeta(t) + H_\epsilon^{x,y}(t), \quad z(0) = 0, \tag{5.1}$$

where

$$H_\epsilon^{x,y}(t) := \frac{1}{\sqrt{\epsilon}} \left[F\left(\bar{u}^x(t), v^y(t/\epsilon) \right) - \bar{F}\left(\bar{u}^x(t) \right) \right].$$

We denote by $\zeta_\epsilon^{x,y}(t)$ its solution. We have:

$$\zeta_\epsilon^{x,y}(t) = \Gamma_\epsilon^{x,y}(t) + \int_0^t e^{(t-s)A} D\bar{F}\left(\bar{u}^x(s) \right) \zeta_\epsilon^{x,y}(s)\, ds.$$

Proposition 5.1. *Under Hypotheses* 1–4*, for any* $T > 0$ *and* $\alpha > 0$ *and for any* $x \in D((-A)^\alpha)$ *and* $y \in H$*, the sequence* $\{\zeta_\epsilon^{x,y}\}_{\epsilon>0}$ *weakly converges in* $C([0,T];H)$ *to the solution of the problem*:

$$z(t) = \Gamma^x(t) + \int_0^t e^{(t-s)A} D\bar{F}\big(\bar{u}^x(s)\big)z(s)\,ds, \quad t \in [0,T]. \tag{5.2}$$

Proof. For any $v, z \in C([0,T];H)$ we define:

$$\Phi_v(z)(t) := v(t) + \int_0^t e^{(t-s)A} D\bar{F}\big(\bar{u}^x(s)\big)z(s)\,ds, \quad t \in [0,T].$$

It is immediate to check that Φ_v maps $C([0,T];H)$ into itself and if we endow $C([0,T];H)$ with the norm,

$$|z|_{\lambda, C([0,T];H)} := \sup_{t\in[0,T]} e^{-\lambda t}\big|z(t)\big|_H,$$

we have that for λ large enough Φ_v is a contraction on $C([0,T];H)$. This means that it admits a unique fixed point in $C([0,T];H)$ which we denote by $\Phi(v)$. Clearly, $\Phi : C([0,T];H) \to C([0,T];H)$ is linear and bounded.

Now, for any $\epsilon > 0$ we have $\zeta_\epsilon^{x,y} = \Phi(\Gamma_\epsilon^{x,y})$. In the previous section we have proved that $\Gamma_\epsilon^{x,y}$ is weakly convergent in $C([0,T];H)$, as $\epsilon \downarrow 0$, to the Gaussian process Γ^x, then $\zeta_\epsilon^{x,y}$ is weakly convergent in $C([0,T];H)$, as $\epsilon \downarrow 0$, to $\Phi(\Gamma^x)$ which is in fact the solution of problem (5.2). □

Theorem 5.2. *Assume Hypotheses* 1–4 *and for any* $x, y \in H$ *and* $\epsilon > 0$*, define*:

$$z_\epsilon^{x,y}(t) := \frac{u_\epsilon^{x,y}(t) - \bar{u}^x(t)}{\sqrt{\epsilon}}, \quad t \in [0,T].$$

Then, if $x \in D((-A)^\alpha)$*, with* $\alpha > 0$*, we have that* $z_\epsilon^{x,y}$ *weakly converges in* $C([0,T];H)$*, as* $\epsilon \downarrow 0$*, to the solution* z^x *of the linear problem,*

$$z^x(t) = \Gamma^x(t) + \int_0^t e^{(t-s)A} D\bar{F}\big(\bar{u}^x(s)\big)z^x(s)\,ds, \quad t \in [0,T], \tag{5.3}$$

where Γ^x *is the Gaussian process arising from Theorem* 4.6.

Proof. In view of Proposition 5.1, it is sufficient to show that

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} \big|z_\epsilon^{x,y}(t) - \zeta_\epsilon^{x,y}(t)\big|_H^2 = 0. \tag{5.4}$$

If we set $\rho_\epsilon^{x,y}(t) := z_\epsilon^{x,y}(t) - \zeta_\epsilon^{x,y}(t)$, we have:

$$\rho_\epsilon^{x,y}(t) = \int_0^t e^{(t-s)A} D_x F\big(\bar{u}^x(s), v^y(s/\epsilon)\big)\rho_\epsilon^{x,y}(s)\,ds + \int_0^t e^{(t-s)A}\varphi_\epsilon^{x,y}(s)\,ds + \int_0^t e^{(t-s)A}\psi_\epsilon^{x,y}(s)\,ds,$$

where

$$\varphi_\epsilon^{x,y}(t) := \frac{1}{\sqrt{\epsilon}}\big[F\big(\bar{u}^x(t) + \sqrt{\epsilon}z_\epsilon^{x,y}(t), v^y(t/\epsilon)\big) - F\big(\bar{u}^x(t), v^y(t/\epsilon)\big) - D_x F\big(\bar{u}^x(t), v^y(t/\epsilon)\big)\sqrt{\epsilon}z_\epsilon^{x,y}(t)\big], \tag{5.5}$$

and

$$\psi_\epsilon^{x,y}(t) := \big[D_x F\big(\bar{u}^x(t), v^y(t/\epsilon)\big) - D\bar{F}\big(\bar{u}^x(t)\big)\big]\zeta_\epsilon^{x,y}(t). \tag{5.6}$$

Therefore we have:

$$\left|\rho_\epsilon^{x,y}(t)\right|_H^2 \leqslant 3\left|\int_0^t e^{(t-s)A}\varphi_\epsilon^{x,y}(s)\,ds\right|_H^2 + 3\left|\int_0^t e^{(t-s)A}\psi_\epsilon^{x,y}(s)\,ds\right|_H^2 + c_T\int_0^t \left|\rho_\epsilon^{x,y}(s)\right|_H^2\,ds,$$

so that, due to the Gronwall Lemma,

$$\sup_{t\in[0,T]}\left|\rho_\epsilon^{x,y}(t)\right|_H^2 \leqslant c_T\int_0^T\left(\left|\int_0^s e^{(s-r)A}\varphi_\epsilon^{x,y}(r)\,dr\right|_H^2 + \left|\int_0^s e^{(s-r)A}\psi_\epsilon^{x,y}(r)\,dr\right|_H^2\right)ds.$$

This implies (5.4), once we have the following result, whose proof is postponed to Appendix B.

