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Smoothing properties of transition semigroups relative to SDEs with values in Banach spaces

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Abstract. In the present paper we consider the transition semigroup P_t related to some stochastic reaction-diffusion equations with the non-linear term f having polynomial growth and satisfying some dissipativity conditions. We are proving that it has a regularizing effect in the Banach space of continuous functions $C(\overline{\mathcal{O}})$, where $\mathcal{O} \subset \mathbb{R}^d$ is a bounded open set. In $L^2(\mathcal{O})$ the only result proved is the strong Feller property, for $d = 1$. Here we are able to prove that if $f \in C^\infty(\mathbb{R})$ and $d \leq 3$, then $P_t\varphi \in C_b^\infty(C(\overline{\mathcal{O}}))$ for any $\varphi \in B_b(C(\overline{\mathcal{O}}))$ and $t > 0$. An important application is to the study of the ergodic properties of the system. These results are also of interest for some problem in stochastic control.

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1. Introduction

We consider the following reaction-diffusion equation forced by a noise

$$\begin{cases} \frac{\partial u}{\partial t}(t,\xi) = \Delta u(t,\xi) + f(u(t,\xi)) + B\frac{\partial w}{\partial t}(t,\xi), & (t,\xi) \in [0,+\infty[\times[0,1]^d \\ u(t,\xi) = 0, \quad (t,\xi) \in [0,+\infty[\times\partial[0,1]^d \\ u(0,\xi) = x(\xi), \quad \xi \in [0,1]^d \ . \end{cases} \tag{1.1}$$

Here Δ denotes the Laplace operator and $\frac{\partial w}{\partial t}(t,\xi)$ stands for a gaussian random field, white noise in space and time. B is a non negative bounded linear operator from $L^2((0,1)^d)$ to itself and d is an integer less than or equal to 3. f is a real function having polynomial growth and satisfying some dissipativity conditions. Here and in the sequel we will denote by H the Hilbert space $L^2((0,1)^d)$ and by E the Banach space $C([0,1]^d)$.

If $u(t;x)$ is the mild solution of (1.1) in E, the transition semigroup associated to equation (1.1) is defined by

$$P_t\varphi(x) = \mathbb{E}\varphi(u(t;x)) \ , \quad x \in E \quad t \geq 0 \ ,$$

for any φ in $B_b(E)$, the Banach space of bounded Borel functions from E into $\mathbb{R}$. Our aim is to study the regularizing effect of P_t in $B_b(E)$. Namely, we prove that if $f \in C^{k+1}(\mathbb{R})$, then

$$\varphi \in B_b(E) \Rightarrow P_t\varphi \in C_b^k(E)^1 \ .$$

In particular the semigroup P_t is *strongly Feller* on E, that is it maps Borel functions into uniformly continuous functions, for any $t > 0$. Moreover, for any $\varphi \in C_b(E)$ and $t > 0$ we establish the following Bismut-Elworthy type formula for the derivative of $P_t\varphi$

$$\langle D(P_t\varphi)(x), h\rangle_E = \frac{1}{t}\mathbb{E}\left(\varphi(u(t;x))\int_0^t \left\langle B^{-1}D_x u(s;x)h, dw(s)\right\rangle_H\right) \ , \quad (1.2)$$

where $D_x u(s;x)h$ is the mean-square derivative of $u(t;x)$ along the direction $h \in E$ and B^{-1} : Range$(B) \subseteq H \to H$ is the inverse of B, not bounded in general. We also prove the following estimates

$$\sup_{x\in E}\left|D^j(P_t\varphi)(x)\right| \leq c(t\wedge 1)^{-\frac{j(1+\epsilon)}{2}}\sup_{x\in E}|\varphi(x)| \ , \quad (1.3)$$

for $j = 0, 1, \ldots, k$. Here the constant ϵ depends on B and may be taken equal to zero in dimension $d = 1$ and strictly less that one if $d \leq 3$.

The first classical application of our results is to the study of the ergodicity of the system. Actually, in [5] by using the *strong Feller* property of P_t on E proved in the present article, we prove the strong Feller property of P_t on H. Then in [6] we prove the irreducibility of P_t on H so that all the transition probabilities $\{P_t(x,\cdot)\}_{t\geq 0}$ are equivalent on H. Then, from the Doob's theorem, the uniqueness and the *strongly mixing* property of the invariant measure μ follow. Notice that ergodicity for stochastic reaction-diffusion equations with coef-

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[1]If X is a Banach space, for any $k \geq 0$ we denote by $C_b^k(X)$ the set of all real valued functions defined on X which are k times differentiable with bounded and uniformly continuous derivatives, up to the k-th order.

ficients having polynomial growth has been studied in several papers (see [8] for a comprehensive bibliography), both in the Hilbert space H and in the Banach space E. But in these papers it is always assumed that f satisfies the following condition

$$(f(t) - f(s))(t - s) \leq -\omega|t - s|^2, \quad t, s \in \mathbb{R} \ ,$$

for some $\omega > 0$. Here we are able to enlarge the class of the functions f considered at least to all polynomials of odd degree having the leading coefficient strictly negative perturbed by a Lipschitz term (see hypothesis 2.4 and remark 2.6 for more details).

Besides, these results are the basic tools in order to prove that for any $t > 0$ the semigroup P_t maps $B_b(H)$ into $C_b^1(H)$ and it holds a suitable estimate for the *sup-norm* of the derivative (see [5]). This allows us to improve the results proved by Da Prato, Elworthy and Zabczyk in [9] into two respects: firstly they only prove that P_t maps $B_b(H)$ into Lipschitz continuous functions and secondly they consider only the case of dimension $d = 1$ and diffusion operators boundedly invertible. These improvements enable us to study the associated Hamilton-Jacobi equation in the standard case of the Hilbert space $L^2((0,1)^d)$, for $d \leq 3$.

Finally, our results seem to be of interest in the study of existence and uniqueness of strong solutions for the associated Kolmogorov equations and hence for the study of controlled reaction-diffusion equations in the Banach space E.

We remark that in order to get (1.2), we have to overcome some difficulties. First of all, we have to prove that the solution $u(t;x)$ is mean-square differentiable in E. This is not trivial, since the Nemytskii operator F associated with the function f is not Lipschitz continuous in E and we cannot apply the standard method based upon the contraction principle (see [7]). Another difficulty lies in the fact that in the present paper we do not assume one of the typical hypotheses in Bismut-Elworthy formula: boundedness of the inverse of the diffusion term. Thus, in order to give a meaning to the Itô integral appearing in the right side of (1.2), first we have to prove that $D_x u(t;x)h \in D(B^{-1})$ for any $x, h \in E$ and $t > 0$ and then that

$$\mathbb{E} \int_0^t \left|B^{-1} D_x u(s;x)h\right|_H^2 ds < +\infty \ .$$

The main idea we follow throughout the paper is to approximate f by a sequence of functions $\{f_\alpha\}_\alpha$ which are Lipschitz continuous and to introduce the approximating equations relative to the Nemytskii operators F_α corresponding to f_α. Then, if we denote by $u^\alpha(t;x)$ the solutions, we can work with the approximating semigroup $P_t^\alpha\varphi(x) = \mathbb{E}\varphi(u^\alpha(t;x))$ in $C_b(H)$, where we can use the usual Itô calculus and all the results proved in [4]. We get the expected results for $P_t^\alpha\varphi$, we find that some estimates independent of α hold and then we pass to the limit. It is worthwhile to remark that the differentiability of P_t^α is not trivial either. Actually, in general, even if f is assumed to be bounded and of class C^∞, the corresponding Nemytskii operator is never Fréchet differentiable on H, unless f is linear, so that the typical assumption of differentiability of coefficients fails to be true. Moreover, directional derivatives of F of order higher than one do not exist along any directions in H, but only along suitable directions.

2. Notations and preliminary results

Let $\mathcal{O}$ denote the open set $(0,1)^d$, with $d \leq 3$. Throughout the paper H will be the Hilbert space $L^2(\mathcal{O})$ and E the densely embedded Banach space $C(\overline{\mathcal{O}})$. The scalar product and the norm in H will be denoted respectively by $<\cdot,\cdot>_H$ and $|\cdot|_H$, whereas the duality form on $E \times E^*$ and the norm on E will be denoted respectively by $<\cdot,\cdot>_E$ and $|\cdot|_E$. The norm in $L^p(\mathcal{O})$, $p \in (2,+\infty]$, will be denoted by $|\cdot|_p$.

We recall that the realization in H of the Laplace operator with Dirichlet boundary conditions is given by

$$D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}) \ , \quad Ax = \Delta x \ , \quad x \in D(A) \ .$$

A is a negative self-adjoint operator with compact resolvent. It admits a complete orthonormal system of eigenvectors $\{e_k\}$, such that $e_k \in C(\overline{\mathcal{O}})$ for any $k \in \mathbb{N}$, and

$$\sup_{k\in\mathbb{N}} |e_k|_E < +\infty \ . \tag{2.1}$$

For the corresponding set of eigenvalues $\{-\alpha_k\}$, we have $0 < \alpha_1 \leq \alpha_2 \leq \cdots$ and $\alpha_k \sim k^{2/d}$. The fractional powers of the operator $-A$ are defined for any $\delta \in \mathbb{R}$ by

$$\begin{cases} D((-A)^\delta) = \left\{ x \in H : \sum_{k=1}^{+\infty} \langle x, e_k\rangle_H^2 \alpha_k^{2\delta} < +\infty \right\} \\ (-A)^\delta x = \sum_{k=1}^{+\infty} \langle x, e_k\rangle_H \alpha_k^\delta e_k, \quad x \in D((-A)^\delta) \ . \end{cases}$$

The operator A generates an analytic semigroup e^{tA}, $t \geq 0$, of class C_0. Thus for any $\delta \geq 0$ and $t > 0$ we have that Range $(e^{tA}) \subset D((-A)^\delta)$ and

$$\left|(-A)^{\delta}e^{tA}x\right|_{H}\leq M_{\delta}t^{-\delta}|x|_{H},\quad x\in H\ ,\tag{2.2}$$

for a suitable positive constant M_{δ}. Moreover, it is easy to verify that if $\delta_1 \leq \delta \leq \delta_2$ and $x\in D((-A)^{\delta_2})$ the following interpolatory inequality holds

$$\left|(-A)^{\delta}x\right|_{H}\leq c\left|(-A)^{\delta_1}x\right|_{H}^{1-\theta}\left|(-A)^{\delta_2}x\right|_{H}^{\theta},\quad \theta=\frac{\delta-\delta_1}{\delta_2-\delta_1}\ .\tag{2.3}$$

Finally, we recall that the semigroup e^{tA}, $t\geq 0$, is *ultracontractive*. That is, e^{tA} maps H into $L^{\infty}(\mathcal{O})$ for any $t>0$ and it holds that

$$|e^{tA}x|_{\infty}\leq ct^{-d/4}|x|_{H},\quad x\in H\ .\tag{2.4}$$

The part A_E of A in E is the realization of the Laplace operator in E with Dirichlet boundary conditions. It is given by

$$\begin{cases} D(A_E)=\left\{x\in\bigcap_{p\geq 1}W^{2,p}_{\text{loc}}(\mathcal{O}):x,\Delta x\in C(\overline{\mathcal{O}})\ ,x_{|_{\partial\mathcal{O}}}=0\right\}\\ A_Ex=\Delta x,\quad x\in D(A_E)\ .\end{cases}$$

A_E is the infinitesimal generator of an analytic semigroup e^{tA_E} on E and, as it can be easily verified,

$$\overline{D(A_E)}=\left\{x\in E:x_{|_{\partial\mathcal{O}}}=0\right\}\ .$$

This means that the closure of $D(A_E)$ is different from E and hence e^{tA_E} is not strongly continuous. In the sequel, it will not be misleading to denote also A_E and e^{tA_E} by A and e^{tA}. For a comprehensive presentation of the theory of analytic semigroups see e.g. Lunardi [15].

