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AVERAGING PRINCIPLE FOR NONAUTONOMOUS SLOW-FAST SYSTEMS OF STOCHASTIC REACTION-DIFFUSION EQUATIONS: THE ALMOST PERIODIC CASE*

SANDRA CERRAI† AND ALESSANDRA LUNARDI‡

Abstract. We study the validity of an averaging principle for a slow-fast system of stochastic reaction-diffusion equations. We assume here that the coefficients of the fast equation depend on time, so that the classical formulation of the averaging principle in terms of the invariant measure of the fast equation is no longer available. As an alternative, we introduce the time-dependent evolution family of measures associated with the fast equation. Under the assumption that the coefficients in the fast equation are almost periodic, the evolution family of measures is almost periodic. This allows us to identify the appropriate averaged equation and prove the validity of the averaging limit.

Key words. averaging principle, stochastic reaction-diffusion systems, evolution families of measures, almost periodic functions

AMS subject classifications. 60H15, 34K33, 35K57, 37A25, 37L40

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1. Introduction. We deal with a class of systems of stochastic partial differential equations of reaction-diffusion type on a bounded domain D of $\mathbb{R}^d$ with $d \geq 1$:

$$(1.1)\quad \begin{cases} \dfrac{\partial u_\epsilon}{\partial t}(t,\xi) = \mathcal{A}_1 u_\epsilon(t,\xi) + b_1(\xi, u_\epsilon(t,\xi), v_\epsilon(t,\xi)) + g_1(\xi, u_\epsilon(t,\xi))\, \dfrac{\partial w^{Q_1}}{\partial t}(t,\xi), \\[2ex] \dfrac{\partial v_\epsilon}{\partial t}(t,\xi) = \dfrac{1}{\epsilon}\left[(\mathcal{A}_2(t/\epsilon) - \alpha) v_\epsilon(t,\xi) + b_2(t/\epsilon, \xi, u_\epsilon(t,\xi), v_\epsilon(t,\xi))\right] \\[2ex] \qquad\qquad + \dfrac{1}{\sqrt{\epsilon}}\, g_2(t/\epsilon, \xi, v_\epsilon(t,\xi))\, \dfrac{\partial w^{Q_2}}{\partial t}(t,\xi), \\[2ex] u_\epsilon(0,\xi) = x(\xi), \quad v_\epsilon(0,\xi) = y(\xi), \quad \xi \in D, \\[2ex] \mathcal{N}_1 u_\epsilon(t,\xi) = \mathcal{N}_2 v_\epsilon(t,\xi) = 0, \quad t \geq 0, \quad \xi \in \partial D, \end{cases}$$

where ϵ is a small positive parameter and α is a fixed positive constant. The operator $\mathcal{A}_2$ and the functions b_2 and g_2 in the fast equation are allowed to depend on time. We assume that $\mathcal{A}_2$ is periodic, and b_2 and g_2 are almost periodic in time.

In a series of previous papers ([9], [10], and [11]), the validity of an averaging principle for some classes of slow-fast stochastic reaction-diffusion systems has been investigated, in the case where the fast equation coefficients do not depend on time. It has been proved that the slow motion u_ϵ converges in $C([0,T]; L^2(D))$, as $\epsilon \downarrow 0$, to the solution $\bar{u}$ of the so-called averaged equation, obtained by taking the average

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†Department of Mathematics, University of Maryland, College Park, MD 20742 (cerrai@math.umd.edu).
‡Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università di Parma, Parma, 43124, Italy (alessandra.lunardi@unipr.it).

of the coefficients b_1 and g_1 (in the case where both depend on the fast motion) with respect to the invariant measure of the fast motion, with frozen slow component (see formulas (1.2) and (1.3)). Moreover, in [8] the fluctuations of u_ϵ around the averaged motion $\bar{u}$ have been studied. More precisely, it has been proven that, under suitable, more restrictive conditions, the normalized difference $z_\epsilon := (u_\epsilon - \bar{u})/\sqrt{\epsilon}$ is weakly convergent in $C([0,T];L^2(D))$, as $\epsilon \downarrow 0$, to a process z, which is given in terms of a Gaussian process whose covariance is explicitly described. Other aspects of the averaging principle for slow-fast systems of stochastic partial differential equations have been studied by several other authors; see, e.g., [17], [18], [23], [30], [31], and [41].

Unlike in all the above-mentioned papers, where only the time-independent case has been considered, in the present paper we deal with nonautonomous systems of reaction-diffusion equations of Hodgkin–Huxley or Ginzburg–Landau type, perturbed by a Gaussian noise of multiplicative type. Such systems arise in many areas of biology and physics and have attracted considerable attention. In particular, in neurophysiology the Hodgkin–Huxley model, and its simplified version given by the FitzHugh–Nagumo system, are used to describe the activation and deactivation dynamics of a spiking neuron (see, e.g., [37] for a mathematical introduction to this theory). The classical Hodgkin–Huxley model has time-independent coefficients, but, as mentioned by Wainrib in [40], where an analogous problem for finite dimensional systems has been studied, systems with time-dependent coefficients are particularly important for studying models of learning in neuronal activity and, for this reason, are worthy of thorough analysis.

Such analysis does not follow in a straightforward manner from results already available in the mathematical literature. On the contrary, it requires the introduction of some new ideas and techniques.

Actually, in the standard setting of time-independent coefficients, the averaged motion $\bar{u}$ solves the equation

$$(1.2)\qquad \left\{\begin{array}{l} \dfrac{\partial \bar{u}}{\partial t}(t,\xi) = \mathcal{A}_1 \bar{u}(t,\xi) + \bar{B}(\bar{u}(t))(\xi) + g_1(\xi,\bar{u}(t,\xi))\,\dfrac{\partial w^{Q_1}}{\partial t}(t,\xi), \\[3mm] \bar{u}(0,\xi) = x(\xi), \qquad \xi \in D, \qquad \mathcal{N}_1 \bar{u}(t,\xi) = 0, \quad t \geq 0, \quad \xi \in \partial D. \end{array}\right.$$

In the equation above, the averaged coefficient $\bar{B}$ is defined by

$$(1.3)\qquad \bar{B}(x) = \int_{C(\bar{D})} B_1(x,z)\,\mu^x(dz), \qquad x \in C(\bar{D}),$$

where $B_1(x,z)(\xi) = b_1(\xi, x(\xi), z(\xi))$ for any $x, z \in C(\bar{D})$ and $\xi \in \bar{D}$, and where μ^x is the invariant measure of the fast equation with frozen slow component $x \in C(\bar{D})$:

$$(1.4)\qquad \left\{\begin{array}{l} \dfrac{\partial v^{x,y}}{\partial t}(t,\xi) = (\mathcal{A}_2 - \alpha) v^{x,y}(t,\xi) + b_2(\xi, x(\xi), v^{x,y}(t,\xi)) \\[2mm] \qquad\qquad\qquad\qquad + g_2(\xi, x(\xi), v^{x,y}(t,\xi))\,\dfrac{\partial w^{Q_2}}{\partial t}(t,\xi), \\[3mm] v^{x,y}(s,\xi) = y(\xi), \qquad \xi \in D, \qquad \mathcal{N}_2 v^{x,y}(t,\xi) = 0, \quad t \geq 0, \quad \xi \in \partial D. \end{array}\right.$$

Furthermore, because of the ergodicity of μ^x, as proven in [10],

$$(1.5)\qquad \mathbb{E}\left|\frac{1}{T}\int_t^{t+T} B_1(x, v^{x,y}(s))\,ds - \bar{B}(x)\right|_{C(\bar{D})} \leq \alpha(T)\left(1 + |x|^{\kappa_1}_{C(\bar{D})} + |y|^{\kappa_2}_{C(\bar{D})}\right)$$

for some function $\alpha : [0,\infty) \to [0,\infty)$ such that

$$\lim_{T\to\infty} \alpha(T) = 0.$$

In the present paper, as $\mathcal{A}_2$, b_2, and g_2 depend on time, we no longer have an invariant measure μ^x for the fast equation with frozen slow component $x \in C(\bar{D})$. Nevertheless, we can prove that there exists an *evolution system of probability measures* $\{\mu^x_t\,;\, t\in\mathbb{R}\}$ on $C(\bar{D})$ associated with the following fast equation:

$$(1.6)\qquad \begin{cases} \dfrac{\partial v^{x,y}}{\partial t}(t,\xi) = [(\mathcal{A}_2(t)-\alpha)v^{x,y}(t,\xi) + b_2(t,\xi,x(\xi),v^{x,y}(t,\xi))] \\[2ex] \qquad\qquad + \, g_2(t,\xi,v^{x,y}(t,\xi))\,\dfrac{\partial w^{Q_2}}{\partial t}(t,\xi), \\[2ex] v^{x,y}(s,\xi) = y(\xi), \qquad \xi\in D, \qquad \mathcal{N}_2 v^{x,y}(t,\xi) = 0, \qquad t\geq s, \qquad \xi\in\partial D. \end{cases}$$

This means that μ^x_t is a probability measure on $C(\bar{D})$ for any $t\in\mathbb{R}$, and, if $P^x_{s,t}$ is the transition evolution operator associated with (1.6), it holds that

$$\int_{C(\bar{D})} P^x_{s,t}\varphi(y)\,\mu^x_s(dy) = \int_{C(\bar{D})} \varphi(y)\,\mu^x_t(dy), \qquad s<t,$$

for every $\varphi\in C_b(C(\bar{D}))$. Moreover, we show that, under suitable dissipativity conditions,

$$(1.7)\quad \left|P^x_{s,t}\varphi(y) - \int_{C(\bar{D})} \varphi(z)\,\mu^x_t(dz)\right| \leq \|\varphi\|_{C^1_b(C(\bar{D}))}\, e^{-\delta(t-s)}\left(1+|x|_{C(\bar{D})}+|y|_{C(\bar{D})}\right)$$

for some positive constant $\delta>0$.

Now, in order to prove the validity of an averaging principle, the next fundamental step consists in identifying an averaged motion $\bar{u}$ as the solution of a suitable averaged equation. Unfortunately, due to the lack of an invariant measure, we do not have anything like (1.3). Still, due to the assumption that $\mathcal{A}_2$ is periodic and both b_2 and g_2 are almost periodic in time, and to the fact that for any fixed $R>0$ the family of measures

$$\Lambda_R := \left\{\mu^x_t\,;\, t\in\mathbb{R},\ x\in B_R(C(\bar{D}))\right\}$$

is tight in $\mathcal{P}(C(\bar{D}))$, by proceeding as in [13] we can prove that the mapping

$$t\in\mathbb{R}\mapsto \mu^x_t \in \mathcal{P}(C(\bar{D}))$$

is almost periodic for every $x\in C(\bar{D})$.

This allows us to find an alternative way to define $\bar{B}$. Actually, we prove that for any compact set $K\subset C(\bar{D})$ the family of functions

$$(1.8)\qquad \left\{t\in\mathbb{R}\mapsto \int_E B_1(x,z)\,\mu^x_t(dz)\in C(\bar{D})\,:\, x\in K\right\}$$

is uniformly almost periodic. Then, because of almost periodicity, we can define

$$(1.9)\qquad \bar{B}(x) := \lim_{T\to\infty}\frac{1}{T}\int_0^T \int_{C(\bar{D})} B_1(x,y)\,\mu^x_t(dy)\,dt, \qquad x\in C(\bar{D}).$$

Of course, in order to prove that (1.2), with $\bar{B}$ defined as in (1.9), is well posed in $C([0,T];C(\bar{D}))$, we need $\bar{B}$ to satisfy some nice properties. Since B_1 is not Lipschitz continuous, there is no hope that $\bar{B}$ is Lipschitz continuous. Nonetheless, we show that, as a consequence of the monotonicity of B_1 and of some nice properties satisfied by the evolution family of measures $\{\mu_t^x\}_{t\in\mathbb{R}}$, the mapping $\bar{B}: C(\bar{D}) \to C(\bar{D})$ is locally Lipschitz continuous and has some monotonicity properties that guarantee the well posedness of (1.2).

Next, in the same spirit as (1.5), by using (1.7) and (1.9) we show that

(1.10)
$$\mathbb{E}\left|\frac{1}{T}\int_s^{s+T} B_1(x, v^x(t;s,y))\,dt - \bar{B}(x)\right|^2_{C(\bar{D})} \leq \frac{c}{T}\left(1+|x|^{\kappa_1}_{C(\bar{D})}+|y|^{\kappa_2}_{C(\bar{D})}\right)+\alpha(T,x)$$

for some mapping $\alpha : [0,\infty)\times C(\bar{D}) \to [0,+\infty)$ such that

$$\lim_{T\to\infty} \alpha(T,x) = 0. \tag{1.11}$$

This allows us to adapt to the present situation the classical Khasminskii method, based on localization in time, and to prove the main result of this paper, namely that, for any fixed $\eta > 0$,

$$\lim_{\epsilon\to 0} \mathbb{P}\left(\sup_{t\in[0,T]} |u_\epsilon(t)-\bar{u}(t)|_{C(\bar{D})} > \eta\right) = 0, \tag{1.12}$$

where $\bar{u}$ is the solution of the averaged equation (1.2) with $\bar{B}$ defined as in (1.9).

Notice that here, due to the polynomial growth of the coefficients, we have also to proceed with a localization in space, which requires, among other things, a suitable approximation for the family of measures $\{\mu_t^x\}_{t\in\mathbb{R}}$.

Of course, for this procedure to work, we need several technical assumptions on the data. However, we are able to treat slow-fast systems of stochastic reaction-diffusion equations as (1.1), where, for example, the differential operators $\mathcal{A}_1$ and $\mathcal{A}_2(t)$ are given by

$$\mathcal{A}_1 = \Delta, \quad \mathcal{A}_2(t) = \gamma(t)\,\Delta$$

for some continuous periodic function γ with positive infimum, the boundary conditions are of Dirichlet type, the reaction coefficients b_1 and b_2 are given by

$$b_1(\xi,u,v) = -\alpha(\xi)\,u^{2n+1} + \sum_{j=0}^{2n}\alpha_j(\xi)u^j + h_1(\xi,v)$$

and

$$b_2(t,\xi,u,v) = -\beta(t,\xi)v^{2m+1} + \sum_{j=1}^{2m}\beta_j(t,\xi)v^j + h_2(t,\xi,u),$$

where h_1 and h_2 are continuous and bounded functions such that $h_2(\cdot,\xi)$ is almost periodic, uniformly with respect to $\xi\in\bar{D}$, all coefficients α,β,α_j, and β_j are continuous,

$$\inf_{\xi\in\bar{D}}\alpha(\xi) > 0, \qquad \inf_{(t,\xi)\in\mathbb{R}^+\times\bar{D}}\beta(t,\xi) > 0,$$

and all mappings $\beta(\cdot,\xi)$ and $\beta_j(\cdot,\xi)$ are almost periodic, uniformly with respect to $\xi \in \bar{D}$. Moreover, we can take as the diffusion coefficients g_1 and g_2 two bounded continuous functions, with $g_2(\cdot,\xi,v)$ almost periodic, uniformly with respect to $\xi \in \bar{D}$ and v in bounded intervals of $\mathbb{R}$. We would like to stress that these are just simple examples, but in fact we can cover more general situations.