Lemma 5.3. *Assume Hypotheses* 1–4. *Then, for any* $\epsilon > 0$, $x, y \in H$ *and* $T > 0$ *we have*:

$$\mathbb{E}\left(\sup_{s\in[0,T]}\left|\int_0^s e^{(s-r)A}\varphi_\epsilon^{x,y}(r)\,dr\right|_H^2 + \left|\int_0^s e^{(s-r)A}\psi_\epsilon^{x,y}(r)\,dr\right|_H^2\right) \leqslant \lambda_T^{x,y}(\epsilon),$$

with

$$\lim_{\epsilon\to 0}\lambda_T^{x,y}(\epsilon) = 0.$$

Appendix A. Proof of Lemma 4.2

First, we notice that it is sufficient to prove the lemma for j even. Actually, if (4.4) is true for all even integers, for $j = 2n+1$ we have:

$$\left[\mathbb{E}\int_{[s,t]^{2n+1}}\prod_{i=1}^{2n+1}\vartheta_{\alpha,\beta}(t-r_i)\Psi_{\epsilon,h}(r_i/\epsilon)\,dr_1\cdots dr_{2n+1}\right]^2$$

$$\leqslant \mathbb{E}\left(\int_{[s,t]^n}\prod_{i=1}^{n}\vartheta_{\alpha,\beta}(t-r_i)\Psi_{\epsilon,h}(r_i/\epsilon)\,dr_1\cdots dr_n\right)^2\mathbb{E}\left(\int_{[s,t]^{n+1}}\prod_{i=1}^{n+1}\vartheta_{\alpha,\beta}(t-r_i)\Psi_{\epsilon,h}(r_i/\epsilon)\,dr_1\cdots dr_{n+1}\right)^2$$

$$= \mathbb{E}\int_{[s,t]^{2n}}\prod_{i=1}^{2n}\vartheta_{\alpha,\beta}(t-r_i)\Psi_{\epsilon,h}(r_i/\epsilon)\,dr_1\cdots dr_{2n}\,\mathbb{E}\int_{[s,t]^{2(n+1)}}\prod_{i=1}^{2(n+1)}\vartheta_{\alpha,\beta}(t-r_i)\Psi_{\epsilon,h}(r_i/\epsilon)\,dr_1\cdots dr_{2(n+1)}$$

$$\leqslant c\,c_{T,2n}c_{T,2(n+1)}\left(1+|x|_H^{2(2n+1)}+|y|_H^{2(2n+1)}\right)\epsilon^{2n+1}\left(\frac{(t-s)\alpha\wedge 1}{\alpha}\right)^{(1-2\beta)(2n+1)},$$

so that (4.4) is true for $j = 2n+1$. This means that if (4.4) is true for all j even, then it is true for all j odd.

Next, before proceeding with the proof of (4.4) for j even, we prove a preliminary result.

Lemma A.1. *Let us fix* $n \in \mathbb{N}$ *and* $T > 0$ *and for any* $i = 1, \dots, 2n$ *let us define*:

$$\Phi_{\epsilon,i}(r) := G_i\big(\bar{u}^x(\epsilon r), v^y(r)\big) - \bar{G}_i\big(\bar{u}^x(\epsilon r)\big), \quad 0 \leqslant r \leqslant \frac{T}{\epsilon},$$

for some Lipschitz-continuous mapping $G_i : H \times H \to \mathbb{R}$. *Then, under the same hypotheses of Lemma* 4.2, *for any* $1 \leqslant j_1 < j_2 \leqslant n$, $1 \leqslant j \leqslant n$ *and* $0 \leqslant r_1 \leqslant \cdots \leqslant r_{2n} \leqslant T$ *and for any* $\rho \in (0,1)$ *we have*:

$$\left|\mathbb{E}\prod_{i=j_1}^{j_2}\Phi_{\epsilon,i}(r_i)\right| \leqslant c_{T,\rho,n}K_{j_1,j_2}\left(1+|x|_H^{j_2-j_1+1}+|y|_H^{j_2-j_1+1}\right)\left(\frac{e^{-\delta(r_{j_2}-r_{j_2-1})}}{\sqrt{r_{j_2}-r_{j_2-1}}}\right)^{\frac{\rho}{2+\rho}}, \tag{A.1}$$

and

$$J_{\epsilon,j}(r_1,\dots,r_{2n}) := \left|\mathbb{E}\prod_{i=1}^{2n}\Phi_{\epsilon,i}(r_i) - \mathbb{E}\prod_{i=1}^{2j}\Phi_{\epsilon,i}(r_i)\mathbb{E}\prod_{i=2j+1}^{2n}\Phi_{\epsilon,i}(r_i)\right|$$
$$\leqslant c_{T,\rho,n}K_{1,2n}\big(1+|x|_H^{2n}+|y|_H^{2n}\big)\left(\frac{e^{-\delta\hat{r}_j}}{\sqrt{\hat{r}_j}}\right)^{\frac{\rho}{2+\rho}}, \tag{A.2}$$

where

$$K_{j_1,j_2} := \prod_{i=j_1}^{j_2}\|G_i\|_{\mathrm{Lip}},$$

and

$$\hat{r}_j := \max(r_{2n}-r_{2n-1}, r_{2j+1}-r_{2j}).$$

Proof. For any $i=1,\dots,2n$, we have $\Phi_{\epsilon,i}(r)\in\mathcal{H}_r^r(y)$ and, according to (2.12) and (2.26) (with $\alpha=0$), for any $\epsilon>0$, $1\leqslant j_1\leqslant j_2\leqslant 2n$, $p\geqslant 1$ and $r_i\in[0,T/\epsilon]$ we have:

$$\mathbb{E}\left|\prod_{i=j_1}^{j_2}\Phi_{\epsilon,i}(r_i)\right|^p \leqslant c\left(1+\sum_{i=j_1}^{j_2}\big|\bar{u}^x(\epsilon r_i)\big|_H^{(j_2-j_1+1)p}+\sum_{i=j_1}^{j_2}\mathbb{E}\big|v^y(r_i)\big|_H^{(j_2-j_1+1)p}\right)\prod_{i=j_1}^{j_2}\|G_i\|_{\mathrm{Lip}}^p$$
$$\leqslant c_{T,p}\big(1+|x|_H^{(j_2-j_1+1)p}+|y|_H^{(j_2-j_1+1)p}\big)\prod_{i=j_1}^{j_2}\|G_i\|_{\mathrm{Lip}}^p. \tag{A.3}$$