The cylindrical Wiener process $w(t)$ is defined as

$$w(t)=\sum_{k=1}^{+\infty}e_kw_k(t)\ ,\tag{2.5}$$

where $\{e_k\}$ is the complete orthonormal system diagonalizing A which we have introduced before and $\{w_k\}$ is a sequence of mutually independent real Brownian motions defined on a stochastic basis $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ and adapted to the non anticipative filtration $\mathcal{F}_t, t\geq 0$. The series (2.5) defining $w(t)$ does not converge in H, but is convergent in any Hilbert space U such that the embedding $H\subset U$ is Hilbert-Schmidt (see [7], Chapter 4).

Consider the Ornstein-Uhlenbeck equation

$$\begin{cases}\frac{\partial z}{\partial t}(t,\xi)=\Delta z(t,\xi)+B\frac{\partial w}{\partial t}(t,\xi)\\ z(0,\xi)=0,\quad \xi\in\mathcal{O},\quad z(t,\xi)=0,\quad \xi\in\partial\mathcal{O}\quad t\geq 0.\end{cases}$$

It can be shown that if

$$\int_0^t Tr\left[e^{sA}B^2e^{sA^*}\right]ds < \infty, \quad t \geq 0 \ ,$$

then this equation has a unique mild solution $w^A(t)$ which is the mean-square continuous gaussian process with values in H given by

$$w^A(t) = \int_0^t e^{(t-s)A}B\,dw(s) \ .$$

In the present paper we will need $w^A(t)$ to be more regular. Namely, we will require that $w^A(t)$ is an E-valued process, having the p-th moment of its E-norm finite, for p sufficiently large. Thus we assume that B commutes with A.

Hypothesis 2.1 *$B : H \to H$ is a non negative bounded linear operator which is diagonal with respect to the orthonormal basis $\{e_k\}$. Moreover, if $\{\lambda_k\}$ is the corresponding set of eigenvalues, we have*

$$\sum_{k=1}^{+\infty} \frac{\lambda_k^2}{\alpha_k^{1-\gamma}} < +\infty \ , \tag{2.6}$$

for some $\gamma \in (0, 1)$.

In [7] it is proved that under hypothesis 2.1 w^A has a version $w^A(t, \xi)$ which is α-Hölder continuous with respect to $t \geq 0$ and $\xi \in \overline{\mathcal{O}}$, $\mathbb{P}$-a.s. for any $\alpha \in [0, 1/4)$. In particular $w^A(t)$ has a E-valued version with α-Hölder continuous paths. Moreover, it is possible to show that for any $T > 0$ and $p \geq 1$

$$\mathbb{E} \sup_{t \in [0,T]} |w^A(t)|_E^p < +\infty \ . \tag{2.7}$$

Next hypothesis is a non degeneracy condition on B required in order to get the smoothing effect of the semigroup P_t.

Hypothesis 2.2 *There exists $\epsilon < 1$ such that*

$$\text{Range}(B) \supseteq D((-A)^{\epsilon/2}) \ . \tag{2.8}$$

According to hypothesis (2.2), due to the closed graph theorem the operator $\Gamma = B^{-1}(-A)^{-\epsilon/2}$ is bounded. Moreover, by (2.2) $\text{Range}(e^{tA}) \subset \text{Range}(B)$ for any $t > 0$ and, by setting $\Gamma(t) = B^{-1}e^{tA}$, it holds that

$$|\Gamma(t)x|_H = \left|\Gamma(-A)^{\epsilon/2}e^{tA}x\right|_H \leq c\|\Gamma\|t^{-\epsilon/2}|x|_H, \quad x \in H \ . \tag{2.9}$$

Remark 2.3 Let

$$Be_k = \beta_k \alpha_k^{-\delta} e_k, \quad k \in \mathbb{N}$$

for some $\delta \geq 0$ and assume that there exists $v > 0$ such that $v^{-1} \leq \beta_k \leq v$, $k \in \mathbb{N}$. Then if $d \leq 3$ there exists some $\delta \geq 0$ such that the operator B satisfies the hypotheses 2.1 and 2.2.

Indeed, if $\gamma \in (0,1)$, we have

$$\sum_{k=1}^{+\infty} \frac{\beta_k^2}{\alpha_k^{1-\gamma+2\delta}} < +\infty \Longleftrightarrow (1-\gamma+2\delta) > d/2 \Longleftrightarrow \delta > d/4 - (1-\gamma)/2 \ ,$$

and then B satisfies hypothesis 2.1 iff $\delta > (d/2-1)/2$. Moreover, (2.8) is clearly satisfied for any $\delta \leq \epsilon/2$, so that if $d \leq 3$, the hypotheses 2.1 and 2.2 are both satisfied for a suitable $\delta \geq 0$. If $d = 1$, we can choose $\delta = 0$, so that the non degenerate case is covered.

2.1. The Nemytskii operator

As far as f is concerned, we will assume that it satisfies the following conditions

Hypothesis 2.4 1. *There exist $k \geq 1$ and $m \geq 1$ such that $f \in C^{k+1}(\mathbb{R})$ and*

$$\sup_{\sigma \in \mathbb{R}} \frac{|f^{(j)}(\sigma)|}{1+|\sigma|^{2m+1-j}} < +\infty, \quad j = 0, \ldots, k+1 \ .$$

2. *There exists $\gamma \in \mathbb{R}$ such that for any $\sigma \in \mathbb{R}$*

$$\begin{cases} f'(\sigma) \leq \gamma, \\ \sigma^{2m} f'(\sigma) - 2m\sigma^{2m-1} f(\sigma) \leq \gamma \ . \end{cases}$$

We denote by F the Nemytskii operator associated to the function f, that is

$$F(x)(\xi) = f(x(\xi)), \quad \xi \in \overline{\mathcal{O}} \ .$$

F is well defined and continuous from $L^p(\mathcal{O})$ to $L^q(\mathcal{O})$, for any $p, q \geq 1$ such that $p/q = 2m+1$. Hence F is not defined from H to H. Nevertheless $F : E \to E$ is continuous and bounded on bounded subsets of E. Moreover, $F \in C^{k+1}(E;E)$ and for any $x, y_1, \ldots, y_{k+1} \in E$ it holds

$$D^{(j)}F(x)(y_1, \ldots, y_j)(\xi) = f^{(j)}(x(\xi)) y_1(\xi) \cdots y_j(\xi), \quad \xi \in \overline{\mathcal{O}} \ . \qquad (2.10)$$

Then, for $j = 0, \ldots, k+1$

$$\left|D^{(j)}F(x)\right|_{\mathscr{L}^j(E)}\leq c\left(1+|x|_E^{2m+1-j}\right),\quad x\in E\ , \tag{2.11}$$

where $\mathscr{L}^j(E)=\mathscr{L}(E;\mathscr{L}^{j-1}(E))$ and $\mathscr{L}^0(E)=E$. This means that $D^{(j)}F$ is bounded on bounded subsets of E and in particular F and all its derivatives up the k-th order are locally Lipschitz continuous.

From now on, we will denote by $\partial|x|_E$ the subdifferential of the E-norm $|\cdot|_E$ at the point $x\in E$. Assume that there exists $\xi_0\in\overline{\mathscr{O}}$ such that $x(\xi_0)=|x|_E$. It is possible to show that $\delta_{\xi_0}\in\partial|x|_E$. In the same way, if we assume that there exists $\xi_0\in\overline{\mathscr{O}}$ such that $x(\xi_0)=-|x|_E$, it follows that $-\delta_{\xi_0}\in\partial|x|_E$ (for the definitions and the proofs see [18]). Moreover, if $u:[0,T]\to E$ is differentiable at $t_0\in[0,T]$, then the function $t\mapsto|u(t)|_E$ is differentiable on the left (and also on the right) at t_0 and it holds

$$\frac{d^-}{dt}\ |u(t_0)|_E=\min\left\{\langle u'(t_0),x^*\rangle,\ x^*\in\partial|u(t_0)|_E\right\}\ . \tag{2.12}$$

In the sequel, for any $x\in E$ we will set

$$\delta_x=\begin{cases}\delta_{\xi_0} & \text{if } x(\xi_0)=|x|_E\\ \\ -\delta_{\xi_0} & \text{if } x(\xi_0)=-|x|_E\ .\end{cases} \tag{2.13}$$

By applying directly the definition of δ_x, it follows from hypothesis 2.4-2 that for any $x,h\in E$

$$\langle F(x+h)-F(x),\delta_h\rangle_E\leq\gamma|h|_E\ . \tag{2.14}$$

Moreover,

$$\langle DF(x)h,h\rangle_H\leq\gamma|h|_H^2\ . \tag{2.15}$$

In some situation a stronger dissipativity condition will be required.

Hypothesis 2.5 *There exist* $a>0$ *and* $b,c\in\mathbb{R}$ *such that for any* $\sigma\in\mathbb{R}$ *and* $\rho\geq 0$

$$f(\sigma+\rho)-f(\sigma)\leq -a\rho^{2m+1}+b|\sigma|^{2m+1}+c\ . \tag{2.16}$$

According to the definition of the subdifferential δ_x, this implies that for any $x,h\in\mathbb{E}$

$$\langle F(x+h)-F(x),\delta_h\rangle_E\leq -a|h|_E^{2m+1}+b|x|_E^{2m+1}+c\ . \tag{2.17}$$

Remark 2.6 It is not difficult to verify that the conditions of hypotheses 2.4 and 2.5 are all satisfied by any function $f=p+f_0$, where p is a polynomial of odd degree having the leading coefficient strictly negative, i.e.

$$p(\sigma) = -a\sigma^{2m+1} + \sum_{i=0}^{2m} a_i\sigma^i, \quad a > 0$$

and f_0 is a Lipschitz continuous function.