Finally, before concluding this introduction, we would like to say a few words about the almost periodicity assumption for the coefficients of the fast equation. In order to prove the validity of the averaging principle (1.12), estimate (1.10) and limit (1.11) are fundamental and unavoidable. When $\bar{B}(x)$ is defined in terms of the invariant measure μ^x as in the autonomous case, due to the ergodicity of μ^x we obtain (1.10) and (1.11). But here, as we do not have μ^x, it is necessary to define $\bar{B}(x)$ directly by the limit in (1.9), whose existence is guaranteed by the almost periodicity of the family of functions (1.8). Actually, as we recall in Theorem 3.4, the almost periodicity of any mapping $f:\mathbb{R}\to Y$ implies the existence of the limit

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(s)\,ds \in Y.$$

This is why we believe that, in the case of time-dependent coefficients, the assumption of almost periodicity is the natural one.

2. Notations, hypotheses, and a few preliminary results. Let D be a bounded domain of $\mathbb{R}^d$ with $d\geq 1$, having smooth boundary. Throughout the paper, we shall denote by H the separable Hilbert space $L^2(D)$, endowed with the scalar product

$$\langle x,y\rangle_H=\int_D x(\xi)y(\xi)\,d\xi$$

and with the corresponding norm $|\cdot|_H$. We shall denote by $\mathcal{H}$ the product space $H\times H$, endowed with the scalar product

$$\langle x,y\rangle_{\mathcal{H}}=\int_D \langle x(\xi),y(\xi)\rangle_{\mathbb{R}^2}\,d\xi=\langle x_1,y_1\rangle_H+\langle x_2,y_2\rangle_H$$

and the corresponding norm $|\cdot|_{\mathcal{H}}$.

Next, we shall denote by E the Banach space $C(\bar{D})$, endowed with the sup-norm

$$|x|_E=\sup_{\xi\in\bar{D}}|x(\xi)|$$

and the duality $\langle\cdot,\cdot\rangle_E$. The product space $E\times E$ will be endowed with the norm

$$|x|_{E\times E}=\left(|x_1|_E^2+|x_2|_E^2\right)^{\frac{1}{2}}$$

and the corresponding duality $\langle\cdot,\cdot\rangle_{E\times E}$. Finally, for any $\theta\in(0,1)$, we shall denote by $C^\theta(\bar{D})$ the subspace of θ-Hölder continuous functions, endowed with the usual norm

$$|x|_{C^\theta(\bar{D})}=|x|_E+[x]_\theta=|x|_E+\sup_{\substack{\xi,\eta\in\bar{D}\\ \xi\neq\eta}}\frac{|x(\xi)-x(\eta)|}{|\xi-\eta|^\theta}.$$

For any $p\in[1,\infty]$ with $p\neq 2$, the norms in $L^p(D)$ and $L^p(D)\times L^p(D)$ will both be denoted by $|\cdot|_p$. If $\delta>0$ and $p<\infty$, we will denote by $|\cdot|_{\delta,p}$ the norm in $W^{\delta,p}(D)$:

$$|x|_{\delta,p}:=|x|_p+\left(\int_D\int_D\frac{|x(\xi)-x(\eta)|^p}{|\xi-\eta|^{\delta p+d}}\,d\xi\,d\eta\right)^{\frac{1}{p}}. \tag{2.1}$$

Now, we introduce some notations which we will use in what follows (for all details we refer to the reader [14, Appendix D] and also, e.g., [5, Appendix A]). For any $x \in E$, we denote

$$M_x = \left\{ \xi \in \bar{D} \,:\, |x(\xi)| = |x|_E \right\}.$$

Moreover, for any $x \in E \setminus \{0\}$, we set

$$\mathcal{M}_x = \{ \delta_{x,\xi} \in E^\star \,;\, \xi \in M_x \},$$

where

$$\langle \delta_{x,\xi}, y \rangle_E = \frac{x(\xi)y(\xi)}{|x|_E}, \qquad y \in E,$$

and for $x = 0$, we set

$$\mathcal{M}_0 = \{ h \in E^\star \,:\, |h|_{E^\star} = 1 \}.$$

Clearly, we have

$$\mathcal{M}_x \subseteq \partial |x|_E := \{ h \in E^\star \,;\, |h|_{E^\star} = 1,\ \langle h, x \rangle_E = |x|_E \}$$

for every $x \in E$, and, due to the characterization $\partial |x|_E$, it is possible to show that if $\# M_x = 1$, then $\mathcal{M}_x = \partial |x|_E$. In particular, if $u : [0,T] \to E$ is any differentiable mapping, then

$$\frac{d}{dt}^{-} |u(t)|_E \leq \langle u'(t), \delta \rangle_E \tag{2.2}$$

for any $t \in [0,T]$ and $\delta \in \mathcal{M}_{u(t)}$.

Analogously, if $x \in E \times E$, we set

$$M_x = \left\{ \xi = (\xi_1, \xi_2) \in \bar{D} \times \bar{D} \,:\, |x_1(\xi_1)| = |x_1|_E,\ |x_2(\xi_2)| = |x_2|_E \right\}.$$

Moreover, for $x \in E \times E \setminus \{0\}$, we set

$$\mathcal{M}_x = \{ \delta_{x,\xi} \in (E \times E)^\star \,;\, \xi \in M_x \},$$

where

$$\langle \delta_{x,\xi}, y \rangle_{E \times E} = \frac{x_1(\xi_1)y_1(\xi_1) + x_2(\xi_2)y_2(\xi_2)}{|x|_{E \times E}},$$

and for $x = 0$, we set

$$\mathcal{M}_0 = \left\{ h \in (E \times E)^\star \,:\, |h|_{(E \times E)^\star} = 1 \right\}.$$

As above, we have

$$\mathcal{M}_x \subseteq \partial\, |x|_{E \times E} := \left\{ h \in (E \times E)^\star \,;\, |h|_{(E \times E)^\star} = 1,\ \langle h, x \rangle_{E \times E} = |x|_E \right\}$$

and (2.2) holds true, with E replaced by $E \times E$.

Now, let X be any Banach space. We shall denote by $B_b(X)$ the space of bounded Borel functions $\varphi : X \to \mathbb{R}$. $B_b(X)$ is a Banach space, endowed with the sup-norm

$$\|\varphi\|_\infty := \sup_{x \in X} |\varphi(x)|.$$

$UC_b(X)$ will be the subspace of uniformly continuous mappings. Moreover, we shall denote by $\mathcal{L}(X)$ the space of bounded linear operators on X and, in the case where X is a Hilbert space, we shall denote by $\mathcal{L}_2(X)$ the subspace of Hilbert–Schmidt operators, endowed with the norm

$$\|Q\|_{\mathcal{L}_2(X)} = \sqrt{\mathrm{Tr}\,[Q^\star Q]}.$$

The stochastic perturbations in the slow and in the fast motion equations (1.1) are given, respectively, by the Gaussian noises $\partial w^{Q_1}/\partial t(t,\xi)$ and $\partial w^{Q_2}/\partial t(t,\xi)$ for $t \geq 0$ and $\xi \in D$, which are assumed to be white in time and colored in space, in the case of space dimension $d > 1$. Formally, the cylindrical Wiener processes $w^{Q_i}(t,\xi)$ are defined by

$$w^{Q_i}(t,\xi) = \sum_{k=1}^{\infty} Q_i e_k(\xi)\,\beta_k(t), \qquad i = 1,2, \tag{2.3}$$

where $\{e_k\}_{k\in\mathbb{N}}$ is a complete orthonormal basis in H, $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is a sequence of mutually independent standard Brownian motions defined on the same complete stochastic basis $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$, and Q_i is a bounded linear operator on H.

2.1. The operators $\mathcal{A}_1$ and $\mathcal{A}_2(t)$. The operators $\mathcal{A}_1$ and $\mathcal{A}_2(t)$, $t \in \mathbb{R}$, are second order uniformly elliptic operators, having continuous coefficients on $\bar{D}$, and the boundary operators $\mathcal{N}_1$ and $\mathcal{N}_2$ can be either the identity operator (Dirichlet boundary condition) or a first order operator with C^1 coefficients satisfying a uniform nontangentiality condition.

In what follows, we shall assume that the operator $\mathcal{A}_2(t)$ has the form

$$\mathcal{A}_2(t) = \gamma(t)\mathcal{A}_2 + \mathcal{L}(t), \qquad t \in \mathbb{R}, \tag{2.4}$$

where $\mathcal{A}_2$ is a second order uniformly elliptic operator with continuous coefficients on $\bar{D}$, independent of t, and $\mathcal{L}(t)$ is a first order differential operator of the form

$$\mathcal{L}(t,\xi)u(\xi) = \langle l(t,\xi), \nabla u(\xi)\rangle_{\mathbb{R}^d}\,, \qquad t \in \mathbb{R}, \quad \xi \in \bar{D}. \tag{2.5}$$

HYPOTHESIS 2.1.

1. *The function* $\gamma : \mathbb{R} \to \mathbb{R}$ *is continuous and there exist* $\gamma_0, \gamma_1 > 0$ *such that*

$$\gamma_0 \leq \gamma(t) \leq \gamma_1, \qquad t \in \mathbb{R}. \tag{2.6}$$

2. *The function* $l : \mathbb{R} \times \bar{D} \to \mathbb{R}^d$ *is continuous and bounded.*

The realizations A_i with $i = 1,2$ of the differential operators $\mathcal{A}_i$ in the spaces $L^p(D)$ and $C(\overline{D})$, endowed with the domains

$$D(A_i^{(p)}) = \left\{ f \in W^{2,p}(D) \;:\; \mathcal{N}_i f = 0 \text{ at } \partial D \right\}, \qquad i = 1,2,$$

and

$$D(A_i) = \left\{ f \in \bigcap_{q>1} W^{2,q}(D) \;:\; \mathcal{A}_i f \in C(\overline{D}),\ \mathcal{N}_i f = 0 \text{ at } \partial D \right\}, \qquad i = 1,2,$$

generate analytic semigroups in $L^p(D)$, $1 < p < \infty$, and in E, respectively. Since $A_i^{(p)}$ is an extension of A_i and $e^{tA_i^{(p)}}$ is an extension of e^{tA_i}, we shall drop the indices and write A_i and e^{tA_i}, even working in $X = L^p(D)$.

As in [9] and [10], we assume that the operators A_1, A_2 and Q_1, Q_2 satisfy the following conditions.

Hypothesis 2.2. *For $i = 1, 2$, there exist a complete orthonormal system $\{e_{i,k}\}_{k\in\mathbb{N}}$ of H, which is contained in $C^1(\overline{D})$, and two sequences of nonnegative real numbers $\{\alpha_{i,k}\}_{k\in\mathbb{N}}$ and $\{\lambda_{i,k}\}_{k\in\mathbb{N}}$ such that*

$$A_i\, e_{i,k} = -\alpha_{i,k}\, e_{i,k}, \qquad Q_i e_{i,k} = \lambda_{i,k} e_{i,k}, \quad k \geq 1,$$

and

$$\kappa_i := \sum_{k=1}^{\infty} \lambda_{i,k}^{\rho_i}\, |e_{i,k}|_\infty^2 < \infty, \qquad \zeta_i := \sum_{k=1}^{\infty} \alpha_{i,k}^{-\beta_i}\, |e_{i,k}|_\infty^2 < \infty$$

for some constants $\rho_i \in (2, +\infty]$ and $\beta_i \in (0, +\infty)$ such that

$$\frac{\beta_i(\rho_i - 2)}{\rho_i} < 1. \tag{2.7}$$

For comments and examples concerning these assumptions on the operators A_i and Q_i and the eigenfunction $e_{i,k}$, we refer the reader to [9, Remark 2.1] and [24].

For any $t > 0$, $\delta \in [0, 2]$, and $p > 11$, the semigroups e^{tA_i} map $L^p(D)$ into $W^{\delta,p}(D)$ with

$$|e^{tA_i}x|_{\delta,p} \leq c_i\,(t\wedge 1)^{-\frac{\delta}{2}}|x|_p, \qquad x \in L^p(D). \tag{2.8}$$

By the Sobolev embedding theorem, this implies that the semigroups e^{tA_i} map $L^p(D)$ into $L^q(D)$ for any $1 < p \leq q$, and

$$|e^{tA_i}x|_q \leq c_i\,(t\wedge 1)^{-\frac{d(q-p)}{2pq}}|x|_p, \qquad x \in L^p(D). \tag{2.9}$$

Moreover, e^{tA_i} maps $C(\overline{D})$ into $C^\theta(\bar{D})$ for any $\theta(0, 2)$ with

$$|e^{tA_i}x|_{C^\theta(\bar{D})} \leq c_i\,(t\wedge 1)^{-\frac{\theta}{2}}|x|_E. \tag{2.10}$$

Now, we define

$$\gamma(t,s) := \int_s^t \gamma(r)\,dr, \qquad s < t,$$

and, for any $\epsilon > 0$ and $\lambda \geq 0$, we set

$$U_{\lambda,\epsilon}(t,s) = e^{\frac{1}{\epsilon}\gamma(r,\rho)A_2 - \frac{\lambda}{\epsilon}(t-s)}, \qquad s < t. \tag{2.11}$$

In the case $\epsilon = 1$, we write $U_\lambda(t,s)$, and in the case $\epsilon = 1$ and $\lambda = 0$, we write $U(t,s)$.

Next, for any $\epsilon > 0$, $\lambda \geq 0$ and for any $u \in C([s,t]; W_0^{1,p}(D))$ and $r \in [s,t]$, we define

$$\psi_{\lambda,\epsilon}(u;s)(r) = \frac{1}{\epsilon}\int_s^r U_{\lambda,\epsilon}(r,\rho)\mathcal{L}(\rho)u(\rho)\,d\rho, \qquad s < r < t, \tag{2.12}$$

where $\mathcal{L}(\rho)$ is the first order differential operator defined in (2.5). Notice that if u is a solution to

$$u'(t) = \frac{1}{\epsilon}(\mathcal{A}_2(t) - \lambda)u(t), \quad t > s, \quad u(s) = 0,$$

then u satisfies $u(r) = \psi_{\lambda,\epsilon}(u;s)(r)$ for $s < r < t$. For $\epsilon = 1$, we simply write $\psi_\lambda(u;s)(r)$.

