

Fast flow asymptotics for stochastic incompressible viscous fluids in \mathbb{R}^2 and SPDEs on graphs

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Abstract Fast advection asymptotics for a stochastic reaction–diffusion–advection equation are studied in this paper. To describe the asymptotics, we introduce a new class of SPDEs defined on a graph, corresponding to the stream function of the underlying incompressible flow. We prove that, as the advection term becomes faster and faster, the solution of the stochastic reaction–diffusion–advection equation on the plane converges in a suitable sense to the solution of such an SPDE defined on the graph. This result is a consequence of averaging, due to a probabilistic representation of solutions, both for the SPDEs on the plane and for the SPDE on the graph.

Keywords Averaging principle · Markov processes on graphs · Stochastic partial differential equations

Mathematics Subject Classification 60H15 · 70K65 · 60J25

1 Introduction

Consider an incompressible flow in \mathbb{R}^2 , with stream function $-H(x)$, $x \in \mathbb{R}^2$, and let some particles move together with the flow. If we denote by $u(t, x)$ the density of the particles at time $t \geq 0$ and position $x \in \mathbb{R}^2$, then the function $u(t, x)$ satisfies the Liouville equation

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$$\begin{cases} \partial_t u(t, x) = \langle \nabla^\perp H(x), \nabla u(t, x) \rangle, & t > 0, \quad x \in \mathbb{R}^2, \\ u(0, x) = \varphi(x), & x \in \mathbb{R}^2. \end{cases} \tag{1.1}$$

Suppose now that the flow has a small viscosity and the particles take part in a slow chemical reaction, with a deterministic and a stochastic component, as described by the equation

$$\begin{cases} \partial_t \tilde{u}_\epsilon(t, x) = \frac{\epsilon}{2} \Delta \tilde{u}_\epsilon(t, x) + \langle \nabla^\perp H(x), \nabla \tilde{u}_\epsilon(t, x) \rangle + \epsilon b(\tilde{u}_\epsilon(t, x)) \\ \quad + \sqrt{\epsilon} g(\tilde{u}_\epsilon(t, x)) \partial_t \tilde{\mathcal{W}}(t, x), \\ \tilde{u}_\epsilon(0, x) = \varphi(x), \quad x \in \mathbb{R}^2. \end{cases} \tag{1.2}$$

Here, $b, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ are Lipschitz continuous and the stream function $-H : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a generic function, having four continuous derivatives, with bounded second derivative, such that $H(x) \rightarrow +\infty$, as $|x| \uparrow +\infty$. Without any loss of generality, we can assume $\min_{x \in \mathbb{R}^2} H(x) = 0$. The noise $\tilde{\mathcal{W}}(t, x)$ is a spatially homogeneous Wiener process having finite spectral measure. Finally, ϵ is a small positive parameter, included in Eq. (1.2) in such a way that all perturbation terms have strength of the same order, as ϵ goes to zero (see Sects. 2 and 5 and also [7] for all assumptions and details).

It is not difficult to check that under the above conditions, if we take the limit as $\epsilon \downarrow 0$, the solution $\tilde{u}_\epsilon(t, x)$ of Eq. (1.2) converges, on any finite time interval, to the solution $u(t, x)$ of Eq. (1.1), in probability, uniformly with respect to x in a bounded domain of \mathbb{R}^2 . But on large time intervals, growing together with ϵ^{-1} , the difference $\tilde{u}_\epsilon(t, x) - u(t, x)$ can have order 1, as $\epsilon \downarrow 0$.

As a matter of fact, in the present paper, we are interested in the limiting behavior of the solution $\tilde{u}_\epsilon(t, x)$ of Eq. (1.2), as $\epsilon \downarrow 0$, on a time interval $[0, T/\epsilon]$. In particular, we will see that in order to describe the limit of $\tilde{u}_\epsilon(t, x)$, one should consider SPDEs on a non standard setting, where the space variable changes on the graph Γ obtained by identifying all points in each connected component of the level sets of the Hamiltonian H . More precisely, for any $z \geq 0$, we denote by $C(z)$ the level set $\{H(x) = z\}$. The set $C(z)$ may consist of several connected components $C_1(z), \dots, C_{N(z)}(z)$. Thus, if we identify all points in \mathbb{R}^2 belonging to the same connected component of a given level set $C(z)$, we obtain the graph Γ , given by intervals I_1, \dots, I_n and vertices O_1, \dots, O_m . This means that any point (z, k) in Γ denotes the k -th connected component of the level set of H corresponding to the value z . In this way, we can define a projection $\Pi : \mathbb{R}^2 \rightarrow \Gamma$ defined by $\Pi(x) = (H(x), k(x))$, where $k(x)$ is the number of the connected component of $C(H(x))$ containing x (see Fig. 1 and Sect. 2 for all definitions and details).

In the present paper, our final goal is proving that for every $p \geq 1$ and $0 < \tau < T$ it holds

$$\lim_{\epsilon \rightarrow 0} \mathbf{E} \sup_{t \in [\tau, T]} |u_\epsilon(t) - \bar{u}(t) \circ \Pi|_H^p = 0,$$

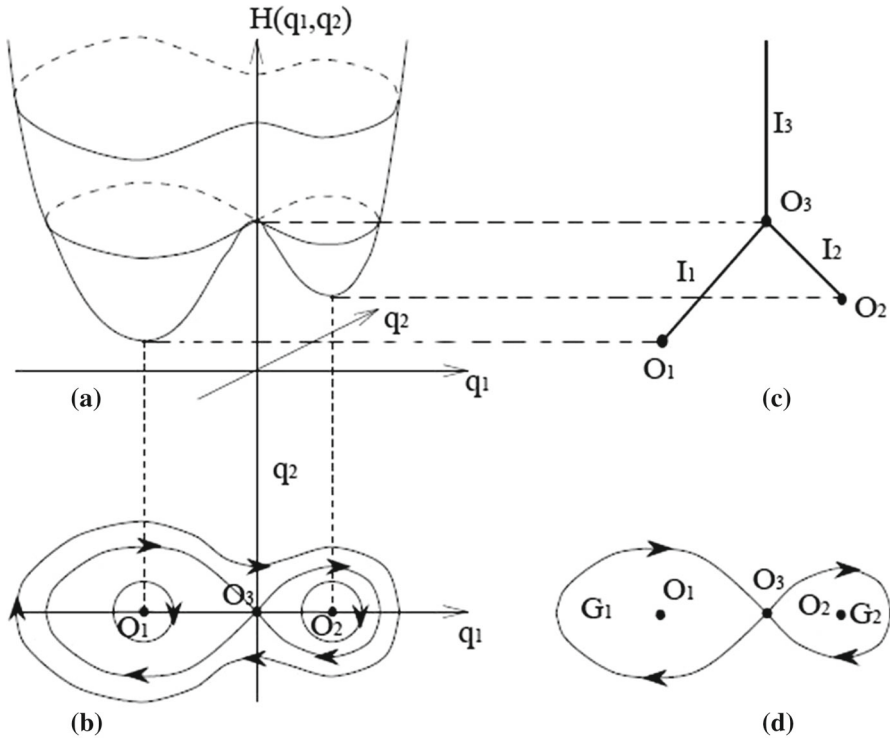


Fig. 1 The Hamiltonian, the level sets, the projection and the graph

where \bar{u} is the solution of an SPDE defined on the graph Γ , and H is a suitable space of weighted L^2 functions on \mathbb{R}^2 .

Before proceeding with the description of the content of this paper, it is worth mentioning that a suitable class of SPDEs on a graph has been already studied in our previous paper [1], where SPDEs defined on a net of narrow channels were studied. In that case, we have tried to understand what happens of the solution of the SPDE defined on a 2-dimensional channel G with many wings and subject to instantaneous reflections at the boundary, when the width of the channel goes to zero. We have proved that the solution converges to the solution of a suitable SPDE, defined on a suitable graph that can be associated with the channel, in $L^p(\Omega; C([\tau, T]; L^2(G)))$, for any $0 < \tau < T$ and $p \geq 1$.

To describe the behavior of the solution \tilde{u}_ϵ of Eq. (1.2) in the time interval $[0, T/\epsilon]$, it is useful to define

$$u_\epsilon(t, x) =: \tilde{u}_\epsilon(t/\epsilon, x), \quad t \geq 0, \quad x \in \mathbb{R}^2.$$

With this change of time, the new function u_ϵ solves the equation

$$\begin{cases} \partial_t u_\epsilon(t, x) = \frac{1}{2} \Delta u_\epsilon(t, x) + \frac{1}{\epsilon} \langle \nabla^\perp H(x), \nabla u_\epsilon(t, x) \rangle \\ \quad + b(u_\epsilon(t, x)) + g(u_\epsilon(t, x)) \partial_t \mathcal{W}(t, x), \\ u_\epsilon(0, x) = \varphi(x), \quad x \in \mathbb{R}^2, \end{cases} \tag{1.3}$$

for some spatially homogeneous Wiener process $\mathcal{W}(t, x)$. Equation (1.3) is a reaction–diffusion–advection equation in \mathbb{R}^2 , where the reaction term has a deterministic and a stochastic component, and the advection term is of order ϵ^{-1} , compared to the diffusion and the reaction part. For every fixed $\epsilon > 0$, the second order differential operator \mathcal{L}_ϵ , defined by

$$\mathcal{L}_\epsilon \varphi(x) = \frac{1}{2} \Delta \varphi(x) + \frac{1}{\epsilon} \langle \nabla^\perp H(x), \nabla \varphi(x) \rangle, \quad x \in \mathbb{R}^2,$$

is the generator of the diffusion process defined by the stochastic equation

$$dX_\epsilon(t) = \frac{1}{\epsilon} \nabla^\perp H(X_\epsilon(t)) dt + dw(t), \quad X_\epsilon(0) = x \in \mathbb{R}^2, \tag{1.4}$$

where $w(t)$ is a Brownian motion in \mathbb{R}^2 , defined on some stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. In particular, u_ϵ is a mild solution to Eq. (1.3) if it satisfies

$$u_\epsilon(t) = S_\epsilon(t)\varphi + \int_0^t S_\epsilon(t-s)B(u_\epsilon(s)) ds + \int_0^t S_\epsilon(t-s) G(u_\epsilon(s)) d\mathcal{W}(s).$$

Here B and G are the composition/multiplication functionals associated with b and g , respectively, (see Sect. 5 for the definition), and $S_\epsilon(t)$ is the Markov transition semigroup associated with Eq. (1.4), that is

$$S_\epsilon(t)\varphi(x) = \mathbb{E}_x \varphi(X_\epsilon(t)), \quad t \geq 0, \quad x \in \mathbb{R}^2,$$

for every $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ Borel and bounded. To study the limiting behavior of $u_\epsilon(t)$, one should first study the limiting behavior of $S_\epsilon(t)\varphi$, for φ in a suitable class of functions. In fact, as our first step, we prove that for every $\varphi \in C_b(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ there exists the limit of $S_\epsilon(\cdot)\varphi(x)$ in $C([\tau, T])$, with $0 < \tau < T$, and then we identify the limit.

To this purpose, in [5, Chapter 8] it is proved that, if Π is the projection of \mathbb{R}^2 onto the graph Γ associated with the Hamiltonian H , then, as a result of averaging, for any $x \in \mathbb{R}^2$ and $T > 0$ the slow process $\Pi(X_\epsilon(\cdot))$ converges, in the sense of weak convergence of distributions in $C([0, T]; \Gamma)$, to a Markov process \bar{Y} on Γ . Namely, for every continuous functional F defined on $C([0, T]; \Gamma)$ and any $x \in \mathbb{R}^2$ it holds

$$\lim_{\epsilon \rightarrow 0} \mathbb{E}_x F(\Pi(X_\epsilon(\cdot))) = \bar{\mathbb{E}}_{\Pi(x)} F(\bar{Y}(\cdot)). \tag{1.5}$$

The generator \bar{L} of the process \bar{Y} is explicitly given, in terms of suitable second order differential operators defined on each edge of the graph and suitable gluing conditions at the vertices. The semigroup generated by \bar{L} is denoted by $\bar{S}(t)$.

In ‘‘Appendix A’’, we prove that for any function $\varphi \in C_b(\mathbb{R}^2)$

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [\tau, T]} |S_\epsilon(t)\varphi(x) - (\bar{S}(t)\varphi^\wedge)(\Pi(x))| = 0, \tag{1.6}$$

with $x \in \mathbb{R}^2$ and $0 < \tau < T$ fixed. Here, if φ is a Borel bounded function on \mathbb{R}^2 , we defined

$$\varphi^\wedge(z, k) := \frac{1}{T_k(z)} \oint_{C_k(z)} \frac{\varphi(x)}{|\nabla H(x)|} dl_{z,k}, \quad (z, k) \in \Gamma,$$

where $dl_{z,k}$ is the length element on the connected component $C_k(z)$ of the level set $C(z) = \{x \in \mathbb{R}^2 : H(x) = z\}$, corresponding to the edge I_k , and

$$T_k(z) = \oint_{C_k(z)} \frac{1}{|\nabla H(x)|} dl_{z,k},$$

(see Sect. 2 for all details).

Note that (1.6) is far from being a straightforward consequence of (1.5). In fact, limit (1.6) follows once we prove that for any $\varphi \in C_b(\mathbb{R}^2)$

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [\tau, T]} |\mathbb{E}_x [\varphi(X_\epsilon(t)) - \varphi^\wedge(\Pi(X_\epsilon(t)))]| = 0, \tag{1.7}$$

and

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [\tau, T]} |\mathbb{E}_x \varphi^\wedge(\Pi(X_\epsilon(t))) - \bar{\mathbb{E}}_{\Pi(x)} \varphi^\wedge(\bar{Y}(t))| = 0. \tag{1.8}$$

A result analogous to (1.6) has been previously studied in [4], under the essential assumption that the projection of the support of the function φ on the graph Γ does not contain any vertex. Actually, assuming that the projection of the support of the initial condition φ on the graph Γ does not contain any vertex, allows to avoid to consider the vertices points, where serious discontinuity problems arise. Limit (1.7) has been obtained in [4] as a consequence of the averaging principle, by using angle-action coordinates, away from the vertices. Limit (1.8) has been obtained in [4] as an immediate consequence of (1.5), since the function φ^\wedge is continuous away from the vertices.

But the assumption that the support of the initial condition φ^\wedge on the graph Γ does not contain any vertex is too restrictive and in several situations it is critical to be able to prove (1.6) for any $\varphi \in C_b(\mathbb{R}^2)$. In particular, this is necessary when dealing with SPDEs, as in this case we cannot assume that the support of the noise satisfies such a condition.

Due to the presence of vertices, our proofs of (1.7) and (1.8) are considerably more delicate than in [4]. Actually, in order to prove (1.7) and (1.8) we have to consider separately the region of \mathbb{R}^2 close to the critical points of the Hamiltonian and the

region far from them and introduce suitable sequences of stopping times that allow to go from one region to the other. By using the fact that the process X_ϵ spends a small amount of time close to the critical points, we obtain suitable nice properties of those stopping times that allow us to conclude the validity of (1.6) for general functions φ .

Once proved the fundamental limit (1.6), we go back to our main object of investigation, SPDE (1.3). Since Eq. (1.3) is defined on the whole space \mathbb{R}^2 , we study it in a space of square integrable functions, with respect to a measure $\gamma^\vee(x) dx$. In fact, the choice of the weight γ^\vee is non-trivial, since we have to choose it in such a way that it is admissible with respect to all semigroups $S_\epsilon(t)$. Moreover, we need that its projection γ on the graph Γ is admissible for the semigroup $\bar{S}(t)$ and the space $L^2(\mathbb{R}^2, \gamma^\vee dx)$ is properly projected to the space $L^2(\Gamma, dv_\gamma)$, where v_γ is the projection on Γ of the measure $\gamma^\vee dx$.

Once identified the right class of weights, we have to introduce the appropriate space-time noise on the graph Γ . To this purpose, we recall that in (1.3) $\mathcal{W}(t, x)$ is a space homogeneous Wiener process, having finite spectral measure μ . This means, that we can represent $\mathcal{W}(t, x)$ as

$$\mathcal{W}(t, x) = \sum_{j=1}^\infty \widehat{u_j \mu}(x) \beta_j(t), \quad t \geq 0,$$

for some complete orthonormal system $\{u_j\}_{j \in \mathbb{N}}$ in $L^2(\mathbb{R}^2; d\mu)$ and a sequence of independent Brownian motions $\{\beta_j\}_{j \in \mathbb{N}}$, all defined on the same stochastic basis (here $\widehat{u_j \mu}$ denotes the Fourier transform of $u_j \mu$). Thus, we define

$$\bar{\mathcal{W}}(t, z, k) = \sum_{j=1}^\infty (\widehat{u_j \mu})^\wedge(z, k) \beta_j(t), \quad t \geq 0 \quad (z, k) \in \Gamma,$$

and we show that $\bar{\mathcal{W}}(t) \in L^2(\mathbf{\Omega}; L^2(\Gamma, dv_\gamma))$, for every $t \geq 0$ (for all properties of the noise $\bar{\mathcal{W}}(t)$ see Sect. 5). In particular, as we can prove that

$$\|\bar{S}(t)\|_{\mathcal{L}(L^2(\Gamma, dv_\gamma))} \leq c_T, \quad t \in [0, T],$$

and g is Lipschitz continuous, we obtain that for any $v \in L^2(\mathbf{\Omega}; C([0, T]; L^2(\Gamma, dv_\gamma)))$ the stochastic convolution

$$\begin{aligned} t \in [0, +\infty) &\mapsto \int_0^t \bar{S}(t-s) G(v(s)) d\bar{\mathcal{W}}(s) \\ &= \sum_{k=1}^\infty \int_0^t \bar{S}(t-s) [G(v(s)) (\widehat{u_k \mu})^\wedge] d\beta_k(s), \end{aligned}$$

is well defined in $L^2(\mathbf{\Omega}; C([0, T]; L^2(\Gamma, dv_\gamma)))$ and the mapping

$$u \mapsto \int_0^\cdot \bar{S}(\cdot - s) G(v(s)) d\bar{\mathcal{W}}(s)$$

is Lipschitz continuous in $L^2(\Omega; C([0, T]; L^2(\Gamma, dv_\gamma)))$. In particular, as also b is Lipschitz continuous, this implies that the SPDE

$$\begin{cases} \partial_t \bar{u}(t, z, k) = \bar{L} \bar{u}(t, z, k) + b(\bar{u}(t, z, k)) + g(\bar{u}(t, z, k)) \partial_t \bar{\mathcal{W}}(t, z, k), & t > 0, \\ \bar{u}(0, z, k) = \varphi^\wedge(z, k), & (z, k) \in \Gamma. \end{cases}$$

is well posed in $L^2(\Omega; C([0, T]; L^2(\Gamma, dv_\gamma)))$.

