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OPTIMAL CONTROL PROBLEMS FOR STOCHASTIC REACTION-DIFFUSION SYSTEMS WITH NON-LIPSCHITZ COEFFICIENTS*

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Abstract. By using the dynamic programming approach, we study a control problem for a class of stochastic reaction-diffusion systems with coefficients having polynomial growth. In the cost functional a non-Lipschitz term appears, and this allows us to treat the quadratic case, which is of interest in the applications. The corresponding Hamilton–Jacobi–Bellman equation is first resolved by a fixed point argument in a small time interval and then is extended to arbitrary time intervals by suitable a priori estimates. The main ingredient in the proof is the smoothing effect of the transition semigroup associated with the uncontrolled system.

Key words. stochastic reaction-diffusion systems, Hamilton–Jacobi–Bellman equations in infinite dimension, stochastic optimal control problems

AMS subject classifications. 60H15, 69J35, 93C20, 93E20

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1. Introduction.

In this paper we consider the following class of stochastic reaction-diffusion systems with distributed parameter controls in bounded domains $\mathcal{O}$ of $\mathbb{R}^d$, with $d \leq 3$:

(1.1)
$$\begin{cases} \dfrac{\partial y_k}{\partial s}(s,\xi) = \mathcal{A}_k\, y_k(s,\xi) + f_k(\xi, y_1(s,\xi),\dots,y_r(s,\xi)) + z_k(s,\xi) + Q_k \dfrac{\partial^2 w_k}{\partial s \partial \xi}(s,\xi), \\ y_k(t,\xi) = x_k(\xi), \quad 0 \leq t < s \leq T, \quad \xi \in \overline{\mathcal{O}}, \\ \mathcal{B}_k\, y_k(s,\xi) = 0, \qquad \xi \in \partial\mathcal{O}, \qquad k = 1,\dots,r. \end{cases}$$

Here $\mathcal{A} = (\mathcal{A}_1,\dots,\mathcal{A}_r)$ is a uniformly elliptic second order differential operator with regular real coefficients, and $\mathcal{B} = (\mathcal{B}_1,\dots,\mathcal{B}_r)$ is a first order differential operator acting on the boundary of $\mathcal{O}$. The reaction term $f = (f_1,\dots,f_r) : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathbb{R}^r$ is continuous, and $f(\xi,\cdot) : \mathbb{R}^r \to \mathbb{R}^r$ is twice differentiable, has polynomial growth together with its derivatives, and fulfills appropriate dissipativity conditions. $Q = (Q_1,\dots,Q_r)$ is a nonnegative bounded linear operator from $H = L^2(\mathcal{O};\mathbb{R}^r)$ into itself, and $\partial^2 w_k/\partial t\,\partial\xi$ are independent space-time white noises defined on a stochastic basis $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbf{P})$. The control $z = (z_1,\dots,z_r)$ is taken in the set of adapted processes of $L^2(\Omega;L^2(0,T;H))$. We remark that the dimension d is taken less than or equal to 3 because the noise should be, in a sense, nondegenerate, and the solution of the system (1.1) has to take value in $E = C(\mathcal{O};\mathbb{R}^r)$.

We are here concerned with the *cost functional*

$$J(t,x;z) = \mathbf{E}\,\varphi(y(T)) + \int_t^T \mathbf{E}\left(g(y(s)) + k(z(s))\right) ds, \tag{1.2}$$

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where $y(s) = y(s,t;x,z)$ is the solution of the problem (1.1), φ and g are bounded and Lipschitz continuous functions from H into $\mathbb{R}$, and $k : H \to]-\infty,+\infty]$ is a measurable function which fulfills suitable conditions. Our aim is to prove that the *value function* corresponding to the cost functional (1.2), which is defined by

$$V(t,x) = \inf \left\{ J(t,x;z) \,;\, z \in L^2(\Omega; L^2(0,T;H)) \,\text{adapted} \right\},$$

satisfies the Hamilton–Jacobi–Bellman equation

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) - K(Du(t,x)) + g(x) = 0, \\[2ex] u(T,x) = \varphi(x), \end{array} \right. \tag{1.3}$$

where $\mathcal{L}$ is the differential operator

$$\mathcal{L}\psi(x) = \frac{1}{2}\text{Tr}\left[Q^2 D^2 u(t,x)\right] + \langle \mathcal{A}x + f(\cdot;x), Du(t,x)\rangle_H$$

and K is the Legendre transform of k. Notice that the hamiltonian K is not assumed to be Lipschitz continuous so that we can cover the important case of quadratic hamiltonians. Moreover, it is important to stress that in the present paper we are only able to treat the case when data φ and g are Lipschitz continuous.

After proving in the first part that there exists a unique mild solution $u(t,x)$ for (1.3), we show that for any adapted control $z \in L^2(\Omega; L^2(0,T;H))$ and for any $x \in H$ and $t \in [0,T]$ the following identity holds:

$$J(t,x;z) = u(t,x)$$
$$+ \int_t^T \mathbf{E}\left[K(Du(s,y(s))) + \langle z(s), Du(s,y(s))\rangle_H + k(z(s))\right] ds.$$

Thus, in particular, we have $V(t,x) \geq u(t,x)$. Now if we could prove the existence of a solution $y^\star(t)$ for the *closed loop* equation

$$dy(t) = (\mathcal{A}y(t) + f(\cdot;y(t)) - DK(Du(t,y(t))))\,dt + Q\,dw(t), \qquad y(0) = x, \tag{1.4}$$

then $z^\star(t) = -DK(Du(t,y^\star(t)))$ would be an optimal control for the minimizing problem related to the functional (1.2). But unfortunately here we are only able to prove C^1 regularity for the solution of the Hamilton–Jacobi–Bellman equation (1.3), so that we cannot prove the existence of a solution for (1.4) which is adapted to the filtration $\mathcal{F}_t$. Actually, as the solution of (1.3) is only C^1, the *closed loop* equation (1.4) admits only martingale solutions, and hence there is no reason why the optimal control which we could get from it is adapted to the filtration we fixed at the beginning. Thus at present we restrict ourselves to the proof of the verification theorem. In the future it will be interesting to check if, by introducing the notion of *relaxed controls* (see [17] and [29] for the definition), it will be possible to prove the existence of an optimal control. However, in dimension $d = 1$ we are able to show that under some additional assumptions, the closed loop equation has a unique solution so that there exists a unique optimal control.

In order to prove the opposite inequality $V(t,x) \leq u(t,x)$, we introduce an approximating cost functional $J_\alpha(t,x;z)$, and we prove that it satisfies a verification theorem and admits a unique optimal control for each $\alpha > 0$. Due to suitable a priori estimates, we show that there exists a subset $\mathcal{M}_R$ of the space of adapted processes in $L^2(\Omega; L^2(0,T;H))$ such that for any $\alpha > 0$

$$V_\alpha(t,x) = \inf \left\{ J_\alpha(t,x;z) \; ; \; z \in \mathcal{M}^2_R(T) \right\}.$$

Moreover, we show that for any $x \in C(\overline{\mathcal{O}};\mathbb{R}^r)$ the functional $J_\alpha(t,x;z)$ converges to $J(t,x;z)$ as α goes to zero, uniformly for $z \in \mathcal{M}_R$, so that

$$\lim_{\alpha \to 0} V_\alpha(t,x) \geq V(t,x).$$

Thus, by showing that

$$\lim_{\alpha \to 0} V_\alpha(t,x) = u(t,x),$$

we have that $u(t,x) \geq V(t,x)$, and the verification theorem holds for $x \in C(\overline{\mathcal{O}};\mathbb{R}^r)$. The general case $x \in H$ follows by further approximation arguments.

Hamilton–Jacobi–Bellman equations in infinite dimensional spaces have been studied by several authors by using both semigroup techniques and the approach of viscosity solutions (see [3], [4], [12], [13], [20], [21], [23], [24], [25], and all references quoted therein). In particular, in [20] and [21] abstract semilinear stochastic problems are studied, and the nonlinear term f is assumed to be Lipschitz continuous. Instead, in the present paper we are able to skip the condition of Lipschitz continuity for f, and we can consider the case of reaction terms which have polynomial growth (and hence are not well defined in H).

In order to solve the problem (1.3), we introduce the transition semigroup P_t associated with the system (1.1) by setting for any bounded Borel function φ from H into $\mathbb{R}$ and for any $x \in H$

$$P_t\varphi(x) = \mathbf{E}\,\varphi(y(t;x)), \qquad t \geq 0,$$

where $y(t;x)$ is the solution of the uncontrolled system (1.1) starting from x at time zero. Due to Itô's formula and the variation of constants formula, we write (1.3) in the mild form

$$u(t,x) = P_t\varphi(x) - \int_0^t P_{t-s}K(Du(s,\cdot))(x)\,ds + \int_0^t P_{t-s}g(x)\,ds,$$

and by using a fixed point argument we show that for any $\varphi, g \in C^1_b(H)$ there exists a unique differentiable solution $u(t,x)$ which is defined only in a small time interval $[0,T_0]$, as K is only locally Lipschitz continuous. We want to emphasize that the crucial point in our argument is given by the smoothing effect of the semigroup P_t. Actually, P_t maps the space of bounded Borel functions defined on H into the space of differentiable functions, and the estimate

$$\sup_{x \in H} |D(P_t\varphi)(x)| \leq c(t \wedge 1)^{-\frac{1+\epsilon}{2}} \sup_{x \in H} |\varphi(x)|$$

holds for some constant $\epsilon < 1$ depending on the dimension $d \leq 3$ (see [7] for the proof). In order to have a global solution we need to obtain some a priori estimates. To this purpose we first approximate the reaction term f by a Lipschitz continuous sequence $\{f_\alpha\}_{\alpha>0}$, and then we consider the approximating Hamilton–Jacobi–Bellman equation, with the nonlinear term f replaced by f_α. By a Galerkin argument we prove some a priori estimates for the corresponding solutions $u_\alpha(t,x)$, and, by taking the limit as α goes to zero, we get the good estimates for $u(t,x)$.

2. Notations and preliminary results. Let $\mathcal{O}$ be a bounded regular open set of $\mathbb{R}^d$, with $d \leq 3$, having a regular boundary. Here and in what follows we denote by H the Hilbert space $L^2(\mathcal{O};\mathbb{R}^r)$, endowed with the scalar product $\langle \cdot,\cdot\rangle_H$ and the norm $|\cdot|_H$. For any $p \geq 1$, $p \neq 2$, we denote by $|\cdot|_p$ the norm in $L^p(\mathcal{O};\mathbb{R}^r)$. Moreover, we denote by E the Banach space $C(\overline{\mathcal{O}};\mathbb{R}^r)$ endowed with the *sup-norm* and the duality pairing $\langle \cdot,\cdot\rangle_E$ in $E \times E^\star$.

If X and Y are two separable Banach spaces, $B_b(X;Y)$ is the Banach space of all bounded Borel functions $\varphi : X \to Y$ endowed with the *sup-norm*

$$\|\varphi\|_0^X = \sup_{x\in X} |\varphi(x)|_Y.$$

$C_b(X;Y)$ is the subspace of uniformly continuous functions. For any integer $k \geq 1$, we denote by $C_b^k(X;Y)$ the subspace of k-times Fréchet differentiable functions, having bounded and uniformly continuous derivatives, up to the kth order. If we set for any $j = 1,\dots,k$

$$[\varphi]_j^X = \sup_{x\in X} |D^j\varphi(x)|_{\mathcal{L}^j(X;Y)},$$

we have that $C_b^k(X;Y)$ is a Banach space endowed with the norm

$$\|\varphi\|_k^X = \|\varphi\|_0^X + \sum_{j=1}^k [\varphi]_j^X.$$

We denote by $\mathrm{Lip}_b(X;Y)$ the subspace of functions $\varphi \in C_b(X;Y)$ such that

$$[\varphi]_{\mathrm{Lip}}^X = \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|_Y}{|x-y|_X} < \infty.$$

$\mathrm{Lip}_b(X;Y)$ is a Banach space endowed with the norm

$$\|\varphi\|_{\mathrm{Lip}}^X = \|\varphi\|_0^X + [\varphi]_{\mathrm{Lip}}^X.$$

When $Y = \mathbb{R}$, we denote $B_b(X;Y)$, $C_b(X;Y)$, $C_b^k(X;Y)$, and $\mathrm{Lip}_b(X;Y)$, respectively, by $B_b(X)$, $C_b(X)$, $C_b^k(X)$, and $\mathrm{Lip}_b(X)$.

2.1. The Nemytskii operator. We assume that for any $k = 1,\dots,r$ there exist two continuous functions $g_k : \overline{\mathcal{O}} \times \mathbb{R} \to \mathbb{R}$ and $h_k : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathbb{R}$ such that for any $\xi \in \overline{\mathcal{O}}$ and $\sigma = (\sigma_1,\dots,\sigma_r) \in \mathbb{R}^r$ it holds that

$$f_k(\xi,\sigma_1,\dots,\sigma_r) = g_k(\xi,\sigma_k) + h_k(\xi,\sigma_1,\dots,\sigma_r).$$

The functions g_k and h_k are assumed to enjoy the following conditions.

HYPOTHESIS 1.

1. *For any* $\xi \in \overline{\mathcal{O}}$ *the function* $h_k(\xi, \cdot)$ *is of class* C^2 *and has bounded derivatives, uniformly with respect to* $\xi \in \overline{\mathcal{O}}$. *Moreover, the mapping* $D^j_\sigma h_k : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathcal{L}^j(\mathbb{R}^r)$ *is continuous for* $j = 1, 2$.
2. *For any* $\xi \in \overline{\mathcal{O}}$, *the function* $g_k(\xi, \cdot)$ *is of class* C^2, *and there exists* $m \geq 0$ *such that*
$$\sup_{\xi \in \overline{\mathcal{O}}} \sup_{t \in \mathbb{R}} \frac{|D^j_t g_k(\xi, t)|}{1 + |t|^{2m+1-j}} < \infty.$$
Moreover, the mapping $D^j_t g_k : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathbb{R}$ *is continuous for* $j = 1, 2$.
3. *There exist* $a > 0$ *and* $c \in \mathbb{R}$ *such that*
$$\sup_{\xi \in \overline{\mathcal{O}}} D_t g_k(\xi, t) \leq -a\, t^{2m} + c, \qquad t \in \mathbb{R}. \tag{2.1}$$

The Nemytskii operator F associated with the function $(\xi, \sigma) \mapsto f(\xi, \sigma)$ is defined as

$$F(x)(\xi) = f(\xi, x(\xi)), \quad \xi \in \mathcal{O}.$$

If we denote

$$p_\star = 2m + 2, \qquad q_\star = \frac{2m+2}{2m+1},$$

it is possible to show that if $m \geq 1$, then F is twice Fréchet differentiable from $L^{p_\star}(\mathcal{O}; \mathbb{R}^r)$ into $L^{q_\star}(\mathcal{O}; \mathbb{R}^r)$, and it holds that

$$|D^j F(x)|_{\mathcal{L}^j(L^{p_\star}, L^{q_\star})} \leq c\left(1 + |x|_{p_\star}^{2m+1-j}\right), \qquad x \in L^{p_\star}(\mathcal{O}; \mathbb{R}^r).$$

From (2.1) we obtain that for any $x, h \in L^{p_\star}(\mathcal{O}; \mathbb{R}^r)$

$$\langle DF(x)h, h\rangle_H \leq -a|x^m h|_H^2 + c|h|_H^2,$$

and, in particular,

$$\langle DF(x)h, h\rangle_H \leq c\,|h|_H^2.$$

Moreover, from (2.1) it follows that for any $\sigma, \rho \in \mathbb{R}^r$

$$\sup_{\xi \in \overline{\mathcal{O}}} \langle f(\xi, \sigma + \rho) - f(\xi, \rho), \sigma\rangle_{\mathbb{R}^r} \leq -a|\sigma|^{2m+2} + c\left(1 + |\rho|^{2m+1}\right)$$

for some constants $a > 0$ and $c \in \mathbb{R}$, possibly different from those introduced in (2.1). This implies that

$$\langle F(x+h) - F(x), h\rangle_H \leq -a\,|h|_{p_\star}^{p^\star} + c\left(1 + |x|_{p_\star}^{p_\star}\right).$$

By using similar arguments, it is immediate to prove that $F : E \to E$ is twice differentiable, and

$$|D^j F(x)|_{\mathcal{L}^j(E)} \leq c\left(1 + |x|_E^{2m+1-j}\right), \qquad x \in E. \tag{2.2}$$

For any $h, y \in E$ we define

$$\langle \delta_h, y\rangle_E = \begin{cases} \dfrac{1}{|h|_E}\displaystyle\sum_{k=1}^r y_k(\xi_k)h_k(\xi_k) & \text{if } h \neq 0,\\ \\ \delta_0 & \text{if } h = 0,\end{cases} \tag{2.3}$$

where $|h_k(\xi_k)| = |h_k|_{C(\overline{\mathcal{O}})}$ for any $k = 1, \ldots, r$, and δ_0 is any element of the unitary ball of $E^\star$. It is possible to show that $\delta_h \in \partial\, |h|_E$ (see [11] and [6] for more details), and for any $x, h \in E$

$$\langle F(x+h) - F(x), \delta_h\rangle_E \leq c\, |h|_E.$$

Remark 2.1. For any $k = 1, \ldots, r$, let us define

$$g_k(\xi, t) = -c_k(\xi)t^{2m+1} + \sum_{j=0}^{2m} c_{kj}(\xi)t^j,$$

where c_k, c_{kj} are bounded continuous functions from $\overline{\mathcal{O}}$ into $\mathbb{R}$. If we assume that

$$\inf_{\xi\in\overline{\mathcal{O}}} c_k(\xi) > 0,$$

then it is possible to check that g_k fulfills parts 2 and 3 of Hypothesis 1.