Therefore, we can apply (3.11) to the random variables $\Phi_{\epsilon,1}(r_1),\dots,\Phi_{\epsilon,2n}(r_{2n})$ and for any $1\leqslant j\leqslant 2n$ and $\rho\in(0,1)$ we have:

$$\left|\mathbb{E}\prod_{i=1}^{2n}\Phi_{\epsilon,i}(r_i)-\mathbb{E}\prod_{i=1}^{j}\Phi_{\epsilon,i}(r_i)\mathbb{E}\prod_{i=j+1}^{2n}\Phi_{\epsilon,i}(r_i)\right|$$
$$\leqslant c_{T,\rho,n}K_{1,2n}\big(1+|x|_H^{2n}+|y|_H^{2n}\big)\left(\frac{e^{-\delta(r_{j+1}-r_j)}}{\sqrt{r_{j+1}-r_j}}\right)^{\frac{\rho}{2+\rho}}. \tag{A.4}$$

Moreover, due to (2.16),

$$\big|\mathbb{E}\Phi_{\epsilon,j_2}(r_{j_2})\big|\leqslant ce^{-2\delta r_{j_2}}\big(1+|y|_H\big)[G_{j_2}]_{\mathrm{Lip}}.$$

Then, in view of (A.3) and (A.4) this implies:

$$\left|\mathbb{E}\prod_{i=j_1}^{j_2}\Phi_{\epsilon,i}(r_i)\right|\leqslant\left|\mathbb{E}\prod_{i=j_1}^{j_2}\Phi_{\epsilon,i}(r_i)-\mathbb{E}\prod_{i=j_1}^{j_2-1}\Phi_{\epsilon,i}(r_i)\mathbb{E}\Phi_{\epsilon,h}(r_{j_2})\right|+\left|\mathbb{E}\prod_{i=j_1}^{j_2-1}\Phi_{\epsilon,i}(r_i)\mathbb{E}\Phi_{\epsilon,h}(r_{j_2})\right|$$
$$\leqslant c_{T,\rho,n}\big(1+|x|_H^{j_2-j_1+1}+|y|_H^{j_2-j_1+1}\big)\left(\frac{e^{-\delta(r_{j_2}-r_{j_2-1})}}{\sqrt{r_{j_2}-r_{j_2-1}}}\right)^{\frac{\rho}{2+\rho}}\prod_{i=j_1}^{j_2}\|G_i\|_{\mathrm{Lip}},$$

so that (A.1) holds.

In particular, we have:

$$\left|\mathbb{E}\prod_{i=1}^{2n}\Phi_{\epsilon,i}(r_i)\right|\leqslant c_{T,n,\rho}K_{1,2n}\big(1+|x|_H^{2n}+|y|_H^{2n}\big)\left(\frac{e^{-\delta(r_{2n}-r_{2n-1})}}{\sqrt{r_{2n}-r_{2n-1}}}\right)^{\frac{\rho}{2+\rho}},$$

and hence, thanks to (A.1) and (A.3), for any $1\leqslant j<2n$,

$$\left|\mathbb{E}\prod_{i=1}^{2n}\Phi_{\epsilon,i}(r_i)-\mathbb{E}\prod_{i=1}^{j}\Phi_{\epsilon,i}(r_i)\mathbb{E}\prod_{i=j+1}^{2n}\Phi_{\epsilon,i}(r_i)\right|$$
$$\leqslant c_{T,\rho,n}K_{1,2n}\big(1+|x|_H^{2n}+|y|_H^{2n}\big)\left(\frac{e^{-\delta(r_{2n}-r_{2n-1})}}{\sqrt{r_{2n}-r_{2n-1}}}\right)^{\frac{\rho}{2+\rho}}. \tag{A.5}$$

Therefore, as the mapping $s\mapsto e^{-\frac{\delta}{2}s}/\sqrt{s}$ is decreasing, combining together (A.4) and (A.5), for any $1\leqslant j<n$ we get (A.2). □

With a change of variables we have:

$$\mathbb{E}\int_{[s,t]^{2n}}\prod_{i=1}^{2n}\vartheta_{\alpha,\beta}(t-r_i)\Psi_{\epsilon,h}(r_i/\epsilon)\,dr_1\cdots dr_{2n}=\epsilon^{2n}\mathbb{E}\int_{[\frac{s}{\epsilon},\frac{t}{\epsilon}]^{2n}}\prod_{i=1}^{2n}\vartheta_{\alpha,\beta}(t-\epsilon r_i)\Psi_{\epsilon,h}(r_i)\,dr_1\cdots dr_{2n}$$
$$=:\epsilon^{2n}\mathbb{E}H_\epsilon(s,t).$$

For any permutation $(\sigma(1),\dots,\sigma(2n))$ we have:

$$\prod_{i=1}^{2n}\vartheta_{\alpha,\beta}(t-\epsilon r_{\sigma(i)})\Psi_{\epsilon,h}(r_{\sigma(i)})=\prod_{i=1}^{2n}\vartheta_{\alpha,\beta}(t-\epsilon r_i)\Psi_{\epsilon,h}(r_i),$$

and then it is immediate to get:

$$H_\epsilon(s,t)=c_n\int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}}\int_{\frac{s}{\epsilon}}^{r_{2n}}\cdots\int_{\frac{s}{\epsilon}}^{r_2}\prod_{i=1}^{2n}\vartheta_{\alpha,\beta}(t-\epsilon r_i)\Psi_{\epsilon,h}(r_i)\,dr_1\cdots dr_{2n},$$

for some constant c_n. In particular, through the rest of the proof we shall assume $s/\epsilon\leqslant r_1\leqslant\cdots\leqslant r_{2n}\leqslant t/\epsilon$.