Therefore, we are able to prove the main result of this paper, namely

$$\lim_{\epsilon \rightarrow 0} \mathbf{E} \sup_{t \in [\tau, T]} |u_\epsilon(t) - \bar{u}(t) \circ \Pi|_{L^2(\mathbb{R}^2, \gamma^\vee(x)dx)}^p = 0.$$

Notice that, in the considerably simpler case H has only one critical point that coincides with the absolute minimum, then $\Gamma = [0, +\infty)$, the projection Π coincides with H and the operator \bar{L} is given by

$$\bar{L}f(z) = \frac{1}{2T(z)} (\alpha f')'(z), \quad z \in (0, +\infty),$$

where

$$T(z) = \oint_{C(z)} \frac{1}{|\nabla H(x)|} dl_z(x), \quad \alpha(z) = \oint_{C(z)} |\nabla H(x)| dl_z(x),$$

(the extremes $z = 0$ and $z = +\infty$ are inaccessible). Moreover, the noise $\bar{\mathcal{W}}(t, z)$ and the initial condition φ^\wedge are obtained from $\mathcal{W}(t, x)$ and φ , respectively, by averaging them with respect to the probability measure

$$\frac{1}{T(z)} \frac{1}{|\nabla H(x)|} dl_z(x), \quad \text{on } C(z).$$

We would like to mention the fact that similar problems have been investigated previously in the deterministic setting. In addition to the above mentioned paper [4], where limit (1.6) has been proved away from the vertex points, the limiting behavior of solutions of the elliptic problem associated with the operator \mathcal{L}_ϵ has been studied in [5, 6]. As a consequence of the limiting result (1.5), in [5, Chapter 8] Freidlin and Wentzell have studied the asymptotic behavior of the solution of the elliptic problem

$$\begin{cases} \frac{1}{2} \Delta f_\epsilon(x) + \frac{1}{\epsilon} \langle \nabla^\perp H(x), \nabla f_\epsilon(x) \rangle = -g, & x \in D, \\ f_\epsilon(x) = \rho(x), & x \in \partial D, \end{cases}$$

where D is a bounded smooth domain in \mathbb{R}^2 and g and ρ are continuous functions on D and ∂D , respectively. They have proved that f_ϵ converges to the solution of the corresponding elliptic equation on the graph, associated with the operator \bar{L} . In

[6], Ishii and Souganidis, by using only deterministic arguments, have proved an analogous result in the more general case the Laplace operator is replaced with the operator $\text{Tr}[A(x)D^2]$, where A is a smooth, symmetric, non-negative and possibly degenerate matrix-valued mapping defined on D . Finally, some SPDEs on graphs have been recently studied in [3].

Before concluding this introduction, we would like to stress the fact that we believe that the techniques we are currently developing to deal with SPDEs on graphs can be further developed to treat more sophisticated and complex situations. For example, one could consider the case there are several different types of particles, instead of only one, as in the present paper. In this case we expect to get a system of SPDEs on a graph. Moreover, if the original flow does not moves on \mathbb{R}^2 , but on a 2-dimensional surface (for example a $2D$ torus), with a positive genus, the underling dynamics can have delays at some vertices. This leads to certain effects for SPDEs on the graph that are worth of investigation. Finally, in the multidimensional case, when several conservation laws are present, we expect will obtain SPDEs on a generalization of a graph, namely an open book (see [5, Chapter 9]).

2 Notations and preliminaries

We consider here the Hamiltonian system

$$\dot{X}(t) = \nabla^\perp H(X(t)), \quad X(0) = x \in \mathbb{R}^2. \tag{2.1}$$

Throughout the present paper, we will assume that the Hamiltonian H is a generic function, with non degenerate critical points, and having quadratic growth, for $|x| \rightarrow \infty$. More precisely

Hypothesis 1 *The mapping $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies the following conditions.*

1. *It is four times continuously differentiable, with bounded second derivative.*
2. *It has only a finite number of critical points x_1, \dots, x_n . The matrix of second derivatives $D^2H(x_i)$ is non degenerate, for every $i = 1, \dots, n$ and $H(x_i) \neq H(x_j)$, if $i \neq j$.*
3. *There exist three positive constants a_1, a_2, a_3 such that $H(x) \geq a_1|x|^2$, $|\nabla H(x)| \geq a_2|x|$ and $\Delta H(x) \geq a_3$, for all $x \in \mathbb{R}^2$, with $|x|$ large enough.*
4. *We have*

$$\min_{x \in \mathbb{R}^2} H(x) = 0.$$

Notice that we can always assume condition 4. without any loss of generality.

Once introduced the Hamiltonian H , for every $z \geq 0$, we denote by $C(z)$ the z -level set

$$C(z) = \left\{ x \in \mathbb{R}^2 : H(x) = z \right\}.$$

The set $C(z)$ may consist of several connected components

$$C(z) = \bigcup_{k=1}^{N(z)} C_k(z),$$

and for every $x \in \mathbb{R}^2$ we have

$$X(0) = x \implies X(t) \in C_{k(x)}(H(x)), \quad t \geq 0,$$

where $C_{k(x)}(x)$ is the connected component of the level set $C(H(x))$, to which the point x belongs. For every $z \geq 0$ and $k = 1, \dots, N(z)$, we shall denote by $G_k(z)$ the domain of \mathbb{R}^2 bounded by the level set component $C_k(z)$.

2.1 The graph Γ and the identification map Π

If we identify all points in \mathbb{R}^2 belonging to the same connected component of a given level set $C(z)$ of the Hamiltonian H , we obtain a graph Γ , given by several intervals I_1, \dots, I_n and vertices O_1, \dots, O_m . The vertices will be of two different types, external and internal vertices. External vertices correspond to local extrema of H , while internal vertices correspond to saddle points of H . Among external vertices, we will also include O_∞ , the endpoint of the only unbounded interval in the graph, corresponding to the point at infinity.

In what follows, we shall denote by $\Pi : \mathbb{R}^2 \rightarrow \Gamma$ the *identification map*, that associates to every point $x \in \mathbb{R}^2$ the corresponding point $\Pi(x)$ on the graph Γ . We have $\Pi(x) = (H(x), k(x))$, where $k(x)$ denotes the number of the interval on the graph Γ , containing the point $\Pi(x)$. If O_i is one of the interior vertices, the second coordinate cannot be chosen in a unique way, as there are three edges having O_i as their endpoint. Notice that both $k(x)$ and $H(x)$ are first integrals (a discrete and a continuous one, respectively) for the Hamiltonian system (2.1).

On the graph Γ , a distance can be introduced in the following way. If $y_1 = (z_1, k)$ and $y_2 = (z_2, k)$ belong to the same edge I_k , then $d(y_1, y_2) = |z_1 - z_2|$. In the case y_1 and y_2 belong to different edges, then

$$d(y_1, y_2) = \min \{d(y_1, O_{i_1}) + d(O_{i_1}, O_{i_2}) + \dots + d(O_{i_j}, y_2)\},$$

where the minimum is taken over all possible paths from y_1 to y_2 , through every possible sequence of vertices O_{i_1}, \dots, O_{i_j} , connecting y_1 to y_2 .

2.2 Some other notations and preliminary facts related to the Hamiltonian

If z is not a critical value, then each $C_k(z)$ consists of one periodic trajectory of the vector field $\nabla^\perp H(x)$. If z is a local extremum of $H(x)$, then, among the components of $C(z)$ there is a set consisting of one point, the rest point of the flow. If $H(x)$ has a saddle point at some point x_0 and $H(x_0) = z$, then $C(z)$ consists of three trajectories,

the equilibrium point x_0 and the two trajectories that have x_0 as their limiting point, as $t \rightarrow \pm\infty$.

We introduce here some other notations related to the Hamiltonian H , that will be used throughout the paper.

- For every edge I_k , we denote

$$G_k = \left\{ x \in \mathbb{R}^2 : \Pi(x) \in \overset{\circ}{I}_k \right\}.$$

- For every $0 \leq z_1 < z_2$ and for every edge I_k , we denote

$$G_k(z_1, z_2) = G_k(z_2, z_1) = \{x \in G_k : z_1 < H(x) < z_2\}.$$

- For every $\delta > 0$, we denote

$$G(\pm\delta) = \bigcup_{i=1}^m G^i(\pm\delta) = \bigcup_{i=1}^m \left\{ x \in \mathbb{R}^2 : H(O_i) - \delta < H(x) < H(O_i) + \delta \right\},$$

(here, with some abuse of notation, we denote by $H(O_i)$ the value of the Hamiltonian H at those $x \in \mathbb{R}^2$ such that $\Pi(x) = O_i$).

- For every vertex O_i and edge I_k , we denote

$$D^i = \left\{ x \in \mathbb{R}^2 : \Pi(x) = O_i \right\}, \quad D_k^i = D^i \cap \bar{G}_k.$$

- For every value $z \geq 0$ taken by the function H on the set \bar{G}_k , we denote

$$C_k(z) = \{x \in \bar{G}_k : H(x) = z\} = C(z) \cap \bar{G}_k.$$

- For every $\delta > 0$ and every edge I_k and vertex O_i such that $I_k \sim O_i$, we denote

$$D(\pm\delta) = \bigcup_{k,i : I_k \sim O_i} D_k^i(\pm\delta) = \bigcup_{k,i : I_k \sim O_i} \{x \in G_k : \text{dist}(\Pi(x), O_i) = \delta\}.$$

Now, for every $(z, k) \in \Gamma$, we define

$$T_k(z) = \oint_{C_k(z)} \frac{1}{|\nabla H(x)|} dl_{z,k}, \quad \alpha_k(z) = \oint_{C_k(z)} |\nabla H(x)| dl_{z,k}, \tag{2.2}$$

where $dl_{z,k}$ is the length element on $C_k(z)$. Notice that $T_k(z)$ is the period of the motion along the level set $C_k(z)$.

In what follows, we assume that the Hamiltonian H , in addition to all conditions in Hypothesis 1, satisfies also the following condition.

Hypothesis 2 *It holds*

$$\frac{dT_k(z)}{dz} \neq 0, \quad (z, k) \in \Gamma. \tag{2.3}$$

Remark 2.1 Condition (2.3) rules out the case $H(x) = |x|^2$. This means that with our method we cannot treat the case $\nabla^\perp H(x)$ is linear in order to prove the main result stated below in (A.11).

As we have seen above, if $X(0) = x \in C_k(z)$, then $X(t) \in C_k(z)$, for every $t \geq 0$. As known, for every $(z, k) \in \Gamma$ the probability measure

$$d\mu_{z,k} := \frac{1}{T_k(z)} \frac{1}{|\nabla H(x)|} dl_{z,k} \tag{2.4}$$

is invariant for system (2.1) on the level set $C_k(z)$. Moreover, it is possible to prove that for any $0 \leq z_1 < z_2$ and $k = 1, \dots, n$

$$\int_{G_k(z_1, z_2)} u(x) dx = \int_{I_k \cap [(z_1, k), (z_2, k)]} \oint_{C_k(z)} \frac{u(x)}{|\nabla H(x)|} dl_{z,k} dz. \tag{2.5}$$

In particular, if we take $u \equiv 1$, we get

$$\text{area}(G_k(z_1, z_2)) = \int_{I_k \cap [(z_1, k), (z_2, k)]} T_k(z) dz,$$

so that

$$T_k(z) = \frac{dS_k(z)}{dz}, \tag{2.6}$$

where $S_k(z)$ is the area of the domain $G_k(z)$ bounded by the level set $C_k(z)$. Finally, by the divergence theorem, it is immediate to check that

$$\alpha_k(z) = \left| \int_{G_k(z)} \Delta H(x) dx \right| = \left| \int_{G_k(z)} \omega_H(x) dx \right|,$$

where ω_H is the vorticity of the flow.

As discussed in [5, section 8.1], since the Hamiltonian H has only non-degenerate critical points, if (z, k) approaches the endpoint of an edge I_k , corresponding to an external vertex $O_i = (H(x_j), k)$, then

$$\lim_{(z,k) \rightarrow O_i} T_k(z) = \frac{2\pi}{\sqrt{\det [\text{Hess}H(x_j)]}} > 0. \tag{2.7}$$

If (z, k) approaches the endpoint of an edge I_k , corresponding to an internal vertex $O_i = (H(x_j), k)$, then

$$T_k(z) = \oint_{C_k(z)} \frac{1}{|\nabla H(x)|} dl_{z,k} \sim \text{const} |\log |z - H(x_i)|| \rightarrow +\infty, \text{ as } (z, k) \rightarrow O_i. \tag{2.8}$$

Finally, if (z, k) approaches O_∞ , we have

$$T_k(z) = O(1), \text{ as } (z, k) \rightarrow O_\infty. \tag{2.9}$$

Actually, for $|x|$ large, we have $H(x) \geq a_1 |x|^2$, so that $S_k(z) = O(z)$ and hence, due to (2.6), (2.9) follows.

Next, we would like to recall that in [5, Lemma 8.1.1] it is proven that if $v \in C^1(\mathbb{R}^2)$, then for every fixed k the mapping

$$z \mapsto \oint_{C_k(z)} v(x) |\nabla H(x)| dl_{z,k},$$

is continuously differentiable with respect to z such that $(z, k) \in \mathring{I}_k$, and

$$\frac{d}{dz} \oint_{C_k(z)} v(x) |\nabla H(x)| dl_{z,k} = \oint_{C_k(z)} \frac{\langle \nabla v(x), \nabla H(x) \rangle + v(x) \Delta H(x)}{|\nabla H(x)|} dl_{z,k}. \tag{2.10}$$

2.3 A limiting result

We consider here the following random perturbation of system (2.1)

$$d\tilde{X}_\epsilon(t) = \nabla^\perp H(\tilde{X}_\epsilon(t)) dt + \sqrt{\epsilon} d\tilde{w}(t), \quad \tilde{X}_\epsilon(0) = x, \tag{2.11}$$

where $\tilde{w}(t)$ is a two dimensional Wiener process and $\epsilon > 0$ is a small parameter. Because of the perturbation, the mapping $t \in [0, +\infty) \mapsto H(\tilde{X}_\epsilon(t)) \in \mathbb{R}$ is not constant any more. Actually, the motion $\tilde{X}_\epsilon(t)$ consists of a fast rotation along the deterministic unperturbed trajectories and a slow motion across them.

In what follows, it will be convenient to do a change of time and, with the time rescaling $t \mapsto t/\epsilon$, the process $\tilde{X}_\epsilon(t/\epsilon)$ will coincide in distribution with the solution of the stochastic equation

$$dX_\epsilon(t) = \frac{1}{\epsilon} \nabla^\perp H(X_\epsilon(t)) dt + dw(t), \quad X_\epsilon(0) = x, \tag{2.12}$$

where $w(t)$ is another two dimensional Wiener process. We will denote by $S_\epsilon(t)$, $t \geq 0$, the Markov transition semigroup associated with Eq. (2.12). Namely, for every Borel bounded function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$S_\epsilon(t)\varphi(x) = \mathbb{E}_x \varphi(X_\epsilon(t)), \quad t \geq 0, \quad x \in \mathbb{R}^2.$$

Now, for every $x \in \mathbb{R}^2$, we consider the process $\Pi(X_\epsilon(t))$, $t \geq 0$, defined on the graph Γ , with $X_\epsilon(0) = x$. In [5, Chapter 8] it has been studied the limiting behavior, as $\epsilon \downarrow 0$, of the process $\Pi(X_\epsilon)$ in the space $C([0, T]; \Gamma)$, for any fixed $T > 0$ and $x \in \mathbb{R}^2$. Namely, in [5, Theorem 8.2.2] it has been proved that the process $\Pi(X_\epsilon)$, which describes the slow motion of the motion $X_{\epsilon, \epsilon}$, converges, in the sense of weak convergence of distributions in the space of continuous Γ -valued functions, to a diffusion process \bar{Y} on Γ .

The process \bar{Y} has been described in [5, Theorem 8.2.1] in terms of its generator \bar{L} , which is given by suitable differential operators \bar{L}_k within each edge I_k of the graph and by certain gluing conditions at the interior vertices O_i of the graph.

For each $k = 1, \dots, n$, the differential operator \bar{L}_k , acting on functions f defined on the edge I_k , has the form

$$\bar{L}_k f(z) = \frac{1}{2 T_k(z)} \frac{d}{dz} \left(\alpha_k \frac{df}{dz} \right) (z), \tag{2.13}$$

where $T_k(z)$ and $\alpha_k(z)$ are the mappings defined in (2.2). The operator \bar{L} , acting on functions f defined on the graph Γ , is defined as

$$\bar{L} f(z, k) = \bar{L}_k f(z), \quad \text{if } (z, k) \text{ is an interior point of the edge } I_k.$$

It is immediate to check that each operator \bar{L}_k can be represented as $\bar{L}_k = (d/dv_k)(d/du_k)$, where

$$v'_k(z) = \oint_{C_k(z)} \frac{1}{|\nabla H(x)|} dl_{z,k}, \quad u'_k(z) = \left(\oint_{C_k(z)} |\nabla H(x)| dl_{z,k} \right)^{-1}.$$

In view of this representation, by studying the limiting behavior of the function u_k and v_k at the vertices of the graph Γ , it is possible to check that the internal vertices, corresponding to the saddle points of H , are accessible, while the external vertices and the vertex O_∞ are inaccessible.