2.2. The approximating Nemytskii operators

For any $\alpha > 0$, let us define

$$f_\alpha \colon \mathbb{R} \to \mathbb{R}, \quad \sigma \mapsto \frac{f(\sigma)}{1 + \alpha\sigma^{2m}} \quad .$$

From hypothesis 2.4-1, f_α has linear growth and

$$|f_\alpha(\sigma)| \le \frac{c}{\alpha}(1 + |\sigma|), \quad \sigma \in \mathbb{R} \quad . \tag{2.18}$$

Clearly $f_\alpha \in C^{k+1}(\mathbb{R})$ and it is easy to check that for any $R > 0$ and $j \le k + 1$

$$\lim_{\alpha \to 0} \sup_{|\sigma| \le R} \left| f_\alpha^{(j)}(\sigma) - f^{(j)}(\sigma) \right| = 0 \quad . \tag{2.19}$$

Moreover, from hypothesis 2.4-2, for any $\alpha \in (0, 1]$ we get

$$f_\alpha'(\sigma) = \frac{f'(\sigma) + \alpha\left(\sigma^{2m} f'(\sigma) - 2m\sigma^{2m-1} f(\sigma)\right)}{\left(1 + \alpha\sigma^{2m}\right)^2} \le 2\gamma, \quad \sigma \in \mathbb{R} \quad . \tag{2.20}$$

Now, for any $\alpha > 0$, let F_α be the Nemytskii operator associated to f_α, that is $F_\alpha(x)(\xi) = f_\alpha(x(\xi))$, $\xi \in \overline{\mathcal{O}}$. As f_α is Lipschitz continuous, then also F_α is Lipschitz continuous as a functional both from H to H and from E to E. Since $f_\alpha \in C^{k+1}(\mathbb{R})$, then it is possible to prove that $F_\alpha \in C^{k+1}(E; E)$ and the j-th Fréchet derivative of F_α, as a mapping from E to $\mathscr{L}^j(E)$, is given by

$$D^j F_\alpha(x)(y_1, \ldots, y_j)(\xi) = f_\alpha^{(j)}(x(\xi)) y_1(\xi) \cdots y_j(\xi), \quad \xi \in \mathcal{O} \quad .$$

Moreover there exists a constant c, which is independent of α, such that for $j = 0, \ldots, k + 1$

$$|D^{(j)} F_\alpha(x)|_{\mathscr{L}^j(E)} \le c\left(1 + |x|_E^{2m+1-j}\right), \quad x \in E \quad , \tag{2.21}$$

so that all $D^{(j)} F_\alpha$ are bounded on bounded subsets of E, uniformly with respect to α. Finally, due to (2.20), proceeding as for F we have

$$\langle F_\alpha(x + h) - F_\alpha(x), \delta_h \rangle_E \le 2\gamma |h|_E, \quad x, h \in E \quad . \tag{2.22}$$

Concerning the differentiability in H, F_α is not Fréchet differentiable, as f_α is not linear. Nevertheless, for any $j \le k + 1$, there exists the j-th

Gâteaux derivative of F_α at any point $x \in H$ and along any directions $y_1, \ldots, y_j \in L^{2j}(\mathcal{O})$ and it is given by

$$D^j_G F_\alpha(x)(y_1, \ldots, y_j)(\xi) = f^{(j)}_\alpha(x(\xi))y_1(\xi) \cdots y_j(\xi), \quad \xi \in \overline{\mathcal{O}} \ ,$$

(for more details see [4]). It is useful to remark that if $x, y_1, \ldots, y_j \in E$ then

$$D^j_G F_\alpha(x)(y_1, \ldots, y_j) = D^j F_\alpha(x)(y_1, \ldots, y_j) \ .$$

Finally, thanks to (2.19), for any $R > 0$ and $j \leq k+1$ we have

$$\begin{cases} \sup\limits_{|x|_E \leq R} |F_\alpha(x) - F(x)|_E \to 0 \\ \sup\limits_{\substack{|x|_E \leq R \\ |y_1|_E, \ldots, |y_j|_E \leq R}} |D^j F_\alpha(x)(y_1, \ldots, y_j) - D^j F(x)(y_1, \ldots, y_j)|_E \to 0 \ , \end{cases} \tag{2.23}$$

as α goes to 0.

2.3. Functional spaces

Let X be a Banach space with norm $|\cdot|_X$. In the present work $B_b(X)$ is the space of all mappings $\varphi : X \to \mathbb{R}$ which are bounded and Borelian. $B_b(X)$, endowed with the norm

$$\|\varphi\|^X_0 = \sup_{x \in X} |\varphi(x)| \ ,$$

is a Banach space. $C_b(X)$ is the subspace of all uniformly continuous functions. For any $k \geq 1$, $C^k_b(X)$ is the subset of $C_b(X)$ of all functions φ which are k times Fréchet differentiable, with bounded and uniformly continuous derivatives up to the k-th order. If we define

$$[\varphi]^X_h = \sup_{x \in X} |D^h \varphi(x)|, \quad h = 1, \ldots, k \ ,$$

then $C^k_b(X)$, endowed with the norm

$$\|\varphi\|^X_k = \|\varphi\|^X_0 + \sum_{h=1}^k [\varphi]^X_h \ ,$$

is a Banach space.

Now, let us investigate the relation between $C_b(H)$ and $C_b(E)$.

Proposition 2.7 *$C_b(H)$ is continuously embedded in $C_b(E)$. On the other hand, for any $\varphi \in C_b(E)$ there exists a sequence $(\varphi_n) \subset C_b(H)$ such that*

$$\begin{cases} \lim_{n\to+\infty} \varphi_n(x) = \varphi(x), \quad \forall x \in E \\ \sup_{n\in\mathbb{N}} \|\varphi_n\|_0^H \leq \|\varphi\|_0^E \ . \end{cases} \tag{2.24}$$

Proof. The first statement is completely trivial since E is continuously embedded in H.

Concerning the second part of the proposition, by using arguments of reflection it is possible to prove that there exists a linear operator $P : H \to L^2(\mathbb{R}^d)$ such that $Px \in E$, for any $x \in E$. Then, for any $x \in H$ and $n \in \mathbb{N}$ let us define

$$x_n(\xi) := n^d \int_{\xi_d}^{\xi_d+1/n} \cdots \int_{\xi_1}^{\xi_1+1/n} Px(\zeta)\, d\zeta_1 \cdots d\zeta_d, \quad \xi \in \overline{\mathcal{O}} \ . \tag{2.25}$$

Our aim is to show that if $\varphi \in C_b(E)$, then for any $n \in \mathbb{N}$ the function φ_n defined by

$$\varphi_n(x) := \varphi(x_n), \quad x \in H$$

belongs to $C_b(H)$ and satisfies (2.24). It is well known that $x_n \in E$, then φ_n is well defined. Moreover, for any $x \in H$ and $\xi \in \overline{\mathcal{O}}$,

$$|x_n(\xi)| \leq n^d \int_{\xi_d}^{\xi_d+1/n} \cdots \int_{\xi_1}^{\xi_1+1/n} |Px(\zeta)|\, d\zeta_1 \cdots d\zeta_d \leq c n^{d/2} |x|_H \ ,$$

so that for any $x, y \in H$ we have

$$|x_n - y_n|_E = |(x-y)_n|_E \leq c n^{d/2} |x-y|_H \ . \tag{2.26}$$

Then, as $\varphi \in C_b(E)$, we can conclude from (2.26) that φ_n is uniformly continuous on H. Finally, from the definition of φ_n we have

$$\|\varphi_n\|_0^H = \sup_{x\in H} |\varphi_n(x)| = \sup_{x\in H} |\varphi(x_n)| \leq \sup_{x\in E} |\varphi(x)| = \|\varphi\|_0^E \ .$$

That is, $\varphi_n \in C_b(H)$ and $\|\varphi_n\|_0^H \leq \|\varphi\|_0^E$, for any $n \in \mathbb{N}$.

To conclude the proof we have to show that for any $x \in E$, $\varphi_n(x) \to \varphi(x)$, as $n \to +\infty$. Actually, since $\varphi \in C_b(E)$, this easily follows if we notice that x_n converges to x in E as $n \to +\infty$, for any $x \in E$. □

Remark 2.8 From the previous proposition we can conclude that $C_b(E)$ is continuously embedded in $B_b(H)$. For any $\varphi \in C_b(E)$, we define

$$i\varphi(x) = \begin{cases} \varphi(x) & \text{if } x \in E \\ 0 & \text{if } x \in H \backslash E \ . \end{cases} \tag{2.27}$$

Then, as E is a Borel subset of H, it follows that $i\varphi \in B_b(H)$ and $\|i\varphi\|_0^H \leq \|\varphi\|_0^E$, so that $C_b(E)$ is continuously embedded in $B_b(H)$.

3. Some a priori estimates for the solution of equation (1.1)

For any $T > 0$ we will denote by $\mathscr{K}_T(E)$ the Banach space of adapted processes in $C(]0,T];L^2(\Omega;E)) \cap L^\infty(0,T;L^2(\Omega;E))$ endowed with the norm

$$\|u\|^2_{\mathscr{K}_T(E)} = \sup_{t\in[0,T]} \mathbb{E}|u(t)|_E^2 \ .$$

$\mathscr{H}_T(E)$ will denote the space of adapted processes $L^2(\Omega;C(]0,T);E)$ $\cap L^\infty(0,T;E))$ endowed with the norm

$$\|u\|^2_{\mathscr{H}_T(E)} = \mathbb{E}\sup_{t\in[0,T]} |u(t)|_E^2 \ .$$

Clearly $\mathscr{H}_T(E)$ is continuously embedded in $\mathscr{K}_T(E)$.

Equation (1.1) can be rewritten as the following abstract stochastic differential equation in E

$$du(t) = (Au(t) + F(u(t)))\,dt + B\,dw(t), \quad u(0) = x \ . \tag{3.1}$$

Definition 3.1 *An E-valued predictable process $u(t;x)$ is a mild solution of the problem* (3.1) *if*

$$u(t;x) = e^{tA}x + \int_0^t e^{(t-s)A}F(u(s;x))\,ds + w^A(t) \ .$$

Proposition 3.2 *Under hypotheses* 2.1 *and* 2.4, *for any $x \in E$, the problem* (3.1) *has a unique mild solution $u(x) \in \mathscr{H}_T(E)$. Moreover,*

$$|u(t;x)|_E \leq e^{\gamma t}|x|_E + h(t), \quad \mathbb{P}-\text{a.s.} \tag{3.2}$$

where h is the process defined by

$$h(t) = ce^{\gamma t}\int_0^t \left(1 + |w^A(s)|_E^{2m+1}\right) ds + \sup_{s\in[0,t]} |w^A(s)|_E \ . \tag{3.3}$$

Proof. Existence and uniqueness of a mild solution $u(t;x)$ are proved in [7], theorem 7.13. It remains to prove estimate (3.2).