Due to Hypothesis 1, there exists $c \in \mathbb{R}$ such that the mapping $\gamma(\xi, \cdot) = f(\xi, \cdot) - cI$ is dissipative for any $\xi \in \overline{\mathcal{O}}$. Then for any $\alpha > 0$ we can define the function

$$\gamma_\alpha : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathbb{R}^r, \qquad (\xi, \sigma) \mapsto \gamma(\xi, J_\alpha(\xi, \sigma)),$$

where

$$J_\alpha(\xi, \sigma) = (I - \alpha\gamma(\xi, \cdot))^{-1}(\sigma), \qquad \sigma \in \mathbb{R}^r.$$

As proved in [9, appendix A], the function $J_\alpha(\xi, \cdot)$ is of class C^2 for any fixed $\xi \in \overline{\mathcal{O}}$. Now if we set

$$f_\alpha(\xi, \sigma) = \gamma_\alpha(\xi, \sigma) + cJ_\alpha(\xi, \sigma),$$

we have that $f_\alpha(\xi, \cdot)$ is Lipschitz continuous, uniformly with respect to $\xi \in \overline{\mathcal{O}}$, is twice differentiable, and

$$\sup_{\xi\in\overline{\mathcal{O}}} \langle f_\alpha(\xi, \sigma+\rho) - f_\alpha(\xi, \sigma), \rho\rangle_{\mathbb{R}^r} \leq c\, |\rho|^2 \tag{2.4}$$

for some constant c independent of α. Moreover, by using well-known properties of the function $J_\alpha(\xi, \sigma)$ (see [11] for the definitions and main results and see [9, chapter 9, appendix A],

$$\sup_{\xi\in\overline{\mathcal{O}}} |f_\alpha(\xi, \sigma) - f(\xi, \sigma)| \leq \alpha\, c(1 + |\sigma|^{4m+1}). \tag{2.5}$$

For any fixed $\xi \in \overline{\mathcal{O}}$ the function $f_\alpha(\xi, \cdot)$ is of class C^2, and for any $R > 0$

$$\lim_{\alpha\to 0} \sup_{\xi\in\overline{\mathcal{O}}} \sup_{|\sigma|\leq R} |D^j_\sigma f_\alpha(\xi, \sigma) - D^j_\sigma f(\xi, \sigma)| = 0. \tag{2.6}$$

Moreover, it is possible to show that

$$\sup_{\xi \in \overline{\mathcal{O}}} \frac{|D^j_\sigma f_\alpha(\xi,\sigma)|}{1+|\sigma|^{2m+1-j}} \leq c < \infty \tag{2.7}$$

for a constant c independent of α.

For each $\alpha > 0$, let F_α be the Nemytskii operator associated with the function f_α. Clearly F_α is Lipschitz continuous both as an operator in E and as an operator in H and is twice Fréchet differentiable in E, and, thanks to (2.4), there exists a constant c independent of α such that if $x, y \in H$,

$$\langle F_\alpha(x) - F_\alpha(y), x - y\rangle_H \leq c\,|x-y|^2_H, \tag{2.8}$$

and if $x, y \in E$,

$$\langle F_\alpha(x) - F_\alpha(y), \delta_{x-y}\rangle_E \leq c\,|x-y|_E, \tag{2.9}$$

where δ_{x-y} is the element in $\partial\,|x-y|_E$ introduced in (2.3). Furthermore, due to (2.6), for each $j = 0, 1, 2$ it holds that

$$\lim_{\alpha\to 0} \sup_{|x|_E \leq R} |D^j F_\alpha(x) - D^j F(x)|_{\mathcal{L}^j(E)} = 0 \tag{2.10}$$

for any $R > 0$, and due to (2.7)

$$|D^j F_\alpha(x)|_{\mathcal{L}^j(E)} \leq c\left(1 + |x|_E^{2m+1-j}\right), \qquad x \in E. \tag{2.11}$$

2.2. The operators A and Q and the stochastic convolution. We shall denote by $\mathcal{A}$ the second order differential operator defined for each $x \in H$ by $\mathcal{A}x = (\mathcal{A}_1 x_1, \dots, \mathcal{A}_r x_r)$. For any $k = 1, \dots, r$ we have

$$\mathcal{A}_k(\xi, D) = \sum_{i,j=1}^d a^{ij}_k(\xi)\frac{\partial^2}{\partial\xi_i\partial\xi_j} + \sum_{i=1}^d b^i_k(\xi)\frac{\partial}{\partial\xi_i}, \qquad \xi \in \overline{\mathcal{O}}.$$

The coefficients a^{ij}_k and b^i_k are assumed to be of class $C^1(\overline{\mathcal{O}})$, and for any $\xi \in \overline{\mathcal{O}}$ the matrix $[a^{ij}_k(\xi)]$ is symmetric and satisfies the uniform ellipticity condition

$$\inf_{\xi\in\overline{\mathcal{O}}} \sum_{i,j=1}^d a^{ij}_k(\xi) h_i h_j \geq \nu |h|^2, \quad h \in \mathbb{R}^d,$$

for some $\nu > 0$. The boundary operator $\mathcal{B}$ is defined by $\mathcal{B}x = (\mathcal{B}_1 x_1, \dots, \mathcal{B}_r x_r)$, and for each $k = 1, \dots, r$ we have

$$\mathcal{B}_k(\xi, D) = I \quad \text{or} \quad \mathcal{B}_k(\xi, D) = \sum_{i,j=1}^d a^{ij}_k(\xi)\nu_j(\xi)\frac{\partial}{\partial\xi_i}, \qquad \xi \in \partial\mathcal{O}. \tag{2.12}$$

We denote by A the realization in H of the elliptic operator $\mathcal{A}$, with the boundary conditions given by $\mathcal{B}$, that is,

$$D(A) = \left\{\, x \in H \,:\, \mathcal{A}x \in H,\ \mathcal{B}x_{|_{\partial D}} = 0 \,\right\}, \quad Ax = \mathcal{A}x,\ x \in D(A).$$

The operator A generates an analytic semigroup e^{tA}. The semigroup e^{tA} is also analytic in each $L^p(\mathcal{O};\mathbb{R}^r)$, for $p \in (1,+\infty]$ (see [26, chapter 3] for all details).

In what follows it will not be restrictive to assume that for any $p \in [2,\infty]$

$$|e^{tA}x|_p \le M|x|_p$$

for some constant $M > 0$ independent of p (see also [9, chapter 4]). In particular, we will have

$$\langle Ax, x\rangle_H \le 0, \qquad x \in H. \tag{2.13}$$

Finally, if we denote by A the realization in E of the operator $\mathcal{A}$ with the boundary conditions given by $\mathcal{B}$, we have that A generates an analytic semigroup $e^{t\,A}$ of negative type; that is, for any $x \in E$ and $\delta_x \in \partial\,|x|_E$ defined as in (2.3)

$$\langle Ax, \delta_x\rangle_E \le 0. \tag{2.14}$$

Now for any $k = 1,\dots,r$ we define

$$\mathcal{G}_k(\xi, D) = \sum_{i=1}^d \left(b_k^i(\xi) - \sum_{j=1}^d \frac{\partial a_k^{ij}}{\partial \xi_j}(\xi)\right)\frac{\partial}{\partial \xi_i}, \qquad \xi \in \mathcal{O},$$

and by difference we define $\mathcal{C}_k = \mathcal{A}_k - \mathcal{G}_k$. The second order elliptic operators $\mathcal{C} = (\mathcal{C}_1,\dots,\mathcal{C}_r)$, generate a negative analytic semigroup e^{tC} in each $L^p(\mathcal{O};\mathbb{R}^r)$ for any $p \in (1,\infty]$ and in E. The semigroup e^{tC} enjoys the same properties as $e^{t\,A}$ and, due to the boundary conditions (2.12), is self-adjoint in H. Moreover, for any $\delta \in \mathbb{R}$ we have that $D((-A)^\delta) = D((-C)^\delta)$ and

$$c_1\,|(-A)^\delta x|_H \le |(-C)^\delta x|_H \le c_2\,|(-A)^\delta x|_H \tag{2.15}$$

for suitable positive constants c_1 and c_2 depending only on δ.

Concerning the realization of the operator $\mathcal{G}$, as the coefficients a_k^{ij} and b_k^i are assumed to be smooth, it is easy to check that $D(G^\star) \subset D((-C)^{1/2})$ and

$$|G^\star x|_H \le c\,|(-C)^{1/2}x|_H, \qquad x \in D(G^\star). \tag{2.16}$$

Finally, we denote by Q the bounded linear operator of components $(Q_1,\dots,Q_r)$. In what follows we shall assume that the operators Q and C fulfill the following conditions.

HYPOTHESIS 2.

1. *There exists a complete orthonormal basis $\{e_k\}$ in H which diagonalizes C such that $\sup_{k\in\mathbb{N}} |e_k|_E < \infty$. The corresponding set of eigenvalues is denoted by $\{-\alpha_k\}$.*
2. *The bounded linear operator $Q : H \to H$ is nonnegative and diagonal with respect to the complete orthonormal basis $\{e_k\}$ which diagonalizes C. Moreover, if $\{\lambda_k\}$ is the corresponding set of eigenvalues, we have*

$$\sum_{k=1}^\infty \frac{\lambda_k^2}{\alpha_k^{1-\gamma}} < +\infty$$

for some $\gamma > 0$.

3. *There exists* $\epsilon < 1$ *such that*

$$D((-C)^{\frac{\epsilon}{2}}) \subset D(Q^{-1}). \tag{2.17}$$

Remark 2.2. It is known (see, for example, the book by Agmon [1]) that when the elliptic operator $\mathcal{A}$ with the boundary conditions $\mathcal{B}$ is smooth enough, then

$$\alpha_k \asymp k^{2/d}.$$

In this case, it is possible to prove that if $d \leq 3$, then there exists an operator Q which fulfills the conditions of parts 2 and 3 of Hypothesis 2.

Actually, if we assume that

$$\lambda_k \asymp \alpha_k^{-\rho}$$

for some $\rho > (d-2)/4$, then

$$\frac{\lambda_k^2}{\alpha_k^{1-\gamma}} \asymp \alpha_k^{-(1-\gamma+2\rho)} \asymp k^{-\frac{2(1-\gamma+2\rho)}{d}}.$$

As $1 + 2\rho > d/2$, we can fix $\gamma > 0$ such that $1 - \gamma + 2\rho > d/2$, and this implies that

$$\sum_{k=1}^{\infty} \frac{\lambda_k^2}{\alpha_k^{1-\gamma}} \asymp \sum_{k=1}^{\infty} k^{-\frac{2(1-\gamma+2\rho)}{d}} < \infty.$$

On the other hand, if $\rho \leq \epsilon/2$, then (2.17) holds. This means that if $d \leq 3$, it is possible to find ρ such that Q enjoys conditions 2 and 3 in Hypothesis 2. Notice that in dimension $d = 1$ one can take $\epsilon = 0$.

Let $\{w_k(t)\}$ be a sequence of mutually independent real-valued Brownian motions defined on a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ and adapted to the nonanticipative filtration $\mathcal{F}_t$, $t \geq 0$. We define the cylindrical Wiener process $w(t)$ as

$$w(t) = \sum_{k=1}^{\infty} e_k w_k(t),$$

where $\{e_k\}$ is the complete orthonormal system of H introduced in part 1 of Hypothesis 2. The series above defining $w(t)$ does not converge in H, but it is convergent in any Hilbert space U such that the embedding $H \subset U$ is Hilbert–Schmidt (see [15, chapter 4]).

Now we consider the Ornstein–Uhlenbeck problem associated with the system (1.1)

$$dv(t) = Av(t)\,dt + Q\,dw(t), \qquad v(s) = 0,$$

for $0 \leq s \leq t \leq T$. Due to parts 1 and 2 of Hypothesis 2, such a problem admits a unique solution $w^A(t,s)$, which is the mean-square Gaussian process with values in H given by

$$w^A(t,s) = \int_s^t e^{(t-r)\,A} Q\,dw(r) \tag{2.18}$$

(see, e.g., [15] for a proof). Moreover, as shown in [8], $w^A(\cdot,s) \in C([s,T]\times\overline{\mathcal{O}})$, $\mathbf{P}$-almost surely (a.s.), and for any $p \geq 1$ it holds that

$$\mathbf{E}\,|w^A(\cdot,s)|^p_{C([s,T];E)} < \infty. \tag{2.19}$$

For any $n \in \mathbb{N}$ we define

$$A_n = P_n A P_n, \qquad C_n = C P_n, \qquad G_n = P_n G P_n, \qquad Q_n = Q P_n,$$

where P_n is the projection of H onto the finite dimensional space H_n generated by the eigenfunctions $\{e_1,\dots,e_n\}$. If we denote by $w^A_n(t,s)$ the solution of the problem

$$dv(t) = A_n v(t)\,dt + Q_n\,dw(t), \qquad v(s) = 0,$$

by using a factorization argument (see [15]) it is not difficult to prove that for any $p \geq 1$

$$\lim_{n\to+\infty} \mathbf{E}\,|w^A(\cdot,s) - w^A_n(\cdot,s)|^p_{C([s,T];H)} = 0.$$

3. The state equation. By using the notations introduced in the previous section, the controlled system (1.1) can be rewritten in the abstract form

$$dy(t) = (Ay(t) + F(y(t)) + z(t))\,dt + Q\,dw(t), \qquad y(s) = x, \tag{3.1}$$

for $0 \leq s \leq t \leq T$.

DEFINITION 3.1.

1. *Let us fix an adapted process $z \in L^2(\Omega; L^2(0,T;E))$ and $x \in E$. An E-valued predictable process $y(t) = y(t,s;x,z)$ is a* mild *solution for the problem* (3.1) *if*

$$y(t) = e^{(t-s)\,A}x + \int_s^t e^{(t-r)\,A}\,(F(y(r)) + z(r))\,dr + w^A(t,s),$$

where the process $w^A(t,s)$ is given by (2.18).

2. *Let us fix an adapted process $z \in L^2(\Omega; L^2(0,T;H))$ and $x \in H$. A H-valued process $y(t,s;x,z)$ is a* generalized *solution for the problem* (3.1) *if, for any sequences $\{x_n\} \subset E$ converging to x in H and $\{z_n\} \subset L^2(\Omega; L^2(0,T;E))$ converging to z in $L^2(\Omega; L^2(0,T;H))$, the corresponding sequence of mild solutions $\{y(\cdot,s;x_n,z_n)\}$ converges to $y(\cdot,s;x,z)$ in $C([s,T];H)$, $\mathbf{P}$-a.s.*

In [15] (see also [6] and [7]) the following existence and uniqueness result is proved for the uncontrolled system. When $z \neq 0$, the proof is analogous, and we do not repeat it.