The domain $D(\bar{L})$ is defined as the set of continuous functions on the graph Γ , that are twice continuously differentiable in the interior part of each edge of the graph, such that for any vertex $O_i = (z_i, k_{i_1}) = (z_i, k_{i_2}) = (z_i, k_{i_3})$ there exist finite

$$\lim_{(z, k_{i_j}) \rightarrow O_i} \bar{L} f(z, k_{i_j}), \quad j = 1, 2, 3,$$

and the limits do not depend on the edge $I_{k_{i_j}} \sim O_i$. Moreover, for each interior vertex O_i the following gluing condition is satisfied

$$\sum_{j=1}^3 \pm \alpha_{k_{i_j}}(z_i) d_{k_{i_j}} f(z_i, k_{i_j}) = 0, \tag{2.14}$$

where $d_{k_{i_j}}$ is the differentiation along $I_{k_{i_j}}$ and the sign $+$ is taken if the H -coordinate increases along $I_{k_{i_j}}$ and the sign $-$ is taken otherwise.

The operator $(\bar{L}, D(\bar{L}))$ is a non-standard operator, because it is a differential operator on a graph, endowed with suitable gluing conditions and because it is degenerate at the vertices of the graph. Nevertheless in [5, Theore 8.2.1] it is shown that it is the generator of a Markov process \bar{Y} on the graph Γ . In what follows, we shall denote by $\bar{\mathbb{P}}_{(z,k)}$ and $\bar{\mathbb{E}}_{(z,k)}$ the probability and the expectation associated with \bar{Y} , starting

from $(z, k) \in \Gamma$. Moreover, we shall denote by $\bar{S}(t)$ the semigroup associated with \bar{Y} , defined by

$$\bar{S}(t) f(z, k) = \bar{\mathbb{E}}_{(z,k)} f(\bar{Y}(t)),$$

for every bounded Borel function $f : \Gamma \rightarrow \mathbb{R}$.

As we mentioned above, in [5, Theorem 8.2.2] it has been proved that the process $\Pi(X_\epsilon(\cdot))$ is weakly convergent to \bar{Y} in $C([0, T]; \Gamma)$, for every $T > 0$ and $x \in \mathbb{R}^2$. Namely, for every continuous functional F defined on $C([0, T]; \Gamma)$ and $x \in \mathbb{R}^2$

$$\lim_{\epsilon \rightarrow 0} \mathbb{E}_x F(\Pi(X_\epsilon(\cdot))) = \bar{\mathbb{E}}_{\Pi(x)} F(\bar{Y}(\cdot)). \tag{2.15}$$

3 Functions and operators defined on the graph Γ

We fix here a continuous mapping $\gamma : \Gamma \rightarrow (0, +\infty)$ such that

$$\sum_{k=1}^n \int_{I_k} \gamma(z, k) T_k(z) dz < \infty, \tag{3.1}$$

where, we recall,

$$T_k(z) = \oint_{C_k(z)} \frac{1}{|\nabla H(x)|} dl_{z,k}.$$

If we define

$$\gamma^\vee(x) = \gamma(\Pi(x)), \quad x \in \mathbb{R}^2,$$

we have that $\gamma^\vee : \mathbb{R}^2 \rightarrow (0, +\infty)$ is a bounded continuous function in $L^1(\mathbb{R}^2)$. Actually, due to (2.5) we have

$$\begin{aligned} \int_{\mathbb{R}^2} \gamma^\vee(x) dx &= \sum_{k=1}^n \int_{I_k} \oint_{C_k(z)} \frac{\gamma^\vee(x)}{|\nabla H(x)|} dl_{z,k} dz \\ &= \sum_{k=1}^n \int_{I_k} \gamma(z, k) \oint_{C_k(z)} \frac{1}{|\nabla H(x)|} dl_{z,k} dz \\ &= \sum_{k=1}^n \int_{I_k} \gamma(z, k) T_k(z) dz < \infty. \end{aligned}$$

In what follows, we shall define

$$H_\gamma = \left\{ u : \mathbb{R}^2 \rightarrow \mathbb{R} : \int_{\mathbb{R}^2} |u(x)|^2 \gamma^\vee(x) dx < \infty \right\} = L^2(\mathbb{R}^2, \gamma^\vee(x) dx),$$

and

$$\bar{H}_\gamma = \left\{ f : \Gamma \rightarrow \mathbb{R} : \sum_{k=1}^n \int_{I_k} |f(z, k)|^2 \gamma(z, k) T_k(z) dz < \infty \right\} = L^2(\Gamma, \nu_\gamma),$$

where the measure ν_γ is defined as

$$\nu_\gamma(A) := \sum_{k=1}^n \int_{I_k} \mathbb{I}_A(z, k) \gamma(z, k) T_k(z) dz, \quad A \subseteq \mathcal{B}(\Gamma).$$

Now, for every $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ we define

$$u^\wedge(z, k) = \frac{1}{T_k(z)} \oint_{C_k(z)} \frac{u(x)}{|\nabla H(x)|} dl_{z,k} = \oint_{C_k(z)} u(x) d\mu_{z,k}, \quad (z, k) \in \Gamma,$$

and for every $f : \Gamma \rightarrow \mathbb{R}$ we define

$$f^\vee(x) = f(\Pi(x)), \quad x \in \mathbb{R}^2.$$

Proposition 3.1 *Assume the Hamiltonian H satisfies Hypothesis 1 and $\gamma : \Gamma \rightarrow (0, +\infty)$ is a weight function satisfying condition (3.1). Then, the following holds.*

1. For every $u \in H_\gamma$ we have $u^\wedge \in \bar{H}_\gamma$ and

$$|u^\wedge|_{\bar{H}_\gamma} \leq |u|_{H_\gamma}. \tag{3.2}$$

2. For every $f \in \bar{H}_\gamma$ we have $f^\vee \in H_\gamma$ and

$$|f^\vee|_{H_\gamma} = |f|_{\bar{H}_\gamma}. \tag{3.3}$$

Proof Let $u \in H_\gamma$. Recalling how the probability measure $\mu_{z,k}$ has been defined in (2.4), due to (2.5) we have

$$\begin{aligned} |u^\wedge(x)|_{\bar{H}_\gamma}^2 &= \sum_{k=1}^n \int_{I_k} |u^\wedge(z, k)|^2 \gamma(z, k) T_k(z) dz \\ &= \sum_{k=1}^n \int_{I_k} \left| \oint_{C_k(z)} u(x) d\mu_{z,k} \right|^2 \gamma(z, k) T_k(z) dz \\ &\leq \sum_{k=1}^n \int_{I_k} \oint_{C_k(z)} |u(x)|^2 d\mu_{z,k} \gamma(z, k) T_k(z) dz \\ &= \sum_{k=1}^n \int_{I_k} \oint_{C_k(z)} \frac{|u(x)|^2 \gamma^\vee(x)}{|\nabla H(x)|} dl_{z,k} dz = \int_{\mathbb{R}^2} |u(x)|^2 \gamma^\vee(x) dx = |u|_{H_\gamma}^2. \end{aligned}$$

This implies that $u^\wedge \in \bar{H}_\gamma$ and (3.2) holds.

Concerning the second part of the Proposition, by using again (2.5) for every $f \in \bar{H}_\gamma$ we have

$$\begin{aligned} |f^\vee|_{H_\gamma}^2 &= \int_{\mathbb{R}^2} |f^\vee(x)|^2 \gamma^\vee(x) dx = \int_{\mathbb{R}^2} |f(\Pi(x))|^2 \gamma(\Pi(x)) dx \\ &= \sum_{k=1}^n \int_{I_k} \oint_{C_k(z)} \frac{|f(z, k)|^2 \gamma(z, k)}{|\nabla H(x)|} dl_{z,k} dz \\ &= \sum_{k=1}^n \int_{I_k} |f(z, k)|^2 \gamma(z, k) T_k(z) dz = |f|_{\bar{H}_\gamma}^2. \end{aligned}$$

This allows to conclude that $f^\vee \in H_\gamma$ and (3.3) holds. □

For every $u \in H_\gamma$ and $f \in \bar{H}_\gamma$, we have

$$\langle f, u^\wedge \rangle_{\bar{H}_\gamma} = \langle f^\vee, u \rangle_{H_\gamma}. \tag{3.4}$$

Actually, we have

$$\begin{aligned} \langle f, u^\wedge \rangle_{\bar{H}_\gamma} &= \sum_{k=1}^n \int_{I_k} f(z, k) u^\wedge(z, k) \gamma(z, k) T_k(z) dz \\ &= \sum_{k=1}^n \int_{I_k} f(z, k) \oint_{C_k(z)} u(x) d\mu_{z,k} \gamma(z, k) T_k(z) dz \\ &= \sum_{k=1}^n \int_{I_k} \oint_{C_k(z)} \frac{f(\Pi(x)) u(x) \gamma^\vee(x)}{|\nabla H(x)|} dl_{z,k} dz, \end{aligned}$$

so that, thanks to (2.5), we can conclude

$$\langle f, u^\wedge \rangle_{\bar{H}_\gamma} = \int_{\mathbb{R}^2} f(\Pi(x)) u(x) \gamma^\vee(x) dx = \langle f^\vee, u \rangle_{H_\gamma}.$$

Moreover, for every $f \in \bar{H}_\gamma$ and $u \in H_\gamma$, we have

$$\begin{aligned} (f^\vee u)^\wedge(z, k) &= \oint_{C_k(z)} f^\vee(x) u(x) d\mu_{z,k} = \oint_{C_k(z)} f(\Pi(x)) u(x) d\mu_{z,k} \\ &= f(z, k) \oint_{C_k(z)} u(x) d\mu_{z,k} = f(z, k) u^\wedge(z, k), \end{aligned}$$

so that

$$(f^\vee u)^\wedge = f u^\wedge. \tag{3.5}$$

As a consequence of (3.4) and (3.5), we conclude that for any $f, g \in \bar{H}_\gamma$

$$\langle f, g \rangle_{\bar{H}_\gamma} = \langle (f^\vee)^\wedge, g \rangle_{\bar{H}_\gamma} = \langle f^\vee, g^\vee \rangle_{H_\gamma}.$$

In particular, if $\{f_n\}_{n \in \mathbb{N}}$ is an orthonormal system in \bar{H}_γ , it follows that the system $\{(f_n)^\vee\}_{n \in \mathbb{N}}$ is orthonormal in H_γ .

Lemma 3.2 *Assume $u \in C^1(\mathbb{R}^2)$. Then, under Hypothesis 1, the mapping u^\wedge is continuously differentiable with respect to z in the set $\bigcup_k \mathring{I}_k$. Moreover, for every fixed $\delta > 0$ and $M > 0$*

$$\sup_{(z,k) \in \bigcup_h \left[\mathring{I}_h \cap \Pi(G(\pm\delta))^c \right]_{z \leq M}} \left| \frac{\partial u^\wedge}{\partial z}(z, k) \right| =: c_{\delta, M} < \infty. \tag{3.6}$$

Proof If $u \in C^1(\mathbb{R}^2)$, then the mapping

$$v(x) = \frac{u(x)}{|\nabla H(x)|^2}, \quad x \in \bigcup_k \mathring{I}_k,$$

is continuously differentiable. As we have seen in Sect. 2.2, due to [5, Lemma 8.1.1], this implies that the mapping

$$z \mapsto \oint_{C_k(z)} \frac{u(x)}{|\nabla H(x)|} dl_{z,k} = \oint_{C_k(z)} v(x) |\nabla H(x)| dl_{z,k}$$

is continuously differentiable for every z such that $(z, k) \in \mathring{I}_k$. Moreover, thanks to (2.10), for every $\delta > 0$ the derivative is uniformly bounded with respect to z such that $(z, k) \in \Pi(G(\pm\delta))^c \cap \mathring{I}_k$. In particular, if we take $u = 1$, we get that the mapping $z \mapsto T_k(z)$ is continuously differentiable and the derivative is uniformly bounded with respect to z such that $(z, k) \in \Pi(G(\pm\delta))^c \cap \mathring{I}_k$. Therefore, as

$$u^\wedge(z, k) = \oint_{C_k(z)} u(x) d\mu_{z,k} = \frac{1}{T_k(z)} \oint_{C_k(z)} \frac{u(x)}{|\nabla H(x)|} dl_{z,k},$$

and $T_k(z)$ remains uniformly bounded from zero for $(z, k) \in \Pi(G(\pm\delta))^c \cap \mathring{I}_k$, we can conclude. □

For every $Q \in \mathcal{L}(H_\gamma)$ and $f \in \bar{H}_\gamma$, we define

$$Q^\wedge f = (Qf^\vee)^\wedge.$$

Thanks to Proposition 3.1, we have that

$$\|Q^\wedge\|_{\mathcal{L}(\bar{H}_\gamma)} \leq \|Q\|_{\mathcal{L}(H_\gamma)}.$$

In an analogous way, for any $A \in \mathcal{L}(\bar{H}_\gamma)$ and $u \in H_\gamma$, we define

$$A^\vee u = (Au^\wedge)^\vee.$$

As above, due to Proposition 3.1, we have that $A^\vee \in \mathcal{L}(H_\gamma)$ and

$$\|A^\vee\|_{\mathcal{L}(H_\gamma)} \leq \|A\|_{\mathcal{L}(\bar{H}_\gamma)}.$$

Moreover, due to (3.5), we can check immediately that $(A^\vee)^\wedge = A$.

4 The semigroup $S_\epsilon(t)$ in the weighted space H_γ

Since $\operatorname{div} \nabla^\perp H = 0$, it is immediate to check that the Lebesgue measure is invariant for the semigroup $S_\epsilon(t)$, for every fixed $\epsilon > 0$. In particular, $S_\epsilon(t)$ can be extended to a contraction semigroup on $L^p(\mathbb{R}^p)$, for every $p \geq 1$ and $\epsilon > 0$.

In what follows, we shall make the following fundamental assumption about the behavior of $S_\epsilon(t)$ in the weighted space H_γ

Hypothesis 3 *The semigroup $S_\epsilon(t)$ is well defined on H_γ , for every $\epsilon > 0$. Moreover, for every fixed $T > 0$, there exists $c_T > 0$ such that*

$$\|S_\epsilon(t)\|_{\mathcal{L}(H_\gamma)} \leq c_T, \quad t \in [0, T], \quad \epsilon > 0. \tag{4.1}$$

In next proposition we will show that, in fact, Hypothesis 3 is fulfilled in some relevant cases.

Proposition 4.1 *Assume that the Hamiltonian H satisfies Hypothesis 1. Then, there exists a strictly positive decreasing function $h \in C^2([0, +\infty))$, such that the function $\gamma : \Gamma \rightarrow (0, +\infty)$ defined by $\gamma(z, k) = h(z)$, for every $(z, k) \in \Gamma$, satisfies (3.1) and Hypothesis 3 holds.*

Proof According to condition 3. in Hypothesis 1, we can fix

$$z_0 > \max_{i=1, \dots, n} H(x_i)$$

such that

$$H(x) \geq z_0 \implies H(x) \geq a_1 |x|^2.$$

Once fixed z_0 , we take a positive decreasing function $h \in C^2([0, +\infty))$, such that

$$h(t) = \begin{cases} \frac{1}{2} \exp(-\lambda (\sqrt{t} - \sqrt{2z_0})), & t \geq 2z_0, \\ 1, & t \leq z_0, \end{cases}$$

for some arbitrary $\lambda > 0$, and define

$$\gamma(z, k) = h(z), \quad (z, k) \in \Gamma.$$

In view of (2.7) and (2.8), it is immediate to check that for every interval I_k not having O_∞ as its end point,

$$\int_{I_k} T_k(z) dz < \infty,$$

so that, as $|h(t)| \leq 1$, we get

$$\int_{I_k} \gamma(z, k) T_k(z) dz < \infty. \quad (4.2)$$

Moreover, if $I_k \sim O_\infty$, due to (2.9), since $h(t)$ has exponential decay, we have that (4.2) holds as well and hence we can conclude that (3.1) holds.