We first remark that if we define

$$v(t;x) = u(t;x) - w^A(t), \quad t \geq 0 \ , \tag{3.4}$$

then $v(t;x)$ is the unique mild solution of the problem

$$\frac{d}{dt}v(t) = Av(t) + F(v(t) + w^A(t)), \quad v(0) = x \ . \tag{3.5}$$

For any $\lambda \in \mathbb{N}$ we introduce the problem

$$\frac{d}{dt}v^\lambda(t) = Av^\lambda(t) + F(v^\lambda(t) + w^A(t)), \quad v^\lambda(0) = \lambda(\lambda - A)^{-1}x. \tag{3.6}$$

By using the same arguments used in [7], theorem 7.13, the problem (3.6) has a unique solution $v^\lambda(t;x)$ which belongs to $C([0,+\infty[;E)$, $\mathbb{P}$–a.s. Now let us fix $T > 0$ and let us define the function

$$f^\lambda(t) = F(v^\lambda(t) + w^A(t)), \quad t \in [0,T] \ .$$

As $f^\lambda \in C([0,T];E)$ and $v^\lambda(0) = \lambda(\lambda - A)^{-1}x \in D(A)$, then v^λ is a strong solution. This means that there exists a sequence $(v^{\lambda,n}) \subset C^1([0,T];E) \cap C([0,T];D(A))$ such that

$$v^{\lambda,n} \to v^\lambda, \ \ \frac{d}{dt}v^{\lambda,n} - Av^{\lambda,n} = f^{\lambda,n} \to f_\lambda, \quad \text{in } C([0,T];E) \tag{3.7}$$

as $n \to +\infty$ (for a proof see [15], proposition 4.1.8).

Now, for any $t \in [0,T]$ according to (2.12) we have

$$\begin{aligned}\frac{d^-}{dt}|v^{\lambda,n}(t)|_E &\leq \langle Av^{\lambda,n}(t), \delta^{\lambda,n}(t)\rangle_E + \langle F(v^{\lambda,n}(t) + w^A(t)) \\ &- F(w^A(t)), \delta^{\lambda,n}(t) >_E + \langle F(w^A(t)), \delta^{\lambda,n}(t)\rangle_E \\ &+ \langle f^\lambda(t) - F(v^{\lambda,n}(t) + w^A(t)), \delta^{\lambda,n}(t)\rangle_E + \langle f^{\lambda,n}(t) - f^\lambda(t), \delta^{\lambda,n}(t)\rangle_E \ ,\end{aligned} \tag{3.8}$$

where $\delta^{\lambda,n}(t) = \delta_{v^{\lambda,n}(t)}$ is defined as in (2.13). Then, since e^{tA} is a contraction semigroup and (2.11) and (2.14) hold, we get

$$\begin{aligned}\frac{d^-}{dt}|v^{\lambda,n}(t)|_E &\leq \gamma|v^{\lambda,n}(t)|_E + |F(w^A(t))|_E + |f^\lambda(t) - F(v^{\lambda,n}(t) + w^A(t))|_E \\ &+ |f^{\lambda,n}(t) - f^\lambda(t)|_E \leq \gamma|v^{\lambda,n}(t)|_E + c(1 + |w^A(t)|_E^{2m+1}) \\ &+ \|f^\lambda - F(v^{\lambda,n} + w^A)\|_{C([0,t];E)} + \|f^{\lambda,n} - f^\lambda\|_{C([0,t];E)} \ .\end{aligned}$$

This implies that

$$\begin{aligned}|v^{\lambda,n}(t)|_E &\leq e^{\gamma t}|v^{\lambda,n}(0)|_E + c\int_0^t e^{\gamma(t-s)}(1 + |w^A(s)|_E^{2m+1})\,ds \\ &+ e^{\gamma t}t\Big(\|f^\lambda - F(v^{\lambda,n} + w^A)\|_{C([0,t];E)} + \|f^{\lambda,n} - f^\lambda\|_{C([0,t];E)}\Big) \ .\end{aligned}$$

By taking the limit as $n \to +\infty$, since F is locally Lipschitz and (3.7) holds and since $|\lambda(\lambda - A)^{-1}x|_E \leq |x|_E$, we have

$$|v^\lambda(t;x)|_E \leq e^{\gamma t}|x|_E + ce^{\gamma t}\int_0^t (1+|w^A(s)|_E^{2m+1})\,ds \ . \tag{3.9}$$

Now set

$$r(x) = \sup_{\lambda\in\mathbb{N}} \|v^\lambda(x)\|_{C([0,t];E)} + \|v(x)\|_{C([0,t];E)} + \|w^A\|_{C([0,t];E)} \ .$$

Since F is locally Lipschitz continuous, there exists $c_{r(x)} > 0$ (we remark that both $r(x)$ and $c_{r(x)}$ are random) such that

$$\begin{aligned}\left|v^\lambda(t;x) - v(t;x)\right|_E \leq & \left|\lambda(\lambda-A)^{-1}e^{tA}x - e^{tA}x\right|_E \\ & + \int_0^t \left|e^{(t-s)A}\left(F(v^\lambda(s;x)+w^A(s)) - F(v(s;x)+w^A(s))\right)\right|_E ds \\ & \leq \left|\lambda(\lambda-A)^{-1}e^{tA}x - e^{tA}x\right|_E + c_{r(x)}\int_0^t \left|v^\lambda(s;x) - v(s;x)\right|_E ds \ .\end{aligned}$$

It follows that

$$\left|v^\lambda(t;x) - v(t;x)\right|_E \leq e^{c_{r(x)}t}\left|\lambda(\lambda-A)^{-1}e^{tA}x - e^{tA}x\right|_E, \quad \mathbb{P}-\text{a.s.}$$

Therefore, if $t > 0$, it follows that $v^\lambda(t;x) \to v(t;x)$ in E as $\lambda \to +\infty$, $\mathbb{P}$-a.s. and by (3.9)

$$|v(t,x)|_E \leq e^{\gamma t}|x|_E + ce^{\gamma t}\int_0^t (1+|w^A(s)|_E^{2m+1})\,ds, \quad \mathbb{P}-\text{a.s.} \tag{3.10}$$

Finally, recalling that $u(t;x) = v(t;x) + w^A(t)$, (3.10) gives

$$|u(t;x)|_E \leq e^{\gamma t}|x|_E + ce^{\gamma t}\int_0^t \left(1+|w^A(s)|_E^{2m+1}\right)ds + \sup_{s\in[0,t]}|w^A(s)|_E \ .$$

In particular, due to (2.7), $u(x) \in \mathscr{H}_T(E)$, for any $T > 0$. □

Remark 3.3 We recall that in [7] (theorem 7.14) it has been proved that for any $x \in H$ the problem (3.1) has a unique generalized solution $u(t;x)$. That is, there exists a sequence $\{x_n\} \in E$ converging to x in H such that the corresponding sequence of mild solutions $\{u(t;x_n)\}$ converges $\mathbb{P}$-a.s. to $u(t;x)$ in $C([0,T];H)$, for any $T > 0$.

By using the stronger dissipativity condition for f described in hypothesis 2.5, we get a stronger estimate for $|u(t;x)|_E$, uniform with respect to $x \in E$.

Proposition 3.4 *Let* $u(t;x)$ *be the unique mild solution of* (3.1). *Under hypotheses* 2.1, 2.4 *and* 2.5, *for any* $t > 0$ *we have*

$$\sup_{x\in E}|u(t;x)|_E \le k(t)t^{-1/2m}, \quad \mathbb{P}-\text{a.s.} \tag{3.11}$$

where the random process $k(t)$ is defined by

$$k(t) = c\left(1 + \sup_{s\in[0,t]}|w^A(s)|_E\right) . \tag{3.12}$$

Proof. Let $(v^{\lambda,n}) \subset C^1([0,T];E) \cap C([0,T];D(A))$ be the approximating sequence of strict solutions introduced in the proof of proposition 3.2. Using (2.17) and (3.8) gives

$$\frac{d^-}{dt}|v^{\lambda,n}(t)|_E \le -a|v^{\lambda,n}(t)|_E^{2m+1} + b|w^A(t)|_E^{2m+1} + c + \left|F(w^A(t))\right|_E$$
$$+ \left|f^\lambda(t) - F(v^{\lambda,n}(t) + w^A(t))\right|_E + \left|f^{\lambda,n}(t) - f^\lambda(t)\right|_E ,$$

so that

$$\frac{d^-}{dt}|v^{\lambda,n}(t)|_E \le -a|v^{\lambda,n}(t)|_E^{2m+1} + C\left(1 + \sup_{s\in[0,t]}|w^A(s)|_E^{2m+1}\right)$$
$$+ \|f^\lambda - F(v^{\lambda,n} + w^A)\|_{C([0,t];E)} + \|f^{\lambda,n} - f^\lambda\|_{C([0,t];E)} .$$

Then, if we proceed as in [3] (see lemma 3.4), for any $t > 0$ by comparison we get

$$|v^{\lambda,n}(t)|_E \le c\left(1 + \sup_{s\in[0,t]}|w^A(s)|_E + \gamma^{\lambda,n}(t)\right)t^{-1/2m}, \quad \mathbb{P}-\text{a.s.}$$

where

$$\gamma^{\lambda,n}(t) := \left(\|f^\lambda - F(v^{\lambda,n} + w^A)\|_{C([0,t];E)} + \|f^{\lambda,n} - f^\lambda\|_{C([0,t];E)}\right)^{1/(2m+1)} .$$

Therefore, by taking the limit as $n \to +\infty$ it follows that

$$|v^\lambda(t)|_E \le c\left(1 + \sup_{s\in[0,t]}|w^A(s)|_E\right)t^{-1/2m}, \quad \mathbb{P}-\text{a.s.}$$

and since $v^\lambda(t;x) \to v(t;x)$ in E as $\lambda \to +\infty$ for any $t > 0$, the same inequality holds for $|v(t;x)|_E$. This implies that for any $t > 0$

$$\sup_{x\in E}|u(t;x)|_E \le \sup_{x\in E}|v(t;x)|_E + |w^A(t)|_E \le k(t)t^{-1/2m}, \quad \mathbb{P}-\text{a.s.}$$

with $k(t)$ defined by (3.12). □

Remark 3.5 It is important to remark hat, due to (2.7), for any $p \geq 1$

$$\mathbb{E}|k(t)|^p < +\infty, \quad t \geq 0. \tag{3.13}$$

3.1. The approximating problem

For any $\alpha > 0$ let us consider the approximating problem

$$du^\alpha(t) = (Au^\alpha(t) + F_\alpha(u^\alpha(t)))\,dt + B\,dw(t), \quad u^\alpha(0) = x\ , \tag{3.14}$$

where F_α is the approximating Nemytskii operator introduced in subsection 2.2. We recall that F_α is Lipschitz continuous in H and then the problem (3.14) admits a unique mild solution $u^\alpha(t;x) \in L^2(\Omega; C([0,+\infty[; H))$, for any $x \in H$. It is useful to note that if $x \in E$ then $u^\alpha(t;x)$ is an E-valued process, so that we can look at the problem (3.14) both as a problem in H and as a problem in E. Moreover, due to (2.22), we can apply the same proof of proposition 3.2 to $u^\alpha(t;x)$ and for each $\alpha > 0$ and $t \geq 0$ we have

$$|u^\alpha(t;x)|_E \leq e^{2\gamma t}|x|_E + h(t), \quad \mathbb{P}-\text{a.s.} \tag{3.15}$$

where the random process h is the same as (3.3) (with γ replaced by 2γ).