LEMMA 2.3. *For any $s < t$, the operator $e^{\gamma(t,s)A_2}\mathcal{L}(s)$ can be extended as a linear operator both in $L^p(D)$ with $1 < p < \infty$ and in E. Moreover, for any $\eta > 0$, its extension (still denoted by $e^{\gamma(t,s)A_2}$) satisfies*

$$\|e^{\gamma(t,s)A_2}\mathcal{L}(s)\|_{\mathcal{L}(E)} \le c_\eta \left((t-s)\wedge 1\right)^{-(\frac{1}{2}+\eta)}. \tag{2.13}$$

Proof. Let $f \in W_0^{1,p}(D)$. For any $0 < s < t$ and $\varphi \in L^{p'}(D)$, since $e^{\gamma(t,s)A_2}$ is self-adjoint we have

$$\int_D \left(e^{\gamma(t,s)A_2}\mathcal{L}(s)f\right)(x)\varphi(x)\,dx = \int_D \mathcal{L}(s)f(x)\,e^{\gamma(t,s)A_2}\varphi(x)\,dx.$$

Therefore, if we integrate by parts, due to (2.8) (with $\delta = 1$) and (2.6), we get

$$\left|\int_D \left(e^{\gamma(t,s)A_2}\mathcal{L}(s)f\right)(x)\varphi(x)\,dx\right| = \left|\int_D f(x)\,D_i\left(l_i(s,\cdot)e^{\gamma(t,s)A_2}\varphi\right)(x)\,dx\right|$$

$$\le c\,\left((t-s)\wedge 1\right)^{-\frac{1}{2}}\,|f|_{L^p(D)}\,|\varphi|_{L^{p'}(D)}.$$

Due to the arbitrariness of $\varphi \in L^{p'}(D)$, this yields

$$\left|e^{\gamma(t,s)A_2}\mathcal{L}(s)f\right|_{L^p(D)} \le c\,\left((t-s)\wedge 1\right)^{-\frac{1}{2}}\,|f|_{L^p(D)}.$$

Due to the density of $W_0^{1,p}(D)$ in $L^p(D)$, the operator $e^{\gamma(t,s)A_2}\mathcal{L}(s)$ has a bounded linear extension to $\mathcal{L}^p(D)$ (still denoted by $e^{\gamma(t,s)A_2}$) that satisfies

$$\|e^{\gamma(t,s)A_2}\mathcal{L}(s)\|_{\mathcal{L}(L^p(D))} \le c\,\left((t-s)\wedge 1\right)^{-\frac{1}{2}}. \tag{2.14}$$

Now, we fix $\delta \in (0,1)$ and $p > d/\delta$ so that $W^{\delta,p}(D)$ is continuously embedded in $C(\bar{D})$. For any $0 < s < t$, we write

$$e^{\gamma(t,s)A_2}\mathcal{L}(s) = e^{\gamma(t,(t-s)/2)A_2}e^{\gamma((t-s)/2,s)A_2}\mathcal{L}(s).$$

The operator $e^{\gamma(t,(t-s)/2)A_2}$ maps $L^p(D)$ into $W^{\delta,p}(D)$ with

$$\|e^{\gamma(t,(t-s)/2)A_2}\|_{\mathcal{L}(L^p(D),W^{\delta,p}(D))} \le c((t-s)\wedge 1)^{-\frac{\delta}{2}}.$$

Using the semigroup law and (2.14), we obtain that $e^{\gamma(t,s)A_2}\mathcal{L}(s)$ maps $L^p(D)$ into $W^{\delta,p}(D)$ with

$$\|e^{\gamma(t,s)A_2}\mathcal{L}(s)\|_{\mathcal{L}(L^p(D),W^{\delta,p}(D))} \le c\,\left((t-s)\wedge 1\right)^{-\frac{1+\delta}{2}}.$$

Now, as $C(\bar{D})$ is continuously embedded in any $L^p(D)$ and $W^{\delta,p}(D)$ is continuously embedded in $C(\bar{D})$ for $p > d/\delta$, we can conclude. ∎

As a consequence of (2.13), if we proceed as in [5, pages 176–177], we can show that $\psi_{\lambda,\epsilon}(\cdot;s)$ is a bounded linear operator in $C([s,t];E)$ and there exists a continuous increasing function c_λ with $c_\lambda(0) = 0$ such that, for any $s < t$,

$$|\psi_{\lambda,\epsilon}(u;s)|_{C([s,t];E)} \le c_\lambda((t-s)/\epsilon)|u|_{C([s,t];E)}. \tag{2.15}$$

Moreover, if $\lambda > 0$, then $c_\lambda \in L^\infty([0,+\infty))$ and

$$\lim_{\lambda\to\infty} |c_\lambda|_\infty = 0. \tag{2.16}$$

LEMMA 2.4. *For every $\eta \in (0,1)$ and $p \geq 1$, there exists $\bar{k} \geq 1$ such that, for every $k \geq \bar{k}$, $s < t$, $0 < \delta < \lambda$, and $u \in C([s,t];E)$,*

$$e^{\delta k r}|\psi_\lambda(u;s)(r)|_{\eta,p}^k \leq c_k(\lambda-\delta)\int_s^r e^{-(\lambda-\delta)(r-\rho)}e^{\delta k\rho}\,|u(\rho)|_E^k\,d\rho, \quad s<r<t, \tag{2.17}$$

for some continuous decreasing function c_k such that

$$\lim_{\gamma\to\infty} c_k(\gamma) = 0.$$

Proof. Due to (2.8) and (2.13), for any $\eta \in (0,1)$ and $p \geq 1$, we have

$$|\psi_\lambda(u;s)(r)|_{\eta,p} \leq c\int_s^r e^{-\lambda(r-\rho)}((r-\rho)\wedge 1)^{-\frac{1+\eta}{2}}|u(\rho)|_{L^p(D)}\,d\rho$$

$$\leq c\,e^{-\delta r}\int_s^r e^{-(\lambda-\delta)(r-\rho)}((r-\rho)\wedge 1)^{-\frac{1+\epsilon}{2}}\,e^{\delta\rho}\,|u(\rho)|_E\,d\rho.$$

Therefore, if we take $\bar{k}$ such that $\bar{k}(1+\eta)/2(\bar{k}-1) < 1$, for any $k \geq \bar{k}$, we have

$$e^{\delta k r}|\psi_\lambda(u;s)(r)|_{\eta,p}^k \leq c_k\left(\int_0^{r-s} e^{-(\lambda-\delta)\rho}(\rho\wedge 1)^{-\frac{(1+\eta)k}{2(k-1)}}\,d\rho\right)^{k-1}$$

$$\cdot\int_s^t r e^{-(\lambda-\delta)(r-\rho)}e^{\delta k\rho}\,|u(\rho)|_E^k\,d\rho.$$

This implies (2.17), if we set

$$c_k(\gamma) = c_k\left(\int_0^{+\infty} e^{-(\lambda-\delta)\rho}(\rho\wedge 1)^{-\frac{(1+\eta)k}{2(k-1)}}\,d\rho\right)^{k-1}.$$

∎

Due to the Sobolev embedding theorem, if we pick $\bar{p}$ large enough such that $\eta\bar{p} > d$, we have that, for any $k \geq \bar{k}$,

$$e^{\delta k r}|\psi_\lambda(u;s)(r)|_{C^\theta(\bar{D})}^k \leq c_k(\lambda-\delta)\int_s^r e^{-(\lambda-\delta)(r-\rho)}e^{\delta k\rho}\,|u(\rho)|_E^k\,d\rho, \quad s<r<t, \tag{2.18}$$

where $\theta = \eta - d/\bar{p}$. In particular, for any $k \geq \bar{k}$,

$$e^{\delta k r}|\psi_\lambda(u;s)(r)|_E^k \leq c_k(\lambda-\delta)\int_s^r e^{-(\lambda-\delta)(r-\rho)}e^{\delta k\rho}\,|u(\rho)|_E^k\,d\rho, \quad s<r<t. \tag{2.19}$$

LEMMA 2.5. *For any $u \in L^k(s,t;E)$ with $k \geq 1$, and for any $\epsilon > 0$ and $\lambda \geq 0$, it holds that*

$$|\psi_{\lambda,\epsilon}(u;s)|_{L^k(s,t;E)} \leq c_{\lambda,k}((t-s)/\epsilon)|u|_{L^k(s,t;E)}.$$

Moreover, if $\lambda > 0$, then $c_{\lambda,k} \in L^\infty(0,\infty)$ and

$$\lim_{\lambda\to\infty}|c_{\lambda,k}|_\infty = 0.$$

Proof. As in the proof of Lemma 2.4, for any $\eta \in (0,1)$ and $p \geq 1$, we have

$$|\psi_{\lambda,\epsilon}(u;s)(r)|_{\eta,p} \leq \frac{c}{\epsilon}\int_s^r e^{-\frac{\lambda}{\epsilon}(r-\rho)}((r-\rho)/\epsilon\wedge 1)^{-\frac{1+\eta}{2}}|u(\rho)|_E\,d\rho.$$

Therefore, if we pick $\bar{p}$ large enough so that $\eta\bar{p} > d$, for any $k \geq 1$, we have, by the Young inequality,

$$\int_s^t |\psi_{\lambda,\epsilon}(u;s)(r)|_E^k\,dr \leq \frac{c_k}{\epsilon^k}\int_s^t \left(\int_s^r e^{-\frac{\lambda}{\epsilon}(r-\rho)}((r-\rho)/\epsilon\wedge 1)^{-\frac{1+\eta}{2}}|u(\rho)|_E\,d\rho\right)^k dr$$

$$\leq \frac{c_k}{\epsilon^k}\int_s^t |u(r)|_E^k\,dr\left(\int_0^{t-s} e^{-\frac{\lambda}{\epsilon}r}(r/\epsilon\wedge 1)^{-\frac{1+\eta}{2}}\,dr\right)^k.$$

Since

$$\frac{1}{\epsilon^k}\left(\int_0^{t-s} e^{-\frac{\lambda}{\epsilon}r}(r/\epsilon\wedge 1)^{-\frac{1+\eta}{2}}\,dr\right)^k = \left(\int_0^{(t-s)/\epsilon} e^{-\lambda r}(r\wedge 1)^{-\frac{1+\eta}{2}}\,dr\right)^k,$$

we conclude by taking

$$c_{\lambda,k}(\gamma) := \left(\int_0^{\gamma} e^{-\lambda r}(r\wedge 1)^{-\frac{1+\eta}{2}}\,dr\right)^k.$$

∎

2.2. The coefficients b_i and g_i. As far as the reaction coefficient $b_1 : \bar{D}\times\mathbb{R}^2 \to \mathbb{R}$ in the slow equation is concerned, we assume the following conditions, which are the same as those in the paper [10].

HYPOTHESIS 2.6.

1. *The mapping* $b_1 : \bar{D}\times\mathbb{R}^2 \to \mathbb{R}$ *is continuous and there exists* $m_1 \geq 1$ *such that*

$$\sup_{\xi\in\bar{D}} |b_1(\xi,\sigma)| \leq c\,(1+|\sigma_1|^{m_1}+|\sigma_2|), \qquad \sigma=(\sigma_1,\sigma_2)\in\mathbb{R}^2. \tag{2.20}$$

2. *There exists* $\theta \geq 0$ *such that*

$$\sup_{\xi\in\bar{D}} |b_1(\xi,\sigma)-b_1(\xi,\rho)| \leq c\left(1+|\sigma|^\theta+|\rho|^\theta\right)|\sigma-\rho|, \qquad \sigma,\rho\in\mathbb{R}^2. \tag{2.21}$$

3. *There exists* $c > 0$ *such that, for any* $\sigma, h \in \mathbb{R}^2$,

$$\sup_{\xi\in\bar{D}} \left(b_1(\xi,\sigma+h)-b_1(\xi,\sigma)\right)h_1 \leq c\,|h_1|\,(1+|\sigma|+|h|)\,. \tag{2.22}$$

Example 2.7 (from [10]). Let $h : \bar{D}\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that $h(\xi,\cdot) : \mathbb{R}\to\mathbb{R}$ is locally Lipschitz continuous, uniformly with respect to $\xi\in\bar{D}$. Assume that

$$\sup_{\xi\in\bar{D}} |h(\xi,s)| \leq c\,(1+|s|^m), \qquad s\in\mathbb{R}, \tag{2.23}$$

and

$$h(\xi,s_1)-h(\xi,s_2) = \rho(\xi,s_1,s_2)(s_1-s_2), \qquad \xi\in\bar{D}, \quad s_1,s_2\in\mathbb{R}, \tag{2.24}$$

for some $\rho : \bar{D}\times\mathbb{R}^2\to\mathbb{R}$ such that

$$\sup_{\substack{\xi\in\bar{D}\\ s_1,s_2\in\mathbb{R}}} \rho(\xi,s_1,s_2) < \infty.$$

Moreover, let $k : \bar{D} \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that $k(\xi, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ has linear growth and is locally Lipschitz continuous, uniformly with respect to $\xi \in \bar{D}$.

Now, we fix any continuous function $f : \bar{D} \times \mathbb{R} \to \mathbb{R}$ such that $f(\xi, \cdot)$ is of class C^1 for any $\xi \in \bar{D}$, and

$$0 \leq \frac{\partial f}{\partial s}(\xi, s) \leq c, \qquad (\xi, s) \in \bar{D} \times \mathbb{R}, \tag{2.25}$$

for some $c > 0$. If we define

$$b_1(\xi, \sigma) = f(\xi, h(\xi, \sigma_1) + k(\xi, \sigma_1, \sigma_2)),$$

it is not difficult to check that conditions 1 and 3 in Hypothesis 2.6 are satisfied. Moreover, if we assume that h and k are differentiable and their derivatives have polynomial growth, then condition 2 is also satisfied.

Next, let β and β_i be continuous functions from $\bar{D}$ into $\mathbb{R}$ for $i = 1, \ldots, 2k$, and assume

$$\inf_{\xi \in \bar{D}} \beta(\xi) > 0.$$

Then, it is possible to check that the function

$$h(\xi, s) := -\beta(\xi) s^{2k+1} + \sum_{i=1}^{2k} \beta_i(\xi) s^i$$

satisfies conditions (2.23) and (2.24). □

For the reaction term $b_2 : \mathbb{R} \times \bar{D} \times \mathbb{R}^2 \to \mathbb{R}$ in the fast equation, we assume the following conditions.

Hypothesis 2.8.

1. *The mapping* $b_2 : \mathbb{R} \times \bar{D} \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and there exists* $m_2 \geq 1$ *such that*

$$\sup_{(t,\xi) \in \mathbb{R} \times \bar{D}} |b_2(t, \xi, \sigma)| \leq c\left(1 + |\sigma_1| + |\sigma_2|^{m_2}\right), \qquad \sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2. \tag{2.26}$$

2. *The mapping* $b_2(t, \xi, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ *is locally Lipschitz continuous, uniformly with respect to* $(t, \xi) \in \mathbb{R} \times \bar{D}$.
3. *There exists* $c > 0$ *such that, for any* $\sigma, h \in \mathbb{R}^2$,

$$\sup_{(t,\xi) \in \mathbb{R} \times \bar{D}} \left(b_2(t, \xi, \sigma + h) - b_2(t, \xi, \sigma)\right) h_2 \leq c\,|h_2|\left(1 + |\sigma| + |h|\right). \tag{2.27}$$

4. *For every* $(t, \xi) \in \mathbb{R} \times \bar{D}$*, we have*

$$b_2(t, \xi, \sigma_1, \sigma_2) - b_2(t, \xi, \rho_1, \sigma_2) = \theta(t, \xi, \sigma_1, \rho_1, \sigma_2) \tag{2.28}$$

for some continuous function $\theta : \mathbb{R} \times \bar{D} \times \mathbb{R}^3 \to \mathbb{R}$ *such that*

$$\inf_{\substack{(t,\xi)\in\mathbb{R}\times\bar{D}\\(\sigma_1,\sigma_2)\in\mathbb{R}^2,\ h>0}}\theta(t,\xi,\sigma_1,\sigma_1+h,\sigma_2)\quad\sup_{\substack{(t,\xi)\in\mathbb{R}\times\bar{D}\\(\sigma_1,\sigma_2)\in\mathbb{R}^2,\ h>0}}\theta(t,\xi,\sigma_1,\sigma_1+h,\sigma_2)\geq 0, \tag{2.29}$$

and such that for any $R>0$ *there exists* $L_R>0$ *with*

$$\sigma_1,\rho_1\in B_{\mathbb{R}}(R)\Longrightarrow\sup_{\substack{(t,\xi)\in\mathbb{R}\times\bar{D}\\\sigma_2\in\mathbb{R}}}|\theta(t,\xi,\sigma_1,\rho_1,\sigma_2)|\leq L_R\,|\sigma_1-\rho_1|. \tag{2.30}$$

5. *For any* $\sigma_1,\sigma_2,\rho_2\in\mathbb{R}$, *we have*

$$b_2(t,\xi,\sigma_1,\sigma_2)-b_2(t,\xi,\sigma_1,\rho_2)=-\lambda(t,\xi,\sigma_1,\sigma_2,\rho_2)(\sigma_2-\rho_2) \tag{2.31}$$

for some measurable function $\lambda:\mathbb{R}\times\bar{D}\times\mathbb{R}^3\to[0,+\infty)$.