Now, recalling that $\gamma^\vee(x) = \gamma(\Pi(x))$, we have that

$$\gamma^\vee(x) = h(H(x)), \quad x \in \mathbb{R}^2.$$

Since $h \in C^2([0, +\infty))$, this implies that $\gamma^\vee \in C^2(\mathbb{R}^2)$. Moreover,

$$\Delta \gamma^\vee(x) = h''(H(x)) |\nabla H(x)|^2 + h'(H(x)) \Delta h(x), \quad x \in \mathbb{R}^2.$$

Hence, as for any $t \geq 2z_0$ we have

$$h'(t) = -\frac{\lambda}{2} h(t) t^{-\frac{1}{2}}, \quad h''(t) = \frac{\lambda}{4} h(t) \left(\lambda t^{-1} + t^{-\frac{3}{2}} \right),$$

we obtain

$$\Delta \gamma^\vee(x) = \frac{\lambda}{4} \gamma^\vee(x) \left[\frac{|\nabla H(x)|^2}{H(x)} \left(\lambda + \frac{1}{H(x)^{\frac{1}{2}}} \right) - 2 \frac{\Delta H(x)}{H(x)^{\frac{1}{2}}} \right].$$

Since $|\nabla H(x)| \leq c|x|$ and $|\Delta H(x)| \leq c$, for every $x \in \mathbb{R}^2$, and $H(x) \geq a_1|x|^2$, if $H(x) \geq z_0$, we have

$$\sup_{\substack{x \in \mathbb{R}^2, \\ H(x) \geq z_0}} \left| \frac{|\nabla H(x)|^2}{H(x)} \left(\lambda + \frac{1}{H(x)^{\frac{1}{2}}} \right) - 2 \frac{\Delta H(x)}{H(x)^{\frac{1}{2}}} \right| =: \kappa_\lambda < \infty,$$

so that

$$H(x) \geq 2z_0 \implies \Delta \gamma^\vee(x) \leq \frac{\lambda \kappa_\lambda}{4} \gamma^\vee(x). \quad (4.3)$$

On the other hand, since $\gamma^\vee \in C^2(\mathbb{R}^2)$, we have that

$$\sup_{\substack{x \in \mathbb{R}^2, \\ H(x) < 2z_0}} |\Delta \gamma^\vee(x)| =: M < \infty,$$

and hence

$$H(x) < 2z_0 \implies \Delta \gamma^\vee(x) \leq M \leq 2h(H(x))M = 2\gamma^\vee(x)M.$$

Together with (4.3), this implies that there exists some $c > 0$ such that

$$\Delta \gamma^\vee(x) \leq c\gamma^\vee(x), \quad x \in \mathbb{R}^2. \tag{4.4}$$

The generator of the semigroup $S_\epsilon(t)$ is the second order differential operator

$$\mathcal{L}_\epsilon = \frac{1}{2}\Delta + \frac{1}{\epsilon}\nabla^\perp H(x) \cdot \nabla.$$

As $\operatorname{div} \nabla^\perp H(x) = 0$, it is immediate to check that the adjoint \mathcal{L}_ϵ^* of the operator \mathcal{L}_ϵ in $L^2(\mathbb{R}^2)$ is given by

$$\mathcal{L}_\epsilon^* = \frac{1}{2}\Delta - \frac{1}{\epsilon}\nabla^\perp H(x) \cdot \nabla.$$

This implies that the adjoint semigroup $S_\epsilon^*(t)$ is the Markov transition semigroup associated with the equation

$$dY_\epsilon(t) = -\frac{1}{\epsilon}\nabla^\perp H(Y_\epsilon(t)) dt + d\hat{w}(t), \quad Y_\epsilon(0) = x,$$

for some 2d Brownian motion $\hat{w}(t)$. Now, by using the Itô formula, we have

$$\begin{aligned} \gamma^\vee(Y_\epsilon(t)) &= \gamma^\vee(x) - \frac{1}{\epsilon} \int_0^t \langle \nabla \gamma^\vee(Y_\epsilon(s)), \nabla^\perp H(Y_\epsilon(s)) \rangle ds \\ &\quad + \int_0^t \langle \nabla \gamma^\vee(Y_\epsilon(s)), dw(s) \rangle + \frac{1}{2} \int_0^t \Delta \gamma^\vee(Y_\epsilon(s)) ds, \end{aligned}$$

and then, since

$$\nabla \gamma^\vee(x) = h'(H(x)) \nabla H(x), \quad x \in \mathbb{R}^2,$$

it follows

$$\gamma^\vee(Y_\epsilon(t)) = \gamma^\vee(x) + \int_0^t \langle \nabla \gamma^\vee(Y_\epsilon(s)), dw(s) \rangle + \frac{1}{2} \int_0^t \Delta \gamma^\vee(Y_\epsilon(s)) ds.$$

In view of (4.4), by taking the expectation of both sides above, we get

$$\begin{aligned} S_\epsilon^*(t)\gamma^\vee(x) &= \hat{\mathbb{E}}_x \gamma^\vee(Y_\epsilon(t)) \leq \gamma^\vee(x) + c \int_0^t \hat{\mathbb{E}}_x \gamma^\vee(Y_\epsilon(s)) ds \\ &= \gamma^\vee(x) + c \int_0^t S_\epsilon^*(s)\gamma^\vee(x) ds, \end{aligned}$$

and, thanks to the Gronwall lemma, this allows to conclude that

$$S_\epsilon^*(t)\gamma^\vee(x) \leq e^{ct}\gamma^\vee(x), \quad x \in \mathbb{R}^2, \quad t \geq 0. \tag{4.5}$$

Now, for any $u \in H_\gamma$, we have

$$\begin{aligned} |S_\epsilon(t)u|_{H_\gamma}^2 &= \int_{\mathbb{R}^2} |S_\epsilon(t)u(x)|^2 \gamma^\vee(x) dx \leq \int_{\mathbb{R}^2} S_\epsilon(t)|u|^2(x) \gamma^\vee(x) dx \\ &= \langle S_\epsilon(t)|u|^2, \gamma^\vee \rangle_{L^2(\mathbb{R}^2)} = \langle |u|^2, S_\epsilon^*(t)\gamma^\vee \rangle_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Then, from (4.5), we have

$$|S_\epsilon(t)u|_{H_\gamma}^2 \leq e^{cT} \langle |u|^2, \gamma^\vee \rangle_{L^2(\mathbb{R}^2)} = e^{cT} |u|_{H_\gamma}^2, \quad t \in [0, T],$$

which implies (4.1), with $c_T = e^{cT}$. □

5 From the SPDE on \mathbb{R}^2 to the SPDE on the graph

We are interested here in the equation

$$\begin{cases} \partial_t u_\epsilon(t, x) = \mathcal{L}_\epsilon u_\epsilon(t, x) + b(u_\epsilon(t, x)) + g(u_\epsilon(t, x))\partial_t \mathcal{W}(t, x), & t > 0, \\ u_\epsilon(0, x) = \varphi(x), & x \in \mathbb{R}^2. \end{cases} \tag{5.1}$$

\mathcal{L}_ϵ is the second order differential operator defined by

$$\mathcal{L}_\epsilon u(x) = \frac{1}{2} \Delta u(x) + \frac{1}{\epsilon} \langle \bar{\nabla} H(x), \nabla u(x) \rangle, \quad x \in \mathbb{R}^2,$$

associated with Eq. (2.12) and with Markov transition semigroup $S_\epsilon(t)$. The Hamiltonian H satisfies Hypotheses 1 and 3 and the nonlinearities $b, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ are assumed to be Lipschitz continuous.

Concerning the random forcing \mathcal{W} , we assume that it is a *spatially homogeneous Wiener process*, with finite spectral measure μ (see e.g. [7] and [2] for all details). This means that there exists a Gaussian random field on $[0, +\infty) \times \mathbb{R}^2$, that we also denote by \mathcal{W} , defined on some stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$, such that

1. the mapping $(t, x) \mapsto \mathcal{W}(t, x)$ is continuous with respect to t and measurable with respect to both variables, \mathbf{P} -almost surely;

2. for each $x \in \mathbb{R}^2$, the process $\mathcal{W}(t, x), t \geq 0$, is a one-dimensional Wiener process;
3. for every $t, s \geq 0$ and $x, y \in \mathbb{R}^2$

$$\mathbf{E} \mathcal{W}(t, x)\mathcal{W}(s, y) = (t \wedge s) \Lambda(x - y), \tag{5.2}$$

where Λ is the Fourier transform of the spectral measure μ , that is

$$\Lambda(x) = \int_{\mathbb{R}^2} e^{i\langle x, \lambda \rangle} \mu(d\lambda), \quad x \in \mathbb{R}^2.$$

Notice that, with this definition, $\mathcal{W}(t, \cdot) \in L^2(\Omega; H_\gamma)$. Actually, we have

$$\begin{aligned} \mathbf{E}|\mathcal{W}(t, \cdot)|_{H_\gamma}^2 &= \mathbf{E} \int_{\mathbb{R}^2} |\mathcal{W}(t, x)|^2 \gamma^\vee(x) dx \\ &= \int_{\mathbb{R}^2} \mathbf{E}|\mathcal{W}(t, x)|^2 \gamma^\vee(x) dx = t \Lambda(0) \int_{\mathbb{R}^2} \gamma^\vee(x) dx. \end{aligned}$$

In what follows, we denote by $L^2_{(s)}(\mathbb{R}^2, d\mu)$ the subspace of the Hilbert space $L^2(\mathbb{R}^2, d\mu; \mathbb{C})$ consisting of all functions φ such that $\varphi_{(s)} = \varphi$, where

$$\varphi_{(s)}(x) = \overline{\varphi(-x)}, \quad x \in \mathbb{R}^2.$$

Moreover, we denote by RK the *reproducing kernel* Hilbert space of the Wiener process \mathcal{W} (see [2] for the definition).

As shown in [7, Proposition 1.2], an orthonormal basis for the reproducing kernel RK is given by $\{\widehat{u_j \mu}\}_{j \in \mathbb{N}}$, where $\{u_j\}_{j \in \mathbb{N}}$ is a complete orthonormal basis of the Hilbert space $L^2_{(s)}(\mathbb{R}^2, d\mu)$ and

$$\widehat{u_j \mu}(x) = \int_{\mathbb{R}^2} e^{i\langle x, \lambda \rangle} u_j(\lambda) \mu(d\lambda), \quad x \in \mathbb{R}^2.$$

This means, in particular, that $\mathcal{W}(t, x)$ can be represented as

$$\mathcal{W}(t, x) = \sum_{j=1}^{\infty} \widehat{u_j \mu}(x) \beta_j(t), \quad t \geq 0, \tag{5.3}$$

for some sequence of independent Brownian motions $\{\beta_j\}_{j \in \mathbb{N}}$, all defined on the same stochastic basis. Moreover, in [7, Lemma 4.1] it is also shown that

$$\sum_{j=1}^{\infty} |\widehat{u_j \mu}|_{H_\gamma}^2 < \infty. \tag{5.4}$$

For every $u \in H_\gamma$ and v in the reproducing kernel of \mathcal{W} , we shall denote

$$B(u)(x) = b(u(x)), \quad G(u)v(x) = [g(u)v](x), \quad x \in \mathbb{R}^2.$$

Since we are assuming b to be Lipschitz continuous, we have that $B : H_\gamma \rightarrow H_\gamma$ is Lipschitz continuous. Moreover, as proved in [7, Lemma 4.1], G maps H_γ into $\mathcal{L}_2(RK, H_\gamma)$, where $\mathcal{L}_2(RK, H_\gamma)$ is the space of Hilbert-Schmidt operators defined on RK with values in H_γ . Furthermore, for every $u_1, u_2 \in H_\gamma$

$$|G(u_1) - G(u_2)|_{\mathcal{L}_2(RK, H_\gamma)} \leq c |u_1 - u_2|_{H_\gamma}. \tag{5.5}$$

By using a stochastic factorization argument and (4.1), this implies that

$$\begin{aligned} & \mathbf{E} \sup_{t \in [0, T]} \left| \int_0^t S_\epsilon(t-s) [G(u_1(s)) - G(u_2(s))] d\mathcal{W}(s) \right|_{H_\gamma}^p \\ & \leq c_p(T) \mathbf{E} \sup_{t \in [0, T]} |u_1(t) - u_2(t)|_{H_\gamma}^p. \end{aligned} \tag{5.6}$$

Definition 5.1 A predictable process $u_\epsilon(t)$, taking values in H_γ is a mild solution to Eq. (5.1) if it satisfies the following integral equation

$$u_\epsilon(t) = S_\epsilon(t)\varphi + \int_0^t S_\epsilon(t-s)B(u_\epsilon(s)) ds + \int_0^t S_\epsilon(t-s) G(u_\epsilon(s)) d\mathcal{W}(s). \tag{5.7}$$

In [7, Theorem 2.1] it is proved that, under our assumptions on the Hamiltonian H , the coefficients b and g and the noise \mathcal{W} , for every $\varphi \in H_\gamma$ and every $p \geq 1$ and $T > 0$ there exists a unique mild solution u_ϵ for Eq. (5.1) in $L^p(\Omega; C([0, T]; H_\gamma))$. Moreover, we have

$$\mathbb{E} \sup_{t \in [0, T]} |u_\epsilon(t)|_{H_\gamma}^p = c_p(T), \quad \epsilon > 0. \tag{5.8}$$

Our purpose is studying the limiting behavior of u_ϵ in the space $L^p(\Omega; C([0, T]; H_\gamma))$ or, equivalently, the limit of u_ϵ^\wedge in the space $L^p(\Omega; C([0, T]; \bar{H}_\gamma))$, as $\epsilon \downarrow 0$.

The limiting process will be the solution \bar{u} of the following SPDE on the graph Γ

$$\begin{cases} \partial_t \bar{u}(t, z, k) = \bar{L} \bar{u}(t, z, k) + b(\bar{u}(t, z, k)) + g(\bar{u}(t, z, k)) \partial_t \bar{\mathcal{W}}(t, z, k), & t > 0, \\ \bar{u}(0, z, k) = \varphi^\wedge(z, k), & (z, k) \in \Gamma. \end{cases} \tag{5.9}$$

Here \bar{L} is the differential operator on the graph Γ , introduced in Sect. 2.3, generator of the limiting Markov process $\bar{Y}(t)$. Concerning the noisy forcing $\bar{\mathcal{W}}$, it is defined by

$$\bar{\mathcal{W}}(t, z, k) = \sum_{j=1}^\infty (\widehat{u_j \mu})^\wedge(z, k) \beta_j(t), \quad t \geq 0 \quad (z, k) \in \Gamma,$$

where $\{\beta_j\}_{j \in \mathbb{N}}$ is the sequence of independent Brownian motions introduced in (5.3). We have

$$\begin{aligned} \mathbf{E} \bar{\mathcal{W}}(t, z_1, k_1) \bar{\mathcal{W}}(s, z_2, k_2) &= \mathbf{E} \oint_{C_{k_1}(z_1)} \mathcal{W}(t, x) d\mu_{z_1, k_1} \oint_{C_{k_2}(z_2)} \mathcal{W}(t, y) d\mu_{z_2, k_2} \\ &= \oint_{C_{k_1}(z_1)} \oint_{C_{k_2}(z_2)} \mathbf{E} \mathcal{W}(t, x) \mathcal{W}(t, y) d\mu_{z_1, k_1} d\mu_{z_2, k_2}. \end{aligned}$$

Thanks to (5.2), this gives

$$\begin{aligned} \mathbf{E} \bar{\mathcal{W}}(t, z_1, k_1) \bar{\mathcal{W}}(s, z_2, k_2) &= (t \wedge s) \oint_{C_{k_1}(z_1)} \oint_{C_{k_2}(z_2)} \int_{\mathbb{R}^2} e^{i\langle \lambda, x-y \rangle} \mu(d\lambda) d\mu_{z_1, k_1} d\mu_{z_2, k_2} \\ &= (t \wedge s) \int_{\mathbb{R}^2} \left[\oint_{C_{k_1}(z_1)} e^{i\langle \lambda, x \rangle} d\mu_{z_1, k_1} \oint_{C_{k_2}(z_2)} e^{-i\langle \lambda, y \rangle} d\mu_{z_2, k_2} \right] \mu(d\lambda) \\ &= (t \wedge s) \int_{\mathbb{R}^2} \left(e^{i\langle \lambda, \cdot \rangle} \right)^\wedge (z_1, k_1) \left(e^{-i\langle \lambda, \cdot \rangle} \right)^\wedge (z_2, k_2) \mu(d\lambda). \end{aligned} \tag{5.10}$$

It is immediate to check that $(\bar{u})^\wedge = \overline{u^\wedge}$, for every $u : \mathbb{R}^2 \rightarrow \mathbb{C}$. Hence, thanks to (5.10) and (3.2), this allows to conclude

$$\begin{aligned} \mathbf{E} |\bar{\mathcal{W}}(t)|_{\bar{H}_\gamma}^2 &= \sum_{k=1}^n \int_{I_k} \mathbf{E} |\bar{\mathcal{W}}(t, z, k)|^2 T_k(z) \gamma(z, k) dz \\ &= t \sum_{k=1}^n \int_{I_k} \int_{\mathbb{R}^2} \left(e^{i\langle \lambda, \cdot \rangle} \right)^\wedge \left(e^{-i\langle \lambda, \cdot \rangle} \right)^\wedge (z, k) \mu(d\lambda) T_k(z) \gamma(z, k) dz \\ &= t \int_{\mathbb{R}^2} \left[\sum_{k=1}^n \int_{I_k} \left| \left(e^{i\langle \lambda, \cdot \rangle} \right)^\wedge \right|^2 (z, k) T_k(z) \gamma(z, k) dz \right] \mu(d\lambda) \\ &= t \int_{\mathbb{R}^2} \left| \left(e^{i\langle \lambda, \cdot \rangle} \right)^\wedge \right|_{\bar{H}_\gamma}^2 \mu(d\lambda) \\ &\leq t \int_{\mathbb{R}^2} |e^{i\langle \lambda, \cdot \rangle}|_{H_\gamma}^2 \mu(d\lambda) = t \mu(\mathbb{R}^2) \int_{\mathbb{R}^2} \gamma^\vee(x) dx < \infty. \end{aligned}$$

This means that $\bar{\mathcal{W}}(t) \in L^2(\Omega; \bar{H}_\gamma)$, for every $t \geq 0$.

Moreover, due to (3.2) and (5.4), we have

$$\sum_{j=1}^\infty |\widehat{u_j \mu}^\wedge|_{\bar{H}_\gamma}^2 \leq \sum_{j=1}^\infty |\widehat{u_j \mu}|_{H_\gamma}^2 < \infty.$$

Then, in view of (B.4), we have that for every $v \in L^p(\Omega; C([0, T]; \bar{H}_\gamma))$ the stochastic convolution

$$\begin{aligned} t \in [0, +\infty) &\mapsto \int_0^t \bar{S}(t-s) G(v(s)) d\bar{W}(s) \\ &= \sum_{k=1}^\infty \int_0^t \bar{S}(t-s) [G(v(s))(\widehat{u_j \mu}^\wedge)] d\beta_j(s), \end{aligned}$$

is well defined in $L^2(\Omega; \bar{H}_\gamma)$ and if $v_1, v_2 \in L^p(\Omega; C([0, T]; \bar{H}_\gamma))$, we have

$$\begin{aligned} \mathbf{E} \sup_{t \in [0, T]} \left| \int_0^t \bar{S}(t-s) [G(v_1(s)) - G(v_2(s))] d\bar{W}(s) \right|_{\bar{H}_\gamma}^2 \\ \leq c(T) \mathbf{E} \sup_{t \in [0, T]} |v_1(t) - v_2(t)|_{\bar{H}_\gamma}^2. \end{aligned} \tag{5.11}$$

Definition 5.2 A predictable process $\bar{u}(t)$, taking values in \bar{H}_γ , is a mild solution to Eq. (5.9) if it satisfies the following integral equation

$$\bar{u}(t) = \bar{S}(t)\varphi^\wedge + \int_0^t \bar{S}(t-s) B(\bar{u}(s)) ds + \int_0^t \bar{S}(t-s) G(\bar{u}(s)) d\bar{W}(s). \tag{5.12}$$

Since the mapping B can be extended to \bar{H}_γ , as a Lipschitz continuous mapping, due to (5.11) we can conclude that there exists a unique mild solution \bar{u} to Eq. (5.9) in $L^p(\Omega, C([0, T]; \bar{H}_\gamma))$, for any $T > 0$ and $p \geq 1$. Moreover

$$\mathbf{E} \sup_{t \in [0, T]} |\bar{u}(t)|_{\bar{H}_\gamma}^p < +\infty. \tag{5.13}$$

Theorem 5.3 Assume Hypotheses 1, 3 and 2. Then, for any $\varphi \in H_\gamma$, $p \geq 1$ and $0 < \tau < T$ we have

$$\lim_{\epsilon \rightarrow 0} \mathbf{E} \sup_{t \in [\tau, T]} |u_\epsilon(t) - \bar{u}(t)^\vee|_{H_\gamma}^p = \lim_{\epsilon \rightarrow 0} \mathbf{E} \sup_{t \in [\tau, T]} |u_\epsilon(t)^\wedge - \bar{u}(t)|_{\bar{H}_\gamma}^p = 0. \tag{5.14}$$

Proof As u_ϵ and \bar{u} are mild solutions to Eqs. (5.1) and (5.9), respectively, we have

$$\begin{aligned} u_\epsilon(t) - \bar{u}(t)^\vee &= [S_\epsilon(t)\varphi - \bar{S}(t)^\vee\varphi] \\ &\quad + \left[\int_0^t S_\epsilon(t-s) B(u_\epsilon(s)) ds - \int_0^t \bar{S}(t-s)^\vee B(\bar{u}(s)^\vee) ds \right] \\ &\quad + [\Theta_\epsilon(t) - \bar{\Theta}^\vee(t)] =: \sum_{i=1}^3 I_{\epsilon, i}(t), \end{aligned}$$

where, for the sake of brevity, we have defined

$$\Theta_\epsilon(t) := \int_0^t S_\epsilon(t-s) G(u_\epsilon(s)) d\mathcal{W}(s), \quad \bar{\Theta}(t) := \int_0^t \bar{S}(t-s) G(\bar{u}(s)) d\bar{\mathcal{W}}(s).$$

In order to conclude the proof of (5.14), we need a couple of Lemmas, whose proof is postponed to the end of this section.