THEOREM 3.2. *Assume Hypotheses* 1 *and* 2, *and fix $0 \leq s \leq T$.*

1. *For any $z \in L^2(\Omega; L^p(s,T;H))$ with $p > 4/(4-d)$ and for any $x \in E$, the problem* (3.1) *admits a unique mild solution $y(\cdot,s;x,z) \in L^2(\Omega; C((s,T];E) \cap L^\infty(s,T;E))$ such that*

$$|y(t,s;x,z)|_E \leq c_T\Big(|x|_E + |z|^{2m+1}_{L^p(s,t;H)} + \sup_{r\in[s,t]} |w^A(r,s)|^{2m+1}_E\Big), \qquad \mathbf{P}-a.s. \tag{3.2}$$

2. *For any* $z \in L^2(\Omega; L^2(s,T;H))$ *and* $x \in H$, *the problem* (3.1) *admits a unique generalized solution* $y(\cdot, s; x, z) \in L^2(\Omega; C([s,T]; H))$ *such that*

(3.3)
$$|y(t,s;x,z)|_H \leq c_T\Big(|x|_H + |z|^{2m+1}_{L^2(s,T;H)} + \sup_{r\in[s,t]} |w^A(r,s)|_E^{2m+1}\Big), \qquad \mathbf{P}-a.s.$$

3. *The unique generalized solution* $y(\cdot, s; x, z)$ *belongs to* $L^{p_\star}(t,T; L^{p_\star}(\mathcal{O}))$, $\mathbf{P}$-*a.s., and*
$$y(t,s;x,z) = e^{(t-s)\,A}x + \int_s^t e^{(t-r)\,A}\,(F(y(r,s;x,z)) + z(r))\,dr + w^A(t,s).$$

4. *For any* $x_1, x_2 \in H$ *and* $z_1, z_2 \in L^2(\Omega; L^2(s,T;H))$, *we have*

(3.4)
$$|y(t,s;x_1,z_1) - y(t,s;x_2,z_2)|_H \leq c_T\left(|x_1 - x_2|_H + |z_1 - z_2|_{L^2(s,T;H)}\right).$$

For any $\alpha > 0$ we consider the approximating problem

$$dy(t) = (Ay(t) + F_\alpha(y(t)) + z(t))\,dt + Q\,dw(t), \qquad y(s) = x, \tag{3.5}$$

$s \leq t \leq T$. Clearly an existence theorem analogous to Theorem 3.2 holds for (3.5). Actually, for each $x \in E$ and $z \in L^2(\Omega; L^p(0,T;H))$, with $p > 4/(4-d)$, there exists a unique mild solution $y_\alpha(\cdot, s; x, z)$ in $L^2(\Omega; C((s,T];E) \cap L^\infty(s,T;E))$, and for each $x \in H$ and $z \in L^2(\Omega; L^2(0,T;H))$ there exists a unique generalized solution $y_\alpha(\cdot, s; x, z)$ which belongs to $L^2(\Omega; C([s,T];H))$. Moreover, estimates analogous to (3.2) and (3.3) hold for every $\alpha > 0$.

LEMMA 3.3. *Under Hypotheses* 1 *and* 2, *if* $x \in E$ *and* $z \in L^p(\Omega; L^\infty(0,T;H))$ *for* p *sufficiently large, then for any fixed* $q \geq 1$

$$\lim_{\alpha\to 0} \mathbf{E}\,|y_\alpha(t,s;x,z) - y(t,s;x,z)|_E^q = 0, \tag{3.6}$$

uniformly with respect to $t \in [0,T]$, x *in bounded subsets of* E *and* z *in the set*
$$\mathcal{M}_R = \Big\{\, z \in L^2(\Omega; L^2(0,T;H))\ ;\ \sup_{t\in[0,T]} |z(t)|_H \leq R,\ \mathbf{P}-a.s. \Big\}$$
for any $R > 0$.

Proof. If we set $v_\alpha(t) = y_\alpha(t) - y(t)$, we have that v_α is the unique solution of the problem
$$\frac{dv}{dt}(t) = Av(t) + F_\alpha(y_\alpha(t)) - F(y(t)), \qquad v(s) = 0.$$
Thus, by using classical properties of the subdifferential of the norm in E introduced in (2.3) (see [11] for all properties), if $\delta_{v_\alpha(t)} \in \partial\,|v_\alpha(t)|_E$, we have
$$\frac{d}{dt}^{-}|v_\alpha(t)|_E \leq \left\langle Av_\alpha(t), \delta_{v_\alpha(t)}\right\rangle_E + \left\langle F_\alpha(y_\alpha(t)) - F(y(t)), \delta_{v_\alpha(t)}\right\rangle_E.$$
From (2.9) and (2.14) this easily implies that
$$\frac{d}{dt}^{-}|v_\alpha(t)|_E \leq c\,|v_\alpha(t)|_E + |F_\alpha(y(t)) - F(y(t))|_E$$

so that, due to the Gronwall lemma and (2.5), we have

$$|v_\alpha(t)|_E \leq \alpha\, c \int_s^t e^{c(t-r)} \left(1 + |y(r)|_E^{4m+1}\right)\, dr.$$

This implies (3.6), as from (2.19) and (3.2) for any $q \geq 1$ we have

$$\sup_{z \in \mathcal{M}_R} \mathbf{E}\, |y(\cdot, s; x, z)|^q_{C([s,T];E)} < \infty. \qquad \square \tag{3.7}$$

Next, for any $n \in \mathbb{N}$ and $\alpha > 0$ we define

$$F_{\alpha,n}(x) = P_n F_\alpha(P_n x), \qquad x \in H.$$

It is immediate to check that for any $x, y \in H$ it holds that

$$\langle F_{\alpha,n}(x) - F_{\alpha,n}(y), x - y \rangle_H \leq c\, |x - y|_H^2 \tag{3.8}$$

for a constant c independent of n and α. Moreover,

$$|F_{\alpha,n}(x) - F_{\alpha,n}(y)|_H \leq c_\alpha\, |x - y|_H \tag{3.9}$$

for some constant c_α independent of n. In correspondence with each $n \in \mathbb{N}$, $\alpha > 0$, and $0 \leq s \leq T$, we consider the approximating problem

$$dy(t) = (A_n y(t) + F_{\alpha,n}(y(t)) + z_n(t))\, dt + Q_n\, dw(t), \qquad y(s) = P_n x, \tag{3.10}$$

where $z_n(t) = P_n z(t)$ and z is an adapted process in $L^2(\Omega; L^2(s,T;H))$. Such a problem is a finite dimensional problem with Lipschitz coefficients. Thus for any $x \in H$ there exists a unique strong solution $y_{\alpha,n}(\cdot, s; x, z) \in L^2(\Omega; C([s,T];H))$.

LEMMA 3.4. *Let z be an adapted process in $L^2(\Omega; L^2(s,T;H))$. If $y_{\alpha,n}(\cdot, s; x, z)$ is the unique solution of the approximating problem* (3.10)*, it holds that*

$$\lim_{n \to +\infty} y_{\alpha,n}(\cdot, s; x, z) = y_\alpha(\cdot, s; x, z) \qquad \text{in } L^2(\Omega; C([s,T];H)), \tag{3.11}$$

uniformly for x in bounded subsets of H.

Proof. For each $n, k \in \mathbb{N}$, we consider the problem

$$dy(t) = (A_n y(t) + F_{\alpha,n}(y(t)) + z_{n \wedge k}(t))\, dt + Q_{k \wedge n}\, dw(t), \qquad y(0) = P_n x. \tag{3.12}$$

By using a factorization argument, we have that for any $p \geq 1$

$$\lim_{k \to +\infty} \sup_{n \in \mathbb{N}} \mathbf{E} \sup_{t \in [s,T]} |w^A_{n,k}(t,s)|_H^p = 0,$$

and, since $z \in L^2(\Omega; L^2(s,T;H))$, we have

$$\lim_{k \to +\infty} \sup_{n \in \mathbb{N}} \mathbf{E}\, |z_n - z_{n \wedge k}|^2_{L^2(s,T;H)} = 0.$$

Thus, by some calculations, if we denote by $y^k_{\alpha,n}(t)$ the solution of (3.12), we have

$$\lim_{k \to +\infty} \sup_{n \in \mathbb{N}} |y^k_{\alpha,n}(\cdot, s; x, z) - y_{\alpha,n}(\cdot, s; x, z)|_{L^2(\Omega; C([s,T];H))} = 0. \tag{3.13}$$

Now for any $k \in \mathbb{N}$ we consider the problem

$$dy(t) = (Ay(t) + F_\alpha(y(t)) + z_k(t))\, dt + Q_k\, dw(t), \qquad y(s) = x.$$

It is immediate to check that $w_k^A \in L^p(\Omega, C((s,T]; D((-A)^\delta)))$ for any $\delta \in \mathbb{R}$ and $p \geq 1$. Hence, by straightforward computations, thanks to (2.15) we have that such a problem admits a unique mild solution $y_\alpha^k(\cdot, s; x, z)$ such that

$$\begin{aligned} &|y_\alpha^k(t)|_H^2 + \int_0^t |(-C)^{1/2} y_\alpha^k(s)|_H^2\, ds \\ &\leq c_T \Big(|x|_H^2 + \sup_{t \in [s,T]} |w_k^A(t,s)|^2_{D((-A)^{1/2})} + |z|^2_{L^2(s,T;H)} \Big). \end{aligned} \tag{3.14}$$

Moreover, it is possible to show that for any fixed $k \in \mathbb{N}$

$$\lim_{n \to +\infty} y_{\alpha,n}^k(\cdot, s; x, z) = y_\alpha^k(\cdot, s; x, z) \qquad \text{in } L^2(\Omega; C([s,T]; H)). \tag{3.15}$$

Finally, we have that

$$\lim_{k \to +\infty} y_\alpha^k(\cdot, s; x, z) = y_\alpha(\cdot, s; x, z) \qquad \text{in } L^2(\Omega; C([s,T]; H)). \tag{3.16}$$

Indeed, if we define $v_\alpha^k(t) = y_\alpha(t) - y_\alpha^k(t) - w^A(t,s) + w_k^A(t,s)$, we have that $v_\alpha^k(t)$ is the unique solution for the problem

$$\frac{dv}{dt}(t) = Av(t) + F_\alpha(y_\alpha(t)) - F_\alpha(y_\alpha^k(t)) + z(t) - z_k(t), \qquad v(s) = 0.$$

Thus, by multiplying each side by $v_\alpha^k(t)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|v_\alpha^k(t)|_H^2 + |(-A)^{1/2} v_\alpha^k(t)|_H^2 \\ &= \left\langle F_\alpha(y_\alpha(t)) - F_\alpha(y_\alpha^k(t)), v_\alpha^k(t) \right\rangle_H + \left\langle z(t) - z_k(t), v_\alpha^k(t) \right\rangle_H \end{aligned}$$

so that, as F_α is Lipschitz continuous, we easily get

$$\frac{1}{2}\frac{d}{dt}|v_\alpha^k(t)|_H^2 \leq c_\alpha\, |v_\alpha^k(t)|_H^2 + c_\alpha\, |w^A(t,s) - w_k^A(t,s)|_H^2 + c\, |z(t) - z_k(t)|_H^2.$$

By applying the Gronwall lemma, by taking the supremum over $t \in [s,T]$, and, finally, by taking the expectation, we get

$$\mathbf{E} \sup_{t \in [s,T]} |v_\alpha^k(t)|_H^2 \leq c_{\alpha,T} \int_s^T \mathbf{E} \left(|w^A(t,s) - w_k^A(t,s)|_H^2 + |z(t) - z_k(t)|_H^2 \right) dt,$$

and this immediately implies (3.16).

Now we can conclude. Actually, due to (3.13) and (3.16), for any $\epsilon > 0$ there exists $k_\epsilon \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ it holds that

$$\mathbf{E} \sup_{t \in [s,T]} \left(|y_{\alpha,n}^{k_\epsilon}(t) - y_{\alpha,n}(t)|_H^2 + |y_\alpha^{k_\epsilon}(t) - y_\alpha(t)|_H^2 \right) < \epsilon/2.$$

Besides, due to (3.15) there exists $n_\epsilon \in \mathbb{N}$ such that

$$\mathbf{E} \sup_{t \in [s,T]} |y_{\alpha,n}^{k_\epsilon}(t) - y_\alpha^{k_\epsilon}(t)|_H^2 < \epsilon/2$$

for any $n \geq n_\epsilon$ so that (3.11) follows. □

4. The first variation equation. Here and in what follows, we shall denote, respectively, by $y(t;x)$, $y_\alpha(t;x)$ and $y_{\alpha,n}(t;x)$ the mild solutions of the problems (3.1), (3.5), and (3.10) when $z=0$ and $s=0$.

In the present section we study the first variation equation associated with the problem (3.1):

$$\frac{dv}{dt}(t) = Av(t) + DF(y(t;x))v(t), \qquad v(0)=h. \tag{4.1}$$

In [6, Theorem 4.2] we have proved that for any $x,h\in E$ there exists a unique mild solution for the problem (4.1), and for any $t\geq 0$ and $x,h\in E$ such a solution is given by $Dy(t;x)h$, the Fréchet derivative of the mapping

$$E\to L^2(\Omega;E), \qquad x\mapsto y(t;x),$$

along the direction h. Moreover, in [7, Proposition 4.1] we have proved that if $x,h\in H$, then the problem (4.1) admits a unique generalized solution $v(x,h)$.

As proved in [5], under Hypotheses 1 and 2 the solution $y_\alpha(t;x)$ is twice mean-square differentiable with respect to $x\in H$. Moreover, the first derivative $Dy_\alpha(t;x)h$ is the unique solution of the first variation equation corresponding to the problem (3.5), which is

$$\frac{dv}{dt}(t) = Av(t) + DF_\alpha(y_\alpha(t))v(t), \qquad v(0)=h.$$

We have the following approximation result.

LEMMA 4.1. *Under Hypotheses* 1 *and* 2*, for any $x\in E$ and $t\geq 0$, it holds that*

$$\lim_{\alpha\to 0}\mathbf{E}\sup_{|h|_H\leq 1}|Dy_\alpha(\cdot;x)h - Dy(\cdot;x)h)|^2_{L^\infty(0,T;H)\cap L^2(0,T;D((-A)^{1/2}))} = 0, \tag{4.2}$$

uniformly for x in bounded sets of E.

Proof. As proved in [6], for any $h\in H$

$$\sup_{x\in H}\left(|Dy(t;x)h|_H^2 + \int_0^t |(-A)^{1/2}Dy(s;x)h|_H^2\right)\leq c_T|h|_H^2, \qquad \mathbf{P}-\text{a.s.} \tag{4.3}$$

If we define $v_\alpha(t) = Dy_\alpha(t;x)h - Dy(t;x)h$, we have that $v_\alpha(t)$ is the unique solution to the problem

$$\frac{dv}{dt}(t) = Av(t) + DF_\alpha(y_\alpha(t;x))Dy_\alpha(t;x)h - DF(y(t;x))Dy(t;x)h, \quad v(0)=0.$$

Thus we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v_\alpha(t)|_H^2 + |(-A)^{1/2}v_\alpha(t)|_H^2\\
&= \langle DF_\alpha(y_\alpha(t))v_\alpha(t), v_\alpha(t)\rangle_H + \langle\left(DF_\alpha(y_\alpha(t)) - DF(y(t))\right)Dy(t;x)h, v_\alpha(t)\rangle_H\\
&\leq c\,|v_\alpha(t)|_H^2 + c\,e^{cT}\,|DF_\alpha(y_\alpha(t)) - DF(y(t))|_E^2|h|_H^2,
\end{aligned}$$

the last inequality following from (2.8), (4.3), and the Young inequality. Since F_α verifies the estimate (2.11), for any $x,y\in E$ we have

$$\begin{aligned}
&|DF_\alpha(x) - DF(y)|_E \leq |DF_\alpha(x) - DF_\alpha(y)|_E + |DF_\alpha(y) - DF(y)|_E\\
&\leq c\left(1 + |x|_E^{2m-1} + |y|_E^{2m-1}\right)|x-y|_E + |DF_\alpha(y) - DF(y)|_E.
\end{aligned}$$

Therefore, thanks to the Gronwall lemma and the above inequality, we have

$$|v_\alpha(t)|_H^2 \leq c_T \int_0^t \left(1+|y_\alpha(s)|_E^{4m-2}+|y(s)|_E^{4m-2}\right)|y_\alpha(s)-y(s)|_E^2\,ds|h|_H^2$$

$$+c_T \int_0^t |DF_\alpha(y(s))-DF(y(s))|_E\,ds|h|_H^2.$$

Due to (2.9) it is immediate to check that

$$\sup_{\alpha>0}|y_\alpha(t;x)|_E \leq c_T\Big(|x|_E+\sup_{t\in[0,T]}|w^A(t,0)|_E\Big), \qquad \mathbf{P}-\text{a.s.}, \tag{4.4}$$

and hence, by using (2.10), (3.2), and (3.6), we have

$$\lim_{\alpha\to 0}\mathbf{E}\sup_{|h|_H\leq 1}|v_\alpha(t)|_H^2=0.$$

This immediately yields

$$\lim_{\alpha\to 0}\mathbf{E}\sup_{|h|_H\leq 1}\int_0^t |(-A)^{1/2}v_\alpha(s)|_H^2\,ds=0,$$

and (4.2) holds true. ∎

Due to part 2 of Hypothesis 2, and the closed graph theorem, we have that the operator $\Gamma_\epsilon = Q^{-1}(-A)^{-\epsilon/2}$ is bounded in H. Thus, for any $x \in D((-A)^{1/2})$, by interpolation we get

$$|Q^{-1}x|_H \leq c\,|(-A)^{1/2}x|_H^\epsilon\,|x|_H^{1-\epsilon}. \tag{4.5}$$

Therefore, from (4.2) we get

$$\lim_{\alpha\to 0}\mathbf{E}\sup_{|h|_H\leq 1}\int_0^t |Q^{-1}(Dy_\alpha(s;x)h-Dy(s;x)h)|_H^2\,ds=0, \tag{4.6}$$

uniformly for x in bounded sets of E.