Now we can prove (4.4) by induction on n. Thanks to (A.1), for $n=1$ we have:

$$\big|\mathbb{E}H_\epsilon(s,t)\big|=c_2\left|\int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}}\int_{\frac{s}{\epsilon}}^{r_2}\vartheta_{\alpha,\beta}(t-\epsilon r_1)\vartheta_{\alpha,\beta}(t-\epsilon r_2)\mathbb{E}\Psi_{\epsilon,h}(r_1)\Psi_{\epsilon,h}(r_2)\,dr_1\,dr_2\right|$$
$$\leqslant c_{T,\rho,2}\big(1+|x|_H^2+|y|_H^2\big)|h|_H^2\int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}}\int_{\frac{s}{\epsilon}}^{r_2}\vartheta_{\alpha,\beta}(t-\epsilon r_1)\vartheta_{\alpha,\beta}(t-\epsilon r_2)\left(\frac{e^{-\delta(r_2-r_1)}}{\sqrt{r_2-r_1}}\right)^{\frac{\rho}{2+\rho}}dr_1\,dr_2.$$

As $\rho/(2+\rho)<1/2$, with a change of variables we get:

$$\int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}}\int_{\frac{s}{\epsilon}}^{r_2}\vartheta_{\alpha,\beta}(t-\epsilon r_1)\vartheta_{\alpha,\beta}(t-\epsilon r_2)\left(\frac{e^{-\delta(r_2-r_1)}}{\sqrt{r_2-r_1}}\right)^{\frac{\rho}{2+\rho}}dr_1\,dr_2$$
$$=\epsilon^{-2\beta}\int_0^{\frac{t-s}{\epsilon}}r_2^{-\beta}e^{-\epsilon r_2\alpha}\int_{r_2}^{\frac{t-s}{\epsilon}}r_1^{-\beta}e^{-\epsilon r_1\alpha}\left(\frac{e^{-\delta(r_1-r_2)}}{\sqrt{r_1-r_2}}\right)^{\frac{\rho}{2+\rho}}dr_1\,dr_2$$
$$\leqslant\epsilon^{-2\beta}\int_0^{\frac{t-s}{\epsilon}}r_2^{-2\beta}e^{-2\epsilon r_2\alpha}\int_0^{+\infty}\left(\frac{e^{-\delta r_1}}{\sqrt{r_1}}\right)^{\frac{\rho}{2+\rho}}dr_1\,dr_2\leqslant\epsilon^{-2\beta}\int_0^{\frac{t-s}{\epsilon}}r^{-2\beta}e^{-2\epsilon r\alpha}\,dr.$$

Then, with another change of variables:

$$\epsilon^2\big|\mathbb{E}H_\epsilon(s,t)\big| \leqslant c_T\big(1+|x|_H^2+|y|_H^2\big)\epsilon^{2-2\beta}(\epsilon\alpha)^{-(1-2\beta)}\int_0^{\alpha(t-s)} r^{-2\beta}e^{-2r}\,dr$$
$$\leqslant c_T\big(1+|x|_H^2+|y|_H^2\big)\epsilon^{2-2\beta}(\epsilon\alpha)^{-(1-2\beta)}\big((t-s)\alpha\wedge 1\big)^{1-2\beta}$$

and this implies (4.4) for $j=2$.

Next, assume that (4.4) is true for any even integer $j<2n$. For any $r=(r_1,\dots,r_{2n})\in[s,t]^{2n}$, with $s\leqslant r_1\leqslant\cdots\leqslant r_{2n}\leqslant t$, we denote by $j(r)$ the integer such that

$$\max_{j=1,\dots,n-1}(r_{2j+1}-r_{2j}) = r_{2j(r)+1}-r_{2j(r)}.$$

Then, with the notations introduced in Lemma A.1, we have:

$$\begin{aligned}\big|\mathbb{E}H_\epsilon(s,t)\big| &\leqslant c_n\int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}}\int_{\frac{s}{\epsilon}}^{r_{2n}}\cdots\int_{\frac{s}{\epsilon}}^{r_2}\prod_{i=1}^{2n}\vartheta_{\alpha,\beta}(t-\epsilon r_i)J_{\epsilon,j(r)}(r_1,\dots,r_{2n})\,dr_1\cdots dr_{2n}\\
&\quad + c_n\sum_{j=1}^{n-1}\int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}}\int_{\frac{s}{\epsilon}}^{r_{2j}}\cdots\int_{\frac{s}{\epsilon}}^{r_2}\prod_{i=1}^{2j}\vartheta_{\alpha,\beta}(t-\epsilon r_i)\left|\mathbb{E}\prod_{i=1}^{2j}\Psi_{\epsilon,h}(t,r_i)\right|dr_1\cdots r_{2j}\\
&\quad\times\int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}}\int_{\frac{s}{\epsilon}}^{r_{2(n-j)}}\cdots\int_{\frac{s}{\epsilon}}^{r_2}\prod_{i=1}^{2(n-j)}\vartheta_{\alpha,\beta}(t-\epsilon r_i)\left|\mathbb{E}\prod_{i=1}^{2(n-j)}\Psi_{\epsilon,h}(t,r_i)\right|dr_1\cdots r_{2(n-j)}\\
&=: I_{1,\epsilon}+I_{2,\epsilon}.\end{aligned}$$

If we apply (A.2), with $\Phi_{\epsilon,i}=\Psi_{\epsilon,h}$ for any $i=1,\dots,2n$, we obtain:

$$J_{\epsilon,j(r)}(r_1,\dots,r_{2n}) \leqslant c_{T,\rho,n}\big(1+|x|_H^{2n}+|y|_H^{2n}\big)|h|_H^{2n}\frac{e^{-\delta_n(r_{2n}-r_{2n-1}+\sum_{i=1}^{n-1}r_{2i+1}-r_{2i})}}{(r_{2n}-r_{2n-2})^{\bar\rho}},$$

where

$$\delta_n := \frac{\delta\rho}{n(2+\rho)},\qquad \bar\rho=\frac{\rho}{2(2+\rho)}.$$