Lemma 5.4 *For every $p \geq 1, T > 0$ and $0 < \tau \leq t \leq T$*

$$\mathbf{E} \sup_{s \in [0,t]} |I_{\epsilon,2}(s)|_{H_Y}^p \leq c_p(T) \int_\tau^t \mathbf{E} \sup_{r \in [\tau,s]} |u_\epsilon(r) - \bar{u}(r)^\vee|_{H_Y}^p ds + R_{T,p}(\tau, \epsilon), \tag{5.15}$$

for some constant $R_{T,p}(\tau, \epsilon)$ such that

$$\lim_{\epsilon, \tau \rightarrow 0} R_{T,p}(\tau, \epsilon) = 0. \tag{5.16}$$

Lemma 5.5 *For every $p \geq 1, T > 0$ and $0 < \tau \leq t \leq T$*

$$\mathbf{E} \sup_{s \in [0,t]} |I_{\epsilon,3}(s)|_{H_Y}^p \leq c_p(T) \int_\tau^t \mathbf{E} \sup_{r \in [\tau,s]} |u_\epsilon(r) - \bar{u}(r)^\vee|_{H_Y}^p ds + S_{T,p}(\tau, \epsilon), \tag{5.17}$$

for some constant $S_{T,p}(\tau, \epsilon)$ such that

$$\lim_{\epsilon, \tau \rightarrow 0} S_{T,p}(\tau, \epsilon) = 0. \tag{5.18}$$

Now, because of Lemmas 5.4 and 5.5, we have

$$\begin{aligned} & \mathbf{E} \sup_{s \in [\tau,t]} |u_\epsilon(s) - \bar{u}^\vee(s)|_{H_Y}^p \\ & \leq c \left(\sup_{s \in [\tau,T]} |S_\epsilon(s)\varphi - \bar{S}(s)^\vee\varphi|_{H_Y}^p + R_{T,p}(\tau, \epsilon) + S_{T,p}(\tau, \epsilon) \right) \\ & \quad + c_p(T) \int_\tau^t \mathbf{E} \sup_{r \in [\tau,s]} |u_\epsilon(r) - \bar{u}^\vee(r)|_{H_Y}^p ds. \end{aligned}$$

Then, thanks the Gronwall lemma

$$\begin{aligned} & \mathbf{E} \sup_{s \in [\tau,T]} |u_\epsilon(s) - \bar{u}^\vee(s)|_{H_Y}^p \\ & \leq c_p e^{c_p(T)(T-\tau)} \left[\sup_{s \in [\tau,T]} |S_\epsilon(s)\varphi - \bar{S}(s)^\vee\varphi|_{H_Y}^p + R_{T,p}(\tau, \epsilon) + S_{T,p}(\tau, \epsilon) \right]. \end{aligned}$$

and we can conclude, thanks to (B.1), (5.16) and (5.18). □

Remark 5.6 In the proof of Theorem 5.3 we have assumed that the coefficients $b, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ are both Lipschitz-continuous. As a matter of fact, in reaction–diffusion setting, b has to have a specific form, which does not satisfy the Lipschitz condition. We believe that we could extend our result to this type of non-Lipschitz setting by using a localization argument, integrated with suitable a-priori bounds for u_ϵ .

6 Proof of Lemmas 5.4 and 5.5

Proof of Lemma 5.4 We have

$$\begin{aligned}
 I_{\epsilon,2}(t) &= \int_0^t S_\epsilon(t-s) [B(u_\epsilon(s)) - B(\bar{u}(s)^\vee)] ds \\
 &\quad + \int_0^t [S_\epsilon(t-s) - \bar{S}(t-s)^\vee] B(\bar{u}(s)^\vee) ds =: J_{\epsilon,1}(t) + J_{\epsilon,2}(t).
 \end{aligned}
 \tag{6.1}$$

If we fix $0 < \tau \leq t \leq T$, , we have

$$\begin{aligned}
 |J_{\epsilon,1}(t)|_{H_\gamma}^p &\leq c_p(T) \left(\int_0^\tau (|u_\epsilon(s)|_{H_\gamma}^p + |\bar{u}(s)^\vee|_{H_\gamma}^p + 1) ds \right. \\
 &\quad \left. + \int_\tau^t |u_\epsilon(s) - \bar{u}(s)^\vee|_{H_\gamma}^p ds \right) \\
 &\leq c_p(T) \sup_{s \in [0,T]} (|u_\epsilon(s)|_{H_\gamma}^p + |\bar{u}(s)^\vee|_{H_\gamma}^p + 1) \tau \\
 &\quad + c_p(T) \int_\tau^t \sup_{r \in [\tau,s]} |u_\epsilon(r) - \bar{u}(r)^\vee|_{H_\gamma}^p ds.
 \end{aligned}$$

Hence, from (5.8) and (5.13) we can conclude

$$\mathbf{E} \sup_{s \in [0,t]} |J_{\epsilon,1}(s)|_{H_\gamma}^p \leq c_p(T) \tau + c_p(T) \int_\tau^t \mathbf{E} \sup_{r \in [\tau,s]} |u_\epsilon(r) - \bar{u}(r)^\vee|_{H_\gamma}^p ds. \tag{6.2}$$

Concerning $J_{\epsilon,2}(t)$, we have

$$\begin{aligned}
 |J_{\epsilon,2}(t)|_{H_\gamma}^p &\leq c_p(T) \int_0^{t-\tau} |[S_\epsilon(t-s) - \bar{S}(t-s)^\vee] B(\bar{u}(s)^\vee)|_{H_\gamma}^p ds \\
 &\quad + c_p(T) \int_{t-\tau}^t (1 + |\bar{u}(s)^\vee|_{H_\gamma}^p) ds
 \end{aligned}$$

$$\begin{aligned} &\leq c_p(T) \int_0^T \sup_{r \in [\tau, T]} \left| [S_\epsilon(r) - \bar{S}(r)^\vee] B(\bar{u}(s)^\vee) \right|_{H_Y}^p ds \\ &\quad + c_p(T) \left(1 + \sup_{s \in [0, T]} |\bar{u}(s)^\vee|_{H_Y}^p \right) \tau. \end{aligned}$$

Then, according to (5.13), we conclude

$$\mathbf{E} \sup_{t \in [0, T]} |J_{\epsilon, 2}(t)|_{H_Y}^p \leq c_p(T) \left(\tau + \mathbf{E} \int_0^T \sup_{r \in [\tau, T]} \left| [S_\epsilon(r) - \bar{S}(r)^\vee] B(\bar{u}(s)^\vee) \right|_{H_Y}^p ds \right).$$

This, together with (6.2), implies (5.15), with

$$R_{T, p}(\tau, \epsilon) := c_p(T)\tau + \mathbf{E} \int_0^T \sup_{r \in [\tau, T]} \left| [S_\epsilon(r) - \bar{S}(r)^\vee] B(\bar{u}(s)^\vee) \right|_{H_Y}^p ds.$$

Moreover, (5.16) follows as a consequence of Theorem A.2 and the dominated convergence theorem, due to the equi-boundedness of the norm of $S_\epsilon(t)$ and $\bar{S}(t)$, and to (5.13). □

Proof of Lemma 5.5 First of all, we notice that, due to (3.5),

$$\bar{\Theta}^\vee(t) := \int_0^t \bar{S}(t-s)^\vee G(\bar{u}(s)^\vee) d\mathcal{W}(s).$$

Then

$$\begin{aligned} I_{\epsilon, 3}(t) &= \int_0^t S_\epsilon(t-s) [G(u_\epsilon(s)) - G(\bar{u}(s)^\vee)] d\mathcal{W}(s) \\ &\quad + \int_0^t [S_\epsilon(t-s) - \bar{S}(t-s)^\vee] G(\bar{u}(s)^\vee) d\mathcal{W}(s) =: J_{\epsilon, 1}(t) + J_{\epsilon, 2}(t). \end{aligned}$$

By using a factorization argument, for every $t \in [0, T]$ and $\alpha \in (0, 1/2)$ we have

$$J_{\epsilon, 1}(t) = \frac{\sin \pi \alpha}{\pi} \int_0^t (t-s)^{\alpha-1} S_\epsilon(t-s) Y_\alpha(s) ds,$$

where

$$Y_\alpha(s) = \int_0^s (s-\sigma)^{-\alpha} S_\epsilon(s-\sigma) [G(u_\epsilon(\sigma)) - G(\bar{u}(\sigma)^\vee)] d\mathcal{W}(\sigma).$$

Therefore, thanks to (4.1), for every $p \geq 1/\alpha$

$$\mathbf{E} \sup_{s \in [0, t]} |J_{\epsilon, 1}(s)|_{H_Y}^p \leq c_{p, \alpha}(T) \int_0^t \mathbf{E} |Y_\alpha(s)|_{H_Y}^p ds. \tag{6.3}$$

In view of (4.1) and (5.5), we have

$$\begin{aligned}
 & \mathbf{E} |Y_\alpha(s)|_{H_\gamma}^p \\
 &= c_p \mathbf{E} \left(\int_0^s (s - \sigma)^{-2\alpha} \sum_{j=1}^\infty |S_\epsilon(s - \sigma) [G(u_\epsilon(\sigma)) - G(\bar{u}(\sigma)^\vee)] \widehat{u_j \mu}|_{H_\gamma}^2 d\sigma \right)^{\frac{p}{2}} \\
 &\leq c_p(T) \mathbf{E} \left(\int_0^s (s - \sigma)^{-2\alpha} \sum_{j=1}^\infty \|[G(u_\epsilon(\sigma)) - G(\bar{u}(\sigma)^\vee)] \widehat{u_j \mu}\|_{H_\gamma}^2 d\sigma \right)^{\frac{p}{2}} \\
 &= c_p(T) \mathbf{E} \left(\int_0^s (s - \sigma)^{-2\alpha} \|G(u_\epsilon(\sigma)) - G(\bar{u}(\sigma)^\vee)\|_{\mathcal{L}_2(RK, H_\gamma)}^2 d\sigma \right)^{\frac{p}{2}} \\
 &\leq c_p(T) \mathbf{E} \left(\int_0^s (s - \sigma)^{-2\alpha} |u_\epsilon(\sigma) - \bar{u}(\sigma)^\vee|_{H_\gamma}^2 d\sigma \right)^{\frac{p}{2}}.
 \end{aligned}$$

Hence, thanks to the Young inequality, due to (5.8) and (5.13), for any $0 < \tau < t < T$

$$\begin{aligned}
 & \int_0^t \mathbf{E} |Y_\alpha(s)|_{H_\gamma}^p ds \leq c_p(T) \left(\int_0^t s^{-2\alpha} ds \right)^{\frac{p}{2}} \int_0^t \mathbf{E} |u_\epsilon(s) - \bar{u}(s)^\vee|_{H_\gamma}^p ds \\
 &\leq c_p(T) \tau \left(\mathbf{E} \sup_{s \in [0, T]} |u_\epsilon(s)|_{H_\gamma}^p + \mathbf{E} \sup_{s \in [0, T]} |\bar{u}(s)|_{H_\gamma}^p \right) \\
 &\quad + c_p(T) \int_\tau^t \mathbf{E} \sup_{r \in [\tau, s]} |u_\epsilon(r) - \bar{u}(r)^\vee|_{H_\gamma}^p ds \\
 &\leq c_p(T) \left(\tau + \int_\tau^t \mathbf{E} \sup_{r \in [\tau, s]} |u_\epsilon(r) - \bar{u}(r)^\vee|_{H_\gamma}^p ds \right).
 \end{aligned}$$

According to (6.3), this yields

$$\mathbf{E} \sup_{s \in [0, t]} |J_{\epsilon, 1}(s)|_{H_\gamma}^p \leq c_p(T) \left(\tau + \int_\tau^t \mathbf{E} \sup_{r \in [\tau, s]} |u_\epsilon(r) - \bar{u}(r)^\vee|_{H_\gamma}^p ds \right). \tag{6.4}$$

Next, by using again a factorization argument, for every $t \in [0, T]$ and $\alpha \in (0, 1/2)$ we have

$$\begin{aligned}
 J_{\epsilon, 2}(t) &= \frac{\sin \pi \alpha}{\pi} \int_0^t (t - s)^{\alpha-1} S_\epsilon(t - s) Y_{\alpha, 1}(s) ds \\
 &\quad + \frac{\sin \pi \alpha}{\pi} \int_0^t (t - s)^{\alpha-1} [S_\epsilon(t - s) - \bar{S}(t - s)^\vee] Y_{\alpha, 2}(s) ds,
 \end{aligned}$$

where

$$Y_{\alpha,1}(s) = \int_0^s (s - \sigma)^{-\alpha} [S_\epsilon(s - \sigma) - \bar{S}(s - \sigma)^\vee] G(\bar{u}(\sigma)^\vee) d\mathcal{W}(\sigma),$$

and

$$Y_{\alpha,2}(s) = \int_0^s (s - \sigma)^{-\alpha} \bar{S}(s - \sigma)^\vee G(\bar{u}(\sigma)^\vee) d\mathcal{W}(\sigma).$$

Thus, by proceeding as above for $J_{\epsilon,1}(t)$, thanks to (4.1) we get

$$\begin{aligned} \mathbf{E} \sup_{s \in [0,t]} |J_{\epsilon,2}(s)|_{H_Y}^p &\leq c_{p,\alpha}(T) \int_0^t \mathbf{E} |Y_{\alpha,1}(s)|_{H_Y}^p ds \\ &\quad + c_{p,\alpha}(T) \mathbf{E} \sup_{s \in [0,t]} \int_0^s |[S_\epsilon(s - r) - \bar{S}(s - r)^\vee] Y_{\alpha,2}(r)|_{H_Y}^p dr \\ &=: H_{\epsilon,1}(t) + H_{\epsilon,2}(t). \end{aligned}$$

According to (4.1) and (B.4), we have

$$\begin{aligned} &\mathbf{E} |Y_{\alpha,1}(s)|_{H_Y}^p \\ &\leq c_p \mathbf{E} \left(\sum_{j=1}^\infty \int_0^s (s - \sigma)^{-2\alpha} |[S_\epsilon(s - \sigma) - \bar{S}(s - \sigma)^\vee] G(\bar{u}(\sigma)^\vee) \widehat{u}_{j\mu}|_{H_Y}^2 d\sigma \right)^{\frac{p}{2}} \\ &\leq c_p \mathbf{E} \left(\sum_{j=1}^\infty \int_0^s (s - \sigma)^{-2\alpha} |G(\bar{u}(\sigma)^\vee) \widehat{u}_{j\mu}|_{H_Y}^2 d\sigma \right)^{\frac{p}{2}}. \end{aligned}$$

Since

$$\left(\sum_{j=1}^\infty \int_0^s (s - \sigma)^{-2\alpha} |G(\bar{u}(\sigma)^\vee) \widehat{u}_{j\mu}|_{H_Y}^2 d\sigma \right)^{\frac{p}{2}} \leq c_p(T) \left(1 + \sup_{\sigma \in [0,s]} |\bar{u}(\sigma)^\vee|_{H_Y}^p \right),$$

due to (5.13) and the dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \mathbf{E} \left(\sum_{j=n+1}^\infty \int_0^s (s - \sigma)^{-2\alpha} |G(\bar{u}(\sigma)^\vee) \widehat{u}_{j\mu}|_{H_Y}^2 d\sigma \right)^{\frac{p}{2}} = 0.$$

Therefore, for every $\eta > 0$ we can fix $n_\eta \in \mathbb{N}$ such that

$$\mathbf{E}|Y_{\alpha,1}(s)|_{H_\nu}^p \leq \eta + c_p(T) \mathbf{E} \left(\sum_{j=n_\eta+1}^\infty \int_0^s (s-\sigma)^{-2\alpha} \|[S_\epsilon(s-\sigma) - \bar{S}(s-\sigma)^\vee] G(\bar{u}(\sigma)^\vee) \widehat{u}_{j\mu}\|_{H_\nu}^2 d\sigma \right)^{\frac{p}{2}}.$$

Once fixed n_η , due to (B.1) and the dominated convergence theorem, we have

$$\lim_{\epsilon \rightarrow 0} \mathbf{E} \left(\sum_{j=n_\eta+1}^\infty \int_0^s (s-\sigma)^{-2\alpha} \|[S_\epsilon(s-\sigma) - \bar{S}(s-\sigma)^\vee] G(\bar{u}(\sigma)^\vee) \widehat{u}_{j\mu}\|_{H_\nu}^2 d\sigma \right)^{\frac{p}{2}} = 0,$$

and this, due to the arbitrariness of η , allows us to conclude that

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [0, T]} H_{\epsilon,1}(t) = 0. \tag{6.5}$$