For each $n\in\mathbb{N}$ and $\alpha>0$, the solution of (3.10) is twice mean-square differentiable with respect to $x\in H$. In the next lemma we show that we can approximate in a suitable sense $Dy_\alpha(t;x)h$ by means of $Dy_{\alpha,n}(t;x)h$.

LEMMA 4.2. *Assume that Hypotheses* 1 *and* 2 *hold. Then*

$$\lim_{n\to+\infty}\mathbf{E}\sup_{|h|_H\leq 1}|Dy_{\alpha,n}(\cdot;x)h-Dy_\alpha(\cdot;x)P_nh|^2_{L^\infty(0,T;H)\cap L^2(0,T;D((-A)^{1/2}))}=0, \tag{4.7}$$

uniformly for x in bounded subsets of H.

Proof. If we set $v_{\alpha,n}(t)=Dy_{\alpha,n}(t;x)h-Dy_\alpha(t;x)P_nh$, we have that $v_{\alpha,n}(t)$ is the unique solution of the problem

$$\frac{dv}{dt}(t)=Cv(t)+G_nDy_{\alpha,n}(t)h-GDy_\alpha(t)P_nh$$

$$+DF_{\alpha,n}(y_{\alpha,n}(t;x))Dy_{\alpha,n}(t)h-DF_\alpha(y_\alpha(t;x))Dy_\alpha(t)P_nh, \qquad v(0)=0.$$

By using (2.15), (2.16), and (3.9) by some computations, we get

(4.8)
$$\frac{d}{dt}|v_{\alpha,n}(t)|_H^2+|(-C)^{1/2}v_{\alpha,n}(t)|_H^2\leq c_\alpha\,|v_{\alpha,n}(t)|_H^2$$
$$+c_{\alpha,T}\left(\|P_n-I\|^2_{\mathcal{L}(D((-C)^{1/2});H)}+|y_{\alpha,n}-y_\alpha|^2_{C([0,T];H)}\right)|Dy_\alpha(t;x)P_nh|^2_{D((-A)^{1/2})}.$$

In [6] it is proved that for each $h\in H$

$$\sup_{x\in H}\int_0^t|Dy_\alpha(s;x)h|^2_{D((-A)^{1/2})}\,ds\leq c_T\,|h|_H^2,\qquad \mathbf{P}-\text{a.s.},$$

and then, by using the Gronwall lemma, this yields

$$|v_{\alpha,n}(t)|_H^2\leq c_{\alpha,T}\left(\|P_n-I\|^2_{\mathcal{L}(D((-C)^{1/2});H)}+|y_{\alpha,n}-y_\alpha|^2_{C([0,T];H)}\right)|h|_H^2.$$

Thus, as

$$\lim_{n\to+\infty}\|P_n-I\|_{\mathcal{L}(D((-C)^{1/2});H)}=0,$$

from Lemma 3.4 we get

$$\lim_{n\to+\infty}\mathbf{E}\sup_{|h|_H\leq 1}\sup_{t\in[0,T]}|Dy_{\alpha,n}(t;x)h-Dy_\alpha(t;x)P_nh|_H^2=0.$$

Thanks to (4.8), from the limit above we get

$$\lim_{n\to+\infty}\mathbf{E}\sup_{|h|_H\leq 1}\int_0^t|(-A)^{1/2}(Dy_{\alpha,n}(s;x)h-Dy_\alpha(s;x)P_nh)|_H^2\,ds=0$$

so that (4.7) follows. □

By using the interpolation inequality (4.5), we get

$$\lim_{n\to+\infty}\mathbf{E}\sup_{|h|_H\leq 1}\int_0^t\left|Q^{-1}\left(Dy_{\alpha,n}(s;x)h-Dy_\alpha(s;x)P_nh\right)\right|_H^2\,ds=0. \tag{4.9}$$

5. The transition semigroup. The transition semigroup P_t associated with the system (1.1) is defined for any $\varphi\in B_b(H)$ and $x\in H$ by

$$P_t\varphi(x)=\mathbf{E}\,\varphi(y(t;x)),\qquad t\geq 0,$$

where $y(t;x)$ is the solution of the problem (1.1), with $z=0$, starting from x at time zero.

As proved in [7], P_t is a contraction semigroup on $C_b(H)$. In general, P_t is not strongly continuous in $C_b(H)$. Nevertheless, $y(\cdot;x)\in L^2(\Omega;C([0,T];H))$ for any fixed $x\in H$ so that the mapping

$$[0,+\infty)\to\mathbb{R},\qquad t\mapsto P_t\varphi(x),$$

is continuous for any $\varphi\in C_b(H)$.

In [7, Theorem 5.1] we have also proved that P_t has a smoothing effect. Namely, we have shown that $P_t : B_b(H) \to C_b^1(H)$ for any $t > 0$, and if ϵ is the constant introduced in part 3 of Hypothesis 2.

$$\|P_t\varphi\|_j^H \le c_0\,(t\wedge 1)^{-\frac{(j-i)(1+\epsilon)}{2}}\|\varphi\|_i^H, \qquad i \le j \le 0,1, \tag{5.1}$$

for some constant $c_0 > 0$. Moreover, if $\varphi \in C_b(H)$, for any $x, h \in H$ it holds that

$$\langle D(P_t\varphi)(x), h\rangle_H = \frac{1}{t}\,\mathbf{E}\,\varphi(y(t;x))\int_0^t \left\langle Q^{-1}v(s;x,h), dw(s)\right\rangle_H, \tag{5.2}$$

where $v(s;x,h)$ is the unique generalized solution of the problem (4.1). The formula above is a generalization to the degenerate case of the Bismut–Elworthy formula (see [2] and [16] for the finite dimension and [27] for the infinite dimension).

Now for any $\alpha > 0$ we define P_t^α as the transition semigroup corresponding to the approximating problem (3.5) with $z = 0$. As proved in [5], the semigroup P_t^α maps $B_b(H)$ into $C_b^2(H)$ for any $t > 0$, and if $\varphi \in C_b(H)$, it holds that

$$\langle D(P_t^\alpha\varphi)(x), h\rangle_H = \frac{1}{t}\,\mathbf{E}\,\varphi(y_\alpha(t;x))\int_0^t \left\langle Q^{-1}Dy_\alpha(s;x)h, dw(s)\right\rangle_H$$

for all $x, h \in H$. Moreover, for $i \le j = 0,1,2$

$$\|P_t^\alpha\varphi\|_j^H \le c_\alpha\,(t\wedge 1)^{-\frac{(j-i)(1+\epsilon)}{2}}\|\varphi\|_i^H. \tag{5.3}$$

Due to (2.8), by proceeding as in [6] it is possible to show that

$$\sup_{x\in H}\left(|Dy_\alpha(t;x)h|_H^2 + \int_0^t |(-A)^{1/2}Dy_\alpha(s;x)h|_H^2\right) \le c_T|h|_H^2, \qquad \mathbf{P}-\text{a.s.}, \tag{5.4}$$

for a constant c_T independent of α. Thus if $j = 1$, for each $i = 0,1$ we have

$$\|P_t^\alpha\varphi\|_1^H \le c\,(t\wedge 1)^{-\frac{(1-i)(1+\epsilon)}{2}}\|\varphi\|_i^H, \tag{5.5}$$

and the constant c is independent of α.

From Lemma 3.3, we easily have that for any $\varphi \in C_b(H)$ it holds that

$$\lim_{\alpha\to 0} P_t^\alpha\varphi(x) = P_t\varphi(x), \tag{5.6}$$

uniformly with respect to $t \in [0,T]$ and x in bounded subsets of E. Moreover, from Lemma 4.1, we have that

$$\lim_{\alpha\to 0} |D(P_t^\alpha\varphi)(x) - D(P_t\varphi)(x)|_H = 0, \qquad t > 0, \tag{5.7}$$

uniformly for x in bounded sets of E. Actually, for each $\alpha > 0$ it holds that

$$\langle D(P_t^\alpha\varphi)(x), h\rangle_H = \frac{1}{t}\mathbf{E}\,\varphi(y_\alpha(t;x))\int_0^t \left\langle Q^{-1}Dy_\alpha(s;x)h, dw(s)\right\rangle_H,$$

and then by easy calculations we obtain

$$\begin{aligned}
&|\langle D(P_t^\alpha\varphi)(x) - D(P_t\varphi)(x), h\rangle_H|\\
&\le \frac{\|\varphi\|_1^H}{t}\left(\mathbf{E}\,|y_\alpha(t,x) - y(t;x)|_H^2\right)^{1/2}\left(\mathbf{E}\int_0^t |Q^{-1}Dy_\alpha(s;x)h|_H^2\,ds\right)^{1/2}\\
&+\frac{\|\varphi\|_1^H}{t}\left(\mathbf{E}\int_0^t |Q^{-1}(Dy_\alpha(s;x)h - Dy(s;x)h)|_H^2\,ds\right)^{1/2}.
\end{aligned}$$

Thus (5.7) follows from (3.6) and (4.6).

In correspondence of each $n \in \mathbb{N}$, we can introduce the transition semigroup $P_t^{\alpha,n}$ associated with the system (3.10). The semigroup $P_t^{\alpha,n}$ fulfills all the regularizing properties described above for P_t^{α}. In particular, due to (3.9) it is not difficult to check that for $i = 0, 1$

$$\|P_t^{\alpha,n}\|_1^H \leq c\,(t \wedge 1)^{-\frac{(1-i)(1+\epsilon)}{2}} \|\varphi\|_i^H, \qquad t > 0, \tag{5.8}$$

for a constant c which does not depend on n and α. In the next theorem we prove that it is possible to approximate $P_t^{\alpha}\varphi$ and its first derivative by means of $P_t^{\alpha,n}$ and its first derivative.

Proposition 5.1. *Under Hypotheses* 1 and 2*, for any* $\varphi \in C_b(H)$ *we have*

$$\lim_{n\to+\infty} P_t^{\alpha,n}\varphi(x) = P_t^{\alpha}\varphi(x), \tag{5.9}$$

uniformly for x in bounded sets of H and $t \in [0,T]$. Moreover,

$$\lim_{n\to+\infty} |D(P_t^{\alpha,n}\varphi)(x) - D(P_t^{\alpha}\varphi)(x)|_H = 0, \tag{5.10}$$

uniformly for x in bounded sets of H and $t \in [\delta,T]$, with $\delta > 0$.

Proof. The limit (5.9) follows directly from Lemma 3.4. As far as the limit (5.10) is concerned, we have

$$\begin{aligned}
&\langle D(P_t^{\alpha}\varphi)(x) - D(P_t^{\alpha,n}\varphi)(x), P_n h\rangle_H \\
&= \frac{1}{t}\mathbf{E}\left(\varphi(y_\alpha(t;x)) - \varphi(y_{\alpha,n}(t;x))\right)\int_0^t \left\langle Q^{-1}Dy_\alpha(s;x)P_n h, dw(s)\right\rangle_H \\
&+\frac{1}{t}\mathbf{E}\,\varphi(y_{\alpha,n}(t;x))\int_0^t \left\langle Q^{-1}(Dy_\alpha(s;x)P_n h - Dy_{\alpha,n}(s;x)h), dw(s)\right\rangle_H .
\end{aligned}$$

Thus we get

$$\begin{aligned}
&|\langle D(P_t^{\alpha}\varphi)(x) - D(P_t^{\alpha,n}\varphi)(x), P_n h\rangle_H|^2 \\
&\leq \frac{2}{t^2}\mathbf{E}\,|\varphi(y_\alpha(t;x)) - \varphi(y_{\alpha,n}(t;x))|^2\,\mathbf{E}\int_0^t |Q^{-1}Dy_\alpha(s;x)P_n h|_H^2\,ds \\
&+\frac{2}{t^2}\|\varphi\|_0^2\,\mathbf{E}\int_0^t |Q^{-1}(Dy_\alpha(s;x)P_n h - Dxy_{\alpha,n}(s;x)h)|_H^2\,ds.
\end{aligned}$$

By taking the supremum over $|h|_H \leq 1$, due to (3.11), (4.6), and (5.4), it follows that

$$\lim_{n\to+\infty} |P_n D(P_t^{\alpha}\varphi)(x) - D(P_t^{\alpha,n}\varphi)(x)|_H = 0,$$

and, as

$$\lim_{n\to+\infty} |P_n D(P_t^{\alpha}\varphi)(x) - D(P_t^{\alpha}\varphi)(x)|_H = 0,$$

we obtain (5.10). □

6. The Hamilton–Jacobi–Bellman equation. We are here concerned with the infinite dimensional Cauchy problem

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}u(t,x) - K(Du(t,x)) + g(x), \qquad u(0,x) = \varphi(x), \tag{6.1}$$

where $\mathcal{L}$ is the differential operator defined by

$$\mathcal{L}\psi(x) = \frac{1}{2}\mathrm{Tr}\,[Q^2 D^2\psi(x)] + \langle Ax + F(x), D\psi(x)\rangle_H\,, \qquad x \in D(A) \cap D(F).$$

In addition to Hypotheses 1 and 2, the following condition shall be assumed.

HYPOTHESIS 3. *The hamiltonian $K : H \to \mathbb{R}$ is Fréchet differentiable and locally Lipschitz continuous together with its derivative. Moreover, $K(0) = 0$.*

Notice that the requirement $K(0) = 0$ is not restrictive, as we can substitute g by $g - K(0)$.

The problem (6.1) can be rewritten in the *mild* form

$$u(t,x) = P_t\varphi(x) - \int_0^t P_{t-s}\left(K(Du(s,\cdot))\right)(x)\,ds + \int_0^t P_{t-s}g(x)\,ds. \tag{6.2}$$

As we noticed in the previous section, the semigroup P_t is not strongly continuous in $C_b(H)$ in general. Nevertheless, the mapping

$$[0,+\infty) \to \mathbb{R}, \qquad t \mapsto P_t\varphi(x)$$

is continuous for any fixed $\varphi \in C_b(H)$ and $x \in H$. Thus the integrals in the formula (6.2) have a meaning only for fixed $x \in H$.

We define $\mathcal{V}_T^1$ as the space of all continuous and bounded functions $u : [0,T]\times H \to \mathbb{R}$, such that $u(t,\cdot) \in C_b^1(H)$ for all $t \in (0,T]$, and the mapping

$$(0,T] \times H \to H, \qquad (t,x) \mapsto Du(t,x)$$

is bounded and measurable. It is easy to check that $\mathcal{V}_T^1$, endowed with the norm

$$\|u\|_{\mathcal{V}_T^1} = \sup_{t\in[0,T]} \|u(t,\cdot)\|_0^H + \sup_{t\in(0,T]} \|Du(t,\cdot)\|_0^H,$$

is a Banach space.