Example 2.9. Let $h:\mathbb{R}\times\bar{D}\times\mathbb{R}\to\mathbb{R}$ be such that $h(t,\cdot)$ satisfies the same conditions as in Example 2.7, uniformly with respect to $t\in\mathbb{R}$. Assume that the function ρ in (2.24) depends also on $t\in\mathbb{R}$ and satisfies

$$\sup_{\substack{(t,\xi)\in\mathbb{R}\times\bar{D}\\s\in\mathbb{R}}}\rho(t,\xi,s)\leq 0. \tag{2.32}$$

Moreover, assume that the mapping $k:\mathbb{R}\times\bar{D}\times\mathbb{R}^2\to\mathbb{R}$ is continuous, the mapping $k(t,\xi,\cdot):\mathbb{R}^2\to\mathbb{R}$ has linear growth and is locally Lipschitz continuous, uniformly with respect to $(t,\xi)\in\mathbb{R}\times\bar{D}$, and the mapping $k(t,\xi,\cdot,\sigma_2):\mathbb{R}\to\mathbb{R}$ is monotone and locally Lipschitz continuous, uniformly with respect to $(t,\xi)\in\mathbb{R}\times\bar{D}$ and $\sigma_2\in\mathbb{R}$.

Then all the conditions in Hypothesis 2.8 are fulfilled if we define

$$b_2(t,\xi,\sigma)=f(t,\xi,h(t,\xi,\sigma_2)+k(t,\xi,\sigma)),\quad(t,\xi)\in\mathbb{R}\times\bar{D},\quad\sigma\in\mathbb{R}^2,$$

for any $f:\mathbb{R}\to\mathbb{R}$ satisfying (2.25). Notice that (2.32) holds for

$$h(t,\xi,s)=-\beta(t,\xi)s^{2k+1}+\sum_{j=1}^{2k}\beta_j(t,\xi)s^j-\lambda s$$

with λ large enough.

Concerning the diffusion coefficients g_1 and g_2, we assume that they satisfy the following conditions.

Hypothesis 2.10.

1. *The mappings* $g_1:\bar{D}\times\mathbb{R}\to\mathbb{R}$ *and* $g_2:\mathbb{R}\times\bar{D}\times\mathbb{R}\to\mathbb{R}$ *are continuous and the mappings* $g_1(\xi,\cdot):\mathbb{R}\to\mathbb{R}$ *and* $g_2(t,\xi,\cdot):\bar{D}\times\mathbb{R}\to\mathbb{R}$ *are Lipschitz continuous, uniformly with respect to* $\xi\in\bar{D}$ *and* $(t,\xi)\in\mathbb{R}\times\bar{D}$, *respectively.*
2. *It holds that*

$$\sup_{\xi\in\bar{D}}|g_1(\xi,\sigma)|\leq c\left(1+|\sigma|^{\frac{1}{m_1}}\right),\quad\sigma\in\mathbb{R}, \tag{2.33}$$

and

$$\sup_{(t,\xi)\in\mathbb{R}\times\bar{D}}|g_2(t,\xi,\sigma)|\leq c\left(1+|\sigma|^{\frac{1}{m_2}}\right),\quad\sigma\in\mathbb{R}, \tag{2.34}$$

where m_1 *and* m_2 *are the constants introduced in* (2.20) *and* (2.26).

Remark 2.11. We are assuming here that the diffusion coefficient g_2 in the fast equation does not depend on the slow variable because of what is required in the proof of Proposition 5.4. If the coefficient b_2 in the fast equation had linear growth, then we could allow g_2 to depend also on the slow variable.

In what follows, for any $t \in \mathbb{R}$ and $x, y \in E$, we shall set

$$B_1(x,y)(\xi) := b_1(\xi, x(\xi), y(\xi)), \qquad B_2(t,x,y)(\xi) := b_2(t,\xi,x(\xi),y(\xi)), \qquad \xi \in \bar{D},$$

and

$$B(t) := (B_1, B_2(t)), \qquad t \in \mathbb{R}.$$

Due to Hypotheses 2.6 and 2.8, the mappings B_1 and B_2 are well defined and continuous from $E\times E$ and $\mathbb{R}\times E\times E$, respectively, to E, so that $B : \mathbb{R}\times E\times E \to E\times E$ is well defined and continuous. As the mappings b_1 and b_2 have polynomial growth, $B(t)$ is not well defined in $\mathcal{H}$.

In view of (2.20) and (2.26), for any $x, y \in E$ and $t \in \mathbb{R}$, we have

$$|B_1(x,y)|_E \le c\,(1 + |x|_E^{m_1} + |y|_E)\,, \quad |B_2(t,x,y)|_E \le c\,(1 + |x|_E + |y|_E^{m_2}) \tag{2.35}$$

so that

$$|B(t,x,y)|_{E\times E} \le c\,(1 + |x|_E^{m_1} + |y|_E^{m_2})\,, \quad x, y \in E, \quad t \in \mathbb{R}. \tag{2.36}$$

As a consequence of (2.22) and (2.27), it is immediate to check that, for any $x, y, h, k \in E$, any $t \in \mathbb{R}$, and any $\delta \in \mathcal{M}_h$,

$$\langle B_1(x+h, y+k) - B_1(x,y), \delta\rangle_E \le c\,(1 + |h|_E + |k|_E + |x|_E + |y|_E) \tag{2.37}$$

and

$$\langle B_2(t, x+h, y+k) - B_2(t,x,y), \delta\rangle_E \le c\,(1 + |h|_E + |k|_E + |x|_E + |y|_E)$$

so that, for any $(x,y), (h,k) \in E \times E$, any $t \in \mathbb{R}$, and any $\delta \in \mathcal{M}_{(h,k)}$,

$$\langle B(t, x+h, y+k) - B(t,x,y), \delta\rangle_{E\times E} \le c\,(1 + |(h,k)|_{E\times E} + |(x,y)|_{E\times E})\,. \tag{2.38}$$

Moreover, from (2.31), we have

$$\langle B_2(t,x,y+k) - B_2(t,x,y), \delta\rangle_E \le 0 \tag{2.39}$$

for every $\delta \in \mathcal{M}_k$. Finally, in view of (2.21), we have

$$\begin{aligned} &|B_1(x_1,y_1) - B_1(x_2,y_2)|_E \\ &\le c\,\left(1 + |(x_1,y_1)|_{E\times E}^{\theta} + |(x_2,y_2)|_{E\times E}^{\theta}\right)(|x_1 - x_2|_E + |y_1 - y_2|_E)\,. \end{aligned} \tag{2.40}$$

Next, for any $x, y, z \in E$ and $t \in \mathbb{R}$, we define

$$[G_1(x)z](\xi) = g_1(\xi, x(\xi))z(\xi), \qquad [G_2(t,y)z](\xi) := g_2(t,\xi,y(\xi))z(\xi), \qquad \xi \in \bar{D}.$$

Due to Hypothesis 2.10, the mappings

$$G_1 : E \to \mathcal{L}(E)$$

and, for any fixed $t \in \mathbb{R}$,

$$G_2(t,\cdot) : E \to \mathcal{L}(E)$$

are Lipschitz continuous, so the same is true for the mapping $G(t) = (G_1, G_2(t))$ defined on $E \times E$ with values in $\mathcal{L}(E \times E)$.

3. Almost periodic functions. We recall here some definitions and results about almost periodic functions. For all details, we refer to the monographs [2] and [19] and the paper [3].

In what follows, (X, d_X) and (Y, d_Y) denote two complete metric spaces. For any bounded function $f : \mathbb{R} \to Y$ and $\epsilon > 0$, we define

$$T(f, \epsilon) = \{\tau \in \mathbb{R} \,:\, d_Y(f(t+\tau), f(t)) < \epsilon, \text{ for all } t \in \mathbb{R}\}.$$

$T(f, \epsilon)$ is called an ϵ-translation set of f.

Definition 3.1.

1. *A continuous function $f : \mathbb{R} \to Y$ is said to be* almost periodic *if, for all $\epsilon > 0$, the set $T(f, \epsilon)$ is* relatively dense *in $\mathbb{R}$; that is, there exists a number $l_\epsilon > 0$ such that $[a, a + l_\epsilon] \cap T(f, \epsilon) \neq \emptyset$ for every $a \in \mathbb{R}$. The number l_ϵ is called the* inclusion length.
2. *Let $F \subset X$ and, for any $x \in F$, let $f(\cdot, x) : \mathbb{R} \to Y$ be an almost periodic function. The family of functions $\{f(\cdot, x)\}_{x\in F}$ is said to be* uniformly almost periodic *if, for any $\epsilon > 0$,*

$$T(F, f, \epsilon) := \bigcap_{x\in F} T(f(\cdot, x), \epsilon)$$

is relatively dense in $\mathbb{R}$ and includes an interval around 0.

In what follows, if $f : \mathbb{R} \to Y$ or $f : \mathbb{R}\times X \to Y$, and if $\gamma = \{\gamma_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$, we shall use the notation $T_\gamma f = g$ to say, respectively, that

$$\lim_{n\to\infty} f(t + \gamma_n) = g(t) \quad \text{in } Y$$

and

$$\lim_{n\to\infty} f(t + \gamma_n, x) = g(t, x) \quad \text{in } Y$$

for any $t \in \mathbb{R}$ and $x \in X$.

We recall here some characterization of uniformly almost periodic families of functions.

Theorem 3.2. *Let $F \subset X$ and let $f(\cdot, x) : \mathbb{R} \to Y$ be a continuous function for any $x \in F$. The following statements are equivalent.*

1. *The family $\{f(\cdot, x)\}_{x\in F}$ is uniformly almost periodic.*
2. *For any sequence $\gamma' = \{\gamma'_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, there exists a subsequence $\gamma \subset \gamma'$ and a continuous function $g : \mathbb{R} \times X \to Y$ such that $T_\gamma f = g$, uniformly on $\mathbb{R} \times F$.*
3. *For every two sequences γ' and β' in $\mathbb{R}$, there exist common subsequences $\gamma \subset \gamma'$ and $\beta \subset \beta'$ such that $T_{\gamma+\beta} f = T_\gamma T_\beta f$, uniformly on $\mathbb{R} \times F$.*

Notice that if $f : \mathbb{R} \to X$ is a continuous periodic function with period τ, then for any sequence $\gamma \subset \mathbb{R}$ there exists $r_\gamma \in [0, \tau]$ such that $T_\gamma f(t) = f(t + r_\gamma)$, uniformly with respect $t \in \mathbb{R}$. In fact, if we denote by $H(f)$ the *hull* of f, that is, the set of functions $\{T_\gamma f \;:\; \gamma = \{\gamma_n\} \subset \mathbb{R}\}$, we have that f is periodic if and only if $H(f) = \{f(\tau + \cdot) \,:\, \tau \in \mathbb{R}\}$.

In the case of a function $f : \mathbb{R} \to Y$, we have the following characterization of almost periodicity.

Theorem 3.3. *A continuous function $f : \mathbb{R} \to Y$ is almost periodic if and only if, for every two sequences γ' and β' in $\mathbb{R}$, there exist common subsequences $\gamma \subset \gamma'$ and $\beta \subset \beta'$ such that $T_{\gamma+\beta} f = T_\gamma T_\beta f$, pointwise on $\mathbb{R}$.*

Finally, in [2, Theorem I.3.2], the following important result for almost periodic functions is shown.

THEOREM 3.4.

1. *There exists the* mean value *in Y of any almost periodic function $f : \mathbb{R} \to Y$, that is,*

$$\exists \lim_{T\to\infty} \frac{1}{T} \int_0^T f(s)\, ds \in Y.$$

Moreover, for every $t \in \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} f(s)\, ds = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(s)\, ds,$$

uniformly with respect to $t \in \mathbb{R}$.

2. *If $\{f(\cdot, x)\}_{x\in F}$ is a uniformly almost periodic family of functions with $F \subset X$, then*

$$\exists \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} f(s, x)\, ds = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(s, x)\, ds,$$

uniformly with respect to $t \in \mathbb{R}$ and $x \in F$.

Remark 3.5. The proof of [2, Theorem I.3.2] is given for a single almost periodic function f. Nevertheless, it is easy to adapt the arguments used in that proof to the case of uniformly almost periodic families of functions, as stated in the second part of Theorem 3.4.

4. The slow-fast system. With the notations introduced in section 2, system (1.1) can be rewritten in the following abstract form:

$$(4.1)\qquad \begin{cases} du_\epsilon(t) = [A_1 u_\epsilon(t) + B_1(u_\epsilon(t), v_\epsilon(t))]\, dt + G_1(u_\epsilon(t))\, dw^{Q_1}(t), \\ dv_\epsilon(t) = \dfrac{1}{\epsilon}\, [(A_2(t/\epsilon) - \alpha) v_\epsilon(t) + B_2(t/\epsilon, u_\epsilon(t), v_\epsilon(t))]\, dt \\ \qquad\qquad + \dfrac{1}{\sqrt{\epsilon}}\, G_2(t/\epsilon, v_\epsilon(t))\, dw^{Q_2}(t) \end{cases}$$

with initial conditions $u_\epsilon(0) = x \in E$ and $v_\epsilon(0) = y \in E$.

In [6, Theorem 5.3], a system analogous to (4.1) has been studied for the case of coefficients independent of time. Thanks to Lemma 2.4, since all estimates satisfied by the coefficients in Hypotheses 2.2, 2.6, 2.8, and 2.10 are uniform with respect to $t \in \mathbb{R}$, the arguments used in the proof of [6, Theorem 5.3] can be adapted to the present situation and it is possible to show that, under Hypotheses 2.1, 2.2, 2.6, 2.8, and 2.10, for any $\epsilon > 0$ and $x, y \in E$, there exists a unique adapted mild solution to problem (4.1) in $L^p(\Omega; C_b((s, T]; E \times E))$ with $s < T$ and $p \geq 1$.