As far $H_{\epsilon,2}(t)$ is concerned, due to (4.1) and (B.4) for every $0 < \tau \leq s \leq t$ we have

$$\begin{aligned} & \int_0^s \|[S_\epsilon(s-r) - \bar{S}(s-r)^\vee] Y_{\alpha,2}(r)\|_{H_\nu}^p dr \\ &= \int_0^{s-\tau} \|[S_\epsilon(s-r) - \bar{S}(s-r)^\vee] Y_{\alpha,2}(r)\|_{H_\nu}^p dr \\ & \quad + \int_{s-\tau}^s \|[S_\epsilon(s-r) - \bar{S}(s-r)^\vee] Y_{\alpha,2}(r)\|_{H_\nu}^p dr \\ & \leq \int_0^T \sup_{\rho \in [\tau, T]} \|[S_\epsilon(\rho) - \bar{S}(\rho)^\vee] Y_{\alpha,2}(r)\|_{H_\nu}^p dr \\ & \quad + c_p(T) \sqrt{\tau} \left(\int_0^T |Y_{\alpha,2}(r)|_{H_\nu}^{2p} dr \right)^{\frac{1}{2}}. \end{aligned}$$

Moreover, if $0 \leq s \leq \tau$, we have

$$\int_0^s \|[S_\epsilon(s-r) - \bar{S}(s-r)^\vee] Y_{\alpha,2}(r)\|_{H_\nu}^p dr \leq c_p(T) \sqrt{\tau} \left(\int_0^T |Y_{\alpha,2}(r)|_{H_\nu}^{2p} dr \right)^{\frac{1}{2}}.$$

Since, for every $q \geq 1$

$$\mathbf{E} \int_0^T |Y_{\alpha,2}(r)|_{H_\nu}^q dr < \infty, \tag{6.6}$$

this implies

$$H_{\epsilon,2}(t) \leq \mathbf{E} \int_0^T \sup_{\rho \in [\tau, T]} \left| [S_\epsilon(\rho) - \bar{S}(\rho)^\vee] Y_{\alpha,2}(r) \right|_{H_Y}^p dr + c_p(T) \sqrt{\tau}.$$

Now, let us fix any $\eta > 0$ and $\tau_\eta > 0$ such that $c_p(T) \sqrt{\tau_\eta} < \eta$. Because of (B.1), we have

$$\lim_{\epsilon \rightarrow 0} \sup_{\rho \in [\tau_\eta, T]} \left| [S_\epsilon(\rho) - \bar{S}(\rho)^\vee] Y_{\alpha,2}(r) \right|_{H_Y}^p = 0.$$

Moreover, as

$$\sup_{\rho \in [\tau_\eta, T]} \left| [S_\epsilon(\rho) - \bar{S}(\rho)^\vee] Y_{\alpha,2}(r) \right|_{H_Y}^p \leq c_p(T) |Y_{\alpha,2}(r)|_{H_Y}^p,$$

from (6.6) and the dominated convergence theorem, we get

$$\liminf_{\epsilon \rightarrow 0} \sup_{t \in [0, T]} H_{\epsilon,2}(t) \leq \lim_{\epsilon \rightarrow 0} \mathbf{E} \int_0^T \sup_{\rho \in [\tau, T]} \left| [S_\epsilon(\rho) - \bar{S}(\rho)^\vee] Y_{\alpha,2}(r) \right|_{H_Y}^p dr + \eta = \eta.$$

Due to the arbitrariness of $\eta > 0$, together with (6.5), this implies that

$$\lim_{\epsilon \rightarrow 0} \mathbf{E} \sup_{s \in [0, t]} |J_{\epsilon,2}(s)|_{H_Y}^p = 0,$$

and this, together with (6.4), implies (5.17). □

A The limiting result for the linear deterministic problem

For every $\epsilon > 0$, we consider the linear parabolic Cauchy problem associated with the second order differential operator \mathcal{L}_ϵ

$$\begin{cases} \partial_t v_\epsilon(t, x) = \mathcal{L}_\epsilon v_\epsilon(t, x), & t > 0, \quad x \in \mathbb{R}^2, \\ v_\epsilon(0, x) = g(x), & x \in \mathbb{R}^2. \end{cases} \tag{A.1}$$

The solution of problem (A.1) has a probabilistic representation in terms of the Markov transition semigroup $S_\epsilon(t)$ associated with Eq. (2.12), that, we recall, is defined for any bounded Borel function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$S_\epsilon(t)\varphi(x) = \mathbb{E}_x \varphi(X_\epsilon(t)), \quad x \in \mathbb{R}^2.$$

Actually, as a consequence of Itô’s formula, if the initial condition φ is taken in $C_b^2(\mathbb{R}^2)$, then

$$\frac{d}{dt} [S_\epsilon(\cdot)\varphi(x)](t) = \mathcal{L}_\epsilon (S_\epsilon(t)\varphi)(x), \quad t \geq 0, \quad x \in \mathbb{R}^2. \tag{A.2}$$

Moreover, as the Hamiltonian H is assumed to be of class $C^4(\mathbb{R})$, with bounded second derivative, we have that the semigroup $S_\epsilon(t)$ has a smoothing effect, namely it maps Borel bounded functions into $C_b^3(\mathbb{R}^2)$, for any $t > 0$. Thanks to the semigroup law, this allows to conclude that (A.2) is satisfied on \mathbb{R}^2 , for any Borel bounded function φ and for all $t > 0$.

Our purpose here is to study the asymptotic behavior of $S_\epsilon(t)\varphi(x)$, and hence of $v_\epsilon(t, x)$, as $\epsilon \downarrow 0$. As a consequence of (2.15), if the Hamiltonian H satisfies Hypothesis 1, then for any continuous mapping $\psi : \Gamma \rightarrow \mathbb{R}$ we have

$$\lim_{\epsilon \rightarrow 0} S_\epsilon(t)\psi^\vee(x) = \bar{S}(t)\psi(\Pi(x)), \quad t \geq 0, \quad x \in \mathbb{R}^2. \tag{A.3}$$

As a first thing, we are going to prove that the limit above is uniform with respect to $t \in [0, T]$, for every fixed $T > 0$. To this purpose, we introduce some notation.

For any $\eta > 0$ we can fix

$$z_\eta \geq \max_{i=1, \dots, n} H(x_i) + 1, \tag{A.4}$$

(we recall x_1, \dots, x_n are the critical points of the Hamiltonian H), such that, for all $\epsilon > 0$ sufficiently small,

$$\mathbb{P}_x(\rho_{\epsilon, \eta} \leq T) \leq \eta, \quad \bar{\mathbb{P}}_{\Pi(x)}(\rho_\eta \leq T) \leq \eta, \tag{A.5}$$

where

$$\rho_{\epsilon, \eta} := \inf \{t \geq 0 : H(X_\epsilon^x(t)) \geq z_\eta\}, \quad \rho_\eta := \inf \{t \geq 0 : \Pi_1 \bar{Y}(t) \geq z_\eta\}, \tag{A.6}$$

(here we have denoted $\pi_1(z, k) = z$). Actually, as proved in [5, Lemma 3.2] the family $\{\Pi(X_\epsilon(\cdot))\}_{\epsilon \in (0, \epsilon_0)}$ is tight in $C([0, T]; \mathbb{R})$. This means that there exists some $z_\eta > 0$ as in (A.4) such that

$$\mathbb{P}_x \left(\sup_{t \leq T} H(X_\epsilon(t)) \geq z_\eta \right) \leq \eta,$$

and (A.5) follows.

In the next proposition, we show that limit (1.5) is in fact uniform with respect to $t \in [0, T]$, for every $T > 0$.

Proposition A.1 *For every $T > 0$ and $\psi \in C_b(\Gamma)$, we have*

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [0, T]} |\mathbb{E}_x \psi^\vee(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} \psi(\bar{Y}(t))| = 0. \tag{A.7}$$

Proof For every $\epsilon > 0$, let us define

$$f_\epsilon(t) := \mathbb{E}_x \psi^\vee(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} \psi(\bar{Y}(t)).$$

For every fixed $t \geq 0$, we have

$$\lim_{\epsilon \rightarrow 0} f_\epsilon(t) = 0. \tag{A.8}$$

If we prove that the family of functions $\{f_\epsilon\}_{\epsilon>0}$ is equibounded and equicontinuous in $C([0, T])$, by the Ascoli-Arzelà Theorem we have that the limit in (A.8) is uniform.

Now, the equiboundedness of $\{f_\epsilon\}_{\epsilon>0}$ follows from the fact that ψ is bounded. In order to prove the equicontinuity of $\{f_\epsilon\}_{\epsilon>0}$, first of all we notice that we may assume that both ψ and ψ^\vee are uniformly continuous. Actually, due to (A.5) for every $\eta > 0$ we have

$$\begin{aligned} & \left| \mathbb{E}_x \psi^\vee(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} \psi(\bar{Y}(t)) \right| \leq \left| \mathbb{E}_x (\psi^\vee(X_\epsilon(t)); \rho_{\epsilon,\eta} \leq T) \right| \\ & \quad + \left| \bar{\mathbb{E}}_{\Pi(x)} (\psi(\bar{Y}(t)); \rho_\eta \leq T) \right| \\ & \quad + \left| \mathbb{E}_x (\psi^\vee(X_\epsilon(t)); \rho_{\epsilon,\eta} > T) - \bar{\mathbb{E}}_{\Pi(x)} (\psi(\bar{Y}(t)); \rho_\eta > T) \right| \\ & \leq 2 \|\psi\|_\infty \eta + \left| \mathbb{E}_x (\psi^\vee(X_\epsilon(t)); \rho_{\epsilon,\eta} > T) - \bar{\mathbb{E}}_{\Pi(x)} (\psi(\bar{Y}(t)); \rho_\eta > T) \right|. \end{aligned}$$

Therefore, as ψ^\vee is uniformly continuous on $\{H(x) \leq z_\eta\}$ and ψ is uniformly continuous on $\{z \leq z_\eta\}$, due to the arbitrariness of η we can conclude.

Now, if ψ are uniformly continuous, for every $\eta > 0$ there exists $\delta_\eta > 0$ such that

$$\begin{aligned} & \left| \Pi(x_1) - \Pi(x_2) \right| < \delta_\eta, \quad \text{dist}((z_1, k_1), (z_2, k_2)) < \delta_\eta \\ & \implies |\psi^\vee(x_1) - \psi^\vee(x_2)| + |\psi(z_1, k_1) - \psi(z_2, k_2)| < \frac{\eta}{2}. \end{aligned} \tag{A.9}$$

Moreover, as $\{\Pi(X_\epsilon)\}_{\epsilon>0} \subset C([0, T]; \Gamma)$ is tight, there exists a compact set $K_\eta \subset C([0, T]; \Gamma)$, such that

$$\mathbb{P}_x \left(\Pi(X_\epsilon) \in K_\eta^c \right) + \bar{\mathbb{P}}_{\Pi(x)} \left(\bar{Y} \in K_\eta^c \right) \leq \frac{\eta}{4\|\psi\|_\infty}. \tag{A.10}$$

Since functions in K_η are equicontinuous, there exists $\theta_\eta > 0$ such that on K_η

$$|t - s| < \theta_\eta \implies |\Pi(X_\epsilon(t)) - \Pi(X_\epsilon(s))| < \delta_\eta, \quad |\bar{Y}(t) - \bar{Y}(s)| < \delta_\eta.$$

Therefore, thanks to (A.9) and (A.10), for every $t, s \in [0, T]$ such that $|t - s| < \theta_\eta$

$$\begin{aligned} & |f_\epsilon(t) - f_\epsilon(s)| \\ & \leq \left| \mathbb{E}_x (\psi^\vee(X_\epsilon(t)) - \psi^\vee(X_\epsilon(s)); \Pi(X_\epsilon) \in K_\eta) \right| \\ & \quad + \left| \bar{\mathbb{E}}_{\Pi(x)} (\psi(\bar{Y}(t)) - \psi(\bar{Y}(s)); \bar{Y} \in K_\eta) \right| \\ & \quad + 2 \|\psi\|_\infty \left(\mathbb{P}_x \left(\Pi(X_\epsilon) \in K_\eta^c \right) + \bar{\mathbb{P}}_{\Pi(x)} \left(\bar{Y} \in K_\eta^c \right) \right) \\ & \leq \frac{\eta}{2} + 2 \|\psi\|_\infty \frac{\eta}{4\|\psi\|_\infty} = \eta. \end{aligned}$$

□

In what follows, we want to show that, in fact, under suitable conditions, limit (A.7) is also true if ψ^\vee is replaced by $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ and ψ is replaced by u^\wedge . Namely, we want to prove the following result.

Theorem A.2 *Assume that the Hamiltonian H satisfies all conditions in Hypotheses 1 and 2. Then, for any $u \in C_b(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, and for any $0 < \tau \leq T$, we have*

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \sup_{t \in [\tau, T]} |S_\epsilon(t)u(x) - \bar{S}(t)^\vee u(x)| \\ &= \lim_{\epsilon \rightarrow 0} \sup_{t \in [\tau, T]} |\mathbb{E}_x u(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} u^\wedge(\bar{Y}(t))| = 0. \end{aligned} \tag{A.11}$$

A.1 A preliminary result

Before proceeding with the proof of Theorem A.2 and of all required preliminary results, we introduce some notations. For every $\epsilon, \eta > 0$ and $0 < \delta' < \delta$, by using the notations introduced in Sect. 2.2 we define

$$\begin{aligned} \sigma_n^{\epsilon, \eta, \delta, \delta'} &= \min \left\{ t \geq \tau_n^{\epsilon, \eta, \delta, \delta'} : X_\epsilon(t) \in G(\pm\delta)^c \right\}, \\ \tau_n^{\epsilon, \eta, \delta, \delta'} &= \min \left\{ t \geq \sigma_{n-1}^{\epsilon, \eta, \delta, \delta'} : X_\epsilon(t) \in D(\pm\delta') \cup C(z_\eta) \right\}, \end{aligned} \tag{A.12}$$

with $\tau_0^{\epsilon, \eta, \delta, \delta'} = 0$ and z_η defined as in (A.4). Clearly, after the process $X_\epsilon(t)$ reaches $C(z_\eta)$, all $\tau_n^{\epsilon, \eta, \delta, \delta'}$ and $\sigma_n^{\epsilon, \eta, \delta, \delta'}$ coincide with the stopping time $\rho_{\epsilon, \eta}$ introduced in (A.6), and for any $n \in \mathbb{N}$

$$X_\epsilon \left(\sigma_n^{\epsilon, \eta, \delta, \delta'} \right) \in D(\pm\delta) \cup C(z_\eta), \quad X_\epsilon \left(\tau_n^{\epsilon, \eta, \delta, \delta'} \right) \in D(\pm\delta') \cup C(z_\eta). \tag{A.13}$$

Moreover, if $X_\epsilon(0) \in G(\pm\delta)^c$, we have that $\sigma_0^{\epsilon, \eta, \delta, \delta'} = 0$ and $\tau_1^{\epsilon, \eta, \delta, \delta'}$ is the first time the process X_ϵ touches $D(\pm\delta') \cup C(z_\eta)$. In particular, if $X_\epsilon(0) \geq z_\eta$, then $\tau_1^{\epsilon, \eta, \delta, \delta'}$ is the first time the process X_ϵ touches $C(z_\eta)$ and all successive stopping times coincide with $\rho_{\epsilon, \eta}$.

Lemma A.3 *Assume that the same assumptions of Theorem A.2 are verified. Then, for every $u \in C_b(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, and for every $0 < \tau < T$*

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [\tau, T]} |\mathbb{E}_x (u^\wedge)^\vee(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} u^\wedge(\bar{Y}(t))| = 0. \tag{A.14}$$

Proof Thanks to (A.7), if u^\wedge were a continuous function on Γ , then (A.14) would follow immediately. Unfortunately, because of the presence of the interior vertices, even if u is continuous on \mathbb{R}^2 we cannot conclude that u^\wedge is continuous on Γ , in general. This means that we have to treat separately the internal vertices and the rest of the points of the graph Γ .

First of all, we fix $\delta > 0$ and $f_\delta \in C_b(\Gamma)$ such that

$$\|f_\delta\|_\infty \leq \|u^\wedge\|_\infty \leq \|u\|_\infty, \quad f_\delta = u^\wedge, \text{ on } \Pi(G(\pm\delta/2)^c).$$

Thus, for every $\delta \in (0, 1)$ and $t \geq 0$, we can write

$$\begin{aligned} \mathbb{E}_x (u^\wedge)^\vee(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} u^\wedge(\bar{Y}(t)) &= \mathbb{E}_x \left[(u^\wedge)^\vee(X_\epsilon(t)) - f_\delta^\vee(X_\epsilon(t)) \right] \\ &\quad + \left[\mathbb{E}_x f_\delta^\vee(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} f_\delta(\bar{Y}(t)) \right] + \bar{\mathbb{E}}_{\Pi(x)} \left[f_\delta(\bar{Y}(t)) - u^\wedge(\bar{Y}(t)) \right] \\ &=: I_1^{\epsilon,\delta}(t) + I_2^{\epsilon,\delta}(t) + I^\delta(t). \end{aligned}$$

If we prove that for any $\eta > 0$ there exist $\delta_\eta, \epsilon_\eta > 0$ such that

$$\sup_{t \in [\tau, T]} \left(|I_1^{\epsilon,\delta_\eta}(t)| + |I^{\delta_\eta}(t)| \right) < \eta, \quad \epsilon < \epsilon_\eta, \tag{A.15}$$

then

$$\begin{aligned} &\sup_{t \in [\tau, T]} \left| \mathbb{E}_x (u^\wedge)^\vee(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} u^\wedge(\bar{Y}(t)) \right| \\ &\leq \eta + \sup_{t \in [\tau, T]} \left| \mathbb{E}_x f_{\delta_\eta}^\vee(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} f_{\delta_\eta}(\bar{Y}(t)) \right|, \quad \epsilon < \epsilon_\eta. \end{aligned}$$

Since $f_{\delta_\eta} \in C_b(\Gamma)$, due to (A.7) and the arbitrariness of $\eta > 0$, this implies (A.14).