Moreover, we define $\mathcal{Z}_T^1$ as the space of bounded continuous functions $y : [0,T] \times H \to \mathbb{R}$, such that $y(t,\cdot) \in C_b^1(H)$ for all $t \in (0,T]$, and the mapping

$$(0,T] \times H \to H, \qquad (t,x) \mapsto t^{\frac{1+\epsilon}{2}} Dy(t,x)$$

is bounded and measurable. It is easy to check that $\mathcal{Z}_T^1$, endowed with the norm

$$\|u\|_{\mathcal{Z}_T^1} = \sup_{t\in[0,T]} \|y(t,\cdot)\|_0^H + \sup_{t\in(0,T]} (t\wedge 1)^{\frac{1+\epsilon}{2}} \|Dy(t,\cdot)\|_0^H,$$

is a Banach space.

Finally, we say that a function $y \in \mathcal{V}_T^1$ belongs to the space $\mathcal{Z}_T^2$ if $y(0,\cdot) \in C_b^1(H)$, the function $y(t,\cdot)$ is in $C_b^2(H)$ for any $t > 0$, and the mapping

$$(0,T] \times H \to \mathcal{L}(H), \qquad (t,x) \mapsto (t\wedge 1)^{\frac{1+\epsilon}{2}} D^2y(t;x)$$

is bounded and measurable. $\mathcal{Z}^2_T$, endowed with the norm

$$\|u\|_{\mathcal{Z}^2_T} = \sup_{t\in[0,T]} \|y(t,\cdot)\|_1^H + \sup_{t\in(0,T]} (t\wedge 1)^{\frac{1+\epsilon}{2}} \|D^2 y(t,\cdot)\|_0^H,$$

is a Banach space.

A proof of the following lemma, in the case when $F=0$, can be found in [20, Lemmas 4.8 and 4.12]. Such a proof completely adapts to our case where $F\neq 0$; thus we do not repeat it.

LEMMA 6.1. *Let us fix $T>0$, and, for $\psi:[0,T]\times H\to\mathbb{R}$ bounded and measurable, let us define*

$$\lambda(\psi)(t,x) = \int_0^t P_{t-s}\psi(s,\cdot)(x)\,ds.$$

Then $\lambda(\psi)$ is continuous and bounded, $\lambda(\psi)(t,\cdot)\in C^1_b(H)$ for any $t\geq 0$, and

$$\sup_{t\in[0,T]} \|\lambda(\psi)(t,\cdot)\|_1^H < \infty.$$

It is immediate to check that Lemma 6.1 adapts to the approximating semigroups P^α_t and $P^{\alpha,n}_t$.

For each $\alpha>0$, we consider the approximating problem

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}_\alpha u(t,x) - K(Du(t,x)) + g(x), \qquad u(0,x)=\varphi(x),$$

where

$$\mathcal{L}_\alpha\psi(x) = \frac{1}{2}\mathrm{Tr}\,[Q^2D^2\psi(x)] + \langle Ax+F_\alpha(x), D\psi(x)\rangle_H\,.$$

In mild form it can be rewritten as

$$u(t,x) = P^\alpha_t\varphi(x) - \int_0^t P^\alpha_{t-s}\left(K(Du(s,\cdot))\right)(x)\,ds + \int_0^t P^\alpha_{t-s}g(x)\,ds. \tag{6.3}$$

The first part of the following theorem was proved in [7], under the assumption of Lipschitz continuity for the hamiltonian K. Here the proof is more delicate, as K is only locally Lipschitz.

THEOREM 6.2. *Assume that Hypotheses* 1, 2, *and* 3 *hold, and fix $T>0$. Then for any $\varphi,g\in\mathrm{Lip}_b(H)$,* (6.2) *admits a unique solution $u(t,x)$ in $\mathcal{V}^1_T$.*

If $u_\alpha(t,x)$ denotes the unique mild solution for the approximating problem (6.3), *we have*

$$\lim_{\alpha\to 0} |u_\alpha(t,x)-u(t,x)| + |Du_\alpha(t,x)-Du(t,x)|_H = 0, \tag{6.4}$$

uniformly for t in compact subsets of $(0,T]$ and for x in bounded subsets of E. Moreover, if $\varphi,g\in C^1_b(H)$, then the limit (6.4) *is uniform for $t\in[0,T]$ and for x in bounded subsets of E.*

We first prove some preliminary results.

LEMMA 6.3. *Fix $\varphi,g\in C^1_b(H)$ and $R\geq 2c_0\|\varphi\|_1^H$, where c_0 is the constant introduced in* (5.1). *Then the problem* (6.2) *admits a unique local solution $u(t,x)$ in $[0,\tau_R]$ for some constant*

$$\tau_R = \tau_R\left(\|\varphi\|_1^H, \|g\|_0^H, \|K\|_1^{B^H_R}\right).$$

Proof. For any $\tau > 0$, we define $\Lambda_R(\tau)$ as the set of all bounded continuous functions $u : [0,\tau] \times H \to \mathbb{R}$ such that $u(t,\cdot) \in C_b^1(H)$ for all $t \in [0,\tau]$, the mapping

$$[0,\tau] \times H \to H, \qquad (t,x) \mapsto Du(t,x)$$

is bounded and measurable, and

$$\sup_{t\in[0,\tau]} \|u(t,\cdot)\|_1^H \leq R.$$

We claim that for some τ_R sufficiently small, the operator Γ defined by

$$\Gamma(v)(t,x) = P_t\varphi(x) - \int_0^t P_{t-s}\left(K(Dv(s,\cdot))\right)(x)\,ds + \int_0^t P_{t-s}g(x)\,ds$$

maps $\Lambda_R(\tau_R)$ into itself as a contraction. Due to Lemma 6.1, $\Gamma(v)(t,x)$ is well defined for any x and t. Due to (5.1) we have

$$\|P_t\varphi\|_1^H \leq c_0\,\|\varphi\|_1^H \leq \frac{R}{2}.$$

Moreover, if we set

$$\Gamma_1(v)(t,x) = -\int_0^t P_{t-s}\left(K(Dv(s,\cdot))\right)(x)\,ds + \int_0^t P_{t-s}g(x)\,ds,$$

we have

$$\|\Gamma_1(v)(t,\cdot)\|_0^H \leq \int_0^t \|K(Dv(s,\cdot))\|_0^H\,ds + t\|g\|_0^H \leq \tau\Big(\sup_{x\in B_R^H}|K(x)| + \|g\|_0^H\Big).$$

Concerning the derivative, due to the estimate (5.1) it holds that

$$\|D(\Gamma_1(v))(t,\cdot)\|_0^H \leq c_0\int_0^t (t-s)^{-\frac{1+\epsilon}{2}}\left(\|K(Dv(s,\cdot))\|_0^H + \|g\|_0^H\right)ds$$

$$\leq c_0\,\tau^{\frac{1-\epsilon}{2}}\Big(\sup_{x\in B_R^H}|K(x)| + \|g\|_0^H\Big).$$

This implies that

$$\sup_{t\in[0,\tau]}\|\Gamma(v)(t,\cdot)\|_1^H \leq \frac{R}{2} + c\left(\tau + \tau^{\frac{1-\epsilon}{2}}\right)\Big(\sup_{x\in B_R^H}|K(x)| + \|g\|_0^H\Big)$$

so that it is possible to find $\bar{\tau}_R$ sufficiently small such that

$$\sup_{t\in[0,\bar{\tau}_R]}\|\Gamma(v)(t,\cdot)\|_1^H \leq R.$$

In a completely analogous way it is possible to show that Γ is a contraction on $\Lambda_R(\tau_R)$ for some $\tau_R \leq \bar{\tau}_R$. This allows us to conclude that there exists a unique fixed point u for Γ in $\Lambda_R(\tau_R)$, which is the unique solution of (6.2) in $[0,\tau_R]$. ∎

Remark 6.4. In an identical way it is possible to prove that for each $\alpha > 0$ the mapping

$$\Gamma_\alpha(v)(t,x) = P_t^\alpha\varphi(x) - \int_0^t P_{t-s}^\alpha\left(K(Dv(s,\cdot))\right)(x)\,ds + \int_0^t P_{t-s}^\alpha g(x)\,ds$$

is a contraction in $\Lambda_R(\tau_R)$, where τ_R is the same as in Lemma 6.3. This implies that there exists a unique solution $u_\alpha(t,x)$ for the problem (6.3).

Moreover, it is useful to remark that thanks to (5.5) the contraction constant of Γ_α in $\Lambda_R(\tau_R)$ can be taken as the same for all $\alpha > 0$.

LEMMA 6.5. *If $u(t,x)$ and $u_\alpha(t,x)$ are, respectively, the solutions of the problems* (6.2) *and* (6.3) *with $\varphi, g \in C_b^1(H)$, we have*

$$\lim_{\alpha\to 0} |u_\alpha(t,x) - u(t,x)| + |Du_\alpha(t,x) - Du(t,x)|_H = 0, \tag{6.5}$$

uniformly for t in $[0,\tau_R]$ and for x in bounded subsets of E.

Proof. In order to prove the existence of the solutions $u(t,x)$ and $u_\alpha(t,x)$ for the problems (6.2) and (6.3), we have applied a contraction theorem. Hence, due to Lemma 6.3 and Remark 6.4, for each $\epsilon > 0$ there exists $k_\epsilon \in \mathbb{N}$ such that

$$\sup_{t\in[0,\tau_R]} \left(\|u_\alpha(t,\cdot) - \Gamma_\alpha^{k_\epsilon}(0)(t,\cdot)\|_1^H + \|u(t,\cdot) - \Gamma^{k_\epsilon}(0)(t,\cdot)\|_1^H \right) \leq \epsilon/2 \tag{6.6}$$

for each $\alpha > 0$. Now, from Proposition 5.1, by using an induction argument we can prove that for each $k \in \mathbb{N}$

$$\lim_{\alpha\to 0} \left|\Gamma_\alpha^k(0)(t,x) - \Gamma^k(0)(t,x)\right| + \left|D\Gamma_\alpha^k(0)(t,x) - D\Gamma^k(0)(t,x)\right|_H = 0, \tag{6.7}$$

uniformly for (t,x) in bounded subsets of $[0,\tau_R]\times E$. Actually, for $k=1$, (6.7) follows directly from (5.6) and (5.7). Now assume that (6.7) holds for some $k \geq 1$. We have

$$\Gamma_\alpha^{k+1}(0)(t,x) - \Gamma^{k+1}(0)(t,x) = \Gamma_\alpha(\Gamma_\alpha^k(0))(t,x) - \Gamma(\Gamma^k(0))(t,x)$$

$$= P_t^\alpha\varphi(x) - P_t\varphi(x) + \int_0^t \left(P_{t-s}^\alpha g(x) - P_{t-s}g(x)\right) ds$$

$$- \int_0^t \left(P_{t-s}^\alpha \left[K(D(\Gamma_\alpha^k(0))(s,\cdot))\right](x) - P_{t-s}\left[K(D(\Gamma^k(0))(s,\cdot))\right](x)\right) ds.$$

Since $\Gamma_\alpha^k(0)$ and $\Gamma^k(0)$ belong to $\Lambda_R(\tau_R)$ and (6.7) holds for k, by using (5.6) and the boundedness of K on bounded subsets of H, from the dominated convergence theorem it follows that

$$\lim_{\alpha\to 0} \int_0^t \left(P_{t-s}^\alpha \left[K(D(\Gamma_\alpha^k(0))(s,\cdot))\right](x) - P_{t-s}\left[K(D(\Gamma^k(0))(s,\cdot))\right](x)\right) ds = 0,$$

uniformly on bounded sets of $[0,\tau_R]\times E$. By using (5.6) once more, we have

$$\lim_{\alpha\to 0} P_t^\alpha\varphi(x) - P_t\varphi(x) + \int_0^t \left(P_{t-s}^\alpha g(x) - P_{t-s}g(x)\right) ds = 0,$$

uniformly on bounded sets of $[0,\tau_R]\times E$, so that we get

$$\lim_{\alpha\to 0} \Gamma_\alpha^{k+1}(0)(t,x) - \Gamma^{k+1}(0)(t,x) = 0.$$

The second part of the limit (6.7) for $k+1$ follows by analogous arguments. By induction we can conclude that (6.7) holds for any $k \in \mathbb{N}$.

Now, from (6.6) we have that

$$|u_\alpha(t,x)-u(t,x)|+|Du_\alpha(t,x)-Du(t,x)|_H$$

$$\leq \sup_{t\in[0,\tau_R]}\|u_\alpha(t,\cdot)-\Gamma^{k_\epsilon}_\alpha(0)(t,\cdot)\|^H_1+|\Gamma^{k_\epsilon}_\alpha(0)(t,x)-\Gamma^{k_\epsilon}(0)(t,x)|$$

$$+|D\Gamma^{k_\epsilon}_\alpha(0)(t,x)-D\Gamma^{k_\epsilon}(0)(t,x)|_H+\sup_{t\in[0,\tau_R]}\|u(t,\cdot)-\Gamma^{k_\epsilon}(0)(t,\cdot)\|^H_1$$

$$\leq \epsilon/2+|\Gamma^{k_\epsilon}_\alpha(0)(t,x)-\Gamma^{k_\epsilon}(0)(t,x)|+\left|D\Gamma^{k_\epsilon}_\alpha(0)(t,x)-D\Gamma^{k_\epsilon}(0)(t,x)\right|_H,$$

and due to (6.7) we can conclude that (6.5) holds. ▯

Proof of Theorem 6.2. Let us fix $T>0$ and $\varphi,g\in C^1_b(H)$, and let us define

$$R=2c\,(1+T)\,e^{c\,T}\left((1+2c_0)\|\varphi\|^H_1+\|g\|^H_1\right).$$

Due to Lemma 6.3, there exists a mild solution $u(t,x)$ defined for $t\in[0,\tau_R]$. Moreover, from Lemma 6.5 we have that (6.4) holds, uniformly with respect to $t\in[0,\tau_R]$ and x in bounded sets of E.

From Proposition A.3, we have that

$$\sup_{t\in[0,\tau_\star]}\|u_\alpha(t,\cdot)\|^H_1\leq c\,(1+T)e^{c\,T}\left(\|\varphi\|^H_1+\|g\|^H_1\right).$$

According to Lemma 6.5, this implies that for any $t\in[0,\tau_\star]$ and $x\in E$

$$|u(t,x)|+|Du(t,x)|_H\leq c\,(1+T)e^{c\,T}\left(\|\varphi\|^H_1+\|g\|^H_1\right),$$

and, since $u(t,\cdot)\in C^1_b(H)$ for $t\in[0,\tau_\star]$, we have

$$\sup_{t\in[0,\tau_\star]}\|u(t,\cdot)\|^H_1\leq c\,(1+T)e^{c\,T}\left(\|\varphi\|^H_1+\|g\|^H_1\right). \tag{6.8}$$

In particular, due to the definition of R we have that

$$\|u(\tau_\star,\cdot)\|^H_1\leq\frac{R}{2}.$$

This allows us to repeat all of the same arguments we have been using until now in the intervals $[\tau_\star,2\tau_\star]$, $[2\tau_\star,3\tau_\star]$, and so on, up to time T, and hence to get a global solution.

Now, assume that $\varphi,g\in \mathrm{Lip}_b(H)$. It is possible to find two bounded sequences $\{\varphi_k\}$ and $\{g_k\}$ in $C^1_b(H)$ converging, respectively, to φ and g in $C_b(H)$. In correspondence with each k, there exists a unique solution $u_k(t,x)$ to the problem

$$u_k(t,x)=P_t\varphi_k(x)-\int_0^t P_{t-s}\left(K(Du_k(s,\cdot))\right)(x)\,ds+\int_0^t P_{t-s}g_k(x)\,ds.$$

Our aim is to show that $\{u_k\}$ is a Cauchy sequence in $\mathcal{Z}^1_T$ and that the limit u fulfills (6.2).