Proposition 3.6 *For any* $R, T > 0$ *we have*

$$\lim_{\alpha \to 0} \sup_{\substack{t \in [0,T] \\ |x|_E \leq R}} |u^\alpha(t;x) - u(t;x)|_E = 0, \quad \mathbb{P}-\text{a.s.} \tag{3.16}$$

Hence, according to (3.2) *and* (3.15), $u^\alpha(x)$ *converges to* $u(x)$ *in* $\mathscr{H}_T(E)$ *as* $\alpha \to 0$, *uniformly with respect to* $x \in B_E(0,R) = \{x \in E : |x|_E \leq R\}$.

Proof. For any $\alpha > 0$ and $x \in E$, we set $z^\alpha(t;x) = u^\alpha(t;x) - u(t;x)$. The process $z^\alpha(t;x)$ is the unique mild solution of the problem

$$\frac{d}{dt} z(t) = Az(t) + F_\alpha(u^\alpha(t;x)) - F(u(t;x)), \quad z(0) = 0\ . \tag{3.17}$$

We can assume that $z^\alpha(t;x)$ is a strict solution of the problem (3.17), otherwise we proceed as in the proof of proposition 3.2, approximating $z^\alpha(t;x)$ by means of a sequence of more regular processes. If $\delta^\alpha(t) = \delta_{z^\alpha(t;x)}$ is defined as in (2.13), by (2.12) we have

$$\begin{aligned}
\frac{d^-}{dt}\ |z^\alpha(t)|_E &\leq \langle Az^\alpha(t) + F_\alpha(u^\alpha(t;x)) - F(u(t;x)), \delta^\alpha(t)\rangle_E \\
&\leq \langle F_\alpha(u^\alpha(t;x)) - F(u^\alpha(t;x)), \delta^\alpha(t)\rangle_E + \langle F(u^\alpha(t;x)) - F(u(t;x)), \delta^\alpha(t)\rangle_E \\
&\leq |F_\alpha(u^\alpha(t;x)) - F(u^\alpha(t;x))|_E + \gamma|z^\alpha(t)|_E\ ,
\end{aligned}$$

so that by the Gronwall lemma

$$|z^\alpha(t;x)|_E \le \int_0^t e^{\gamma(t-s)}|F_\alpha(u^\alpha(s;x)) - F(u^\alpha(s;x))|_E\,ds, \quad \mathbb{P}-\text{a.s.}$$

Due to (3.2) and (2.23), from the first part of (26) it follows that

$$\lim_{\alpha\to 0} \sup_{\substack{t\in[0,T]\\ |x|_E\le R}} |F_\alpha(u^\alpha(t;x)) - F(u^\alpha(t;x))|_E = 0, \quad \mathbb{P}-\text{a.s.}$$

and this implies (3.16). □

4. Differential dependence on initial datum for the solution of problem (3.1)

For any $\alpha > 0$, $F_\alpha : E \to E$ is Lipschitz continuous and $k+1$ times Fréchet differentiable. Then, by a standard contraction argument (see e.g. [7]), we have that the solution $u^\alpha(t;x)$ of the approximating problem (3.14) is $k+1$ times mean-square differentiable in E with respect to $x \in E$. Moreover, for all $j = 1,\dots,k+1$ the derivatives $D^j_x u^\alpha(t;x)(h_1,\dots,h_j)$ are solutions of suitable PDE's having random coefficients. Our aim is to prove that also the solution of the problem (3.1) is mean-square differentiable with respect to $x \in E$ and its derivatives are the limit in $\mathscr{H}_T(E)$ of the derivatives of $u^\alpha(t;x)$.

The first derivative $D_x u^\alpha(t;x)h$ at a point $x \in E$ and along a direction $h \in E$ is the unique mild solution of the problem

$$\frac{d}{dt}v(t) = Av(t) + DF_\alpha(u^\alpha(t;x))v(t), \quad v(0) = h \ . \tag{4.1}$$

Then $D_x u^\alpha(x)h \in L^2(\Omega; C(]0,+\infty[;E) \cap L^\infty_{\text{loc}}([0,+\infty[;E))$ and, proceeding as in the proof of proposition 3.2, from (2.22) it follows that for any $\alpha > 0$

$$\sup_{x\in E}|D_x u^\alpha(t;x)h|_E \le e^{2\gamma t}|h|_E, \quad \mathbb{P}-\text{a.s.} \tag{4.2}$$

Now, for any $x, h \in H$, let us consider the derivative equation relative to the problem (3.1)

$$\frac{d}{dt}v(t) = Av(t) + DF(u(t;x))v(t), \quad v(0) = h \ . \tag{4.3}$$

We recall that DF is bounded on bounded subsets of E, then due to (2.11) for $j = 1$ and to (3.2), for any $y \in E$ we have

$$\sup_{s\in[0,t]}|DF(u(s;x))y|_E \le c(t,x)|y|_E, \quad \mathbb{P}-\text{a.s.} \tag{4.4}$$

where $c(t,x) > 0$ is a suitable random process having finite moments of any order and increasing with respect to t. Then (4.3) has a unique mild solution in $L^2(\Omega; C(]0,+\infty[;E) \cap L^\infty_{\text{loc}}([0,+\infty[;E))$ which we will denote by $v^1(t;x,h)$. Moreover, with the same arguments used before for $D_x u^\alpha(t;x)h$, we have

$$\sup_{x\in E}\left|v^1(t;x,h)\right|_E \leq e^{\gamma t}|h|_E, \quad \mathbb{P}-\text{a.s.} \tag{4.5}$$

Concerning the second derivative $D_x^2 u^\alpha(t;x)(h_1,h_2)$ of $u^\alpha(t;x)$, it is the unique mild solution of the problem

$$\begin{cases} \frac{d}{dt}v^\alpha(t) = Av^\alpha(t) + DF_\alpha(u^\alpha(t;x))v^\alpha(t) \\ \qquad\qquad + D^2F_\alpha(u^\alpha(t;x))(D_x u^\alpha(t;x)h_1, D_x u^\alpha(t;x)h_2) \\ v^\alpha(0) = 0\ . \end{cases} \tag{4.6}$$

Hence, due to (2.21), (2.22), (3.15) and (4.2), we have that $D_x^2 u^\alpha(x)(h_1,h_2)$ belongs to $L^2(\Omega; C([0,+\infty[;E))$ and for any $R,T>0$ and $p\geq 1$ we have

$$\mathbb{E}\sup_{\substack{t\in[0,T]\\ |x|_E\leq R}}\left|D_x^2 u^\alpha(t;x)(h_1,h_2)\right|_E^p \leq c|h_1|_E|h_2|_E\ , \tag{4.7}$$

where $c = c(R,T,p)$ is a constant independent of α.

Now, for $x,h_1,h_2\in E$, let us consider the second derivative equation

$$\begin{cases} \frac{d}{dt}v(t) = Av(t) + DF(u(t;x))v(t) + D^2F(u(t;x))\left(v^1(t;x,h_1), v^1(t;x,h_2)\right) \\ v(0) = 0\ . \end{cases} \tag{4.8}$$

From estimates (2.11) and (2.14) and from estimates (3.2) and (4.5), respectively on $|u(t;x)|_E$ and $|v^1(t;x,h)|_E$, it is easy to verify that this problem has a unique mild solution $v^2(t;x,h_1,h_2)$ in $L^2(\Omega; C([0,+\infty[;E))$. Proceeding as for the second derivative $D_x^2 u^\alpha(t;x)(h_1,h_2)$, it is possible to show that for $v^2(t;x,h_1,h_2)$ an estimate analogous to (4.7) holds.

In a similar way, for any $j\leq k$ and $x,h_1,\ldots,h_k\in E$, we will denote by $D_x^j u^\alpha(t;x)(h_1,\ldots,h_j)$ the j-th mean-square derivative of $u^\alpha(t;x)$. By recurrence it is possible to show that for any $R,T>0$ and $p\geq 1$

$$\mathbb{E}\sup_{\substack{t\in[0,T]\\ |x|_E\leq R}}\left|D_x^j u^\alpha(t;x)(h_1,\ldots,h_j)\right|_E^p \leq c\prod_{i=1}^j |h_i|_E, \quad \mathbb{P}-\text{a.s} \tag{4.9}$$

where $c = c(j,R,T,p)$. Clearly, we can repeat the same arguments for the solution of the j-th derivative equation, $j\leq k+1$, giving

$$\mathbb{E} \sup_{\substack{t\in[0,T] \\ |x|_E \le R}} \left| v^j(t;x)(h_1,\dots,h_j) \right|_E^p \le c \prod_{i=1}^{j} |h_i|_E, \quad \mathbb{P}-\text{a.s.} \tag{4.10}$$

where $c = c(j, R, T, p)$.

Proposition 4.1 *For any $j \le k+1$ we have*

$$\lim_{\alpha\to 0} \sup_{\substack{t\in[0,T] \\ |x|_E, |h_i|_E \le R}} \left| D_x^j u^\alpha(t;x)(h_1,\dots,h_j) - v^j(t;x,h_1,\dots,h_j) \right|_E = 0, \quad \mathbb{P}-\text{a.s.} \ , \tag{4.11}$$

for any $R, T > 0$. Then from (4.9) and (4.10), we get that $D_x^j u^\alpha(t;x)(h_1,\dots,h_j)$ converges to $v^j(t;x,h_1,\dots,h_j)$ in $\mathscr{H}_T(E)$ as $\alpha \to 0$, uniformly with respect to $x, h_i \in B_E(0,R)$, $i \le j$.