Thus, let us prove (A.15). If we take $\eta' = \eta/8\|u\|_\infty$, due to (A.5) we have

$$\begin{aligned} I_1^{\epsilon,\delta}(t) &= \mathbb{E}_x \left((u^\wedge)^\vee(X_\epsilon(t)) - f_\delta^\vee(X_\epsilon(t)); t < \rho_{\epsilon,\eta'} \right) \\ &\quad + \mathbb{E}_x \left((u^\wedge)^\vee(X_\epsilon(t)) - f_\delta^\vee(X_\epsilon(t)); t \geq \rho_{\epsilon,\eta'} \right) \\ &\leq \mathbb{E}_x \left((u^\wedge)^\vee(X_\epsilon(t)) - f_\delta^\vee(X_\epsilon(t)); t < \rho_{\epsilon,\eta'} \right) + 2\|u\|_\infty \mathbb{P}_x(t \geq \rho_{\epsilon,\eta'}) \\ &\leq \mathbb{E}_x \left((u^\wedge)^\vee(X_\epsilon(t)) - f_\delta^\vee(X_\epsilon(t)); t < \rho_{\epsilon,\eta'} \right) + \frac{\eta}{4} =: J_1^{\epsilon,\eta',\delta}(t) + \frac{\eta}{4}. \end{aligned} \tag{A.16}$$

Recalling that $f_\delta = u^\wedge$ on $\Pi(G(\pm\delta/2))^c$, we have that $f_\delta^\vee = (u^\wedge)^\vee$ on $G(\pm\delta/2)^c$, so that

$$\begin{aligned} J_1^{\epsilon,\eta',\delta}(t) &= \mathbb{E}_x \left((u^\wedge)^\vee(X_\epsilon(t)) - f_\delta^\vee(X_\epsilon(t)); t \in \bigcup_{n=0}^\infty \left[\tau_n^{\epsilon,\eta',\delta,\delta/2}, \sigma_n^{\epsilon,\eta',\delta,\delta/2} \right) \right) \\ &= \sum_{n=0}^\infty \mathbb{E}_x \left((u^\wedge)^\vee(X_\epsilon(t)) - f_\delta^\vee(X_\epsilon(t)); t \in \left[\tau_n^{\epsilon,\eta',\delta,\delta/2}, \sigma_n^{\epsilon,\eta',\delta,\delta/2} \right) \right) \\ &=: \sum_{n=0}^\infty J_{1,n}^{\epsilon,\eta',\delta}(t). \end{aligned} \tag{A.17}$$

Due to the strong Markov property, we have

$$\begin{aligned} |J_{1,n}^{\epsilon,\eta',\delta}(t)| &= \left| \mathbb{E}_x \left([(u^\wedge)^\vee(X_\epsilon(t)) - f_\delta^\vee(X_\epsilon(t))] \mathbb{I}_{\{t \geq \tau_n^{\epsilon,\eta',\delta,\delta/2}\}} \mathbb{I}_{\{\sigma_n^{\epsilon,\eta',\delta,\delta/2} > t\}} \right) \right| \\ &\leq \mathbb{E}_x \left(\mathbb{I}_{\{\tau_n^{\epsilon,\eta',\delta,\delta/2} \leq t\}} \left| \mathbb{E}_{X_\epsilon(\tau_n^{\epsilon,\eta',\delta,\delta/2})} \left((u^\wedge)^\vee(X_\epsilon(t)) - f_\delta^\vee(X_\epsilon(t)) ; t < \sigma_0^{\epsilon,\eta',\delta,\delta/2} \right) \right| \right). \end{aligned}$$

Thanks to (A.13), this implies

$$\begin{aligned} |J_{1,n}^{\epsilon,\eta',\delta}(t)| &\leq \mathbb{P}_x \left(\tau_n^{\epsilon,\eta',\delta,\delta/2} \leq t \right) \sup_{y \in D(\pm\delta/2) \cup C(z'_\eta)} \left| \mathbb{E}_y \left((u^\wedge)^\vee(X_\epsilon(t)) - f_\delta^\vee(X_\epsilon(t)) ; t < \sigma_0^{\epsilon,\eta',\delta,\delta/2} \right) \right| \\ &\leq e^t \mathbb{E}_x e^{-\tau_n^{\epsilon,\eta',\delta,\delta/2}} \sup_{y \in D(\pm\delta/2) \cup C(z'_\eta)} \left| \mathbb{E}_y \left((u^\wedge)^\vee(X_\epsilon(t)) - f_\delta^\vee(X_\epsilon(t)) ; t < \sigma_0^{\epsilon,\eta',\delta,\delta/2} \right) \right|. \end{aligned} \tag{A.18}$$

According to what proved in [5, Section 8.3, see (8.3.14)], there exist a constant $c > 0$ and $\delta_1 > 0$ such that for all $x \in \mathbb{R}^2$ and for all $\delta \leq \delta_1$ and $\epsilon > 0$ sufficiently small it holds

$$\sum_{n=0}^\infty \mathbb{E}_x e^{-\tau_n^{\epsilon,\eta',\delta,\delta/2}} = \sum_{n=0}^\infty \mathbb{E}_x \left(e^{-\tau_n^{\epsilon,\eta',\delta,\delta/2}} ; \tau_n^{\epsilon,\eta',\delta,\delta/2} \leq \rho_{\epsilon,\eta'} \right) \leq \frac{c}{\delta}. \tag{A.19}$$

Therefore, from (A.18) we get

$$\sum_{n=0}^\infty |J_{1,n}^{\epsilon,\eta',\delta}(t)| \leq \frac{c}{\delta} \frac{2\|u\|_\infty e^t}{t} \sup_{y \in D(\pm\delta/2)} \mathbb{E}_y \sigma_0^{\epsilon,\eta',\delta,\delta/2}. \tag{A.20}$$

Because of our definition, $\sigma_0^{\epsilon,\eta',\delta,\delta/2}$ is the first exit time of the process $X_\epsilon(t)$ from $G(\pm\delta)$. In [5, Section 8.5, (8.5.17)] it is proved that there exists $\delta_2 > 0$ such that for all $\delta < \delta_2$ and $\epsilon > 0$ sufficiently small

$$\mathbb{E}_y \sigma_0^{\epsilon,\eta',\delta,\delta/2} \leq c \delta^2 |\log \delta|, \quad y \in G(\pm\delta). \tag{A.21}$$

In particular, due to (A.16), (A.17) and (A.20), this implies that for all $\delta < \delta_0 := \delta_1 \wedge \delta_2$ and all ϵ small enough

$$\sup_{t \in [\tau, T]} |I_1^{\epsilon,\delta}(t)| \leq \frac{c}{\delta} \frac{2\|u\|_\infty e^T}{\tau} \delta^2 |\log \delta| + \frac{\eta}{4}.$$

This means that for any $\eta > 0$ fixed, there exist $\epsilon_{1,\eta} > 0$ and $\delta_{1,\eta} > 0$ such that

$$\sup_{t \in [\tau, T]} |I_1^{\epsilon,\delta}(t)| \leq \frac{\eta}{2}, \quad \epsilon < \epsilon_{1,\eta}, \quad \delta < \delta_{1,\eta}. \tag{A.22}$$

Concerning $I^\delta(t)$, recalling how f_δ was defined, for every $\delta > 0$ we have

$$\begin{aligned}
 |I^\delta(t)| &\leq 2 \|u\|_\infty \bar{\mathbb{P}}_{\Pi(x)}(\bar{Y}(t) \in \Pi(G(\pm\delta/2))) \leq 2 \|u\|_\infty \bar{\mathbb{E}}_{\Pi(x)} \psi_\delta(\bar{Y}(t)) \\
 &\leq 2 \|u\|_\infty \left| \bar{\mathbb{E}}_{\Pi(x)} \psi_\delta(\bar{Y}(t)) - \mathbb{E}_x \psi_\delta^\vee(X_\epsilon(t)) \right| \\
 &\quad + 2 \|u\|_\infty \mathbb{E}_x \psi_\delta^\vee(X_\epsilon(t)),
 \end{aligned}
 \tag{A.23}$$

where ψ_δ is a function in $C_b(\Gamma)$ such that

$$\mathbb{1}_{\Pi(G(\pm\delta/2))} \leq \psi_\delta \leq 1, \quad \psi_\delta \equiv 0, \quad \text{on } \Pi(G(\pm(\delta)))^c.$$

Now, by proceeding as in the proof of (A.22), we can find $\epsilon_{2,\eta} > 0$ and $\delta_{2,\eta} > 0$ such that

$$2 \|u\|_\infty \sup_{t \in [\tau, T]} \mathbb{E}_x \psi_\delta^\vee(X_\epsilon(t)) \leq \frac{\eta}{4}, \quad \epsilon \leq \epsilon_{2,\eta}, \quad \delta \leq \delta_{2,\eta}.
 \tag{A.24}$$

Therefore, if we set $\delta_\eta := \delta_{1,\eta} \wedge \delta_{2,\eta}$, from (A.22) and (A.23) we get

$$\begin{aligned}
 &\sup_{t \in [\tau, T]} \left(|I_1^{\epsilon, \delta_\eta}(t)| + |I^{\delta_\eta}(t)| \right) \\
 &\leq \frac{\eta}{2} + 2 \|u\|_\infty \sup_{t \in [\tau, T]} \left| \bar{\mathbb{E}}_{\Pi(x)} \psi_{\delta_\eta}(\bar{Y}(t)) - \mathbb{E}_x \psi_{\delta_\eta}^\vee(X_\epsilon(t)) \right|,
 \end{aligned}$$

for every $\epsilon \leq \epsilon_{1,\eta} \wedge \epsilon_{2,\eta}$. As $\psi_{\delta_\eta} \in C_b(\Gamma)$, due to (A.7), this implies that there exists $\epsilon_\eta \leq \epsilon_{1,\eta} \wedge \epsilon_{2,\eta}$ such that (A.15) holds and hence (A.14) follows. \square

A.2 Proof of Theorem A.2

In Lemma A.3 we have proved that for any $u \in C_b(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ and for any $0 < \tau < T$

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [\tau, T]} \left| \mathbb{E}_x (u^\wedge)^\vee(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} u^\wedge(\bar{Y}(t)) \right| = 0.$$

Thus, in order to prove (A.11), it is sufficient to prove that

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [\tau, T]} \left| \mathbb{E}_x [u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t))] \right| = 0.
 \tag{A.25}$$

In what follows, we shall assume that $u \in C_b^1(\mathbb{R}^2)$. Actually, if this is not the case, we can fix a sequence $\{u_n\}_{n \in \mathbb{N}} \subset C_b^1(\mathbb{R}^2)$ such that

$$\lim_{n \rightarrow \infty} \|u - u_n\|_\infty = 0, \quad \|u_n\|_\infty \leq \|u\|_\infty.$$

Since this also implies that

$$\lim_{n \rightarrow \infty} \|(u^\wedge)^\vee - (u_n^\wedge)^\vee\|_\infty = 0,$$

we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sup_{\epsilon > 0} \sup_{t \in [\tau, T]} |\mathbb{E}_x [u(X_\epsilon(t)) - u_n(X_\epsilon(t))(X_\epsilon(t))]| \\ &= \lim_{n \rightarrow \infty} \sup_{\epsilon > 0} \sup_{t \in [\tau, T]} |\mathbb{E}_x [(u_n^\wedge)^\vee(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t))]| = 0. \end{aligned}$$

Therefore, in order to prove (A.25), we have to prove that for any fixed $n \in \mathbb{N}$

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [\tau, T]} |\mathbb{E}_x [u_n(X_\epsilon(t)) - (u_n^\wedge)^\vee(X_\epsilon(t))]| = 0.$$

To this purpose, let us fix $\alpha > 0$ and take $\epsilon > 0$ small enough so that $\tau - \epsilon^\alpha > 0$. If we fix $\eta > 0$ and take $\eta' = \eta/4\|u\|_\infty$, we have

$$\sup_{t \geq 0} |\mathbb{E}_x (u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t)); t - \epsilon^\alpha \geq \rho_{\epsilon, \eta'})| \leq \frac{\eta}{2}, \quad \epsilon > 0,$$

where $\rho_{\epsilon, \eta'}$ is the stopping time defined in (A.6) and satisfying (A.5). This implies

$$\begin{aligned} & |\mathbb{E}_x (u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t)))| \\ & \leq |\mathbb{E}_x (u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t)); t - \epsilon^\alpha < \rho_{\epsilon, \eta'})| + \frac{\eta}{2}. \end{aligned} \tag{A.26}$$

Now, as in the proof of Lemma A.3, we have

$$\begin{aligned} & \mathbb{E}_x (u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t)); t - \epsilon^\alpha < \rho_{\epsilon, \eta'}) \\ &= \sum_{n \in \mathbb{N}} \mathbb{E}_x (u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t)); t - \epsilon^\alpha \in [\tau_n^{\epsilon, \eta', \delta, \delta/2}, \sigma_n^{\epsilon, \eta', \delta, \delta/2})) \\ & \quad + \sum_{n \in \mathbb{N}} \mathbb{E}_x (u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t)); t - \epsilon^\alpha \in [\sigma_n^{\epsilon, \eta', \delta, \delta/2}, \tau_{n+1}^{\epsilon, \eta', \delta, \delta/2})) \\ &=: \sum_{n \in \mathbb{N}} J_{1,n}^{\epsilon, \eta', \delta}(t) + \sum_{n \in \mathbb{N}} J_{2,n}^{\epsilon, \eta', \delta}(t). \end{aligned} \tag{A.27}$$

As in the proof of Lemma A.3, due to (A.20), we have that there exist $\delta_1 > 0$ and a constant $c > 0$ such that for all ϵ sufficiently small and $\delta < \delta_1$

$$\sum_{n \in \mathbb{N}} J_{1,n}^{\epsilon, \eta', \delta}(t) \leq \frac{c}{\delta} \frac{2\|u\|_\infty e^{t-\epsilon^\alpha}}{t - \epsilon^\alpha} \sup_{y \in D(\pm\delta/2)} \mathbb{E}_y \sigma_0^{\epsilon, \eta', \delta, \delta/2},$$

so that, thanks to (A.21), we get

$$\sup_{t \in [\tau, T]} \sum_{n \in \mathbb{N}} J_{1,n}^{\epsilon, \eta', \delta}(t) \leq c \frac{\|u\|_\infty e^T}{\tau - \epsilon^\alpha} \delta |\log \delta|. \tag{A.28}$$

On the other hand, by using once more the strong Markov property, we have

$$\begin{aligned}
 & |J_{2,n}^{\epsilon,\eta',\delta}(t)| \\
 & \leq \mathbb{E}_x \left(\mathbb{I}_{\{\sigma_n^{\epsilon,\eta',\delta,\delta/2} \leq t - \epsilon^\alpha\}} \left| \mathbb{E}_{X_\epsilon(\sigma_n^{\epsilon,\eta',\delta,\delta/2})} \right. \right. \\
 & \quad \left. \left. \left(u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t)); t - \epsilon^\alpha < \tau_1^{\epsilon,\eta',\delta,\delta/2} \right) \right| \right) \\
 & \leq \mathbb{P}_x \left(\sigma_n^{\epsilon,\eta',\delta,\delta/2} \leq t - \epsilon^\alpha \right) \\
 & \quad \sup_{y \in D(\pm\delta)} \left| \mathbb{E}_y \left(u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t)); t - \epsilon^\alpha < \tau_1^{\epsilon,\eta',\delta,\delta/2} \right) \right| \\
 & \leq e^{t-\epsilon^\alpha} \mathbb{E}_x \left(e^{-\tau_n^{\epsilon,\eta',\delta,\delta/2}} \right) \\
 & \quad \sup_{y \in D(\pm\delta)} \left| \mathbb{E}_y \left(u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t)); t - \epsilon^\alpha < \tau_1^{\epsilon,\eta',\delta,\delta/2} \right) \right|,
 \end{aligned}$$

and thanks to (A.19), this implies that there exists $\delta_2 > 0$, such that for any $\delta \leq \delta_2$

$$\begin{aligned}
 \sum_{n=1}^\infty |J_{2,n}^{\epsilon,\eta',\delta}(t)| & \leq \frac{c e^{t-\epsilon^\alpha}}{\delta} \\
 \sup_{y \in D(\pm\delta)} \left| \mathbb{E}_y \left(u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t)); t - \epsilon^\alpha < \tau_1^{\epsilon,\eta',\delta,\delta/2} \right) \right|.
 \end{aligned}$$

For every $y \in \mathbb{R}^2$, we have

$$\begin{aligned}
 & \mathbb{E}_y \left(u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t)); t - \epsilon^\alpha < \tau_1^{\epsilon,\eta',\delta,\delta/2} \right) \\
 & = \mathbb{E}_y \left(u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t - \epsilon^\alpha)); t - \epsilon^\alpha < \tau_1^{\epsilon,\eta',\delta,\delta/2} \right) \\
 & \quad + \mathbb{E}_y \left((u^\wedge)^\vee(X_\epsilon(t - \epsilon^\alpha)) - (u^\wedge)^\vee(X_\epsilon(t)); t - \epsilon^\alpha < \tau_1^{\epsilon,\eta',\delta,\delta/2}, t < \tau_1^{\epsilon,\eta',\delta,\delta/4} \right) \\
 & \quad + \mathbb{E}_y \left((u^\wedge)^\vee(X_\epsilon(t - \epsilon^\alpha)) - (u^\wedge)^\vee(X_\epsilon(t)); t - \epsilon^\alpha < \tau_1^{\epsilon,\eta',\delta,\delta/2}, t \geq \tau_1^{\epsilon,\eta',\delta,\delta/4} \right) \\
 & =: \sum_{i=1}^3 L_i^{\epsilon,\delta}(t, y).
 \end{aligned}$$

Let us start considering $L_2^{\epsilon,\delta}(t)$. If $y \in D(\pm\delta)$, then $\tau_1^{\epsilon,\eta',\delta,\delta/4}$ is the first time the process X_ϵ touches $D(\pm\delta/4) \cup C(z_{\eta'})$. This means that

$$t < \tau_1^{\epsilon,\eta',\delta,\delta/4} \implies X_\epsilon(s) \in G(\pm\delta/4)^c \cap \{H \leq z_{\eta'}\}, \quad s \leq t.$$

In particular, $\Pi(X_\epsilon(s))$ remains in the interior of the same edge of the graph Γ where y is, for all $s \leq t < \tau_1^{\epsilon,\eta',\delta,\delta/4}$. As we are assuming that $u \in C_b^1(\mathbb{R}^2)$, due to Lemma 3.2 we have that u^\wedge is continuously differentiable on $\Pi(G(\pm\delta/4))^c$, with uniformly bounded derivative.