For each $k,n\in\mathbb{N}$ we have

$$u_k(t,x)-u_h(t,x)=P_t\left(\varphi_k-\varphi_h\right)(x)$$

$$-\int_0^t P_{t-s}\left(K(Du_k(s,\cdot))-K(Du_h(s,\cdot))\right)(x)\,ds+\int_0^t P_{t-s}\left(g_k-g_h\right)(x)\,ds.$$

Due to (6.8) we easily have

$$\sup_{k\in\mathbb{N}}\sup_{t\in[0,T]}\|Du_k(t,\cdot)\|_1^H\leq c\,(1+T)e^{c\,T}\sup_{k\in\mathbb{N}}\left(\|\varphi_k\|_1+\|g_k\|_1^H\right)=c_T. \tag{6.9}$$

If M is the Lipschitz constant of K in $B^H_{c_T}$, we have

$$\begin{aligned}\|u_k(t,\cdot)-u_h(t,\cdot)\|_0^H&\leq\|\varphi_k-\varphi_h\|_0^H\\&+M\int_0^t\|Du_k(s,\cdot)-Du_h(s,\cdot)\|_0^H\,ds+t\,\|g_k-g_h\|_0.\end{aligned} \tag{6.10}$$

Moreover, we have

$$\begin{aligned}&Du_k(t,x)-Du_h(t,x)=DP_t(\varphi_k-\varphi_h)(x)\\&-\int_0^t DP_{t-s}\left(K(Du_k(s,\cdot))-K(Du_h(s,\cdot))\right)(x)\,ds+\int_0^t DP_{t-s}(g_k-g_h)(x)\,ds,\end{aligned}$$

so that, thanks to (5.1), we get

$$\begin{aligned}&\|Du_k(t,\cdot)-Du_h(t,\cdot)\|_0^H\leq c\,t^{-\frac{1+\epsilon}{2}}\|\varphi_k-\varphi_h\|_0^H\\&+c\,M\int_0^t(t-s)^{-\frac{1+\epsilon}{2}}\|Du_k(s,\cdot)-Du_h(s,\cdot)\|_0^H\,ds+c\int_0^t(t-s)^{-\frac{1+\epsilon}{2}}\,ds\|g_k-g_h\|_0^H.\end{aligned}$$

This implies that

$$\begin{aligned}&t^{\frac{1+\epsilon}{2}}\|Du_k(t,\cdot)-Du_h(t,\cdot)\|_0^H\leq c\,\|\varphi_k-\varphi_h\|_0^H\\&+c\,Mt^{\frac{1+\epsilon}{2}}\int_0^t(t-s)^{-\frac{1+\epsilon}{2}}\|Du_k(s,\cdot)-Du_h(s,\cdot)\|_0^H\,ds+c\,t\,\|g_k-g_h\|_0^H.\end{aligned} \tag{6.11}$$

By combining (6.10) and (6.11), we conclude that

$$\begin{aligned}&\|u_k(t,\cdot)-u_h(t,\cdot)\|_0^H+t^{\frac{1+\epsilon}{2}}\|Du_k(t,\cdot)-Du_h(t,\cdot)\|_0^H\\&\leq c\left(\|\varphi_k-\varphi_h\|_0^H+T\|g_k-g_h\|_0^H\right)+M(1+c\,T^{\frac{1+\epsilon}{2}})\int_0^t s^{-\frac{1+\epsilon}{2}}\left((t-s)^{-\frac{1+\epsilon}{2}}+1\right)\\&\left(\|u_k(s,\cdot)-u_h(s,\cdot)\|_0^H+s^{\frac{1+\epsilon}{2}}\|Du_k(s,\cdot)-Du_h(s,\cdot)\|_0^H\right)\,ds.\end{aligned}$$

Thus, from a generalization of the Gronwall lemma, we can say that

$$\|u_k-u_h\|_{\mathcal{Z}^1_T}\leq c_T\left(\|\varphi_k-\varphi_h\|_0^H+T\|g_k-g_h\|_0^H\right) \tag{6.12}$$

for some constant c_T independent of k and h. This implies that $\{u_k\}$ is a Cauchy sequence in $\mathcal{Z}^1_T$, and hence it converges to a limit $u\in\mathcal{Z}^1_T$. Moreover, from (6.9) we have that

$$\sup_{t\in[0,T]}\|Du(t,\cdot)\|_0^H<+\infty,$$

so that $u \in \mathcal{V}^1_T$.

Now, we show that u is the mild solution of the problem (6.2). Actually, for any $s > 0$ and $x \in H$

$$\lim_{k\to+\infty} K(Du_k(s,x)) = K(Du(s,x)).$$

Due to (6.9) we can apply the dominated convergence theorem, and we get

$$\lim_{k\to+\infty} \int_0^t P_{t-s}K(Du_k(s,\cdot))(x)\,ds = \int_0^t P_{t-s}K(Du(s,\cdot))(x)\,ds.$$

Therefore, since

$$\lim_{k\to+\infty} P_t\varphi_k(x) = P_t\varphi(x)$$

and

$$\lim_{k\to+\infty} \int_0^t P_{t-s}g_k(x)\,ds = \int_0^t P_{t-s}g(x)\,ds,$$

we conclude that u is a solution of (6.2).

Finally, uniqueness follows from the Gronwall lemma and local Lipschitzianity of K. Indeed, if u_1 and u_2 are two solutions in $\mathcal{V}^1_T$, we have

$$u_1(t,x) - u_2(t,x) = -\int_0^t P_{t-s}\left(K(Du_1(s,\cdot)) - K(Du_2(s,\cdot))\right)(x)\,ds,$$

and then, if M is the Lipschitz constant of K in $B^H_{c_T}$, where

$$c_T = \|u_1\|_{\mathcal{V}^1_T} + \|u_2\|_{\mathcal{V}^1_T},$$

we have

$$\|u_1 - u_2\|_{\mathcal{V}^1_T} \le M\int_0^t \left(1 + (t-s)^{-\frac{1+\epsilon}{2}}\right) ds \|u_1 - u_2\|_{\mathcal{V}^1_T}.$$

This implies that $u_1 = u_2$. □

7. Application to control. We apply here the results proved in the previous section to a stochastic control problem. Let $k : H \to]-\infty, +\infty]$ be a convex lower semicontinuous function, and let K be its Legendre transform; that is,

$$K(x) = \sup_{z\in H}\left\{-\langle x, z\rangle_H - k(z)\right\}, \qquad x \in H.$$

We assume that k is such that K fulfills Hypothesis 3. We consider here the cost functional

$$J(t,x;z) = \mathbf{E}\int_t^T \left(g(y(s)) + k(z(s))\right) ds + \mathbf{E}\,\varphi(y(T)),$$

where $y(s) = y(s,t;x,z)$ is the unique solution of the controlled system (1.1) at time s, starting from x at time t. We want to minimize the functional J over all adapted controls $z \in L^2(\Omega; L^2([0,T]; H))$.

The *value function* corresponding to the cost functional J is defined by

$$V(t,x) = \inf\left\{ J(t,x;z) \,:\, z \in L^2(\Omega; L^2([0,T];H)) \text{ adapted} \right\}$$

and is related to the Hamilton–Jacobi–Bellman equation (6.1). Namely, we are showing that for every $t \in [0,T]$ and $x \in H$

$$V(t,x) = u(T-t,x),$$

where $u(t,x)$ is the unique mild solution of the problem (6.1).

For any $\alpha > 0$ we introduce the approximating cost functional

$$J_\alpha(t,x;z) = \mathbf{E}\int_t^T (g(y_\alpha(s)) + k(z(s)))\, ds + \mathbf{E}\,\varphi(y_\alpha(T)), \tag{7.1}$$

where $y_\alpha(s) = y_\alpha(s,t;x,z)$ is the unique solution to the problem (3.5). In what follows we will denote by $V_\alpha(t,x)$ the corresponding value function.

LEMMA 7.1. *Assume Hypotheses* 1, 2, *and* 3, *and assume that* $\varphi, g \in \mathrm{Lip}_b(H)$. *If* u *is the mild solution of the problem* (6.2), *for any control* $z \in L^2(\Omega; L^2([0,T];H))$, $x \in H$, *and* $t \in [0,T]$, *the following identity holds:*

$$\begin{aligned} &J(t,x;z) = u(T-t,x)\\ &\quad + \int_t^T \mathbf{E}\,\left(K(Du(T-s,y(s))) + \langle z(s), Du(T-s,y(s))\rangle_H + k(z(s))\right) ds, \end{aligned} \tag{7.2}$$

where $y(s) = y(s,t;x,z)$ *is the solution of the problem* (3.1).

Moreover, the same identity holds with $J(t,x;z)$, $u(t,x)$, $Du(t,x)$, *and* $y(t)$ *replaced, respectively, by* $J_\alpha(t,x;z)$, $u_\alpha(t,x)$, $Du_\alpha(t,x)$, *and* $y_\alpha(t)$.

Proof. We first assume that $\varphi, g \in C^1_b(H)$. Let $u_{\alpha,n}(t,x)$ be the solution of (A.2), and let $y_{\alpha,n}(s) = y_{\alpha,n}(s,t;x,z)$ be the solution to the problem (3.10). Since $u_{\alpha,n}$ is smooth (in fact, $u_{\alpha,n} \in Z^2(T)$) and $y_{\alpha,n}$ is a strong solution, we can apply Itô's formula to the function $s \mapsto u_{\alpha,n}(T-s, y_{\alpha,n}(s))$ for $t \le s \le T$, and we get

$$\begin{aligned} &du_{\alpha,n}(T-s,y_{\alpha,n}(s)) = \langle dy_{\alpha,n}(s), Du_{\alpha,n}(T-s,y_{\alpha,n}(s))\rangle_H\\ &\quad + \left(\frac{1}{2}\mathrm{Tr}\left[Q_n^2 D^2 u_{\alpha,n}(T-s,y_{\alpha,n}(s))\right] - \frac{\partial u_{\alpha,n}}{\partial t}(T-s,y_{\alpha,n}(s))\right) ds. \end{aligned}$$

By integrating with respect to $s \in [t,T]$ and by taking the expectation, we get

$$\begin{aligned} \mathbf{E}\,\varphi(y_{\alpha,n}(T)) - u_{\alpha,n}(T-t,x) &= \mathbf{E}\int_t^T (K(Du_{\alpha,n}(T-s,y_{\alpha,n}(s)))\\ &\quad + \langle z(s), Du_{\alpha,n}(T-s,y_{\alpha,n}(s))\rangle_H - g(y_{\alpha,n}(s)))\, ds. \end{aligned} \tag{7.3}$$

Now, due to Lemma 3.4 and (A.3), we can take the limit as n goes to $+\infty$ in each side of (7.3), and, rearranging all terms, we get

$$\begin{aligned} &\mathbf{E}\,\varphi(y_\alpha(T)) + \mathbf{E}\int_t^T g(y_\alpha(s))\, ds\\ &= u_\alpha(T-t,x) + \mathbf{E}\int_t^T (K(Du_\alpha(T-s,y_\alpha(s))) + \langle z(s), Du_\alpha(T-s,y_\alpha(s))\rangle_H)\, ds. \end{aligned} \tag{7.4}$$

This implies (7.2).

Now, let $\varphi, g \in \mathrm{Lip}_b(H)$. As in the proof of Theorem 6.2, let $\{\varphi_k\}$ and $\{g_k\}$ be two bounded sequences in $C_b^1(H)$ converging, respectively, to φ and g in $C_b(H)$. If we denote by $u_\alpha^k(t,x)$ the solutions of the problem (6.2) corresponding to φ_k and g_k, we have

$$\mathbf{E}\,\varphi_k(y_\alpha(T)) + \mathbf{E}\int_t^T g_k(y_\alpha(s))\,ds = u_\alpha^k(T-t,x)$$

$$+\mathbf{E}\int_t^T \left(K(Du_\alpha^k(T-s,y_\alpha(s))) + \left\langle z(s), Du_\alpha^k(T-s,y_\alpha(s))\right\rangle_H\right)\,ds.$$

It is immediate to check that the sequence $\{u_\alpha^k\}$ fulfills an estimate analogous to (6.12), and then the sequence $\{u_\alpha^k\}$ converges to u_α in $\mathcal{Z}_T^1$, as k goes to infinity. Moreover, due to (2.19), (3.3), and (6.9), we can apply the dominated convergence theorem and, by taking the limit for k going to $+\infty$, we get (7.4) for any $\varphi, g \in \mathrm{Lip}_b(H)$.

Now, if $x \in E$, then $y_\alpha(s) \in E$ and (4.4) holds. Thus, due to (3.11) and (6.4), we can take the limit as α goes to zero in each side of (7.4), and we get (7.2) for $x \in E$. Finally, if $x \in H$, we fix a sequence $\{x_n\} \subset E$ converging to x in H. Thanks to (3.4) we have that $y(s,t;x_n,z)$ converges to $y(s,t;x,z)$ in H, and then, as $u(t,\cdot) \in C_b^1(H)$, we easily get (7.2) for any $x \in H$. ∎

Now we can conclude by giving the main result of this section.

THEOREM 7.2. *Under Hypotheses* 1, 2, *and* 3, *for any* $\varphi, g \in \mathrm{Lip}_b(H)$ *the value function* $V(t,x)$ *coincides with* $u(T-t,x)$*, where* $u(t,x)$ *is the solution of the problem* (6.2). *Moreover,*

$$V(t,x) = \lim_{\alpha\to 0}\, \min\, \left\{\, J_\alpha(t,x;z),\ z \in L^2(\Omega; L^2(0,T;H)),\ \textit{adapted}\,\right\},$$

where $J_\alpha(t,x;z)$ *is the cost functional defined in* (7.1).

Proof. From (7.2) we immediately have that $J(t,x;z) \geq u(T-t,x)$ for any $z \in L^2(\Omega; L^2(0,T;H))$, so that $V(t,x) \geq u(T-t,x)$. Now we prove the opposite inequality.

Since J_α fulfills a formula analogous to (7.2), we have $V_\alpha(t,x) \geq u_\alpha(T-t,x)$. In fact, it holds that $V_\alpha(t,x) = u_\alpha(T-t,x)$. Actually, by a general property of the Legendre transform, for each $t \in [0,T]$ the mapping

$$H \to \mathbb{R}, \qquad z \mapsto \langle z, Du_\alpha(T-t,y(t))\rangle_H + k(z),$$

attains its maximum for

$$z_\alpha(t) = -DK(Du_\alpha(T-t,y(t))), \qquad t \in [0,T].$$

Thus, if we prove that the *closed loop* equation

(7.5)
$$dy(t) = (Ay(t) + F_\alpha(y(t)) - DK(Du_\alpha(T-t,y(t))))\,dt + Q\,dw(t), \qquad y(0) = x,$$

admits a unique solution $y_\alpha^\star(t)$, and if we define

$$z_\alpha^\star(t) = -DK(Du_\alpha(T-t,y_\alpha^\star(t))),$$

due to (7.2) for J_α we have that $J_\alpha(t,x;z_\alpha^\star) = u(T-t,x)$, so that $y_\alpha^\star(t)$ and $z_\alpha^\star(t)$ are, respectively, the unique optimal state and the unique optimal control for the minimizing problem corresponding to the functional J_α.

Assume that $\varphi, g \in C_b^1(H)$. Due to (5.3) it is possible to show that the solution u_α of the problem (6.3) belongs to $Z^2(T)$ and

$$\|u_\alpha\|_{Z^2(T)} \leq c_{\alpha,T} \left(\|\varphi\|_1^H + T\,\|g\|_1^H\right). \tag{7.6}$$

Thus, if we define for $(t,x) \in [0,T] \times H$

$$U_\alpha(t,x) = -DK(Du_\alpha(t,x)),$$

we have that the function U_α fulfills the conditions of Lemma A.1, so that there exists a unique solution $y_\alpha^\star(t)$ for the closed loop equation (7.5).