This implies:

$$\begin{aligned}I_{1,\epsilon} &\leqslant c_{T,\rho,n}\big(1+|x|_H^{2n}+|y|_H^{2n}\big)|h|_H^{2n}\\
&\quad\times\int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}}\int_{\frac{s}{\epsilon}}^{r_{2n}}\cdots\int_{\frac{s}{\epsilon}}^{r_2}\frac{e^{-\delta_n(r_{2n}-r_{2n-2})}}{(r_{2n}-r_{2n-2})^{\bar\rho}}\prod_{i=1}^{2n}\vartheta_{\alpha,\beta}(t-\epsilon r_i)\prod_{i=1}^{n-2}e^{-\delta_n(r_{2i+1}-r_{2i})}\,dr_1\cdots dr_{2n}\\
&=\epsilon^{-2n\beta}c_{T,\rho,n}\big(1+|x|_H^{2n}+|y|_H^{2n}\big)|h|_H^{2n}\\
&\quad\times\int_0^{\frac{t-s}{\epsilon}}e^{-(\epsilon\alpha-\delta_n)r_{2n}}r_{2n}^{-\beta}\int_{r_{2n}}^{\frac{t-s}{\epsilon}}e^{-\epsilon\alpha r_{2n-1}}r_{2n-1}^{-\beta}\int_{r_{2n-1}}^{\frac{t-s}{\epsilon}}\frac{e^{-(\epsilon\alpha+\delta_n)r_{2n-2}}}{(r_{2n-2}-r_{2n})^{\bar\rho}}r_{2n-2}^{-\beta}\\
&\quad\times\int_{r_{2n-2}}^{\frac{t-s}{\epsilon}}\cdots\int_{r_3}^{\frac{t-s}{\epsilon}}\prod_{i=2}^{2n-2}e^{-(\epsilon\alpha+(-1)^i\delta_n)r_i}r_i^{-\beta}\int_{r_2}^{\frac{t-s}{\epsilon}}e^{-\epsilon\alpha r_1}r_1^{-\beta}\,dr_1\cdots dr_{2n}.\end{aligned}\tag{A.6}$$

With a new change of variable, for $k=1,2,3$ and $i=1,3,\dots,2n-1$ we have:

$$\int_{r_{i+1}}^{\frac{t-s}{\epsilon}} e^{-k\epsilon\alpha r_i} r_i^{-k\beta}\,dr_i = (\epsilon\alpha)^{k\beta-1}\int_{\epsilon\alpha r_{i+1}}^{(t-s)\alpha} e^{-kr_i} r_i^{-k\beta}\,dr_i \leqslant c(\epsilon\alpha)^{k\beta-1}\big[(t-s)\alpha\wedge 1\big]^{1-k\beta}. \tag{A.7}$$

Moreover for any $i=2,4,\dots,2n$ we have:

$$\int_{r_{i+1}}^{\frac{t-s}{\epsilon}} e^{-(\epsilon\alpha+\delta_n)r_i} r_i^{-\beta}\,dr_i \leqslant r_{i+1}^{-\beta} e^{-(\epsilon\alpha+\delta_n)r_{i+1}}(\epsilon\alpha+\delta_n)^{-1} \leqslant c_n r_{i+1}^{-\beta} e^{-(\epsilon\alpha+\delta_n)r_{i+1}}. \tag{A.8}$$

Therefore, combining together (A.7) and (A.8), we get:

$$\begin{aligned}
&\int_{r_{2n-2}}^{\frac{t-s}{\epsilon}}\cdots\int_{r_3}^{\frac{t-s}{\epsilon}} \prod_{i=2}^{2n-2} e^{-(\epsilon\alpha+(-1)^i\delta_n)r_i} r_i^{-\beta}\int_{r_2}^{\frac{t-s}{\epsilon}} e^{-\epsilon\alpha r_1} r_1^{-\beta}\,dr_1\cdots dr_{2n-3}\\
&\quad\leqslant (\epsilon\alpha)^{\beta-1}\big[(t-s)\alpha\wedge 1\big]^{1-\beta}(\epsilon\alpha)^{(2\beta-1)(n-2)}\big[(t-s)\alpha\wedge 1\big]^{(1-2\beta)(n-2)}\\
&\quad= (\epsilon\alpha)^{n(2\beta-1)-(3\beta-1)}\big[(t-s)\alpha\wedge 1\big]^{(1-\beta)+(1-2\beta)(n-2)}.
\end{aligned} \tag{A.9}$$

Now, by using again both (A.7) and (A.8) we have:

$$\begin{aligned}
&\int_0^{\frac{t-s}{\epsilon}} e^{-(\epsilon\alpha-\delta_n)r_{2n}} r_{2n}^{-\beta}\int_{r_{2n}}^{\frac{t-s}{\epsilon}} e^{-\epsilon\alpha r_{2n-1}} r_{2n-1}^{-\beta}\int_{r_{2n-1}}^{\frac{t-s}{\epsilon}} \frac{e^{-(\epsilon\alpha+\delta_n)r_{2n-2}}}{(r_{2n-2}-r_{2n})^{\bar\rho}} r_{2n-2}^{-\beta}\,dr_{2n-2}\,dr_{2n-1}\,dr_{2n}\\
&\quad\leqslant c_n\int_0^{\frac{t-s}{\epsilon}} e^{-3\epsilon\alpha r_{2n}} r_{2n}^{-3\beta}\int_{r_{2n}}^{\frac{t-s}{\epsilon}} \frac{e^{-(2\epsilon\alpha+\delta_n)(r_{2n-1}-r_{2n})}}{(r_{2n-1}-r_{2n})^{\bar\rho}}\,dr_{2n-1}\,dr_{2n}\\
&\quad\leqslant c_n\int_0^{\frac{t-s}{\epsilon}} e^{-3\epsilon\alpha r_{2n}} r_{2n}^{-3\beta}\int_0^{\infty} \frac{e^{-\delta_n r_{2n-1}}}{r_{2n-1}^{\bar\rho}}\,dr_{2n-1}\,dr_{2n}\\
&\quad\leqslant c_n(\epsilon\alpha)^{3\beta-1}\big[(t-s)\alpha\wedge 1\big]^{1-3\beta}.
\end{aligned} \tag{A.10}$$

Hence, by putting together (A.9) and (A.10) into (A.6), we obtain:

$$\begin{aligned}
I_{1,\epsilon} &\leqslant \epsilon^{-2n\beta} c_{T,n}\big(1+|x|_H^{2n}+|y|_H^{2n}\big)|h|_H^{2n}(\epsilon\alpha)^{n(2\beta-1)}\big[(t-s)\alpha\wedge 1\big]^{(1-\beta)+(1-2\beta)(n-2)+(1-3\beta)}\\
&= c_{T,n}\big(1+|x|_H^{2n}+|y|_H^{2n}\big)|h|_H^{2n}\epsilon^{-n}\alpha^{-n(1-2\beta)}\big[(t-s)\alpha\wedge 1\big]^{n(1-2\beta)}.
\end{aligned}$$

Finally, due to the inductive hypothesis we have:

$$I_{2,\epsilon} \leqslant c_{T,n}\big(1+|x|_H^{2n}+|y|_H^{2n}\big)|h|_H^{2n}\epsilon^{-n}\alpha^{-n(1-2\beta)}\big[(t-s)\alpha\wedge 1\big]^{n(1-2\beta)},$$

and then we can conclude that (4.4) holds.