Proof. Let us fix $x, h \in E$. If we set $v^\alpha(t) = D_x u^\alpha(t;x)h - v^1(t;x,h)$, we have

$$v^\alpha(t) = \int_0^t e^{(t-s)A} \left(DF_\alpha(u^\alpha(s;x)) D_x u^\alpha(s;x)h - DF(u(s;x)) v^1(s;x,h) \right) ds \ ,$$

so that

$$\begin{aligned}
|v^\alpha(t)|_E &\le \int_0^t \left| DF_\alpha(u^\alpha(s;x)) D_x u^\alpha(s;x)h - DF(u(s;x)) v^1(s;x,h) \right|_E ds \\
&\le \int_0^t |(DF_\alpha(u^\alpha(s;x)) - DF(u^\alpha(s;x))) D_x u^\alpha(s;x)h|_E \, ds \\
&\quad + \int_0^t |(DF(u^\alpha(s;x)) - DF(u(s;x))) D_x u^\alpha(s;x)h|_E \, ds \\
&\quad + \int_0^t |DF(u(s;x)) v^\alpha(s)|_E \, ds \ .
\end{aligned}$$

From (4.4), for any $t \in [0,T]$ we get

$$|v^\alpha(t)|_E \le \gamma^\alpha(t)(x,h) + c(t,x) \int_0^t |v^\alpha(s)|_E \, ds, \quad \mathbb{P}-\text{a.s.}$$

where

$$\begin{aligned}
\gamma^\alpha(t)(x,h) &= \int_0^t |(DF_\alpha(u^\alpha(s;x)) - DF(u^\alpha(s;x))) D_x u^\alpha(s;x)h|_E \, ds \\
&\quad + \int_0^t |(DF(u^\alpha(s;x)) - DF(u(s;x))) D_x u^\alpha(s;x)h|_E \, ds \ .
\end{aligned}$$

Hence, for any t, we have

$$|v^\alpha(t)|_E \leq \int_0^t e^{c(t,x)(t-s)} \sup_{|x|_E, |h|_E \leq R} \gamma^\alpha(s)(x,h)\, ds$$
$$\leq te^{c(t,x)t} \sup_{\substack{s \in, [0,t] \\ |x|_E, |h|_E \leq R}} \gamma^\alpha(s)(x,h), \quad \mathbb{P}-\text{a.s.}$$

Now, according to (3.15) and to the second of (2.23), to (3.16) and to the continuity of DF, as (4.2) holds it is easy to show that

$$\lim_{\alpha \to 0} \sup_{\substack{t \in [0,T] \\ |x|_E, |h|_E \leq R}} \gamma^\alpha(t)(x,h) = 0, \quad \mathbb{P}-\text{a.s.}$$

so that (6.1) follows.

The proof for $j \geq 2$ is almost identical. □

An important consequence of previous convergence results is given by the following

Theorem 4.2 *For any* $T > 0$, *the solution* $u(t;x)$ *of equation* (3.1) *is* k *times differentiable in* $\mathscr{H}_T(E)$ *with respect to* x *and, for any* $h_1, \ldots, h_k \in E$ *and* $p \geq 1$,

$$\mathbb{E} \sup_{\substack{t \in [0,T] \\ x \in E}} \left| D_x^j u(t;x)(h_1, \ldots, h_j) \right|_E^p \leq c \prod_{i=1}^j |h_i|_E^p, \quad \mathbb{P}-\text{a.s.} \tag{4.12}$$

for suitable positive constants $c = c(p,j,T)$, $j = 1, \ldots, k$ and $p \geq 1$.

Proof. $u^\alpha(t;x)$ is mean-square differentiable, then for any $\alpha > 0$, we have

$$u^\alpha(x+h) - u^\alpha(x) = D_x u^\alpha(x)h + \int_0^1 \int_0^1 D_x^2 u^\alpha(x + \rho\theta h)(h,h)\, d\rho\, d\theta \ .$$

According to propositions 3.6 and 4.1, we can take the limit in $\mathscr{H}_T(E)$ as $\alpha \to 0$ and we get

$$u(x+h) - u(x) = v^1(x,h) + \int_0^1 \int_0^1 v^2(x + \rho\theta h, h, h)\, d\rho\, d\theta \ . \tag{4.13}$$

Due to (4.5), the mapping

$$v^1(x) : E \to \mathscr{H}_T(E), \quad h \mapsto v^1(x,h) \ ,$$

is clearly linear and continuous. Then, thanks to (4.10) for $j = 2$

$$\lim_{|h|_E \to 0} \frac{\left| \int_0^1 \int_0^1 v^2(x + \rho\theta h, h, h)\, d\rho\, d\theta \right|_E}{|h|_E} = 0, \quad in\ \mathscr{H}_T(E) \ ,$$

so that $u(t;x)$ is differentiable and its derivative coincides with $v^1(t;x,h)$. Estimate (4.12) for $j=1$ has just been proved. Second and higher order mean-square differentiability can be proved by using the same arguments used before for the first order derivative.

Now, let us prove (4.12) for $j=2$. We can assume that $D_x^2 u(t;x)(h_1,h_2)$ is a strict solution of the second derivative equation, otherwise we approximate it as in the proof of proposition 3.2. It holds that

$$\begin{aligned}
&\frac{d^-}{dt}\,|D_x^2 u(t;x)(h_1,h_2)|_E \\
&\quad \le \gamma|D_x^2 u(t;x)(h_1,h_2)|_E + \left|D^2F(u(t;x))(D_x u(t;x)h_1, D_x u(t;x)h_2)\right|_E \\
&\quad \le \gamma|D_x^2 u(t;x)(h_1,h_2)|_E + c\left(1+|u(t;x)|_E^{2m-1}\right)e^{2\gamma t}|h_1|_E|h_2|_E \ .
\end{aligned}$$

The last inequality follows from (2.11) and (4.5). Therefore, by using (3.11), for any $t>0$ we have

$$\begin{aligned}
\sup_{x\in E}\left|D_x^2 u(t;x)(h_1,h_2)\right|_E &\le c\int_0^t e^{\gamma(t+s)}\left(1+|u(s;x)|_E^{2m-1}\right)ds|h_1|_E|h_2|_E \\
&\le c\int_0^t e^{\gamma(t+s)}\left(1+k(s)^{2m-1}s^{-1+1/2m}\right)ds|h_1|_E|h_2|_E \\
&\le ce^{2\gamma t}\left(t+k(t)^{2m-1}t^{1/2m}\right)|h_1|_E|h_2|_E \ .
\end{aligned}$$

The result now follows from (3.13).

In order to prove (4.12) for general $j\le k$, we proceed by recurrence, by using estimates (4.12) for $i<j$. We have that

$$\begin{aligned}
&\frac{d^-}{dt}\,|D_x^j u(t;x)(h_1,\dots,h_j)|_E \\
&\quad \le \gamma|D_x^j u(t;x)(h_1,\dots,h_j)|_E + c(t)\left(1+|u(t;x)|_E^{2m+1-j}\right)\prod_{i=1}^{j}|h_i|_E \ ,
\end{aligned}$$

for a suitable continuous function $c(t)>0$ increasing with respect to t. Then, by using (3.11) and (3.13), the inequality (4.12) may be concluded for any $j\le k$. □

5. Further properties of the derivatives of $u(t;x)$

Our aim is now to prove that the derivatives of $u(t;x)$ belong to the domain of B^{-1} and satisfy some estimates, uniform with respect to $x\in E$.

Proposition 5.1 *We have that* $D_x^j u(t;x)(h_1,\dots,h_j) \in D(B^{-1})$, *for* $t>0$ *and* $j \leq k$ *and there exists a continuous function* $c(t)>0$ *increasing with respect to* t *such that*

$$\mathbb{E}\sup_{x\in E}\int_0^t |B^{-1}D_x^j u(s;x)(h_1,\dots,h_j)|_H^2\,ds \leq c(t)t^{1-\epsilon}\prod_{i=1}^{j}|h_i|_E^2\ . \tag{5.1}$$

Proof. As seen in previous section, the first derivative $D_x u(t;x)h$ along the direction $h\in E$ is the unique mild solution of the problem

$$\frac{d}{dt}v(t) = Av(t) + DF(u(t;x))v(t), \quad v(0)=h\ .$$

Such an equation can be regarded as an equation in H and due to (2.2) it is easy to prove that for any $\delta<1$ the mild solution $v(t)\in D((-A)^{\delta})$, for any $t>0$. Besides, possibly by approximating $v(t)$ by means of more regular solutions (see the proof of proposition 3.2), we have

$$\frac{1}{2}\frac{d}{dt}|v(t)|_H^2 + \langle(-A)v(t),v(t)\rangle_H = \langle DF(u(t;x))v(t),v(t)\rangle_H\ .$$

Hence, by integrating with respect to t and by using (2.15), we get

$$|v(t)|_H^2 + 2\int_0^t |(-A)^{1/2}v(s)|_H^2\,ds \leq |h|_H^2 + 2\gamma\int_0^t |v(s)|_H^2\,ds\ .$$

By (4.5), this yields

$$\int_0^t |(-A)^{1/2}v(s)|_H^2\,ds \leq c\big(1+\gamma t e^{2\gamma t}\big)|h|_H^2, \quad \mathbb{P}-\text{a.s}\ . \tag{5.2}$$

Now, by the interpolatory inequality given in (2.3) (with $\delta_1=0$, $\delta=\epsilon/2$ and $\delta_2=1/2$) and by the Hölder inequality, we have

$$\begin{aligned}\int_0^t |(-A)^{\epsilon/2}v(s)|_H^2\,ds &\leq \int_0^t |(-A)^{1/2}v(s)|_H^{2\epsilon}|v(s)|_H^{2(1-\epsilon)}\,ds\\ &\leq \left(\int_0^t |(-A)^{1/2}v(s)|_H^2\,ds\right)^{\epsilon}\left(\int_0^t |v(s)|_H^2\,ds\right)^{1-\epsilon},\end{aligned}$$

so that, thanks to (4.5) and (5.2),

$$\int_0^t |(-A)^{\epsilon/2}D_x u(s;x)h|_H^2\,ds \leq c\big(1+\gamma t e^{2\gamma t}\big)^{\epsilon}e^{2(1-\epsilon)\gamma t}t^{1-\epsilon}|h|_H^2, \quad \mathbb{P}-\text{a.s.}$$

Finally, recalling that by hypothesis 2.2 we have $B^{-1}=\Gamma(-A)^{\epsilon/2}$ for a suitable bounded linear operator Γ, it follows that $D_x u(t;x)h\in D(B^{-1})$ for any $t>0$ and that

$$\mathbb{E}\sup_{x\in E}\int_0^t |B^{-1}D_x u(s;x)h|_H^2\,ds$$
$$\leq \|\Gamma\|^2\mathbb{E}\sup_{x\in E}\int_0^t |(-A)^{\epsilon/2}D_x u(s;x)h|_H^2\,ds \leq c(t)t^{1-\epsilon}|h|_H^2 \ , \qquad (5.3)$$

for a continuous function $c(t) > 0$ increasing with respect to t.