As a consequence of Itô’s formula, for every $s < t$ we have

$$H(X_\epsilon(t)) - H(X_\epsilon(s)) = \frac{1}{2} \int_s^t \Delta H(X_\epsilon(r)) dr + \int_s^t \langle \nabla H(X_\epsilon(r)), dw(r) \rangle.$$

Hence, since for $y \in D(\pm\delta)$ and $s < t < \tau_1^{\epsilon, \eta', \delta, \delta/4}$, the process $\Pi(X_\epsilon(s))$ remains in the same edge of the graph Γ , we get

$$\begin{aligned} & \mathbb{E}_y \left(|\Pi(X_\epsilon(t)) - \Pi(X_\epsilon(s))|^2 ; t < \tau_1^{\epsilon, \eta', \delta, \delta/4} \right) \\ &= \mathbb{E}_y \left(|H(X_\epsilon(t)) - H(X_\epsilon(s))|^2 ; t < \tau_1^{\epsilon, \eta', \delta, \delta/4} \right) \\ &\leq c \|\Delta H\|_\infty^2 (t - s)^2 + \sup_{y' \in \{H \leq z_{\eta'}\}} |\nabla H(y')|(t - s) \leq c_{T, \eta'}(t - s). \end{aligned} \tag{A.29}$$

In particular, for every $y \in D(\pm\delta)$

$$\begin{aligned} & |L_2^{\epsilon, \delta}(t, y)| \\ &\leq \mathbb{E}_y \left(\|u^\wedge\|_{C^1(\Pi(G(\pm\delta/4))^c)} |\Pi(X_\epsilon(t - \epsilon^\alpha)) \right. \\ &\quad \left. - \Pi(X_\epsilon(t))\right| ; t - \epsilon^\alpha < \tau_1^{\epsilon, \eta', \delta, \delta/2}, t < \tau_1^{\epsilon, \eta', \delta, \delta/4} \Big) \\ &\leq \|u^\wedge\|_{C^1(\Pi(G(\pm\delta/4))^c)} c_{T, \eta'}^{1/2} \epsilon^{\alpha/2}. \end{aligned} \tag{A.30}$$

Next, concerning $L_3^{\epsilon, \delta}(t, y)$, we have

$$\begin{aligned} & \mathbb{P}_y \left(t - \epsilon^\alpha < \tau_1^{\epsilon, \eta', \delta, \delta/2}, t \geq \tau_1^{\epsilon, \eta', \delta, \delta/4} \right) \\ &\leq \mathbb{P}_y \left(|H \left(X_\epsilon \left(\tau_1^{\epsilon, \eta', \delta, \delta/4} \right) \right) - H(X_\epsilon(t - \epsilon^\alpha))| \right. \\ &\quad \left. \geq \delta/4, t - \epsilon^\alpha < \tau_1^{\epsilon, \eta', \delta, \delta/2}, t \geq \tau_1^{\epsilon, \eta', \delta, \delta/4} \right) \\ &\leq \mathbb{P}_y \left(|H \left(X_\epsilon \left(t \wedge \tau_1^{\epsilon, \eta', \delta, \delta/4} \right) \right) - H \left(X_\epsilon \left((t - \epsilon^\alpha) \wedge \tau_1^{\epsilon, \eta', \delta, \delta/4} \right) \right)| \geq \delta/4 \right) \\ &\leq \frac{16}{\delta^2} \mathbb{E}_y \left(|H \left(X_\epsilon \left(t \wedge \tau_1^{\epsilon, \eta', \delta, \delta/4} \right) \right) - H \left(X_\epsilon \left((t - \epsilon^\alpha) \wedge \tau_1^{\epsilon, \eta', \delta, \delta/4} \right) \right)|^2 \right), \end{aligned} \tag{A.31}$$

and, thanks to (A.29), this yields

$$\sup_{y \in D(\pm\delta)} |L_3^{\epsilon, \delta}(t, y)| \leq \frac{16}{\delta^2} c_{T, \eta'} \epsilon^\alpha. \tag{A.32}$$

Finally, we consider $L_1^{\epsilon, \delta}(t, y)$. As a consequence of the Markov property,

$$\begin{aligned} & \mathbb{E}_y \left(u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t - \epsilon^\alpha)); t - \epsilon^\alpha < \tau_1^{\epsilon, \eta', \delta, \delta/2} \right) \\ &= \mathbb{E}_y \left(\psi_\epsilon(\epsilon^\alpha, X_\epsilon(t - \epsilon^\alpha)); t - \epsilon^\alpha < \tau_1^{\epsilon, \eta', \delta, \delta/2} \right), \end{aligned}$$

where

$$\psi_\epsilon(s, x) = \mathbb{E}_x u(X_\epsilon(s)) - (u^\wedge)^\vee(x).$$

Since the family $\{\Pi(X_\epsilon)\}_{\epsilon > 0}$ is weakly convergent in $C([0, +\infty); \Gamma)$ and $H(x) \uparrow \infty$, as $|x| \uparrow \infty$, we have that for any $\eta > 0$ there exists $M_\eta > 0$ such that

$$\begin{aligned} & \left| \mathbb{E}_y \left(\psi_\epsilon(\epsilon^\alpha, X_\epsilon(t - \epsilon^\alpha)); t - \epsilon^\alpha < \tau_1^{\epsilon, \eta', \delta, \delta/2}, |X_\epsilon(t - \epsilon^\alpha)| > M_\eta \right) \right| \\ & \leq 2 \|u\|_\infty \mathbb{P}_y \left(|X_\epsilon(t - \epsilon^\alpha)| > M_\eta \right) \leq \eta. \end{aligned}$$

Therefore,

$$\begin{aligned} |L_1^{\epsilon, \delta}(t, y)| & \leq \eta + \mathbb{E}_y \left(|\psi_\epsilon(\epsilon^\alpha, X_\epsilon(t - \epsilon^\alpha))|; t - \epsilon^\alpha < \tau_1^{\epsilon, \eta', \delta, \delta/2}, |X_\epsilon(t - \epsilon^\alpha)| \leq M_\eta \right). \end{aligned}$$

As above, we write

$$\begin{aligned} & \mathbb{E}_y \left(|\psi_\epsilon(\epsilon^\alpha, X_\epsilon(t - \epsilon^\alpha))|; t - \epsilon^\alpha < \tau_1^{\epsilon, \eta', \delta, \delta/2}, |X_\epsilon(t - \epsilon^\alpha)| \leq M_\eta \right) \\ &= \mathbb{E}_y \left(|\psi_\epsilon(\epsilon^\alpha, X_\epsilon(t - \epsilon^\alpha))|; t - \epsilon^\alpha < \tau_1^{\epsilon, \eta', \delta, \delta/2}, t < \tau_1^{\epsilon, \eta', \delta, \delta/4}, |X_\epsilon(t - \epsilon^\alpha)| \leq M_\eta \right) \\ & \quad + \mathbb{E}_y \left(|\psi_\epsilon(\epsilon^\alpha, X_\epsilon(t - \epsilon^\alpha))|; t - \epsilon^\alpha < \tau_1^{\epsilon, \eta', \delta, \delta/4}, t < \tau_1^{\epsilon, \eta', \delta, \delta/2}, |X_\epsilon(t - \epsilon^\alpha)| \leq M_\eta \right) \\ &=: I_{\epsilon,1}(t) + I_{\epsilon,2}(t). \end{aligned}$$

Due to (A.29) and (A.31), we have

$$I_{\epsilon,1}(t) \leq 2 \|u\|_\infty c_T \frac{\epsilon^\alpha}{\delta^2}.$$

Now, in [4, Lemma 4.3], it has been proved that, as a consequence of the averaging principle, under the crucial assumption (2.3) given in Hypothesis 2, if $\alpha \in (4/7, 2/3)$, then for every fixed $\delta > 0$

$$\lim_{\epsilon \rightarrow 0} \sup_{x \in K} |\psi_\epsilon(\epsilon^\alpha, x)| = \lim_{\epsilon \rightarrow 0} \sup_{x \in K} \left| \mathbb{E}_x u(X_\epsilon(\epsilon^\alpha)) - (u^\wedge)^\vee(x) \right| = 0, \tag{A.33}$$

for any compact subset K in $G(\pm\delta/2)^c$ and any function u whose support is contained in $G(\pm\delta/4)^c$. Since $X_\epsilon(t) \in G(\pm\delta/4)^c$, if $t < \tau_1^{\epsilon, \eta', \delta, \delta/4}$, (A.33) implies that

$$\lim_{\epsilon \rightarrow 0} \sup_{y \in D(\pm\delta)} \mathbb{E}_y \left(|\psi_\epsilon(\epsilon^\alpha, X_\epsilon(t - \epsilon^\alpha))|; t - \epsilon^\alpha < \tau_1^{\epsilon, \eta', \delta, \delta/2}, t < \tau_1^{\epsilon, \eta', \delta, \delta/4}, |X_\epsilon(t - \epsilon^\alpha)| \leq M_\eta \right) = 0,$$

and, because of the arbitrariness of $\eta > 0$, we conclude that

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [\tau, T]} \sup_{y \in D(\pm\delta)} |L_1^{\epsilon, \delta}(t, y)| = 0.$$

This, together with (A.30) and (A.32), implies that for every $\delta \leq \delta_2$ fixed

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [\tau, T]} \sum_{n=1}^\infty |J_{2,n}^{\epsilon, \eta', \delta}(t)| = 0.$$

Therefore, if in (A.28) we pick $\bar{\delta} \in (0, \delta_2]$ such that

$$\sup_{t \in [\tau, T]} \sum_{n \in \mathbb{N}} J_{1,n}^{\epsilon, \eta', \bar{\delta}}(t) < \frac{\eta}{4},$$

and then we pick $\epsilon_\eta > 0$ such that

$$\sup_{t \in [\tau, T]} \sum_{n=1}^\infty |J_{2,n}^{\epsilon, \eta', \bar{\delta}}(t)| < \frac{\eta}{4}, \quad \epsilon \leq \epsilon_\eta,$$

because of (A.26) and (A.27), we can conclude that

$$\sup_{t \in [\tau, T]} \left| \mathbb{E}_x \left(u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t)); t - \epsilon^\alpha < \rho_{\epsilon, \eta'} \right) \right| < \frac{\eta}{2}, \quad \epsilon < \epsilon_\eta,$$

and (A.25) follows.

B Some consequences of Theorem A.2

The first immediate consequence of Theorem A.2 is that the semigroup $S_\epsilon(t)$ converges to the semigroup $\bar{S}(t)$ in H_γ , as $\epsilon \downarrow 0$.

Corollary B.1 *Under Hypotheses 1, 3 and 2, for every $0 < \tau < T$ and $u \in H_\gamma$ we have*

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [\tau, T]} |S_\epsilon(t)u - \bar{S}(t)^\vee u|_{H_\gamma} = \lim_{\epsilon \rightarrow 0} \sup_{t \in [\tau, T]} |(S_\epsilon(t)u)^\wedge - \bar{S}(t)u^\wedge|_{\bar{H}_\gamma} = 0. \quad (\text{B.1})$$

Proof First of all, we notice that in view of (3.2), the first limit in (B.1) implies the second one. So, we will only prove the first limit. We have

$$|S_\epsilon(t)u - \bar{S}(t)^\vee u|_{H_\gamma}^2 = \int_{\mathbb{R}^2} |S_\epsilon(t)u(x) - \bar{S}(t)^\vee u(x)|^2 \gamma^\vee(x) dx.$$

If $u \in C_b(\mathbb{R}^2)$, we have

$$\sup_{t \geq 0} |S_\epsilon(t)u(x) - \bar{S}(t)^\vee u(x)| \leq 2|u|_\infty.$$

Hence, since $\gamma^\vee \in L^1(\mathbb{R}^2)$, in view of (A.11) and the dominated convergence theorem, we have that the first limit in (B.1) is true for every $u \in C_b(\mathbb{R}^2)$. Moreover, in view of Hypothesis 3, we have that

$$|S_\epsilon(t)u|_{H_\gamma} \leq c_T |u|_{H_\gamma}, \quad t \in [0, T], \quad \epsilon > 0, \tag{B.2}$$

so that $\bar{S}(t)^\vee u \in H_\gamma$ and

$$|\bar{S}(t)^\vee u|_{H_\gamma} \leq c_T |u|_{H_\gamma}, \quad t \in [0, T]. \tag{B.3}$$

Since $C_c(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)$ and we assume the weight γ^\vee to be continuous and strictly positive, we have that $C_c(\mathbb{R}^2)$ is dense in H_γ . Actually, if $u \in H_\gamma$, then $u\sqrt{\gamma^\vee} \in L^2(\mathbb{R}^2)$. Then, if $\{u_n\}_{n \in \mathbb{N}}$ is a sequence in $C_c(\mathbb{R}^2)$, converging to $u\sqrt{\gamma^\vee} \in H_\gamma$ in $L^2(\mathbb{R}^2)$, we have that the sequence $\{u_n/\sqrt{\gamma^\vee}\}_{n \in \mathbb{N}}$ converges to u in H_γ . Moreover, as u_n has compact support and $1/\sqrt{\gamma^\vee}$ is continuous and positive, it follows that $\{u_n/\sqrt{\gamma^\vee}\}_{n \in \mathbb{N}} \subset C_c(\mathbb{R}^2)$.

Thus, for any $u \in H_\gamma$, we can fix a sequence $\{u_n\}_{n \in \mathbb{N}} \subset C_c(\mathbb{R}^2)$ converging to u in H_γ . Thanks to (B.3), we get that the sequence $\{\bar{S}(t)^\vee u_n\}_{n \in \mathbb{N}}$ is Cauchy in H_γ , so that we conclude that

$$\exists \lim_{n \rightarrow \infty} \bar{S}(t)^\vee u_n =: \bar{S}(t)^\vee u \in H_\gamma,$$

the limit does not depend on the sequence $\{u_n\}_{n \in \mathbb{N}}$ and (B.3) holds for every $u \in H_\gamma$.

Finally, since we have

$$\begin{aligned} |S_\epsilon(t)u - \bar{S}(t)^\vee u|_{H_\gamma} &\leq |S_\epsilon(t)(u - u_n)|_{H_\gamma} \\ &\quad + |\bar{S}(t)^\vee(u - u_n)|_{H_\gamma} + |S_\epsilon(t)u_n - \bar{S}(t)^\vee u_n|_{H_\gamma}, \end{aligned}$$

according to (B.2) and (B.3), for every $\eta > 0$ we can find $n_\eta \in \mathbb{N}$ such that

$$\sup_{t \in [\tau, T]} |S_\epsilon(t)u - \bar{S}(t)^\vee u|_{H_\gamma} \leq \eta + \sup_{t \in [\tau, T]} |S_\epsilon(t)u_{n_\eta} - \bar{S}(t)^\vee u_{n_\eta}|_{H_\gamma}.$$

This allows to conclude, as $u_{n_\eta} \in C_c(\mathbb{R}^2)$. □

Remark B.2 From the proof of the corollary above, it is clear that from the pointwise convergence of $S_\epsilon(t)u$ to $\bar{S}(t)^\vee u$, as stated in Theorem A.2, we cannot conclude that limit (B.1) is also true in $L^2(\mathbb{R}^2)$, as the Lebesgue measure in \mathbb{R}^2 is not finite. It is only after introducing a weight that we can prove the convergence in H_γ .

Corollary B.3 *Under Hypotheses 1, 3 and 2, we have that the semigroup $\bar{S}(t)$ is well defined in \bar{H}_γ and for any $T > 0$ there exists $c_T > 0$ such that*

$$\|\bar{S}(t)\|_{\mathcal{L}(\bar{H}_\gamma)} \leq c_T, \quad t \in [0, T]. \quad (\text{B.4})$$

Proof In (B.3) we have seen that $\bar{S}(t)^\vee$ is well defined in H_γ and for any $T > 0$ there exists $c_T > 0$ such that for any $u \in H_\gamma$

$$|\bar{S}(t)^\vee u|_{H_\gamma} \leq c_T |u|_{H_\gamma}, \quad t \in [0, T].$$

Therefore, thanks to (3.3) and (3.2), if $f \in \bar{H}$

$$|\bar{S}(t)f|_{\bar{H}_\gamma} = |(\bar{S}(t)f)^\vee|_{H_\gamma} = |\bar{S}(t)^\vee(f^\vee)|_{H_\gamma} \leq c_T |f^\vee|_{H_\gamma} = c_T |f|_{\bar{H}_\gamma}, \quad t \in [0, T],$$

and this allows to conclude. \square

References

1. Cerrai, S., Freidlin, M.I.: SPDEs on narrow domains and on graphs: an asymptotic result. *Annales de l'Institut Henry Poincaré*. [arXiv:1412.6423](https://arxiv.org/abs/1412.6423)
2. Da Prato, G., Zabczyk, J.: *Stochastic Equations in Infinite Dimensions*, 2nd edn. Cambridge University Press, Cambridge (2013)
3. Fan, W.T.L.: Stochastic PDEs on graphs as scaling limits of discrete interacting systems. [arXiv:1708.01793](https://arxiv.org/abs/1708.01793)
4. Freidlin, M.I.: Reaction–diffusion in incompressible fluid: asymptotic problems. *J. Differ. Equ.* **179**, 44–96 (2002)
5. Freidlin, M.I., Wentzell, A.D.: *Random Perturbations of Dynamical Systems*, 3rd edn. Springer, Berlin (2012)
6. Ishii, H., Souganidis, P.: A PDE approach to small stochastic perturbations of Hamiltonian flows. *J. Differ. Equ.* **252**, 1748–1775 (2012)
7. Peszat, S., Zabczyk, J.: Stochastic evolution equations with spatially homogeneous Wiener process. *Stoch. Process. Appl.* **72**, 187–204 (1997)