Appendix B. Proof of Lemma 5.3

Since we are assuming that $f(\xi,\cdot,\rho_2):\mathbb{R}\to\mathbb{R}$ is twice continuously differentiable with bounded derivatives, uniformly with respect to $\xi\in[0,L]$ and $\rho_2\in\mathbb{R}$, we have that for any fixed $x,y,h,k\in H$ the mapping,

$$s\in\mathbb{R}\mapsto D_xF(x+sk,y)h\in L^1(0,L),$$

is differentiable, and

$$\frac{d}{ds}D_xF(x+sk,y)h_{|_{s=0}} = D_x^2F(x,y)(h,k),$$

where

$$D_x^2 F(x,y)(h,k)(\xi) = \frac{\partial^2 f}{\partial \rho_1^2}\big(\xi, x(\xi), y(\xi)\big)h(\xi)k(\xi), \quad \xi \in [0,L].$$

This means

$$\begin{aligned} &F\big(\bar u^x(t) + \sqrt{\epsilon} z_\epsilon^{x,y}(t), v^y(t/\epsilon)\big) - F\big(\bar u^x(t), v^y(t/\epsilon)\big) - D_x F\big(\bar u^x(t), v^y(t/\epsilon)\big)\sqrt{\epsilon} z_\epsilon^{x,y}(t) \\ &\quad = \epsilon \int_0^1 D_x^2 F\big(\bar u^x(t) + \theta\sqrt{\epsilon} z_\epsilon^{x,y}(t), v^y(t/\epsilon)\big)\big(z_\epsilon^{x,y}(t), z_\epsilon^{x,y}(t)\big)\, d\theta, \end{aligned}$$

so that, recalling how $\varphi_\epsilon^{x,y}(t)$ has been defined in (5.5), we have:

$$\big|\varphi_\epsilon^{x,y}(t)\big|_{L^1(0,L)} \leqslant \sqrt{\epsilon}\big|z_\epsilon^{x,y}(t)\big|_H^2.$$

This implies that

$$\left|\int_0^t e^{(t-s)A}\varphi_\epsilon^{x,y}(s)\, ds\right|_H \leqslant c\int_0^t (t-s)^{-\frac14}\big|\varphi_\epsilon^{x,y}(s)\big|_{L^1(0,L)}\, ds \leqslant c\sqrt{\epsilon}\int_0^t (t-s)^{-\frac14}\big|z_\epsilon^{x,y}(s)\big|_H^2\, ds. \tag{B.1}$$

Now, since we have:

$$z_\epsilon^{x,y}(t) = \Gamma_\epsilon^{x,y}(t) + \frac{1}{\sqrt{\epsilon}}\int_0^t e^{(t-s)A}\big[F\big(u_\epsilon^{x,y}(s), v^y(s/\epsilon)\big) - F\big(\bar u^x(s), v^y(s/\epsilon)\big)\big]\, ds,$$

for any $p \geqslant 1$ we obtain,

$$\begin{aligned} \big|z_\epsilon^{x,y}(t)\big|_H^p &\leqslant c_p\big|\Gamma_\epsilon^{x,y}(t)\big|_H^p + \frac{c_{T,p}}{\epsilon^{\frac p2}}\int_0^t \big|F\big(u_\epsilon^{x,y}(s), v^y(s/\epsilon)\big) - F\big(\bar u^x(s), v^y(s/\epsilon)\big)\big|_H^p\, ds \\ &\leqslant c_p\big|\Gamma_\epsilon^{x,y}(t)\big|_H^p + c_{T,p}\int_0^t \big|z_\epsilon^{x,y}(s)\big|_H^p\, ds. \end{aligned}$$

This yields,

$$\big|z_\epsilon^{x,y}(t)\big|_H^p \leqslant c_p\int_0^t e^{c_{T,p}(t-s)}\big|\Gamma_\epsilon^{x,y}(s)\big|_H^p,$$

and hence, according to (4.3), we get:

$$\sup_{\epsilon>0}\mathbb{E}\sup_{t\in[0,T]}\big|z_\epsilon^{x,y}(t)\big|_H^p \leqslant c_{T,p}\big(1 + |x|_H^p + |y|_H^p\big).$$

By replacing in (B.1), this allows to conclude that

$$\mathbb{E}\left|\int_0^t e^{(t-s)A}\varphi_\epsilon^{x,y}(s)\, ds\right|_H^2 \leqslant c_T\epsilon\,\mathbb{E}\sup_{t\in[0,T]}\big|z_\epsilon^{x,y}(t)\big|_H^4 \leqslant c_T\epsilon\big(1 + |x|_H^4 + |y|_H^4\big). \tag{B.2}$$

Next, we have to estimate:

$$\mathbb{E}\left|\int_0^t e^{(t-s)A}\psi_\epsilon^{x,y}(s)\, ds\right|_H^2.$$

As $\zeta_\epsilon^{x,y}(t)$ solves Eq. (5.1), the process $\rho_\epsilon^{x,y}(t) := \zeta_\epsilon^{x,y}(t) - \Gamma_\epsilon^{x,y}(t)$ solves the equation:

$$\frac{d\rho_\epsilon^{x,y}}{dt}(t) = \left[A + D\bar{F}\left(\bar{u}^x(t)\right)\right]\rho_\epsilon^{x,y}(t) + D\bar{F}\left(\bar{u}^x(t)\right)\Gamma_\epsilon^{x,y}(t), \quad \rho_\epsilon^{x,y}(0) = 0.$$

Therefore, if we denote by $U(t,s)$ the evolution system associated with the time dependent operator $A + D\bar{F}(\bar{u}^x(t))$, we have:

$$\rho_\epsilon^{x,y}(t) = \int_0^t U(t,s) D\bar{F}\left(\bar{u}^x(s)\right)\Gamma_\epsilon^{x,y}(s)\,ds,$$

so that

$$\zeta_\epsilon^{x,y}(t) = \Gamma_\epsilon^{x,y}(t) + \int_0^t U(t,s) D\bar{F}\left(\bar{u}^x(s)\right)\Gamma_\epsilon^{x,y}(s)\,ds.$$