Now, assume that $\varphi, g \in \mathrm{Lip}_b(H)$. As in the proofs of Theorem 6.2 and of Lemma 7.1, we approximate them in $C_b(H)$ by two bounded sequences $\{\varphi_k\}$ and $\{g_k\}$ in $C_b^1(H)$. For each k there exists a unique solution $u_{\alpha,k}$ for the problem (6.3), with data φ_k and g_k. Thus, as proved above, in correspondence with each $u_{\alpha,k}$ there exists a unique solution $y_{\alpha,k}^\star(t)$ for the problem (7.5). Let us define $v_{h,k}^\alpha(t) = y_{\alpha,k}^\star(t) - y_{\alpha,h}^\star(t)$. We have that $v_{h,k}^\alpha(t)$ is the unique solution of the problem

$$\begin{aligned}\frac{dv}{dt}(t) = \quad & Av(t) + F_\alpha(y_{\alpha,k}^\star(t)) - F_\alpha(y_{\alpha,h}^\star(t)) - DK(Du_{\alpha,k}(T-t, y_{\alpha,k}^\star(t)))\\[2mm] & +DK(Du_{\alpha,h}(T-t, y_{\alpha,h}^\star(t))), \qquad v(0) = 0.\end{aligned}$$

Thus, by multiplying each side by $v_{h,k}^\alpha(t)$, due to the Lipschitz continuity of F_α and (2.13) we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v_{h,k}^\alpha(t)|_H^2 \leq c_\alpha\,|v_{h,k}^\alpha(t)|_H^2\\[2mm] &+|DK(Du_{\alpha,k}(T-t, y_{\alpha,k}^\star(t))) - DK(Du_{\alpha,h}(T-t, y_{\alpha,h}^\star(t)))|_H |v_{h,k}^\alpha(t)|_H.\end{aligned} \tag{7.7}$$

Since Proposition A.3 holds, the sequences $\{\varphi_k\}$ and $\{g_k\}$ are bounded in $C_b^1(H)$, and DK is locally Lipschitz continuous, there exists $c > 0$ such that

$$\begin{aligned}&|DK(Du_{\alpha,k}(T-t, y_{\alpha,k}^\star(t))) - DK(Du_{\alpha,h}(T-t, y_{\alpha,h}^\star(t)))|_H\\[2mm] &\leq c\,|Du_{\alpha,k}(T-t, y_{\alpha,k}^\star(t)) - Du_{\alpha,h}(T-t, y_{\alpha,h}^\star(t))|_H.\end{aligned}$$

Now, for any $t > 0$ and $x, y \in H$, due to (7.6) we have

$$\begin{aligned}&|Du_{\alpha,k}(t,x) - Du_{\alpha,k}(t,y)|_H\\[2mm] &\leq c_{\alpha,T} t^{-\frac{1+\epsilon}{2}} \left(\|\varphi_k\|_1^H + T\,\|g_k\|_1^H\right) |x-y|_H \leq c_{\alpha,T} t^{-\frac{1+\epsilon}{2}} |x-y|_H\end{aligned}$$

for some constant independent of k. Moreover, we can repeat all arguments used in the proof of Theorem 6.2, and we have

$$|Du_{\alpha,k}(t,y) - Du_{\alpha,h}(t,y)|_H \leq c_{\alpha,T} t^{-\frac{1+\epsilon}{2}} \left(\|\varphi_k - \varphi_h\|_0^H + T\|g_k - g_h\|_0^H\right).$$

Therefore, we get

$$\begin{aligned}&\left|K(Du_{\alpha,k}(T-t, y_{\alpha,k}^\star(t))) - DK(Du_{\alpha,h}(T-t, y_{\alpha,h}^\star(t)))\right|_H\\[2mm] &\leq c_{\alpha,T}(T-t)^{-\frac{1+\epsilon}{2}} \left(|v_{h,k}^\alpha(t)|_H + \|\varphi_k - \varphi_h\|_0^H + T\|g_k - g_h\|_0^H\right),\end{aligned}$$

so that, from (7.7) we conclude

$$\frac{d}{dt}|v^\alpha_{h,k}(t)|^2_H \leq c_{\alpha,T}\left(1+(T-t)^{-\frac{1+\epsilon}{2}}\right)|v^\alpha_{h,k}(t)|^2_H$$

$$+c_{\alpha,T}(T-t)^{-\frac{1+\epsilon}{2}}\left(\|\varphi_k-\varphi_h\|^H_0+T\|g_k-g_h\|^H_0\right)^2.$$

Due to the Gronwall lemma this yields

$$|y^\star_{\alpha,k}(t)-y^\star_{\alpha,h}(t)|^2_H \leq c_{\alpha,T}\left(\|\varphi_k-\varphi_h\|^H_0+T\|g_k-g_h\|^H_0\right)^2,$$

and the sequence $\{y^\star_{\alpha,k}\}$ converges to some $y^\star_\alpha$ in $C([0,T];H)$, $\mathbf{P}$-a.s. and in mean-square, and clearly $y^\star_\alpha$ is the unique solution of the closed loop (7.5).

Since $z^\star_\alpha(t)=-DK(Du_\alpha(T-t,y^\star_\alpha(t)))$, then due to (7.6) there exists a constant R such that

$$\sup_{\alpha>0}\sup_{t\in[0,T]}|z^\star_\alpha(t)|_H=R, \qquad \mathbf{P}-\text{a.s.}$$

This means that if we define the set $\mathcal{M}_R$ as in the Lemma 3.3, then for any $\alpha>0$

$$V_\alpha(t,x)=\inf\{\,J_\alpha(t,x;z)\ ;\ z\in\mathcal{M}_R\,\}. \tag{7.8}$$

By using (6.4) we have that if $\varphi,g\in C^1_b(H)$, then for any $t\in[0,T]$ and $x\in E$

$$\lim_{\alpha\to 0}V_\alpha(t,x)=u(T-t,x).$$

Moreover,

$$|J_\alpha(t,x;z)-J(t,x;z)|\leq\|\varphi\|^H_1\mathbf{E}\,|y_\alpha(T)-y(T)|_E+\|g\|^H_1\int_0^t\mathbf{E}\,|y_\alpha(s)-y(s)|_E\,ds,$$

so that, due to Lemma (3.3) we have

$$\lim_{\alpha\to 0}\sup_{z\in\mathcal{M}_R}|J_\alpha(t,x;z)-J(t,x;z)|=0.$$

Due to (7.8) we have that

$$u(T-t,x)=\lim_{\alpha\to 0}V_\alpha(t,x)=\inf\{\,J(t,x;z)\ ;\ z\in\mathcal{M}_R\,\}\geq V(t,x),$$

and since $u(T-t,x)\leq V(t,x)$, we conclude that $u(T-t,x)=V(t,x)$ for $\varphi,g\in C^1_b(H)$ and $x\in E$.

Now, if $x\in H$ and $\{x_n\}$ is a sequence in E converging to x in H, by using (3.4) we can prove that

$$\lim_{n\to+\infty}\sup_{z\in\mathcal{M}_R}|J(t,x_n;z)-J(t,x;z)|=0.$$

Therefore, since $u(t,x_n)$ converges to $u(t,x)$, we get the theorem for any $x\in H$. Finally, if $\varphi,g\in\text{Lip}_b(H)$, let $\{\varphi_k\}$ and $\{g_k\}$ be two bounded sequences in $C^1_b(H)$, converging, respectively, to φ and g in $C_b(H)$. We have

$$\lim_{k\to+\infty}\mathbf{E}\,\varphi_k(y(T))+\mathbf{E}\int_0^t(g_k(y(s))+k(z(s)))\,ds=J(t,x;z),$$

uniformly with respect to z, and then, thanks to (6.12), the theorem holds for any $\varphi, g \in \mathrm{Lip}_b(H)$. $\square$

In some particular cases the closed loop equation admits a unique solution, and then there exist a unique optimal control and a unique state for the control problem.

THEOREM 7.3. *Assume the hypotheses of Theorem* 7.2*, and take the space dimension* $d = 1$.

1. *If the constant* m *in Hypothesis* 1 *is less than* 2*, then for any* $\varphi, g \in Lip_b(H)$*, and* $x \in H$ *there exists a unique optimal control for the minimizing problem associated with the functional* J*. Furthermore, the optimal control* $z^\star$ *is related to the corresponding optimal state* $y^\star$ *by the feedback formula*

$$z^\star(t) = -DK(D_x V(T-t, y^\star(t))), \qquad t \in [0,T].$$

2. *If* DK *can be extended as a Lipschitz continuous mapping from* $E^\star$ *into itself, then the same conclusion of* 1 *holds for any* $x \in E$.

Proof. We first prove 1. As we have seen in the proof of the previous theorem, the only thing we have to show is that for any $\varphi, g \in C_b^1(H)$ the derivative with respect to x of the solution u of the problem (6.1) is Lipschitz continuous, and for any $x, y \in H$

$$|Du(t,x) - Du(t,y)|_H \leq c_T\,(t \wedge 1)^{-\frac{1+\epsilon}{2}}\,\left(\|\varphi\|_1^H + T\,\|g\|_1^H\right)|x-y|_H. \tag{7.9}$$

Actually, if we define for $(t,x) \in [0,T] \times H$

$$U(t,x) = -DK(Du(T-t,x)),$$

the function U verifies the conditions of Lemma A.1, and then there exists a unique solution $y^\star(t)$ for the closed loop equation. Thanks to Lemma 7.1 this implies the existence of a unique optimal control and state. Finally, as in the proof of the previous theorem, the general case of $\varphi, g \in \mathrm{Lip}_b(H)$ follows by approximation.

We have seen that u is the unique solution of (6.2) in $C_b^1(H)$ and

$$\|u\|_1^H \leq c_T\,\left(\|\varphi\|_1^H + T\,\|g\|_1^H\right).$$

Clearly, if we show that the function $D(P_t\varphi)$ is Lipschitz continuous for any $\varphi \in C_b^1(H)$ and $t > 0$ and

$$|D(P_t\varphi)(x) - D(P_t\varphi)(y)|_H \leq c\,(t \wedge 1)^{-\frac{1+\epsilon}{2}}\,\|\varphi\|_1^H |x-y|_H, \tag{7.10}$$

then, by using the same arguments of section 6, we have that $u(t,\cdot) \in C_b^1(H)$ and

$$|Du(t,x) - Du(t,y)|_H \leq c\,(t \wedge 1)^{-\frac{1+\epsilon}{2}}|x-y|_H,$$

where the constant c depends only on g, φ, and T. Since $D(P_t\varphi)$ is given by the formula (5.2), (7.10) immediately follows once one proves that for any $x, y \in E$ it holds that

$$|v(t;x,h) - v(t;y,h)|_H^2 + \int_0^t |Q^{-1}v(s;x,h) - v(s;y,h)|_H^2 \leq c_T|h|_H^2|x-y|_H^2, \tag{7.11}$$

P-a.s. Let us define $z(t) = v(t;x,h) - v(t;y,h)$. We have that z is the unique solution of the problem

$$\begin{cases} \dfrac{dz}{dt}(t) = & (Az(t) + DF(y(t;x))z(t))\;dt \\ \\ & +\,(DF(y(t;x)) - DF(y(t;y)))\,v(t;y,h)\,dt, \qquad z(0) = 0. \end{cases}$$

Thus we have

$$\frac{1}{2}\frac{d}{dt}|z(t)|_H^2 + |z(t)|^2_{D((-A)^{1/2})} \leq c\,|z(t)|_H^2$$

$$+\,|\langle (DF(y(t;x)) - DF(y(t;y)))\,v(t;y,h), z(t)\rangle_H|\,.$$

Due to the Sobolev embedding theorem, for any $\delta > 0$ we have

$$|\langle (DF(y(t;x)) - DF(y(t;y)))\,v(t;y,h), z(t)\rangle_H|$$

$$\leq |z(t)|_{D\left((-A)^{(1+\delta)/4}\right)}\,|(DF(y(t;x)) - DF(y(t;y)))\,v(t;y,h)|_1\,.$$

In [6] it is proved that

$$\sup_{x\in E}|y(t;x)|_E \leq k(t)(t\wedge 1)^{-\frac{1}{2m}}, \qquad \mathbf{P}-\text{a.s.},$$

for some process $k(t)$ having all moments finite. Thus, since

$$\sup_{x\in H}|v(t;x,h)|_H \leq c(t)|h|_H,$$

for some continuous increasing function $c(t)$, by interpolation we get

$$|\langle (DF(y(t;x)) - DF(y(t;y)))\,v(t;y,h), z(t)\rangle_H|$$

$$\leq |z(t)|^{(1+\delta)/2}_{D\left((-A)^{1/2}\right)}|z(t)|_H^{(1-\delta)/2}|x-y|_H|h|_H\,c(t)\,(t\wedge 1)^{-(2m-1)/2m}.$$

As we can write

$$(t\wedge 1)^{-(2m-1)/2m} = (t\wedge 1)^{-(1-\delta)/2}(t\wedge 1)^{-\frac{1}{2}(1+\delta-1/m)},$$

thanks to the Young inequality we get

$$\frac{1}{2}\frac{d}{dt}|z(t)|_H^2 + |z(t)|^2_{D((-A)^{1/2})} \leq c\,|z(t)|_H^2 + \frac{1}{2}|z(t)|^2_{D\left((-A)^{1/2}\right)}$$

$$+c\,|x-y|_H^2|h|_H^2(t\wedge 1)^{-(1-\delta)} + c(t)\,(t\wedge 1)^{-2(1+\delta-1/m)/(1-\delta)}\,|z(t)|_H^2,$$

where $c(t)$ is a process having all moments finite. Now, if $m < 2$, it is possible to find some $\delta\in(0,1)$ such that

$$2(1+\delta-1/m)/(1-\delta) < 1,$$

and then, by using the Gronwall lemma, (7.11) follows.

Concerning the proof of 2, we recall that in [6] it has been proved that for any $\varphi\in C_b^1(E)\supset C_b^1(H)$ and $t>0$ it holds that

$$|D(P_t\varphi)(x) - D(P_t\varphi)(y)|_{E^\star} \leq c\,(t\wedge 1)^{-\frac{1+\epsilon}{2}}\,\|\varphi\|_1^E|x-y|_E, \qquad x,y\in E.$$

Then as before we have that $u(t,\cdot)\in C_b^1(E)$ and

$$|Du(t,x) - Du(t,y)|_{E^\star} \leq c\,(t\wedge 1)^{-\frac{1+\epsilon}{2}}\,|x-y|_E, \tag{7.12}$$

where the constant c depends only on g, φ, and T. This makes it possible to prove that the closed loop equation admits a unique mild solution. Actually, due to the Sobolev embedding theorem, as the dimension d equals 1, for any $\epsilon > 0$ we have that $D((-C)^{1/4+\epsilon})$ is continuously embedded into E, and then

$$\left|\int_0^t e^{(t-s)C} DK(u(T-s, y(s;x)))\, ds\right|_E$$

$$\leq c \int_0^t \left|e^{(t-s)C} DK(u(T-s, y(s;x)))\right|_{D((-C)^{1/4+\epsilon})} ds$$

$$\leq c \int_0^t (t-s)^{-1/2-2\epsilon} \left|DK(u(T-s, y(s;x)))\right|_{\left(D((-C)^{1/4+\epsilon})\right)^\star}$$

$$\leq c \int_0^t (t-s)^{-1/2-2\epsilon} \left|DK(u(T-s, y(s;x)))\right|_{E^\star}.$$

Therefore, as DK is Lipschitz continuous on $E^\star$ and (7.12) holds, it is easy to show that the closed loop equation admits a unique mild solution. □

Notice that if $K(x) = |x|_H^2$, then $DK(x) = x$, so that DK can be extended as a Lipschitz continuous mapping from $E^\star$ into itself.

Appendix A. An a priori estimate.

We prove here an a priori C^1 estimate for the solution u_α of the approximating problem (6.3). As in [21] we represent u_α and Du_α by means of the transition semigroups associated with suitable stochastic problems. This allows us to have a maximum principle both for u_α and Du_α.

LEMMA A.1. *Let $U : [0,T] \times H \to H$ be a bounded and measurable mapping, such that $U(t,\cdot)$ is Lipschitz continuous for any $t > 0$ and*

$$\sup_{t\in\,[\epsilon,T]} \|U(t,\cdot)\|_{\text{Lip}}^H < \infty$$

for any $\epsilon > 0$. Then, for any $\alpha > 0$ the stochastic problem

$$dy(t) = (Ay(t) + F_\alpha(y(t)) + U(T-t, y(t)))\, dt + Q\, dw(t), \qquad y(r) = x,$$

admits a unique solution $y_\alpha(t,r;x) \in L^2(\Omega; C([r,T); H) \cap L^\infty(r,T;H))$.