This means that there exist two unique adapted processes u_ϵ and v_ϵ in $L^p(\Omega; C_b((s, T]; E))$ such that

$$u_\epsilon(t) = e^{tA_1} x + \int_s^t e^{(t-r)A_1} B_1(u_\epsilon(r), v_\epsilon(r))\, ds + \int_s^t e^{(t-s)A_1} G_1(u_\epsilon(r))\, dw^{Q_1}(r)$$

and

$$v_\epsilon(t) = U_{\alpha,\epsilon}(t,s)y + \frac{1}{\epsilon}\psi_{\alpha,\epsilon}(v_\epsilon;s)(t) + \frac{1}{\epsilon}\int_s^t U_{\alpha,\epsilon}(t,r)B_2(r,u_\epsilon(r),v_\epsilon(r))\,dr$$

$$+ \frac{1}{\sqrt{\epsilon}}\int_s^t U_{\alpha,\epsilon}(t,r)G_2(r,v_\epsilon(r))\,dw^{Q_2}(r),$$

where, with the same notations as in section 2, for every $\epsilon > 0$,

$$U_{\alpha,\epsilon}(t,s) = e^{\frac{1}{\epsilon}\gamma(t,s)A_2 - \frac{\alpha}{\epsilon}(t-s)}, \qquad s < t,$$

and

$$\psi_{\alpha,\epsilon}(u;s)(r) = \int_s^r U_{\alpha,\epsilon}(r,\rho)\mathcal{L}(\rho)u(\rho)\,d\rho, \qquad r \in [s,t].$$

Recall that in section 2 we have defined

$$U_\alpha(t,s) := U_{\alpha,1}(t,s), \qquad \psi_\alpha(u;s)(r) := \psi_{\alpha,1}(u;s)(r).$$

Thanks to Lemma 2.5, we can adapt to the present situation the arguments used in the proof of [10, Lemma 3.1], and it is possible to show that, for any $p \geq 1$ and $s < T$, there exists a constant $c_{p,s,T} > 0$ such that, for any $x, y \in E$ and $\epsilon \in (0,1]$,

$$\mathbb{E}\sup_{t\in[s,T]}|u_\epsilon(t)|_E^p \leq c_{p,s,T}\,\left(1 + |x|_E^p + |y|_E^p\right) \tag{4.2}$$

and

$$\mathbb{E}\int_s^T |v_\epsilon(t)|_E^p\,dt \leq c_{p,s,T}\,\left(1 + |x|_E^p + |y|_E^p\right) \tag{4.3}$$

for some constants $c_{s,p,T}$ independent of $\epsilon > 0$.

Moreover, as in [10, Proposition 3.2], we can show that there exists $\bar{\theta} > 0$ such that, for any $\theta \in [0,\bar{\theta})$, $x \in C^\theta(\bar{D})$, $y \in E$, and $s < T$,

$$\sup_{\epsilon\in(0,1]}\mathbb{E}\,|u_\epsilon|_{L^\infty(s,T;C^\theta(\bar{\theta}))} \leq c_{s,T}\left(1 + |x|_{C^\theta(\bar{D})} + |y|_E\right). \tag{4.4}$$

Finally, by proceeding as in [9, Proposition 4.4] (see also [10, Proposition 3.3]), we can prove that, for any $\theta > 0$, there exists $\gamma(\theta) > 0$ such that, for any $T > 0$, $p \geq 2$, $x \in C^\theta(\bar{D})$, $y \in E$, and $r_1, r_2 \in [s,t]$,

$$\sup_{\epsilon\in(0,1)}\mathbb{E}\,|u_\epsilon(r_1) - u_\epsilon(r_2)|_E^p \leq c_p(T)\left(1 + |x|_{C^\theta(\bar{D})}^{pm_1} + |y|_E^p\right)|r_1 - r_2|^{\gamma(\theta)p}. \tag{4.5}$$

Due to the Kolmogorov test and the Ascoli–Arzelà theorem, (4.4) and (4.5) imply that the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon\in(0,1]}$, given by the laws of the solutions u_ϵ, is tight in $C([s,T];E)$ for any $x \in C^\theta(\bar{D})$ with $\theta > 0$, and for any $y \in E$. That is, for every $\eta > 0$, there exists a compact set $K_\eta \subset C([s,T];E)$ such that $\mathbb{P}\,(u_\epsilon \in K_\eta) \geq 1 - \eta$ for every $\epsilon \in (0,1]$.

5. An evolution family of measures for the fast equation. For any frozen slow component $x \in E$, any initial condition $y \in E$, and any $s \in \mathbb{R}$, we introduce the problem

$$dv(t) = [(A_2(t) - \alpha)v(t) + B_2(t, x, v(t))]\, dt + G_2(t, v(t))\, d\bar{w}^{Q_2}(t), \quad v(s) = y, \tag{5.1}$$

where $A_2(t) = \gamma(t)A_2 + \mathcal{L}(t)$ and

$$\bar{w}^{Q_2}(t) = \begin{cases} w_1^{Q_2}(t) & \text{if } t \geq 0, \\ \\ w_2^{Q_2}(-t) & \text{if } t < 0 \end{cases}$$

for two independent Q_2-Wiener processes, $w_1^{Q_2}(t)$ and $w_2^{Q_2}(t)$, both defined as in (2.3). An $\{\mathcal{F}_t\}_{t \geq s}$-adapted process $v^x(\cdot; s, y) \in L^p(\Omega; C([s, T]; E))$ is a *mild solution* of (5.1) if

$$v^x(t; s, y) = U_\alpha(t, s)y + \psi_\alpha(v^x(\cdot; s, y); s)(t)$$

$$+ \int_s^t U_\alpha(t, r)\, B_2(r, x, v^x(r; s, y))\, dr + \int_s^t U_\alpha(t, r)\, G_2(r, v^x(r; s, y))\, d\bar{w}^{Q_2}(r),$$

where $\psi_\alpha(\cdot; s)$ is the linear bounded operator defined in (2.12) with $\epsilon = 1$.

Moreover, if $C(\mathbb{R}; E)$ is the space of continuous paths on $\mathbb{R}$ with values in E, endowed with the topology of uniform convergence on bounded intervals, an $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$-adapted process $v^x \in L^p(\Omega; C(\mathbb{R}; E))$ is a *mild solution* of the equation

$$dv(t) = [(A_2(t) - \alpha)v(t) + B_2(t, x, v(t))]\, dt + G_2(t, v(t))\, d\bar{w}^{Q_2}(t) \tag{5.2}$$

in $\mathbb{R}$ if, for every $s < t$,

$$v^x(t) = U_\alpha(t, s)v^x(s) + \psi_\alpha(v^x; s)(t)$$

$$+ \int_s^t U_\alpha(t, r)\, B_2(r, x, v^x(r))\, dr + \int_s^t U_\alpha(t, r)\, G_2(r, v^x(r))\, d\bar{w}^{Q_2}(r).$$

According to (2.15), the mapping $\psi_\alpha(\cdot; s) : C([s, T]; E) \to C([s, T]; E)$ is Lipschitz continuous, so we can adapt the proof of [6, Theorem 5.3] to the present situation, and we have that, for any $x, y \in E$, there exists a unique mild solution $v^x(\cdot; s, y) \in L^p(\Omega; C((s, T]; E) \cap L^\infty((s, T); E))$ with $p \geq 1$ and $s < T$.

All this allows us to introduce, for any fixed $x \in E$, the transition evolution operator

$$P^x_{s,t}\, \varphi(y) = \mathbb{E}\, \varphi(v^x(t; s, y)), \quad s < t, \quad y \in E,$$

where $\varphi \in B_b(E)$.

For any $\lambda > 0$, (5.1) can be rewritten as

$$dv(t) = [(A_2(t) - \lambda)v(t) + B_{2,\lambda}(t, x, v(t))]\, dt + G_2(t, v(t))\, d\bar{w}^{Q_2}(t), \quad v(s) = y,$$

where

$$B_{2,\lambda}(t, x, y) = B_2(t, x, y) + (\lambda - \alpha)\, y.$$

In what follows, for any $x \in E$ and any process $u \in L^p(\Omega; C_b((s, T]; E))$ adapted, we shall set

$$\Gamma_\lambda(u; s)(t) = \int_s^t U_\lambda(t, r)\, G_2(r, u(s))\, d\bar{w}^{Q_2}(s), \quad t > s. \tag{5.3}$$

By proceeding as in the proof of [7, Lemma 7.1], where the case $s = 0$ was considered, it is possible to show that there exists $\bar{p} > 1$ such that, for any $p \geq \bar{p}$ and $0 < \delta < \lambda$, and for any $u, v \in L^p(\Omega; C_b((s,t]; E))$ with $s < t$,

$$\begin{aligned}&\sup_{r\in[s,t]} e^{\delta p(r-s)}\mathbb{E}\,|\Gamma_\lambda(u;s)(r) - \Gamma_\lambda(v;s)(r)|_E^p \\ &\leq c_{p,1}\frac{L_{g_2}^p}{(\lambda-\delta)^{c_{p,2}}}\sup_{r\in[s,t]} e^{\delta p(r-s)}\,\mathbb{E}\,|u(r) - v(r)|_E^p,\end{aligned} \tag{5.4}$$

where L_{g_2} is the Lipschitz constant of g_2, and $c_{p,1}, c_{p,2}$ are two suitable positive constants independent of $\lambda > 0$ and $s < t$.

Moreover, using (2.34), we can show that

$$\sup_{r\in[s,t]} e^{\delta p(r-s)}\mathbb{E}\,|\Gamma_\lambda(u;s)(r)|_E^p \leq c_{p,1}\frac{M_{g_2}^p}{(\lambda-\delta)^{c_{p,2}}}\sup_{r\in[s,t]} e^{\delta p(r-s)}\left(1 + \mathbb{E}\,|u(r)|_E^{\frac{p}{m_2}}\right), \tag{5.5}$$

where

$$M_{g_2} = \sup_{\xi\in\bar{D},\ \sigma\in\mathbb{R}}\frac{|g_2(\xi,\sigma)|}{1 + |\sigma|^{\frac{1}{m_2}}}$$

(see [7, Remark 3.2]). In fact, in [7] it is shown that there exists some $\eta > 0$ such that, for any $p \geq 1$ large enough,

$$\sup_{r\in[s,t]} e^{\delta p(r-s)}\mathbb{E}\,|\Gamma_\lambda(u;s)(r)|_{\eta,p}^p \leq c_{p,1}\frac{M_{g_2}^p}{(\lambda-\delta)^{c_{p,2}}}\sup_{r\in[s,t]} e^{\delta p(r-s)}\left(1 + \mathbb{E}\,|u(r)|_E^{\frac{p}{m_2}}\right).$$

This means that if we pick $\bar{p} \geq 1$ such that $\eta\bar{p} > d$ and define $\theta = \eta - d/\bar{p}$, by the Sobolev embedding theorem we have that, for any $p \geq \bar{p}$,

$$\sup_{r\in[s,t]} e^{\delta p(r-s)}\mathbb{E}\,|\Gamma_\lambda(u;s)(r)|_{C^\theta(\bar{D})}^p \leq c_{p,1}\frac{M_{g_2}^p}{(\lambda-\delta)^{c_{p,2}}}\sup_{r\in[s,t]} e^{\delta p(r-s)}\left(1 + \mathbb{E}\,|u(r)|_E^{\frac{p}{m_2}}\right). \tag{5.6}$$

Now, for any fixed adapted process $u \in L^p(\Omega; C_b((s,T]; E))$, let us introduce the problem

$$dz(t) = (A_2(t) - \lambda)z(t)\,dt + G_2(t, u(t))\,d\bar{w}^{Q_2}(t), \qquad z(s) = 0, \tag{5.7}$$

and let us denote by $\Lambda_\lambda(u;s)$ its unique mild solution in $L^p(\Omega; C_b((s,T]; E))$. This means that $\Lambda_\lambda(u;s)$ solves the equation

$$\Lambda_\lambda(u;s)(t) = \psi_\lambda(\Lambda_\lambda(u;s);s)(t) + \Gamma_\lambda(u;s)(t), \quad s < t < T.$$

Due to Lemma 2.4, for any $0 < \delta < \lambda$ and $p \geq 1$ large enough, and for any two adapted processes u_1 and u_2 in $L^p(\Omega; C_b((s,T]; E))$ with $s < t$, we have

$$\begin{aligned}&e^{\delta p(t-s)}\,\mathbb{E}\,|\Lambda_\lambda(u_1;s)(t) - \Lambda_\lambda(u_2;s)(t)|_E^p \leq c_p\,e^{\delta p(t-s)}\,\mathbb{E}\,|\psi_\lambda(\Lambda_\lambda(u_1;s) - \Lambda_\lambda(u_2;s);s)(t)|_E \\ &\qquad + c_p\,e^{\delta p(t-s)}\,\mathbb{E}\,|\Gamma_\lambda(u_1;s)(t) - \Gamma_\lambda(u_2;s)(t)|_E^p \\ &\leq c_p(\lambda-\delta)\int_0^{t-s} e^{-(\lambda-\delta)\rho}\,d\rho\sup_{\rho\in[s,t]} e^{\delta p(\rho-s)}\,\mathbb{E}\,|\Lambda_\lambda(u_1;s)(\rho) - \Lambda_\lambda(u_2;s)(\rho)|_E^p \\ &\qquad + c_p\,e^{\delta p(t-s)}\,\mathbb{E}\,|\Gamma_\lambda(u_1;s)(t) - \Gamma_\lambda(u_2;s)(t)|_E^p.\end{aligned}$$

Therefore, thanks to (2.16), we can find $\lambda(\delta) > \delta$ large enough such that, for any $\lambda \geq \lambda(\delta)$,

$$\sup_{\rho\in[s,t]} e^{\delta p(\rho-s)}\,\mathbb{E}\,|\Lambda_\lambda(u_1;s)(\rho)-\Lambda_\lambda(u_2;s)(\rho)|_E^p$$
$$\leq c_p \sup_{\rho\in[s,t]} e^{\delta p(\rho-s)}\,\mathbb{E}\,|\Gamma_\lambda(u_1;s)(t)-\Gamma_\lambda(u_2;s)(\rho)|_E^p.$$

Due to (5.4), this yields

$$\begin{aligned}&\sup_{r\in[s,t]} e^{\delta p(r-s)}\mathbb{E}\,|\Lambda_\lambda(u_1;s)(r)-\Lambda_\lambda(u_2;s)(r)|_E^p\\ &\leq c_{p,1}\frac{L_{g_2}^p}{(\lambda-\delta)^{c_{p,2}}}\sup_{r\in[s,t]} e^{\delta p(r-s)}\,\mathbb{E}\,|u_1(r)-u_2(r)|_E^p.\end{aligned} \tag{5.8}$$

In the same way, we get that

$$\sup_{r\in[s,t]} e^{\delta p(r-s)}\mathbb{E}\,|\Lambda_\lambda(u;s)(r)|_E^p \leq c_{p,1}\frac{M_{g_2}^p}{(\lambda-\delta)^{c_{p,2}}}\sup_{r\in[s,t]} e^{\delta p(r-s)}\left(1+\mathbb{E}\,|u(r)|_E^{\frac{p}{m_2}}\right). \tag{5.9}$$