Recalling how $\psi_\epsilon^{x,y}(t)$ has been defined in (5.6), this means that

$$\begin{aligned}\int_0^t e^{(t-s)A}\psi_\epsilon^{x,y}(s)\,ds &= \int_0^t e^{(t-s)A}\left[D_x F\left(\bar{u}^x(s), v^y(s/\epsilon)\right) - D\bar{F}\left(\bar{u}^x(s)\right)\right]\Gamma_\epsilon^{x,y}(s)\,ds \\ &\quad + \int_0^t e^{(t-s)A}\left[D_x F\left(\bar{u}^x(s), v^y(s/\epsilon)\right) - D\bar{F}\left(\bar{u}^x(s)\right)\right]\int_0^s U(s,\sigma) D\bar{F}\left(\bar{u}^x(\sigma)\right)\Gamma_\epsilon^{x,y}(\sigma)\,d\sigma\,ds \\ &=: I_{\epsilon,1}(t) + I_{\epsilon,2}(t).\end{aligned}$$

We have:

$$\begin{aligned}\left|I_{\epsilon,1}(t)\right|_H^2 &= \sum_{j=1}^\infty \left(\int_0^t \left\langle e^{(t-s)A}\left[D_x F\left(\bar{u}^x(s), v^y(s/\epsilon)\right) - D\bar{F}\left(\bar{u}^x(s)\right)\right]\Gamma_\epsilon^{x,y}(s), e_j\right\rangle_H ds\right)^2 \\ &= \sum_{j=1}^\infty \left(\int_0^t e^{-(t-s)\alpha_j}\left\langle \Delta_\epsilon DF(s)\Gamma_\epsilon^{x,y}(s), e_j\right\rangle_H ds\right)^2,\end{aligned}$$

where we have set,

$$\Delta_\epsilon DF(s) := D_x F\left(\bar{u}^x(s), v^y(s/\epsilon)\right) - D\bar{F}\left(\bar{u}^x(s)\right).$$

Then,

$$\begin{aligned}\mathbb{E}\left|I_{\epsilon,1}(t)\right|_H^2 &= 2\sum_{j=1}^\infty \int_0^t \int_0^{s_1} e^{-(t-s_1)\alpha_j} e^{-(t-s_2)\alpha_j} \\ &\quad \times \mathbb{E}\left\langle \Delta_\epsilon DF(s_1)\Gamma_\epsilon^{x,y}(s_1), e_j\right\rangle_H \left\langle \Delta_\epsilon DF(s_2)\Gamma_\epsilon^{x,y}(s_2), e_j\right\rangle_H ds_2\,ds_1\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left|I_{\epsilon,1}(t)\right|_H^2 &= \frac{2}{\epsilon}\sum_{j=1}^\infty \int_0^t ds_1\, e^{-(t-s_1)\alpha_j}\int_0^{s_1} ds_2\, e^{-(t-s_2)\alpha_j}\int_0^{s_1} dr_1 \int_0^{s_2} dr_2 \\ &\quad \times \mathbb{E}\left\langle \Delta_\epsilon DF(s_1) e^{(s_1-r_1)A}\Delta_\epsilon F(r_1), e_j\right\rangle_H \left\langle \Delta_\epsilon DF(s_2) e^{(s_2-r_2)A}\Delta_\epsilon F(r_2), e_j\right\rangle_H \\ &= \frac{2}{\epsilon}\sum_{i,j,l=1}^\infty \int_0^t ds_1\, e^{-(t-s_1)\alpha_j}\int_0^{s_1} ds_2\, e^{-(t-s_2)\alpha_j}\int_0^{s_1} dr_1\, e^{-(s_1-r_1)\alpha_l}\int_0^{s_2} dr_2\, e^{-(s_2-r_2)\alpha_i} \\ &\quad \times \mathbb{E}\left\langle \Delta_\epsilon DF(s_2) e_j, e_i\right\rangle_H \left\langle \Delta_\epsilon F(r_2), e_i\right\rangle_H \left\langle \Delta_\epsilon DF(s_1) e_j, e_l\right\rangle_H \left\langle \Delta_\epsilon F(r_1), e_l\right\rangle_H,\end{aligned}$$

where we have set:

$$\Delta_\epsilon F(s) := F\big(\bar{u}^x(s), v(s/\epsilon)\big) - \bar{F}\big(\bar{u}^x(s)\big).$$

With a change of variables, this yields:

$$\begin{aligned}\mathbb{E}\big|I_{\epsilon,1}(t)\big|_H^2 = 2\epsilon^3 \sum_{i,j,l=1}^{\infty} e^{-2t\alpha_j} \int_0^{\frac{t}{\epsilon}} ds_1\, \mathbb{E} H_{j,l}(\epsilon s_1)\bigg[\int_0^{s_1} H_{j,i}(\epsilon s_2)\int_0^{s_2} K_i(\epsilon r_2)\int_0^{r_2} K_l(\epsilon r_1)\, dr_1\, dr_2\, ds_2 \\ + \int_0^{s_1} H_{j,i}(\epsilon s_2)\int_0^{s_2} K_l(\epsilon r_1)\int_0^{r_1} K_i(\epsilon r_2)\, dr_2\, dr_1\, ds_2 \\ + \int_0^{s_1} K_l(\epsilon r_1)\int_0^{\epsilon r_1} H_{j,i}(\epsilon s_2)\int_0^{s_2} K_i(\epsilon r_2)\, dr_2\, ds_2\, dr_1\bigg],\end{aligned} \tag{B.3}$$

where for any $j, l \in \mathbb{N}$

$$H_{j,l}(s) := e^{(\alpha_j-\alpha_l)s}\big\langle \Delta_\epsilon D_x F(s)e_j, e_l\big\rangle_H, \qquad K_l(s) := e^{\alpha_l s}\big\langle \Delta_\epsilon F(s), e_l\big\rangle_H.$$