As proved in section 4, higher order derivatives $D_x^j u(t;x)(h_1,\dots,h_j)$, $j\geq 2$, are the unique mild solutions of the problems

$$\frac{d}{dt}v^j(t) = Av^j(t) + DF(u(t;x))v^j(t) + \theta^j(t), \quad v^j(0) = 0 \ ,$$

for suitable processes $\theta^j(t)$ depending on all the lower order derivatives. Then, as before for the first derivative, we have that $v^j(t)\in D((-A)^{1/2})$ for $t > 0$ and

$$|v^j(t)|_H^2 + 2\int_0^t |(-A)^{1/2}v^j(s)|_H^2\,ds$$
$$\leq 2\gamma\int_0^t |v^j(s)|_H^2\,ds + 2\int_0^t \left|\langle\theta^j(s),v^j(s)\rangle\right|_H ds \ .$$

By using (4.12) and Hölder inequality, this implies that

$$\mathbb{E}\sup_{x\in E}\int_0^t |(-A)^{1/2}v^j(s)|_H^2\,ds \leq c^j(t)\prod_{i=1}^j |h_i|_E^2 \ , \qquad (5.4)$$

for suitable continuous functions $c^j(t) > 0$, increasing with respect to t. For example, let us prove (5.4) for $j = 2$. We have $\theta^2(t) = D^2F(u(t;x))(D_x u(t;x)h_1, D_x u(t;x)h_2)$ and then, by (3.11)

$$\int_0^t |(-A)^{1/2}v^2(s)|_H^2\,ds$$
$$\leq 2\gamma\int_0^t |v^2(s)|_H^2\,ds + c\int_0^t e^{2\gamma s}\left(1+|u(s;x)|_E^{2m-1}\right)|v^2(s)|_H\,ds|h_1|_E|h_2|_E$$
$$\leq 2\gamma t\sup_{s\in[0,t]}|v^2(s)|_H^2 + ce^{2\gamma t}\left(t + k(t)^{2m-1}t^{1/2m}\right)\sup_{s\in[0,t]}|v^2(s)|_H|h_1|_E|h_2|_E \ .$$

Therefore, by using (3.13) and (4.12) for $j = 2$, (5.4) easily follows. By the same interpolatory arguments used above for the first derivative (5.1) follows for $j = 2$. □

Remark 5.2 The same arguments used before can be used in order to prove that for any $\alpha > 0$ and $x, h_1,\dots,h_k \in H$ we have that $D_x^j u^\alpha(t;x)(h_1,\dots,h_j)\in D(B^{-1})$, $t > 0$. Furthermore for $R > 0$ there exists a continuous function $c^R(t) > 0$ increasing in t such that

$$\mathbb{E} \sup_{|x|_E \le R} \int_0^t |B^{-1} D_x^j u^\alpha(s;x)(h_1,\dots,h_j)|_H^2 \, ds \le c^R(t) t^{1-\epsilon} \prod_{i=1}^j |h_i|_H^2 \ . \quad (5.5)$$

Proposition 5.3 *For any $x, h \in E$,*

$$\lim_{\alpha \to 0} \mathbb{E} \int_0^t \left| B^{-1}(D_x u^\alpha(s;x)h - D_x u(s;x)h) \right|_H^2 ds = 0, \quad t \ge 0 \ . \qquad (5.6)$$

Proof. If we set $z^\alpha(t) = D_x u^\alpha(s;x)h - D_x u(s;x)h$, we have

$$B^{-1} z^\alpha(t) = \int_0^t \Gamma(t-s)(DF_\alpha(u^\alpha(s;x)) D_x u^\alpha(s;x)h - DF(u(s,x)) D_x u(s;x)h) \, ds \ ,$$

where $\Gamma(t) = B^{-1} e^{tA}$. Then, according to (2.9), it easily follows

$$\left| B^{-1} z^\alpha(t) \right|_H^2 \le c t^{1-\epsilon} \int_0^t |DF_\alpha(u^\alpha(s;x)) D_x u^\alpha(s;x)h - DF(u(s;x)) D_x u(s;x)h|_H^2 \, ds \ .$$

By using proposition 4.1 and (3.2), (4.2) and (4.5) it is possible to show that

$$\lim_{\alpha \to 0} \int_0^t |DF_\alpha(u^\alpha(s;x)) D_x u^\alpha(s;x)h - DF(u(s;x)) D_x u(s;x)h|_H^2 \, ds = 0 \ ,$$

$\mathbb{P}$-a.s. and then by dominated convergence (5.6) follows. □

The last preliminary result that we will use in the sequel is the following

Proposition 5.4 *For any $j < k$ and $x, h_1, \dots, h_k \in E$, we have that the process $B^{-1} D_x^j u(t;x)(h_1,\dots,h_j)$ is mean-square differentiable in H along any direction $h_{j+1} \in E$ and that*

$$D_x \big(B^{-1} D_x^j u(t;x)(h_1,\dots,h_j) \big) h_{j+1} = B^{-1} D_x^{j+1} u(t;x)(h_1,\dots,h_{j+1}) \ . \qquad (5.7)$$

Proof. We prove our statement for $j = 1$. For $j > 1$ the proof is quite similar.

For any $\epsilon > 0$ and $x, h, k \in E$ we have

$$\begin{aligned} &B^{-1} D_x u(t; x+\epsilon k)h - B^{-1} D_x u(t;x)h \\ &= \int_0^t \Gamma(t-s)(DF(u(s;x+\epsilon k)) D_x u(s;x+\epsilon k)h - DF(u(s,x)) D_x u(s;x)h) \, ds \ . \end{aligned}$$

Now, $DF(u(s;x))D_xu(s;x)h$ is differentiable in E along any direction $k \in E$ and its derivative is given by

$$D^2F(u(t;x))(D_xu(t;x)h, D_xu(t;x)k) - DF(u(t;x))D_x^2u(t;x)(h,k) \ .$$

Besides, since $\Gamma(t)$ is continuous in H and (2.9) holds, we can conclude that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(B^{-1}D_xu(t;x+\epsilon k)h - B^{-1}D_xu(t;x)h\right) \\
&\quad = \int_0^t \Gamma(t-s)D^2F(u(s;x))(D_xu(s;x)h, D_xu(s;x)k)\,ds \\
&\qquad - \int_0^t \Gamma(t-s)DF(u(s;x))D_x^2u(s;x)(h,k)\,ds = B^{-1}D_x^2u(t;x)(h,k)
\end{aligned}$$

and the limit is in $\mathscr{K}_T(H)$.

Remark 5.5 The same statement of proposition 5.3 can be proved for the derivatives of $u^\alpha(t;x)$. Moreover, since $u^\alpha(t;x)$ is twice mean-square differentiable in H (for a proof see [4]), for any $x, h \in H$ the process $B^{-1}D_xu^\alpha(t;x)h$ is mean-square differentiable in H and for any $k \in H$

$$D_x\left(B^{-1}D_xu^\alpha(t;x)h\right)k = B^{-1}D_x^2u^\alpha(t;x)(h,k) \ .$$

6. Smoothing properties of the transition semigroup

For any $\varphi \in B_b(E)$ we set

$$P_t\varphi(x) = \mathbb{E}(\varphi(u(t;x))), \quad x \in E, \quad t \geq 0 \ .$$

P_t is the Markov transition semigroup associated to the problem (3.1). In [9] the *strong Feller* property has been proved for the restriction of P_t to $B_b(H)$, that is it has been proved that

$$\varphi \in B_b(H) \Rightarrow P_t\varphi \in C_b(H), \quad t > 0 \ .$$

In particular, as $C_b(E)$ is continuously embedded in $B_b(H)$ (see remark 2.8) and $C_b(H)$ is continuously embedded in $C_b(E)$, we have that P_t, $t \geq 0$, maps $C_b(E)$ into itself as a contraction. In particular P_t is a *Feller* semigroup on E.

For any $x \in H$, the approximating problem (3.14) has a unique mild solution $u^\alpha(t;x)$ and if $x \in E$ the solution is an E-valued process. Then the associated transition semigroup P_t^α, $t \geq 0$, can be studied in $B_b(E)$. In [4] regularity properties of P_t^α have been studied for its restriction to $B_b(H)$. It has been proved that $C_b(H)$ is an invariant subspace for P_t^α and P_t^α has a smoothing effect. That is

$$\varphi \in B_b(H) \Rightarrow P_t^\alpha \varphi \in C_b^j(H), \quad t > 0 \ ,$$

for some j depending on the ultracontractivity property of the operator A (see (2.4)). Actually, even if $f \in C^\infty(\mathbb{R})$, we have that $j < 4/d + 1$. Our aim is now to study the smoothing properties of P_t in $B_b(E)$. To this purpose we first remark that for any $\varphi \in C_b(E)$ and $R, T > 0$, we have

$$\lim_{\alpha \to 0} \sup_{\substack{t \in [0,T] \\ |x|_E \leq R}} P_t^\alpha \varphi(x) = P_t \varphi(x) \ . \tag{6.1}$$

Indeed

$$P_t^\alpha \varphi(x) - P_t \varphi(x) = \mathbb{E}(\varphi(u^\alpha(t;x)) - \varphi(u(t;x))) \ ,$$

and then, as (3.16) holds and $\varphi \in C_b(E)$, we get (6.1). Therefore, we will first prove differentiability for P_t^α and then we will proceed by approximation.

Proposition 6.1 *Under hypotheses* 2.1, 2.2, 2.4 *and* 2.5, *for any* $\varphi \in B_b(H)$ *and* $t > 0$ *we have that* $P_t^\alpha \varphi$ *is at least twice differentiable. Moreover, if* $\varphi \in C_b(H)$ *it holds*

$$\left\langle D(P_t^\alpha \varphi)(x), h \right\rangle_H = \frac{1}{t} \mathbb{E}\left(\varphi(u^\alpha(t;x)) \int_0^t \left\langle B^{-1} D_x u^\alpha(s;x) h, \, dw(s) \right\rangle_H \right) \ . \tag{6.2}$$

Proof. For any $\alpha > 0$, the solution $u^\alpha(t;x)$ of equation (3.14) is j-times differentiable with respect to $x \in H$, for any $j < 4/d + 1$. Therefore, if $d \leq 3$ we have that $u^\alpha(t;x)$ is at least twice mean-square differentiable in H, with respect to x and its derivatives are the solutions of equations (4.1) and (4.6), regarded as equations in H. Hence, for any $\varphi \in C_b^2(H)$ the function $(t,x) \mapsto P_t^\alpha \varphi(x)$ belongs to $C^{1,2}([0,+\infty) \times H)$ (for a proof see [7] e.g.) and

$$\varphi(u^\alpha(t;x)) = P_t^\alpha \varphi(x) + \int_0^t \left\langle D_x(P_{t-s}^\alpha \varphi)(u^\alpha(s;x)), B \, dw(s) \right\rangle_H \ . \tag{6.3}$$