Proof. For any $\epsilon > 0$ the function $U(T-t,\cdot)$ is Lipschitz continuous, uniformly for $t \in [0, T-\epsilon]$, and then there exists a unique solution $y_\alpha(t)$ in the interval $[0, T-\epsilon]$ for any $\epsilon > 0$. If we define $v_\alpha(t) = y_\alpha(t) - w^A(t,r)$, we have that $v_\alpha(t)$ is the unique solution of the problem

$$\text{(A.1)} \qquad \frac{dv}{dt}(t) = Av(t) + F_\alpha(y_\alpha(t)) + U(T-t, y_\alpha(t)), \qquad v(r) = x.$$

Thus, by multiplying each side of (A.1) by $v_\alpha(t)$, we get

$$\frac{1}{2}\frac{d}{dt}|v_\alpha(t)|_H^2 + |(-A)^{1/2} v_\alpha(t)|_H^2 = \left\langle F_\alpha(v_\alpha(t) + w^A(t,r)) - F_\alpha(w^A(t,r)), v_\alpha(t)\right\rangle_H$$

$$+ \left\langle F_\alpha(w^A(t,r)), v_\alpha(t)\right\rangle_H + \langle U(T-t, y_\alpha(t)), v_\alpha(t)\rangle_H.$$

Due to the Lipschitz continuity of F_α and the boundedness of U, this implies that

$$\frac{1}{2}\frac{d}{dt}|v_\alpha(t)|_H^2 \leq c_\alpha\,|v_\alpha(t)|_H^2 + c_\alpha\left(|w^A(t,r)|_H^2+1\right).$$

Therefore, by integrating with respect to t, by taking the supremum for $t\in\,[r,T-\epsilon]$, and, finally, by taking the expectation, due to the Gronwall lemma we have

$$\mathbf{E}\sup_{t\in\,[r,T-\epsilon]}|v_\alpha(t)|_H^2 \leq c_{\alpha,T}\Big(|x|_H^2+\mathbf{E}\sup_{t\in\,[r,T]}|w^A(t,r)|_H^2\Big).$$

Thanks to the regularity of $w^A(t,r)$, this allows us to conclude that

$$\mathbf{E}\sup_{t\in\,[r,T-\epsilon]}|y_\alpha(t)|_H^2 \leq 2\sup_{t\in\,[r,T-\epsilon]}\left(|v_\alpha(t)|_H^2+\,|w^A(t,r)|_H^2\right)\leq c_{\alpha,T}\,(|x|_H^2+1).$$

As the constant $c_{\alpha,T}$ does not depend on ϵ, by a uniqueness argument we have that the solution $y_\alpha(t)$ is defined for any $t\in\,[r,T)$ and $y_\alpha(t,r;x)\in\,L^2(\Omega;C([r,T);H)\cap L^\infty(r,T;H))$. □

For each $\alpha>0$ and $n\in\mathbb{N}$, we introduce the approximating Hamilton–Jacobi–Bellman equation

$$\text{(A.2)}\qquad \frac{\partial u}{\partial t}(t,x)=\mathcal{L}_{\alpha,n}u(t,x)-K_n(Du(t,x))+g_n(x),\qquad u(0,x)=\varphi_n(x),$$

where

$$\mathcal{L}_{\alpha,n}\psi(x)=\frac{1}{2}\mathrm{Tr}\left[Q_n^2D^2\psi(x)\right]+\langle A_nx+F_{\alpha,n}(x),D\psi(x)\rangle_H\,.$$

By arguing as for the problem (6.3) (see Remark 6.4), if $\varphi,g\in\,\mathrm{Lip}_b(H)$, the problem (A.2) admits a unique local solution $u_{\alpha,n}$ in $\Lambda_R(\tau_R)$. The solution $u_{\alpha,n}$ is the unique fixed point for the functional

$$\Gamma_{\alpha,n}(v)(t,x)=P_t^{\alpha,n}\varphi(x)-\int_0^t P_{t-s}^{\alpha,n}K(Dv(s,\cdot))(x)\,ds+\int_0^t P_{t-s}^{\alpha,n}g(x)\,ds,$$

and due to (5.8) the contraction constant of $\Gamma_{\alpha,n}$ is independent of n and α. Thus we can proceed as in the proof of Lemma 6.5, and thanks to Proposition 5.1 we conclude that for any $\alpha>0$

$$\text{(A.3)}\qquad \lim_{n\to+\infty}|u_{\alpha,n}(t,x)-u_\alpha(t,x)|+|Du_{\alpha,n}(t,x)-Du_\alpha(t,x)|_H=0,$$

uniformly for $t\in\,[0,\tau_R]$ and for x in bounded sets of H.

According to (5.3) it is possible to show that u_α and $u_{\alpha,n}$ have a stronger regularity.

LEMMA A.2. *If $\varphi,g\in\,C_b^1(H)$, then the solutions u_α and $u_{\alpha,n}$ of the problems* (6.2) *and* (6.3) *belong to $\mathcal{Z}^2_{\tau_\star}$ for some $\tau_\star=\tau_\star(\alpha)\leq T$, which can be taken independent of n.* For a proof we refer to [9, chapter 9].

PROPOSITION A.3. *Let us fix $\varphi,g\in\,C_b^1(H)$, and assume that u_α is the unique solution of the problem* (6.3) *in $\mathcal{Z}^2_{\tau_\star}$, with $\tau_\star=\tau_\star(\alpha)\leq T$. Then, under Hypotheses* 1, 2, *and* 3, *we have*

$$\sup_{\alpha>0}\Big(\sup_{t\in\,[0,\tau_\star]}\|u_\alpha(t,\cdot)\|_1^H\Big)\leq c\,(1+T)e^{c\,T}\left(\|\varphi\|_1^H+\|g\|_1^H\right).$$

Proof. If we define

$$U_{\alpha,n}(t,x)=\int_0^1 DK_n(\lambda Du_{\alpha,n}(t,x))\,d\lambda, \tag{A.4}$$

the problem (A.2) can be rewritten as

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x)=\dfrac{1}{2}\mathrm{Tr}\,[Q_n^2D^2u(t,x)]+\langle A_nx+F_{\alpha,n}(x)+U_{\alpha,n}(t,x),Du(t,x)\rangle+g_n(x), \\ u(0,x)=\varphi_n(x). \end{cases}$$

Since $\varphi, g\in C_b^1(H)$, we have that the solution $u_{\alpha,n}\in \mathcal{Z}^2_{\tau_\star}$ and then $U_{\alpha,n}:[0,\tau_\star]\times H\to H_n$ is continuous and bounded. Moreover, since DK is locally Lipschitz continuous, if we define $M_{\alpha,n}$ as the Lipschitz constant of DK in the ball $\{\,x\in H\,;\,|x|_H\leq \|u_{\alpha,n}\|_{\mathcal{Z}^2_{\tau_\star}}\,\}$ for any $x,y\in H$ and $t>0$, we have that

$$|U_{\alpha,n}(t,x)-U_{\alpha,n}(t,y)|_H\leq\int_0^1|DK_n(\lambda Du_{\alpha,n}(t,x))-DK_n(\lambda Du_{\alpha,n}(t,y))|_H\,d\lambda$$

$$\leq M_{\alpha,n}\,|Du_{\alpha,n}(t,x)-Du_{\alpha,n}(t,y)|_H\leq c\,M_{\alpha,n}\sup_{z\in H}|D^2u_{\alpha,n}(t,z)||x-y|_H$$

$$\leq c\,M_{\alpha,n}\,t^{-\frac{1+\epsilon}{2}}\|u_{\alpha,n}\|_{\mathcal{Z}^2_{\tau_\star}}|x-y|_H.$$

This means that the function $U_{\alpha,n}$ fulfills the hypotheses of Lemma A.1 so that for each $0\leq r<T$ the stochastic problem

$$dy(t)=(A_ny(y)+F_{\alpha,n}(y(t))+U_{\alpha,n}(t,y(t)))\,dt+Q_ndw(t),\qquad y(r)=P_nx, \tag{A.5}$$

admits a unique strong solution $y_{\alpha,n}(t,r;x)\in L^2(\Omega;C([r,\tau_\star);H)\cap L^\infty(r,\tau_\star;H))$.

If we denote by $R^{\alpha,n}_{s,t}$ the corresponding transition semigroup, that is,

$$R^{\alpha,n}_{s,t}\varphi(x)=\mathbf{E}\,\varphi(y_{\alpha,n}(t,s;x)),\qquad 0\leq s\leq t\leq\tau_\star,$$

for $\varphi\in B_b(H)$ and $x\in H$, we have

$$u_{\alpha,n}(t,x)=R^{\alpha,n}_{\tau_\star-t,\tau_\star}\varphi(x)+\int_0^t R^{\alpha,n}_{\tau_\star-t,\tau_\star-s}g(x)\,ds. \tag{A.6}$$

Indeed, since $y_{\alpha,n}(t)$ is a strong solution and $u_{\alpha,n}(t,x)$ is regular, we can apply Itô's formula to the function $s\mapsto u_{\alpha,n}(\tau_\star-s,y_{\alpha,n}(s,\tau_\star-t;x))$, and by integrating with respect to $s\in[\tau_\star-t,\tau_\star]$ and by taking the expectation, we get

$$u_{\alpha,n}(t,x)=\mathbf{E}\,\varphi(y_{\alpha,n}(\tau_\star,\tau_\star-t;x))+\int_0^t\mathbf{E}\,g(y_{\alpha,n}(\tau_\star-s,\tau_\star-t;x))\,ds,$$

which coincides with (A.6). As an immediate consequence this yields

$$\sup_{t\in[0,\tau_\star]}\|u_{\alpha,n}(t,\cdot)\|_0^H\leq\|\varphi\|_0^H+T\|g\|_0^H. \tag{A.7}$$

The proof of the analogous estimate for the derivative of $u_{\alpha,n}(t,x)$ is more complicated but is based on similar arguments.

The problem (A.2) can be rewritten as

$$(\text{A.8})\quad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) = & \dfrac{1}{2}\displaystyle\sum_{k=1}^{n} \lambda_k^2 D_k^2 u(t,x) + \sum_{k,h=1}^{n} a_{k,h} x_h D_k u(t,x) - K_n(Du(t,x)) \\ & + \displaystyle\sum_{k=1}^{n} \langle F_{\alpha,n}(x), e_k \rangle_H D_k u(t,x) + g_n(x), \\ u(0,x) = & \varphi_n(x), \end{cases}$$

where for each $k,h \in \mathbb{N}$ we denote $D_k u = \langle Du, e_k \rangle$ and $a_{k,h} = \langle Ae_k, e_h \rangle$. By differentiating each side of (A.8) with respect to x_j and by setting $v_j = D_j u$, we get

$$\frac{\partial v_j}{\partial t} = \frac{1}{2}\sum_{k=1}^{n} \lambda_k^2 D_k^2 v_j + \sum_{k=1}^{n} a_{k,h} x_h D_k v_j + \sum_{k=1}^{n} a_{k,j} v_k + \sum_{k=1}^{n} \langle F_{\alpha,n}(x), e_k \rangle D_k v_j$$

$$+ \sum_{k=1}^{n} \langle DF_{\alpha,n}(x) e_j, e_k \rangle v_k - \sum_{k=1}^{n} \langle DK_n(Du_{\alpha,n}), e_k \rangle D_k v_j + \langle Dg_n(x), e_j \rangle .$$

By multiplying each side by v_j and by summing up on j, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\sum_{j=1}^{n} v_j^2 = \frac{1}{2}\sum_{k,j=1}^{n} \lambda_k^2 v_j D_k^2 v_j + \sum_{k,h,j=1}^{n} a_{k,h} x_h v_j D_k v_j + \sum_{k,j=1}^{n} a_{k,j} v_k v_j$$

$$+ \sum_{k,j=1}^{n} \langle F_{\alpha,n}(x), e_k \rangle v_j D_k v_j + \sum_{k,j=1}^{n} \langle DF_{\alpha,n}(x) v_j e_j, v_k e_k \rangle$$

$$- \sum_{k,j=1}^{n} \langle DK_n(Du_{\alpha,n}(t,x)), e_k \rangle v_j D_k v_j + \sum_{j=1}^{n} \langle Dg_n(x), v_j e_j \rangle .$$

Now, if we define $2z_{\alpha,n}(t,x) = |Du_{\alpha,n}(t,x)|_H^2$, it holds that

$$\sum_{k,j=1}^{n} \lambda_k^2 v_j D_k^2 v_j = \sum_{k=1}^{n} \lambda_k^2 D_k^2 z_{\alpha,n} - \sum_{k,j=1}^{n} \lambda_k^2 (D_k v_j)^2,$$

$$\sum_{k,h,j=1}^{n} a_{k,h} x_h v_j D_k v_j + \sum_{k,j=1}^{n} a_{k,j} v_k v_j = \langle A_n x, Dz_{\alpha,n} \rangle + \langle A_n Du_{\alpha,n}, Du_{\alpha,n} \rangle ,$$

$$\sum_{k,j=1}^{n} \langle F_{\alpha,n}(x), e_k \rangle v_j D_k v_j = \langle F_{\alpha,n}(x), Dz_{\alpha,n} \rangle ,$$

$$\sum_{k,j=1}^{n} \langle DK_n(Du_{\alpha,n}), e_k \rangle v_j D_k v_j = \langle DK_n(Du_{\alpha,n}), Dz_{\alpha,n} \rangle .$$

Moreover, we have

$$\sum_{k,j=1}^{n} \langle DF_{\alpha,n}(x) v_j e_j, v_k e_k \rangle = \langle DF_{\alpha,n}(x) Du_{\alpha,n}, Du_{\alpha,n} \rangle,$$

$$\sum_{j=1}^{n} \langle Dg_n, v_j e_j \rangle = \langle Dg_n, Du_{\alpha,n} \rangle .$$

Thus, by substituting and by taking into account of (2.13) and (3.8), we can conclude that

$$\frac{\partial z_{\alpha,n}}{\partial t}(t,x) \leq \mathcal{M}_{\alpha,n} z_{\alpha,n}(t,x) + c\, z_{\alpha,n}(t,x) + |Dg_n|_H^2,$$

where the differential operator $\mathcal{M}_{\alpha,n}$ is defined by

$$\mathcal{M}_{\alpha,n}\psi(x) = \frac{1}{2}\mathrm{Tr}\,[Q_n^2 D^2\psi(x)] + \langle A_n x + F_{\alpha,n}(x) - DK_n(Du_{\alpha,n}(t,x)), D\psi(x)\rangle_H .$$

Now we define

$$V_{\alpha,n}(t,x) = -DK_n(Du_{\alpha,n}(t,x)).$$

By arguing as above for the function $U_{\alpha,n}(t,x)$ defined in (A.4), we have that $V_{\alpha,n} : [0,\tau_\star] \times H \to H$ satisfies the hypotheses of Lemma A.1 so that the stochastic problem

$$\text{(A.9)} \quad dy(t) = (A_n y(t) + F_{\alpha,n}(y(t)) + V_{\alpha,n}(t, y(t)))\, dt + Q_n dw(t), \qquad y(r) = P_n x,$$

admits a unique strong solution $y_{\alpha,n}(t,r;x)$ for any $0 \leq r \leq \tau_\star$. If we denote by $S_{s,t}^{\alpha,n}$ the transition semigroup associated with (A.9), by arguing as before for the semigroup associated with the problem (A.5), we have that the solution of the problem

$$\frac{\partial v}{\partial t}(t,x) = \mathcal{M}_{\alpha,n} v(t,x) + cv(t,x) + |Dg_n(x)|_H^2, \qquad v(0,x) = |D\varphi_n(x)|_H^2,$$

is given by

$$v_{\alpha,n}(t,x) = e^{c\,t} S^{\alpha,n}_{\tau_\star - t, \tau_\star} |D\varphi_n|_H^2(x) + \int_0^t e^{c(t-s)} S^{\alpha,n}_{\tau_\star - t, \tau_\star - s} |Dg_n|_H^2(x)\, ds.$$

This yields

$$\sup_{t \in [0,\tau_\star]} \|v_{\alpha,n}(t,\cdot)\|_0^H \leq c\,(1+T)\, e^{c\,T} \left(\|\varphi\|_1^H + \|g\|_1^H \right)^2$$

so that by a comparison argument we conclude

$$\text{(A.10)} \quad \sup_{t \in [0,\tau_\star]} \|Du_{\alpha,n}(t,\cdot)\|_0^H \leq 2 \sup_{t \in [0,T]} \|z_{\alpha,n}(t,\cdot)\|_0^H \leq c\,(1+\sqrt{T})\, e^T \left(\|\varphi\|_1^H + \|g\|_1^H \right).$$

From (A.7) and (A.10), due to (A.3), we conclude that our statement holds. □

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