As $2\bar{\rho} := \rho/(2+\rho) < 1$ for any $\rho > 0$, we have:

$$\sum_{i=1}^{\infty} \frac{1}{\alpha_i^{1-\bar{\rho}}} < \infty.$$

Hence, thanks to (A.1) and (A.2), by proceeding with arguments analogous to those used in the proof of Lemma 4.2 after some computations we get:

$$\mathbb{E}\big|I_{\epsilon,1}(t)\big|_H^2 \leqslant c_{T,\rho}\big(1 + |x|_H^4 + |y|_H^4\big)\epsilon^{\bar{\rho}}. \tag{B.4}$$

Finally, let us estimate $I_{\epsilon,2}(t)$. As for $I_{\epsilon,1}(t)$, we have:

$$\big|I_{\epsilon,2}(t)\big|_H^2 = \sum_{j=1}^{\infty}\bigg(\int_0^t e^{-(t-s)\alpha_j}\bigg\langle \Delta_\epsilon DF(s)\int_0^s U(s,\sigma)D\bar{F}\big(\bar{u}^x(\sigma)\big)\Gamma_\epsilon^{x,y}(\sigma)\, d\sigma, e_j\bigg\rangle_H ds\bigg)^2.$$

For any $0 \leqslant s \leqslant t$,

$$\begin{aligned}&\bigg\langle \Delta_\epsilon DF(s)\int_0^s U(s,\sigma)D\bar{F}\big(\bar{u}^x(\sigma)\big)\Gamma_\epsilon^{x,y}(\sigma)\, d\sigma, e_j\bigg\rangle_H \\ &\quad= \frac{1}{\sqrt{\epsilon}}\bigg\langle \Delta_\epsilon DF(s)\int_0^s U(s,\sigma)D\bar{F}\big(\bar{u}^x(\sigma)\big)\int_0^\sigma e^{(\sigma-r)A}\Delta_\epsilon F(r)\, dr\, d\sigma, e_j\bigg\rangle_H \\ &\quad= \frac{1}{\sqrt{\epsilon}}\int_0^s\int_0^\sigma \big\langle e^{(\sigma-r)A}\Delta_\epsilon F(r), \big[U(s,\sigma)D\bar{F}\big(\bar{u}^x(\sigma)\big)\big]^\star \Delta_\epsilon DF(s)e_j\big\rangle_H\, dr\, d\sigma \\ &\quad= \frac{1}{\sqrt{\epsilon}}\sum_{i=1}^{\infty}\int_0^s e^{r\alpha_i}\big\langle \Delta_\epsilon F(r), e_i\big\rangle_H \big\langle \Delta_\epsilon DF(s)e_j, \Lambda_i^x(s,r)e_i\big\rangle_H\, dr,\end{aligned}$$

where

$$\Lambda_i^x(s,r) := \int_r^s e^{-\sigma\alpha_i}U(s,\sigma)D\bar{F}\big(\bar{u}^x(\sigma)\big)\, d\sigma.$$

Then, we get:

$$\mathbb{E}\big|I_{\epsilon,2}(t)\big|_H^2=\frac{2}{\epsilon}\sum_{i,j,l=1}^{\infty}\int_0^t ds_1\, e^{-(t-s_1)\alpha_j}\int_0^{s_1} ds_2\, e^{-(t-s_2)\alpha_j}\int_0^{s_1} dr_1\, e^{r_1\alpha_l}\int_0^{s_2} dr_2\, e^{r_2\alpha_i}$$
$$\times\,\mathbb{E}\big\langle \Delta_\epsilon DF(s_2)e_j,\, \Lambda_i^x(s_2,r_2)e_i\big\rangle_H\big\langle \Delta_\epsilon F(r_2), e_i\big\rangle_H\big\langle \Delta_\epsilon DF(s_1)e_j,\, \Lambda_i^x(s_1,r_1)e_l\big\rangle_H\big\langle \Delta_\epsilon F(r_1), e_l\big\rangle_H.$$

This implies that, as for $I_{\epsilon,2}(t)$ in (B.3), we can develop the integral above in the following way:

$$\mathbb{E}\big|I_{\epsilon,2}(t)\big|_H^2=2\epsilon^3\sum_{i,j,l=1}^{\infty}e^{-2t\alpha_j}\int_0^{\frac{t}{\epsilon}} ds_1\mathbb{E}H_{j,l}(\epsilon s_1,\epsilon r_1)\Bigg[\int_0^{s_1}H_{j,i}(\epsilon s_2,\epsilon r_2)\int_0^{s_2}K_i(\epsilon r_2)\int_0^{r_2}K_l(\epsilon r_1)\,dr_1\,dr_2\,ds_2$$
$$+\int_0^{s_1}H_{j,i}(\epsilon s_2,\epsilon r_2)\int_0^{s_2}K_l(\epsilon r_1)\int_0^{r_1}K_i(\epsilon r_2)\,dr_2\,dr_1\,ds_2$$
$$+\int_0^{s_1}K_l(\epsilon r_1)\int_0^{r_1}H_{j,i}(\epsilon s_2,\epsilon r_2)\int_0^{s_2}K_i(\epsilon r_2)\,dr_2\,ds_2\,dr_1\Bigg],$$

where for any $j,l\in\mathbb{N}$,

$$H_{j,l}(s,r):=e^{\alpha_j s}\big\langle \Delta_\epsilon D_xF(s)e_j,\, \Lambda_l^x(s,r)e_l\big\rangle_H,\qquad K_l(s):=e^{\alpha_l s}\big\langle \Delta_\epsilon F(s), e_l\big\rangle_H.$$

Now, since $U(t,s):H\to H$ is bounded and $\|U(t,s)\|_0\leqslant c_T$, for any $0\leqslant s\leqslant t\leqslant T$, and $D\bar{F}:H\to\mathcal{L}(H)$ is bounded, we have:

$$\big\|\Lambda_i^x(s,r)\big\|_0\leqslant\frac{c_T}{\alpha_i}e^{-r\alpha_i}.$$

This means that we can use the same arguments we have just used above for $I_{\epsilon,2}(t)$ and we obtain:

$$\mathbb{E}\big|I_{\epsilon,2}(t)\big|_H^2\leqslant c_{T,\rho}\big(1+|x|_H^4+|y|_H^4\big)\epsilon^{\bar{\rho}}. \tag{B.5}$$

Combining together (B.4) and (B.5) we conclude that

$$\lim_{\epsilon\to 0}\Bigg|\int_0^t e^{(t-s)A}\psi_\epsilon^{x,y}(s)\,ds\Bigg|_H^2=0,$$

and the proof of Lemma 5.3 is finished.

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