Now, since (5.5) holds, we can multiply the both sides of (6.3) by the term

$$\int_0^t \left\langle B^{-1} D_x u^\alpha(s;x) h, \, dw(s) \right\rangle_H$$

and by taking expectation we get

$$\mathbb{E}\left(\varphi(u^\alpha(t;x))\int_0^t \left\langle B^{-1}D_x u^\alpha(s;x)h,\, dw(s)\right\rangle_H\right)$$
$$= \mathbb{E}\int_0^t \left\langle BD_x(P^\alpha_{t-s}\varphi)(u^\alpha(s;x)), B^{-1}D_x u^\alpha(s;x)h\right\rangle_H ds$$
$$= \left\langle D_x \int_0^t \mathbb{E}(P^\alpha_{t-s}\varphi)(u^\alpha(s;x))\, ds, h\right\rangle_H \ .$$

From the Markov property we have

$$\mathbb{E}(P^\alpha_{t-s}\varphi)(u^\alpha(s;x)) = \mathbb{E}\varphi(u^\alpha(t;x)) \ ,$$

so that (6.2) follows. The previous formula, which is proved for regular functions, can be extended to functions $\varphi \in C_b(H)$. Indeed, as in [17], for any $\varphi \in C_b(H)$ we define

$$\varphi_n(x) = \int_{\mathbb{R}^n} \rho_n(\xi - \Pi_n x)\varphi(T_n\xi)d\xi, \quad n \in \mathbb{N}\ x \in H \ ,$$

where $\{\rho_n\}$ is a sequence of non negative smooth functions such that

$$\operatorname{supp}(\rho_n) \subset \{\xi \in \mathbb{R}^n \ : \ |\xi| \le 1/n\}, \quad \int_{\mathbb{R}^n} \rho_n(\xi)d\xi = 1$$

and $\Pi_n : H \to \mathbb{R}^n$, $x \mapsto \left(\langle x, e_1\rangle_H, \dots, \langle x, e_n\rangle_H\right)$ and $T_n : \mathbb{R}^n \to H$, $\xi \mapsto \sum_{i=1}^n \xi_i e_i$. It is possible to show that φ_n is smooth and

$$\lim_{n\to+\infty} \varphi_n(x) = \varphi(x), \quad x \in H, \quad \|\varphi_n\|_0^H \le \|\varphi\|_0^H \ . \tag{6.4}$$

Therefore, since we have formula (6.2) for φ_n, thanks to (6.4) we can take the limit as $n \to +\infty$ and we get formula (6.2) for φ.

Moreover, proceeding as in [9] and [17], for any $x, y \in H$ formula (6.2) yields

$$\operatorname{Var}\left(P^\alpha_t(x,\cdot) - P^\alpha_t(y,\cdot)\right) \le c_t|x-y|_H \ ,$$

where $P^\alpha_t(x,\cdot)$ is the law of $u^\alpha(t;x)$ so that for any $\varphi \in B_b(H)$ and $t > 0$, $P^\alpha_t\varphi$ is Lipschitz continuous in H. Thus, since $P^\alpha_t\varphi = P^\alpha_{t/2}(P^\alpha_{t/2}\varphi)$, this implies that $P^\alpha_t\varphi$ is differentiable for any $\varphi \in B_b(H)$. Now, if we set $\psi^\alpha = P^\alpha_{t/2}\varphi$, by remark 5.5 and the semigroup law, we can differentiate each side of (6.2) with respect to x in H to obtain

$$\left\langle D^2(P^\alpha_t\varphi)(x)k, h\right\rangle_H$$
$$= \frac{2}{t}\mathbb{E}\left(\langle D\psi^\alpha(u^\alpha(t/2;x)), D_x u^\alpha(t;x)k\rangle_H \int_0^{t/2} \left\langle B^{-1}D_x u^\alpha(s;x)h,\, dw(s)\right\rangle_H\right)$$
$$+ \frac{2}{t}\mathbb{E}\left(\psi^\alpha(u^\alpha(t/2;x))\int_0^{t/2} \left\langle B^{-1}D^2_x u^\alpha(s;x)(h,k),\, dw(s)\right\rangle_H\right) \ . \tag{6.5}$$

□

Theorem 6.2 *Under hypotheses* 2.1, 2.2, 2.4 *and* 2.5, $P_t\varphi \in C_b^k(E)$ *for any* $\varphi \in B_b(E)$ *and* $t > 0$ *and*

$$\|P_t\varphi\|_h^E \leq c(t \wedge 1)^{-\frac{h(1+\epsilon)}{2}}\|\varphi\|_0^E, \quad h = 1, \ldots, k \ . \tag{6.6}$$

In particular if $\varphi \in C_b(E)$ *for any* $x, h \in E$ *we have*

$$\langle D(P_t\varphi)(x), h\rangle_E = \frac{1}{t}\,\mathbb{E}\left(\varphi(u(t;x))\int_0^t \left\langle B^{-1}D_x u(s;x)h,\, dw(s)\right\rangle_H\right) \ . \tag{6.7}$$

Proof. Let $\varphi \in C_b(H)$. For any $\alpha > 0$ and $x, h \in E$ we have

$$\begin{aligned} &P_t^\alpha\varphi(x+h) - P_t^\alpha\varphi(x) \\ &\quad = \left\langle D(P_t^\alpha\varphi)(x), h\right\rangle_H + \int_0^1 \left\langle D(P_t^\alpha\varphi)(x+\theta h) - D(P_t^\alpha\varphi)(x), h\right\rangle_H d\theta \ , \end{aligned} \tag{6.8}$$

where $D(P_t^\alpha\varphi)$ is defined by (6.2). By (3.15), (3.16) and (5.6) it is not difficult to prove that

$$\lim_{\alpha\to 0}\left\langle D(P_t^\alpha\varphi)(x), h\right\rangle_H = \frac{1}{t}\,\mathbb{E}\left(\varphi(u(t;x))\int_0^t \left\langle B^{-1}D_x u(s;x)h,\, dw(s)\right\rangle_H\right) \ , \tag{6.9}$$

and the limit is uniform with respect to x, h in $B_E(0,R)$, for any $R > 0$. Moreover, from (5.5) we have

$$\begin{aligned} \left|\left\langle D(P_t^\alpha\varphi)(x), h\right\rangle_H\right| &\leq \frac{1}{t}\,\|\varphi\|_0^H \left(\mathbb{E}\int_0^t \left|B^{-1}D_x u^\alpha(s;x)h\right|_H^2 ds\right)^{1/2} \\ &\leq c^{|x|_E}(t)\|\varphi\|_0^H t^{-\frac{1+\epsilon}{2}}|h|_E \ . \end{aligned} \tag{6.10}$$

Then, by taking the limit in (6.8) as $\alpha \to 0$, we get

$$\begin{aligned} P_t\varphi(x+h) - P_t\varphi(x) = \frac{1}{t}\,\mathbb{E}&\left(\varphi(u(t;x))\int_0^t \left\langle B^{-1}D_x u(s;x)h,\, dw(s)\right\rangle_H\right) \\ &+ R_t(x,h) \ , \end{aligned}$$

for a suitable remainder $R_t(x,h)$. Under our hypotheses, $P_t^\alpha\varphi$ is at least twice differentiable, and

$$\begin{aligned} &\int_0^1 \left\langle D(P_t^\alpha\varphi)(x+\theta h) - D(P_t^\alpha\varphi)(x), h\right\rangle_H ds \\ &\quad = \int_0^1 \theta\int_0^1 \left\langle D^2(P_t^\alpha\varphi)(x+\rho\theta h)h, h\right\rangle_H d\rho\, d\theta \ . \end{aligned}$$

Now, since $\left\langle D^2(P_t^\alpha\varphi)(x)h,k\right\rangle_H$ is given by (6.5), by using (4.2) we have that

$$\left|\left\langle D^2(P_t^\alpha\varphi)(x)h,k\right\rangle_H\right| \leq \frac{2}{t}\|D\psi^\alpha\|_0^H e^{\gamma t}|k|_E\left(\mathbb{E}\int_0^t |B^{-1}D_x u^\alpha(s;x)h|_H^2\,ds\right)^{1/2}$$
$$+\frac{2}{t}\|\psi^\alpha\|_0^H\left(\mathbb{E}\int_0^t |B^{-1}D_x^2 u^\alpha(s;x)(h,k)|_H^2\,ds\right)^{1/2}$$

and then, recalling (5.5) and (6.10), we get

$$\left|\left\langle D^2(P_t^\alpha\varphi)(x)h,k\right\rangle_H\right| \leq c^{|x|_E}(t)(t\wedge 1)^{-(1+\epsilon)}\|\varphi\|_0^H|h|_E|k|_E\ . \tag{6.11}$$

Since

$$R_t(x,h) = \lim_{\alpha\to 0}\int_0^1 \theta\int_0^1 \left\langle D^2(P_t^\alpha\varphi)(x+\rho\theta h)h,h\right\rangle_H d\rho\, d\theta\ ,$$

this implies that

$$\frac{|R_t(x,h)|}{|h|_E} \leq c^{|x|_E}(t)(t\wedge 1)^{-(1+\epsilon)}\|\varphi\|_0^H|h|_E \to 0 \tag{6.12}$$

as $|h|_E\to 0$. Hence we have proved that for any $t>0$ and $\varphi\in C_b(H)$, $P_t\varphi$ is differentiable in E and

$$\langle D(P_t\varphi)(x),h\rangle_E = \frac{1}{t}\mathbb{E}\left(\varphi(u(t;x))\int_0^t \left\langle B^{-1}D_x u(s;x)h,\, dw(s)\right\rangle_H\right)\ .$$

Moreover, proceeding as for (6.10), due to (5.1) we get estimate (6.6) uniform with respect to $x\in E$.

Now, let $\varphi\in C_b(E)$ and let $(\varphi_n)\subset C_b(H)$ be the approximating sequence introduced in proposition 2.7. For any $n\in\mathbb{N}$ and $t>0$, we have that $P_t\varphi_n$ is differentiable in E. Besides, since $u(t;x)\in E$, for any $t\ \geq\ 0$ and $x\in E$, for any $x,h\in E$ we have

$$\begin{cases} P_t\varphi_n(x)\to P_t\varphi(x) \\ \langle D(P_t\varphi_n)(x),h\rangle_E \to \frac{1}{t}\mathbb{E}\left(\varphi(u(t;x))\int_0^t\left\langle B^{-1}D_x u(s;x)h,\, dw(s)\right\rangle_H\right)\ , \end{cases}$$

as $n\to+\infty$. Then, proceeding as before, we get that $P_t\varphi$ is differentiable in E and (6.6) and (6.7) hold. Finally, in order to get that $P_t\varphi$ is differentiable for any $\varphi\in B_b(E)$ and $t>0$ we proceed as in the proof of proposition 6.1, by considering the variation of $P_t(x,\cdot)$, for each $t>0$ and $x\in E$.

Higher order differentiability follows by recurrence from formula (6.7), by using theorem 4.2 and proposition 5.4 and the semigroup law.

In order to prove estimate (6.6) for a certain $h\leq k$, we use estimates (4.12) and (5.1) and estimate (6.6) proved for $j<h$. □

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