Proposition 5.1. *Assume Hypotheses* 2.1, 2.2, 2.8, *and* 2.10. *Then, there exists* $\delta > 0$ *such that, for any* $x, y \in E$ *and* $p \geq 1$,

$$\mathbb{E}\,|v^x(t;s,y)|_E^p \leq c_p\left(1+e^{-\delta p(t-s)}\,|y|_E^p+|x|_E^p\right), \qquad s<t. \tag{5.10}$$

Proof. We set $z_\lambda(t) := v^x(t;s,y)-\Lambda_\lambda(t)$, where $\Lambda_\lambda(t)=\Lambda_\lambda(v^x(\cdot;s,y);s)(t)$ is the solution of problem (5.7) with $u=v^x(\cdot;s,y)$ and $\lambda>\alpha$. Thanks to (2.39), for every $\delta\in\mathcal{M}_{z_\lambda(t)}$, we have

$$\begin{aligned}\frac{d}{dt}^-|z_\lambda(t)|_E &\leq \langle (A_2(t)-\lambda)z_\lambda(t),\delta\rangle_E+\langle B_{2,\lambda}(t,x,z_\lambda(t)+\Lambda_\lambda(t))\\ &\quad -B_{2,\lambda}(t,x,\Lambda_\lambda(t)),\delta\rangle_E+\langle B_{2,\lambda}(t,x,\Lambda_\lambda(t)),\delta\rangle_E\\ &\leq -\alpha\,|z_\lambda(t)|_E+c\,(1+|x|_E+|\Lambda_\lambda(t)|_E^{m_2})+(\lambda-\alpha)\,|\Lambda_\lambda(t)|_E\\ &\leq -\alpha\,|z_\lambda(t)|_E+c\,(1+|x|_E+|\Lambda_\lambda(t)|_E^{m_2})+(\lambda-\alpha)^{\frac{m_2}{m_2-1}},\end{aligned}$$

the last estimate following from the Young inequality. By comparison, we get

$$|z_\lambda(t)|_E \leq e^{-\alpha(t-s)}|y|_E+c\left(1+|x|_E+(\lambda-\alpha)^{\frac{m_2}{m_2-1}}\right)+c\int_s^t e^{-\alpha(t-r)}|\Lambda_\lambda(r)|_E^{m_2}\,dr$$

so that, for any $p\geq 1$,

$$\begin{aligned}|v^x(t;s,y)|_E^p &\leq c_p|\Lambda_\lambda(t)|_E^p+c_p\,e^{-\alpha p(t-s)}|y|_E^p\\ &\quad + c_p\left(1+|x|_E^p+(\lambda-\alpha)^{\frac{pm_2}{m_2-1}}\right)+c_p\left(\int_s^t e^{-\alpha(t-r)}|\Lambda_\lambda(r)|_E^{m_2}\,dr\right)^p.\end{aligned}$$

Due to (5.9), this implies that we can proceed as in the proof of [10, Proposition 4.1] (where (5.5) with $s=0$ is used), and (5.10) follows. ∎

The following proposition gives a generalization to the case of multiplicative noise of [12, Lemma 2.2]. The fact that the diffusion coefficient is not constant makes the proof of the result considerably more complicated as compared to [12, Lemma 2.2].

PROPOSITION 5.2. *Under Hypotheses* 2.1, 2.2, 2.8, *and* 2.10, *if* $\alpha > 0$ *is large enough and/or* L_{g_2} *is small enough, for any* $t \in \mathbb{R}$ *and* $x \in E$ *there exists* $\eta^x(t) \in L^p(\Omega; E)$ *for all* $p \geq 1$ *such that*

$$\lim_{s \to -\infty} \mathbb{E}\,|v^x(t; s, y) - \eta^x(t)|_E^p = 0 \tag{5.11}$$

for any $y \in E$ *and* $t \in \mathbb{R}$. *Moreover, for every* $p \geq 1$, *there exists some* $\delta_p > 0$ *such that*

$$\mathbb{E}\,|v^x(t; s, y) - \eta^x(t)|_E^p \leq c_p\, e^{-\delta_p(t-s)}\left(1 + |x|_E^p + |y|_E^p\right). \tag{5.12}$$

Finally, η^x *is a mild solution in* $\mathbb{R}$ *of equation* (5.2).

Proof. If we fix $h > 0$ and define

$$\rho(t) = v^x(t; s, y) - v^x(t; s-h, y), \quad t > s,$$

we have that $\rho(t)$ is the unique mild solution of the problem

$$\left\{\begin{array}{l} d\rho(t) = \left[(A_2(t) - \alpha)\rho(t) + B_2(t, x, v^x(t; s, y)) - B_2(t, x, v^x(t; s-h, y))\right] dt \\[2mm] \qquad\qquad + \left[G_2(t, v^x(t; s, y)) - G_2(t, v^x(t; s-h, y))\right] d\bar{w}^{Q_2}(t), \\[2mm] \rho(s) = y - v^x(s; s-h, y). \end{array}\right. \tag{5.13}$$

According to (2.31), we have

$$B_2(t, x, v^x(t; s, y)) - B_2(t, x, v^x(t; s-h, y)) = -J^x(t)\rho(t),$$

where

$$J^x(t, \xi) = \lambda(t, \xi, x(\xi), v^x(t; s, y)(\xi), v^x(t; s-h, y)(\xi)), \quad \xi \in D.$$

Therefore, if we define

$$K^x(t, \xi) = \frac{g_2(t, \xi, v^x(t; s, y)(\xi)) - g_2(t, \xi, v^x(t; s-h, y)(\xi))}{\rho(t)(\xi)}, \quad \xi \in D,$$

we can rewrite (5.13) as

$$\left\{\begin{array}{l} d\rho(t) = \left[(A_2(t) - \alpha)\rho(t) - J^x(t)\rho(t)\right] dt + K^x(t)\rho(t)\, d\bar{w}^{Q_2}(t), \\[2mm] \rho(s) = y - v^x(s; s-h, y). \end{array}\right. \tag{5.14}$$

Notice that, due to (2.31), we have

$$J^x(t, \xi) \geq 0, \quad (t, \xi) \in \mathbb{R} \times D. \tag{5.15}$$

Moreover, as $g_2(t, \xi, \cdot)$ is assumed to be Lipschitz continuous, uniformly with respect to $(t, \xi) \in \mathbb{R} \times \bar{D}$, we have that

$$\sup_{(t,\xi) \in \mathbb{R} \times \bar{D}} |K^x(t, \xi)| = \sup_{(t,\xi) \in \mathbb{R} \times \bar{D}} [g_2(t, \xi, \cdot)]_{\text{Lip}} < \infty. \tag{5.16}$$

Now, for any $\mathcal{F}_s$-measurable $y_s \in L^2(\Omega; E)$, we introduce the auxiliary problem

$$\left\{ \begin{array}{l} dz(t) = (A_2(t) - \alpha) z(t)\, dt + K^x(t) z(t)\, d\bar{w}^{Q_2}(t), \\ z(s) = y_s, \end{array} \right. \tag{5.17}$$

and we denote by $z(t; s, y_s)$ its solution. By proceeding as in the proof of (5.8), we have that, for any p large enough, there exist two constants $c_{p,1}$ and $c_{p,2}$ such that, for any $0 < \delta < \alpha$,

$$\sup_{r \in [s,t]} e^{\delta p(r-s)} \mathbb{E}\, |z(r; s, y_s)|_E^p \leq c_p\, \mathbb{E}\, |y_s|_E^p + c_{p,1} \frac{L_{g_2}^p}{(\alpha - \delta)^{c_{p,2}}} \sup_{r \in [s,t]} e^{\delta p(r-s)}\, \mathbb{E}\, |z(r; s, y_s)|_E^p.$$

Therefore, if we pick $\alpha > 0$ large enough and/or L_{g_2} small enough so that

$$c_{p,1} \frac{L_{g_2}^p}{\alpha^{c_{p,2}}} < 1,$$

we can find $0 < \bar{\delta}_p < \alpha$ such that

$$c_{p,1} \frac{L_{g_2}^p}{(\alpha - \bar{\delta}_p)^{c_{p,2}}} < 1.$$

This implies that

$$\sup_{r \in [s,t]} e^{p\bar{\delta}_p(r-s)} \mathbb{E}\, |z(r; s, y_s)|_E^p \leq c_p\, \mathbb{E}\, |y_s|_E^p,$$

so that

$$\mathbb{E}\, |z(r; s, y_s)|_E^p \leq c_p\, e^{-\delta_p(r-s)}\, \mathbb{E}\, |y_s|_E^p, \qquad s < r, \tag{5.18}$$

with $\delta_p = p\bar{\delta}_p$.

Next, for any $\mathcal{F}_s$-measurable $y_s \in L^2(\Omega; E)$, we introduce the problem

$$\left\{ \begin{array}{l} dz(t) = \left[(A_2(t) - \alpha) z(t) - J^x(t) z(t)\right]\, dt + K^x(t) z(t)\, d\bar{w}^{Q_2}(t), \\ \\ z(s) = y_s, \end{array} \right. \tag{5.19}$$

and we denote by $\hat{z}(t; s, y_s)$ its solution.

Due to the linearity of (5.19), by a comparison argument (see [16]) we have

$$y_s \geq 0, \quad \mathbb{P}\text{-a.s.} \Longrightarrow \hat{z}(t; s, y_s) \geq 0, \quad s < t, \quad \mathbb{P}\text{-a.s.}$$

Moreover, in view of the sign condition (5.15), again by a comparison argument (see [16]) we have

$$y_s \geq 0, \quad \mathbb{P}\text{-a.s.} \Longrightarrow 0 \leq \hat{z}(t; s, y_s) \leq z(t; s, y_s), \quad s < t, \quad \mathbb{P}\text{-a.s.} \tag{5.20}$$

Thanks to (5.18), this allows us to conclude

$$y_s \geq 0, \quad \mathbb{P}\text{-a.s.} \Longrightarrow \mathbb{E}\, |\hat{z}(t; s, y_s)|_E^p \leq c_p\, e^{-\delta_p(t-s)} |y_s|_E^2, \qquad s < t. \tag{5.21}$$

Now, as a consequence of the linearity of problem (5.19), we have

$$\begin{aligned} & v^x(t; s, y) - v^x(t; s-h, y) = \hat{z}(t; s, y - v^x(s; s-h, y)) \\ & = \hat{z}(t; s, y - v^x(s; s-h, y) \wedge y) - \hat{z}(t; s, v^x(s; s-h, y) - v^x(s; s-h, y) \wedge y). \end{aligned}$$

Then, thanks to (5.10) and (5.21), we can conclude that, for some $\delta_p > 0$,

$$\begin{aligned}\mathbb{E}\,|v^x(t;s,y) - v^x(t;s-h,y)|_E^p &\leq c_p\, e^{-\delta_p(t-s)}\mathbb{E}\,|y - v^x(s;s-h,y)|_E^p \\ &\leq c_p\, e^{-\delta_p(t-s)}\left(|y|_E^p + e^{-\delta_p h}|y|_E^p + |x|_E^p + 1\right).\end{aligned} \tag{5.22}$$

Therefore, if we take the limit as $s \to -\infty$, due to the completeness of $L^p(\Omega;E)$, this implies that, for any $t \in \mathbb{R}$ and $x, y \in E$, there exists $\eta^x(t) \in L^p(\Omega;E)$ such that (5.11) holds. Moreover, if we let $h \to \infty$, we obtain (5.12).

Next, in order to prove that $\eta^x(t)$ does not depend on $y \in E$, we take $y_1, y_2 \in E$ and consider the difference

$$\rho(t) = v^x(t;s,y_1) - v^x(t;s,y_2), \quad t > s.$$

The same arguments, used above for the difference $v^x(t;s,y) - v^x(t;s-h,y)$, can be used here for $\rho(t)$, and we have

$$\mathbb{E}\,|v^x(t;s,y_1) - v^x(t;s,y_2)|_E^p \leq c_p\, e^{-\delta_p(t-s)}\,|y_1 - y_2|_E^p, \quad s < t,$$

so that, by taking the limit above as $s \to -\infty$, we get that the limit $\eta^x(t)$ does not depend on the initial condition $y \in E$.

Finally, let us show that η^x is the mild solution in $\mathbb{R}$ of equation (5.2). For any $s < t$ and $h > 0$, we have

$$\begin{aligned}&v^x(t;s-h,0) = U_\alpha(t,s)v^x(s;s-h,0) + \psi_\alpha(v^x(\cdot;s-h,0);s)(t)\\ &+ \int_s^t U_\alpha(t,r)B_2(r,x,v^x(r;s-h,0))\,dr + \int_s^t U_\alpha(t,r)G_2(r,v^x(r;s-h,0))\,d\bar{w}^{Q_2}(r).\end{aligned}$$

Due to (5.11), we can take the limit as h goes to infinity on both sides, and we get for any $s < t$,

$$\begin{aligned}\eta^x(t) &= U_\alpha(t,s)\eta^x(s) + \psi_\alpha(\eta^x;s)(t)\\ &\quad + \int_s^t U_\alpha(t,r)B_2(r,x,\eta^x(r))\,dr + \int_s^t U_\alpha(t,r)G_2(r,\eta^x(r))\,d\bar{w}^{Q_2}.\end{aligned} \tag{5.23}$$

This means that $\eta^x(t)$ is a mild solution in $\mathbb{R}$ of equation (5.2). ◻

In what follows, for any $t \in \mathbb{R}$ and $x \in E$, we shall denote by μ_t^x the law of the random variable $\eta^x(t)$. Our purpose here is to show that the family $\{\mu_t^x\}_{t\in\mathbb{R}}$ defines an *evolution system of probability measures* on E for equation (5.1), indexed by $t \in \mathbb{R}$. This means that μ_t^x is a probability measure on E for any $t \in \mathbb{R}$, and it holds that

$$\int_E P_{s,t}^x\varphi(y)\,\mu_s^x(dy) = \int_E \varphi(y)\,\mu_t^x(dy), \qquad s < t, \tag{5.24}$$

for every $\varphi \in C_b(E)$.

Notice that, due to (5.11) and (5.10), for any $p \geq 1$, we have

$$\sup_{t\in\mathbb{R}} \mathbb{E}\,|\eta^x(t)|_E^p \leq c_p\,\left(1 + |x|_E^p\right), \quad x \in E, \tag{5.25}$$

so that

$$\sup_{t\in\mathbb{R}} \int_E |y|_E^p\,\mu_t^x(dy) \leq c_p\,\left(1 + |x|_E^p\right). \tag{5.26}$$

PROPOSITION 5.3. *Under Hypotheses* 2.1, 2.2, 2.8, *and* 2.10, *if* $\alpha > 0$ *is large enough and/or* L_{g_2} *is small enough, for any fixed* $x \in E$ *the family of probability measures* $\{\mu_t^x\}_{t\in\mathbb{R}}$ *introduced above defines an evolution family of measure for equation* (5.1) *such that*

$$\lim_{s\to-\infty} P^x_{s,t}\varphi(y) = \int_E \varphi(y)\,\mu_t^x(dy) \tag{5.27}$$

for any $\varphi \in C_b(E)$. *Moreover, if* $\varphi \in C_b^1(E)$, *we have*

$$\left|P^x_{s,t}\varphi(y) - \int_E \varphi(z)\,\mu_t^x(dz)\right| \leq \|\varphi\|_{C_b^1(E)}\, e^{-\delta_1(t-s)}\,(1+|x|_E+|y|_E)\,. \tag{5.28}$$

Finally, if $\{\nu_t^x\}_{t\in\mathbb{R}}$ *is another evolution family of measures for* (5.1) *such that*

$$\sup_{t\in\mathbb{R}} \int_E |y|_E\,\nu_t^x(dy) < \infty, \tag{5.29}$$

then $\nu_t^x = \mu_t^x$ *for all* $t \in \mathbb{R}$ *and* $x \in E$.

Proof. According to (5.11), for any $\varphi \in C_b(E)$ and $y \in E$, we have

$$\lim_{s\to-\infty} P^x_{s,t}\varphi(y) = \lim_{s\to-\infty} \mathbb{E}\,\varphi(v^x(t;s,y)) = \mathbb{E}\varphi(\eta^x(t)) = \int_E \varphi(y)\mu_t^x(dy).$$

Therefore, since for any $s < r < t$ we have

$$P^x_{s,r}P^x_{r,t}\varphi(y) = P^x_{s,t}\varphi(y), \qquad y \in E,$$

by taking the limit above in both sides, as $s \to -\infty$, we obtain

$$\int_E P^x_{r,t}\varphi(y)\,\mu_r^x(dy) = \int_E \varphi(y)\,\mu_t^x(dy),$$

which means that $\{\mu_t^x\}_{t\in\mathbb{R}}$ is an evolution family of measures satisfying (5.27).

In order to prove (5.28), we have

$$\left|P^x_{s,t}\varphi(y) - \int_E \varphi(z)\,\mu_t^x(dz)\right| \leq \mathbb{E}\,|\varphi(v^x(t;s,y)) - \varphi(\eta^x(t))|$$
$$\leq \|\varphi\|_{C_b^1(E)}\mathbb{E}\,|v^x(t;s,y) - \eta^x(t)|_E,$$

so that (5.28) follows from (5.12).

Next, let us prove uniqueness. If we show that, for any $\varphi \in C_b^1(E)$,

$$\lim_{s\to-\infty} \int_E P_{s,t}\varphi(y)\,\nu_s^x(dy) = \int_E \varphi(y)\,\mu_t^x(dt), \tag{5.30}$$

then, recalling that $\{\nu_t^x\}_{t\in\mathbb{R}}$ is an evolution family, we have that, for any $\varphi \in C_b^1(E)$,

$$\int_E \varphi(y)\,\nu_t^x(dy) = \int_E \varphi(y)\,\mu_t^x(dy), \qquad t \in \mathbb{R},$$

which implies that $\mu_t^x = \nu_t^x$ for any $t \in \mathbb{R}$ and $x \in E$.

In order to prove (5.30), we notice that, due to (5.12),

$$\left|\int_E P_{s,t}\varphi(y)\,\nu_s^x(dy) - \int_E \varphi(y)\,\mu_t^x(dt)\right| \leq \int_E \mathbb{E}\,|\varphi(v^x(t;s,y)) - \varphi(\eta^x(t))|\;\nu_s^x(dy)$$

$$\leq \|\varphi\|_{C_b^1(E)}\int_E \mathbb{E}|v^x(t;s,y) - \eta^x(t)|_E\,\nu_s^s(dy)$$

$$\leq \|\varphi\|_{C_b^1(E)}e^{-\delta(t-s)}\left(1 + |x|_E + \int_E |y|_E\,\nu_s^x(dy)\right).$$

Then, as a consequence of condition (5.29), we can conclude that (5.30) holds and, as we have seen, uniqueness follows. ◻

Now, we want to study the dependence of $\eta^x(t)$, and hence of μ_t^x, on the parameter $x \in E$.

PROPOSITION 5.4. *Under Hypotheses* 2.1, 2.2, 2.8, *and* 2.10, *if $\alpha > 0$ is large enough and/or L_{g_2} is small enough, we have that, for any $R > 0$, there exists $c_R > 0$ such that*

$$x_1, x_2 \in B_E(R) \Longrightarrow \sup_{t\in\mathbb{R}}\mathbb{E}\,|\eta^{x_1}(t) - \eta^{x_2}(t)|_E^2 \leq c_R\,|x_1 - x_2|_E^2. \tag{5.31}$$

Proof. In view of (5.11), it is sufficient to show that, for any $R > 0$, there exists $c_R > 0$ such that

$$x_1, x_2 \in B_E(R) \Longrightarrow \sup_{s<t}\mathbb{E}\,|v^{x_1}(t;s,0) - v^{x_2}(t;s,0)|_E^2 \leq c_R\,|x_1 - x_2|_E^2. \tag{5.32}$$

If we define

$$\rho(t) = v^{x_1}(t;s,0) - v^{x_2}(t;s,0), \quad s < t,$$

we have that $\rho(t)$ is the unique mild solution of the problem

(5.33)
$$\left\{\begin{array}{l} d\rho(t) = [(A_2(t) - \alpha)\rho(t) + B_2(t, x_1, v^{x_1}(t;s,0)) - B_2(t, x_2, v^{x_2}(t;s,0))]\;dt \\[2mm] \qquad\qquad + \Big[G_2(t, v^{x^1}(t;s,0)) - G_2(t, v^{x_2}(t;s,0))\Big]\;d\bar{w}^{Q_2}(t), \\[2mm] \rho(s) = 0. \end{array}\right.$$

According to (2.28) and (2.31), we have

$$B_2(t, x_1, v^{x_1}(t;s,0)) - B_2(t, x_2, v^{x_2}(t;s,0)) = -J(t)\rho(t) + I(t),$$

where

$$J(t,\xi) = \lambda(t, \xi, x_1(\xi), v^{x_1}(t;s,0)(\xi), v^{x_2}(t;s,0)(\xi)), \quad \xi \in D,$$

and

$$I(t,\xi) = \theta(t, \xi, x_1(\xi), x_2(\xi), v^{x_2}(t;s,0)(\xi)), \quad \xi \in D.$$

Therefore, if we define

$$K(t,\xi) = \frac{g_2(t, \xi, v^{x_1}(t;s,0)(\xi)) - g_2(t, \xi, v^{x_2}(t;s,0)(\xi))}{\rho(t)}, \quad \xi \in D,$$

we can rewrite (5.33) as

$$\left\{\begin{array}{l} d\rho(t) = [(A_2(t)-\alpha)\rho(t) - J(t)\rho(t) + I(t)]\,dt + K(t)\rho(t)\,d\bar{w}^{Q_2}(t), \\ \\ \rho(s) = 0. \end{array}\right. \tag{5.34}$$

Notice that, due to (2.31), we have

$$J(t,\xi) \geq 0, \quad (t,\xi) \in [s,+\infty) \times D. \tag{5.35}$$

Due to (2.30), we have

$$x_1, x_2 \in B_E(R) \Longrightarrow \sup_{s<t} |I(t)|_E \leq L_R\,|x_1 - x_2|_E. \tag{5.36}$$

Moreover, due to (2.29), we can assume, without any loss of generality, that

$$x_1(\xi) \geq x_2(\xi) \Longrightarrow I(t,\xi) \geq 0, \quad (t,\xi) \in [s,+\infty) \times \bar{D}. \tag{5.37}$$

Finally, as $g_2(t,\xi,\cdot)$ is assumed to be Lipschitz continuous, uniformly with respect to $(t,\xi) \in \mathbb{R} \times \bar{D}$, we have that $K(t)$ satisfies (5.16).

Thanks to (5.37), by a comparison argument we have

$$x_1 \geq x_2 \Longrightarrow \rho(t) \geq 0, \quad \mathbb{P}\text{-a.s.}, \quad s < t.$$

Therefore, again by comparison, due to (5.35) we have

$$x_1 \geq x_2 \Longrightarrow 0 \leq \rho(t) \leq \hat{\rho}(t), \quad \mathbb{P}\text{-a.s.}, \quad s < t, \tag{5.38}$$

where $\hat{\rho}(t)$ is the solution of the problem

$$\left\{\begin{array}{l} d\hat{\rho}(t) = [(A_2(t)-\alpha)\hat{\rho}(t) + I(t)]\,dt + K(t)\hat{\rho}(t)\,d\bar{w}^{Q_2}(t), \\ \\ \hat{\rho}(s) = 0. \end{array}\right.$$

This means that

$$\hat{\rho}(t) = \psi_\alpha(\hat{\rho};s)(t) + \int_s^t U_\alpha(t,r)I(r)\,dr + \int_s^t U_\alpha(t,r)K(r)\hat{\rho}(r)\,d\bar{w}^{Q_2}(r).$$

As a consequence of (5.36), by using the same arguments as in the proof of Proposition 5.1, we get that if α is large enough and/or L_{g_2} is small enough,

$$x_1 \geq x_2, \quad x_1, x_2 \in B_E(R) \implies \sup_{s<t} \mathbb{E}\,|\hat{\rho}(t)|_E^2 \leq c_R\,|x_1 - x_2|_E^2,$$

so that, thanks to (5.38), we have

$$x_1 \geq x_2, \quad x_1, x_2 \in B_E(R) \implies \sup_{s<t} \mathbb{E}\,|v^{x_1}(t;s,0) - v^{x_2}(t;s,0)|_E^2 \leq c_R\,|x_1 - x_2|_E^2.$$

As in the proof of Proposition 5.2, the general estimate (5.32) follows by noticing that

$$\begin{aligned} &|v^{x_1}(t;s,0) - v^{x_2}(t;s,0)|_E^2 \\ &\leq 2\,|v^{x_1}(t;s,0) - v^{x_1 \wedge x_2}(t;s,0)|_E^2 + 2\,|v^{x_1 \wedge x_2}(t;s,0) - v^{x_2}(t;s,0)|_E^2. \end{aligned}$$ □

6. Almost periodicity of the evolution family of measures. In what follows, we shall assume the following conditions on the operator $A_2(t)$ and the coefficients $b_2(t,\xi,\sigma)$ and $g_2(t,\xi,\sigma)$.

HYPOTHESIS 6.1.
1. *The functions* $\gamma : \mathbb{R} \to (0,\infty)$ *and* $l : \mathbb{R} \times \bar{D} \to \mathbb{R}^d$ *are both periodic, with the same period.*
2. *The families of functions*

$$\mathcal{B}_R := \left\{b_2(\cdot,\xi,\sigma) \,:\, \xi \in \bar{D},\ \sigma \in B_{\mathbb{R}^2}(R)\right\},\\ \mathcal{G}_R := \left\{g_2(\cdot,\xi,\sigma) \,:\, \xi \in \bar{D},\ \sigma \in B_{\mathbb{R}}(R)\right\}$$

are both uniformly almost periodic for any $R > 0$.

LEMMA 6.2. *Under Hypothesis* 6.1, *for any* $R > 0$, *the family of functions*

$$\{B_2(\cdot,x,y) \,:\, (x,y) \in B_{E\times E}(R)\}, \quad \{G_2(\cdot,y) \,:\, y \in B_E(R)\}$$

are both uniformly almost periodic.

Proof. Due to the uniform almost periodicity of the family $\mathcal{B}_R$, for any $\epsilon > 0$ there exists $l_{\epsilon,R} > 0$ such that in any interval of $\mathbb{R}$ of length $l_{\epsilon,R}$ we can find $\tau > 0$ such that

$$|b_2(t+\tau,\xi,\sigma) - b_2(t,\xi,\sigma)| < \epsilon, \qquad (t,\xi,\sigma) \in \mathbb{R} \times \bar{D} \times B_{\mathbb{R}^2}(R).$$

This implies that

$$|B_2(t+\tau,x,y) - B_2(t,x,y)|_E$$
$$= \sup_{\xi\in\bar{D}} |b_2(t+\tau,x(\xi),y(\xi)) - b_2(t,x(\xi),y(\xi))| < \epsilon, \quad (t,x,y) \in \mathbb{R} \times B_{E\times E}(R).$$

In a completely analogous way, we can show that the family $\{G_2(\cdot,y) \,:\, y \in B_E(R)\}$ is uniformly almost periodic. ∎

Now, for any $\mu,\nu \in \mathcal{P}(E)$, we define

$$d(\mu,\nu) = \sup\left\{\left|\int_E f(y)\,(\mu-\nu)(dy)\right|,\ |f|_{\text{Lip}} \leq 1\right\},$$

where

$$|f|_{\text{Lip}} = |f|_E + [f]_{\text{Lip}} = |f|_E + \sup_{\xi\neq\eta}\frac{|f(\xi)-f(\eta)|}{|\xi-\eta|}.$$

It is known that the space $(\mathcal{P}(E),d)$ is a complete metric space and the distance d generates the weak topology on $\mathcal{P}(E)$.

In [13] it is proven that if $A_2(\cdot)$ is periodic, the family of functions

$$\{B_2(\cdot,x,y) \,:\, (x,y) \in B_{E\times E}(R)\}, \quad \{G_2(\cdot,y) \,:\, y \in B_E(R)\}$$

are both uniformly almost periodic for any $R > 0$ and the family of measures $\{\mu^x_t\}_{t\in\mathbb{R}}$ is tight in $\mathcal{P}(E)$, then the mapping $t \in \mathbb{R} \mapsto \mu^x_t \in \mathcal{P}(E)$ is almost periodic. The proof in [13] is based on Theorem 3.3. Actually, it is proved that, for every two sequences γ' and β' in $\mathbb{R}$, there exist common subsequences $\gamma \subset \gamma'$ and $\beta \subset \beta'$ such that $T_{\gamma+\beta}\mu^x_\cdot = T_\gamma T_\beta \mu^x_\cdot$, pointwise on $\mathbb{R}$.

Unlike in this paper, in [13] it is assumed that the coefficients are Lipschitz continuous and the covariance Q_2^2 of the noise is trace class. But all the arguments used in [13] can be adapted to the present situation without major difficulties. Therefore, in view of Lemma 6.2, if we prove that the family of measures $\{\mu_t^x\}_{t\in\mathbb{R}}$ is tight in $\mathcal{P}(E)$, we obtain the following result.

THEOREM 6.3. *Under Hypotheses* 2.1, 2.2, 2.8, 2.10, *and* 6.1, *if* $\alpha > 0$ *is large enough and/or* L_{g_2} *is small enough, we have that the mapping*

$$t \in \mathbb{R} \mapsto \mu_t^x \in \mathcal{P}(E)$$

is almost periodic for any fixed $x \in E$.

Thus, it only remains to prove tightness.

LEMMA 6.4. *Under Hypotheses* 2.1, 2.2, 2.8, *and* 2.10, *if* α *is sufficiently large and/or* L_{g_2} *is sufficiently small, there exists* $\theta > 0$ *such that, for any* $p \geq 1$ *and any* $x \in E$,

$$\sup_{t\in\mathbb{R}} \mathbb{E}\,|\eta^x(t)|^p_{C^\theta(\bar{D})} \leq c_p\,(1 + |x|^p_E)\,. \tag{6.1}$$

In particular, the family of measures

$$\Lambda_R := \{\mu_t^x\,;\, t \in \mathbb{R},\; x \in B_E(R)\}$$

is tight in $\mathcal{P}(E)$ *for any* $R > 0$.

Proof. Due to (5.10) and (5.12), with $y = 0$, we have that, for any $p \geq 1$,

$$\sup_{t\in\mathbb{R}} \mathbb{E}\,|\eta^x(t)|^p_E \leq c_p\,(1 + |x|^p_E). \tag{6.2}$$

Moreover, thanks to (2.10) and (5.23), for every $t \in \mathbb{R}$ and $\theta > 0$,

$$|\eta^x(t)|_{C^\theta(\bar{D})} \leq c\,|\eta^x(t-1)|_E + |\psi_\alpha(\eta^x; t-1)(t)|_{C^\theta(\bar{D})}$$

$$+ \int_{t-1}^t |U_\alpha(t,r)B_2(r,x,\eta^x(r))|_{C^\theta(\bar{D})}\,dr + |\Gamma_\alpha^x(\eta^x, t-1)(t)|_{C^\theta(\bar{D})}\,.$$

According to (2.18), (5.6), and (2.10), this implies that, for some $\theta > 0$ and any $0 < \delta < \alpha$ and $p \geq 1$,

$$e^{\delta p}\mathbb{E}\,|\eta^x(t)^p|_{C^\theta(\bar{D})} \leq c_p\,\mathbb{E}|\eta^x(t-1)|^p_E + c_p \sup_{r\in[t-1,t]} e^{\delta p(r-t+1)}\mathbb{E}\,|\psi_\alpha(\eta^x; t-1)(r)|^p_{C^\theta(\bar{D})}$$

$$+ c_p \int_{t-1}^t |U_\alpha(t,r)B_2(r,x,\eta^x(r))|^p_{C^\theta(\bar{D})}\,dr$$

$$+ c_p \sup_{r\in[t-1,t]} e^{\delta p(r-t+1)}\mathbb{E}\,|\Gamma_\alpha^x(\eta^x; t-1)(r)|^p_{C^\theta(\bar{D})}$$

$$\leq c_p\,\mathbb{E}|\eta^x(t-1)|^p_E + c_p \sup_{r\in[t-1,t]} e^{\delta p(r-t+1)}\mathbb{E}\,|\eta^x(r)|^p_E$$

$$+ c_p \left(\int_{t-1}^t (t-r)^{-\frac{\theta}{2}}\,(1 + |x|_E + \mathbb{E}\,|\eta^x(r)|^{m_2}_E)\,dr\right)^p$$

$$+ c_p \sup_{r\in[t-1,t]} e^{\delta p(r-t+1)}\left(1 + \mathbb{E}\,|\eta^x(r)|^{\frac{p}{m_2}}_E\right)$$

so that

$$e^{\delta p}\mathbb{E}\,|\eta^x(t)|^p_{C^\theta(\bar{D})} \leq c_p\,\left(\mathbb{E}|\eta^x(t-1)|^p_E + 1 + |x|^p_E\right) + c_p \sup_{r\in\,[t-1,t]} e^{\delta p(r-t+1)}\mathbb{E}\,|\eta^x(r)|^p_E$$
$$+c_p\left(\int_{t-1}^t (t-r)^{-\frac{\theta p}{2(p-1)}}\,dr\right)^{p-1}\int_{t-1}^t \mathbb{E}\,|\eta^x(r)|^{m_2 p}_E\,dr.$$

If $p \geq 2$, then for any $\theta < 1$ we have that $\theta p/(p-1) < 2$. Then, thanks to (6.2), we can conclude that (6.1) holds true for any $p \geq 2$. Due to the Hölder inequality, (6.1) holds for any $p \geq 1$. ∎

7. The averaged equation.

For any fixed $x \in E$, the mapping $B_1(x,\cdot) : E \to E$ is continuous and

$$|B_1(x,y)|_E \leq c\,(1 + |x|^{m_1}_E + |y|_E)\,. \tag{7.1}$$

B_1 is unbounded and only locally Lipschitz continuous, but, as a consequence of Proposition 6.3, it is still possible to prove the following result.

LEMMA 7.1. *Under the same hypotheses as Proposition* 6.3*, for every compact set* $K \subset E$*, the family of functions*

$$\left\{t \in \mathbb{R} \mapsto \int_E B_1(x,z)\,\mu^x_t(dz) \in E\,:\, x \in K\right\} \tag{7.2}$$

is uniformly almost periodic.

Proof. For every $n \in \mathbb{N}$, we define

$$b_{1,n}(\xi,\sigma_1,\sigma_2) := \begin{cases} b_1(\xi,\sigma_1,\sigma_2) & \text{if } |\sigma_2| \leq n, \\ \\ b_1(\xi,\sigma_1,\sigma_2 n/|\sigma_2|) & \text{if } |\sigma_2| > n, \end{cases}$$

and we set

$$B_{1,n}(x,y)(\xi) = b_{1,n}(\xi, x(\xi), y(\xi)), \quad \xi \in \bar{D}.$$

Clearly, we have that $B_{1,n}(x,\cdot) : E \to E$ is Lipschitz continuous and bounded for any fixed $x \in E$, and $B_{1,n}(x,y) = B_1(x,y)$ if $|y|_E \leq n$. Moreover, for any $R > 0$,

$$\sup_{|x|_E \leq R} |B_{1,n}(x,\cdot)|_{\mathrm{Lip}_b(E)} := c_{n,R} < \infty. \tag{7.3}$$

Now, for any $n \in \mathbb{N}$, we have

$$\int_E B_1(x,z)\,\mu^x_t(dz) = \int_E B_{1,n}(x,z)\,\mu^x_t(dz) + \int_{\{|z|_E > n\}} \left(B_1(x,z) - B_{1,n}(x,z)\right)\,\mu^x_t(dz).$$

According to (5.26) and (7.1), we have

$$\sup_{t\in\mathbb{R}}\left|\int_{\{|z|_E>n\}} \left(B_1(x,z) - B_{1,n}(x,z)\right)\,\mu^x_t(dz)\right|$$
$$\leq c\sup_{t\in\mathbb{R}}\int_{\{|z|_E>n\}} \left(1 + |x|^{m_1}_E + |z|_E\right)\,\mu^x_t(dz) \leq \frac{c}{n}\left(1 + |x|^{m_1+1}_E\right).$$

This implies that, for any $\epsilon > 0$ and $R > 0$, we can find $\bar{n} = \bar{n}(\epsilon, R) \in \mathbb{N}$ such that

$$\sup_{\substack{x \in B_E(R)\\ t \in \mathbb{R}}} \left| \int_{\{|z|_E > \bar{n}\}} (B_1(x,z) - B_{1,\bar{n}}(x,z))\ \mu_t^x(dz) \right| \leq \frac{\epsilon}{4},$$

so that, for any $t, \tau \in \mathbb{R}$ and $x \in B_E(R)$,

$$\left| \int_E B_1(x,z)\, \mu_{t+\tau}^x(dz) - \int_E B_1(x,z)\, \mu_t^x(dz) \right|$$
$$\leq \left| \int_E B_{1,\bar{n}}(x,z)\, \mu_{t+\tau}^x(dz) - \int_E B_{1,\bar{n}}(x,z)\, \mu_t^x(dz) \right| + \frac{\epsilon}{2}.$$

Now, let us define

$$f(t,x) = \int_E B_{1,\bar{n}}(x,z)\, \mu_t^x(dz), \qquad (t,x) \in \mathbb{R} \times E.$$

If we show that, for any compact set $K \subset E$, the family $\{f(\cdot, x) : x \in K\}$ is uniformly almost periodic we have concluded our proof.

Since, for any $t, \tau \in \mathbb{R}$, we have

$$|f(t+\tau, x) - f(t,x)|_E \leq |B_{1,\bar{n}}(x,\cdot)|_{\mathrm{Lip}_b(E)}\, d(\mu_{t+\tau}^x, \mu_t^x),$$

in view of Theorem 6.3 and (7.3), the function $f(\cdot, x)$ is almost periodic for any $x \in E$. Moreover, f is continuous in $x \in K$, uniformly with respect to $t \in \mathbb{R}$. Actually, thanks to (2.40), we have

$$|f(t,x) - f(t,y)|_E \leq \mathbb{E}\, |B_{1,\bar{n}}(x, \eta^x(t)) - B_{1,\bar{n}}(y, \eta^y(t))|_E$$
$$\leq c\, \mathbb{E} \left(1 + |x|_E^\theta + |y|_E^\theta + |\eta^x(t)|_E^\theta + |\eta^y(t)|_E^\theta\right) (|x-y|_E + |\eta^x(t) - \eta^y(t)|_E)\,.$$

Now, as K is compact it is bounded, so that there exists $R > 0$ such that $K \subset B_E(R)$. Therefore, due to Proposition 5.4 and (5.25), we can conclude that, for any $x, y \in K$,

$$\sup_{t \in \mathbb{R}} |f(t,x) - f(t,y)|_E \leq c_R \left(|x-y|_H + \sup_{t \in \mathbb{R}} \left(\mathbb{E}\, |\eta^x(t) - \eta^y(t)|_E^2\right)^{\frac{1}{2}} \right) \leq c_R\, |x-y|_E,$$

and this implies that the family of functions $\{f(t,\cdot) : t \in \mathbb{R}\}$ is equicontinuous. In [19, Theorem 2.10], it is proven that this implies the uniform almost periodicity of the family $\{f(\cdot, x) : x \in K\}$. ∎

Due to the almost periodicity of the family of mappings (7.2), according to Theorem 3.4 we can define

$$\bar{B}(x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_E B_1(x,y)\, \mu_t^x(dy)\, dt, \qquad x \in E.$$

Thanks to (5.26) and (7.1), we have that

$$|\bar{B}(x)|_E \leq c\, (1 + |x|_E^{m_1})\,. \tag{7.4}$$

Actually, in view of (7.1), we have

$$\left| \frac{1}{T} \int_0^T \int_E B_1(x,y)\, \mu_t^x(dy)\, dt \right|_E \leq c\, \frac{1}{T} \int_0^T \int_E (1 + |x|_E^{m_1} + |y|_E)\ \mu_t^x(dy)\, dt$$

and then, thanks to (5.26), we have

$$|\bar{B}(x)|_E \leq c\,(1+|x|_E^{m_1}) + c_1\,(1+|x|_E)\,,$$

which implies (7.4).

As a consequence of (5.12), we have the following crucial result.

LEMMA 7.2. *Under Hypotheses* 2.1 *to* 6.1*, if α is sufficiently large and/or L_{g_2} is sufficiently small, there exist some constants $\kappa_1, \kappa_2 \geq 0$ such that, for any $T > 0$, $s \in \mathbb{R}$, and $x, y \in E$,*

$$\mathbb{E}\left|\frac{1}{T}\int_s^{s+T} B_1(x, v^x(t;s,y))\,dt - \bar{B}(x)\right|_E^2 \leq \frac{c}{T}\,(1+|x|_E^{\kappa_1}+|y|_E^{\kappa_2}) + \alpha(T,x) \tag{7.5}$$

for some mapping $\alpha : [0,\infty)\times E \to [0,+\infty)$ such that

$$\sup_{T>0}\,\alpha(T,x) \leq c\,(1+|x|_E^{m_1})\,, \qquad x \in E,$$

and, for any compact set $K \subset E$,

$$\lim_{T\to\infty}\,\sup_{x\in K}\,\alpha(T,x) = 0.$$

Proof. For any fixed $\Lambda \in E^\star$ and $x \in E$, we denote by $\Pi^x_\Lambda B_1$ the mapping

$$(t,y)\in \mathbb{R}\times E \mapsto \Pi^x_\Lambda B_1(t,y) := \left\langle B_1(x,y),\Lambda\right\rangle_E - \int_E \left\langle B_1(x,z),\Lambda\right\rangle_E\,\mu^x_t(dz) \in \mathbb{R}.$$

By proceeding as in the proof of Lemma 2.3 in [9] and the proof of Lemma 5.1 in [10], we have

(7.6)

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{T}\int_s^{s+T}\left[\left\langle B_1(x,v^x(t;s,y)),\Lambda\right\rangle_E - \int_E \left\langle B_1(x,z),\Lambda\right\rangle_E\,\mu^x_t(dz)\right]dt\right)^2\\
&= \frac{2}{T^2}\int_s^{s+T}\int_r^{s+T}\mathbb{E}\left[\Pi^x_\Lambda B_1(r,v^x(r;s,y))\,P^x_{r,t}\Pi^x_\Lambda B_1(r,v^x(r;s,y))\right]dt\,dr\\
&\leq \frac{2}{T^2}\int_s^{s+T}\int_r^{s+T}\left(\mathbb{E}\,|\Pi^x_\Lambda B_1(r,v^x(r;s,y))|^2\right)^{\frac{1}{2}}\,\left(\mathbb{E}\,|P^x_{r,t}\Pi^x_\Lambda B_1(r,v^x(r;s,y))|^2\right)^{\frac{1}{2}}dt\,dr.
\end{aligned}$$

Due to (2.35), (5.10), and (5.26), we have

$$\begin{aligned}
&\mathbb{E}\,|\Pi^x_\Lambda B_1(r,v^x(r;s,y))|^2 \leq c\left(1+|x|_E^{2m_1}+\mathbb{E}\,|v^x(r;s,y)|_E^2\right)|\Lambda|^2_{E^\star}\\
&\leq c\left(1+|x|_E^{2m_1}+e^{-2\delta(r-s)}|y|_E^2\right)|\Lambda|^2_{E^\star}.
\end{aligned} \tag{7.7}$$

Moreover, due to (2.40), we have

$$|\left\langle B_1(x,y),\Lambda\right\rangle_E - \left\langle B_1(x,z),\Lambda\right\rangle_E| \leq c\,|y-z|_E\left(1+|x|_E^\theta+|y|_E^\theta+|z|_E^\theta\right)|\Lambda|_{E^\star}$$

so that, thanks to (5.12), we have

$$\mathbb{E}\,|P^x_{r,t}\Pi^x_\Lambda B_1(r,v^x(r;s,y))|^2 \leq c\left(1+|x|_E^{2(\theta\vee 1)}+|y|_E^{2(\theta\vee 1)}\right)|\Lambda|^2_{E^\star}e^{-2\delta(t-r)}.$$

Therefore, if we plug the estimate above and estimate (7.7) into (7.6), we get

$$\begin{aligned}&\mathbb{E}\left|\frac{1}{T}\int_s^{s+T}\left[B_1(x,v^x(t;s,y))-\int_E B_1(x,z)\,\mu_t^x(dz)\right]\,dt\right|_E^2\\ &\le c\,\left(1+|x|_E^{m_1}+|y|_E\right)\left(1+|x|_E^{\theta\vee 1}+|y|_E^{\theta\vee 1}\right)\frac{1}{T^2}\int_s^{s+T}\int_r^{s+T}e^{-\delta(t-r)}\,dt\,dr\\ &\le c\,\left(1+|x|_E^{m_1}+|y|_E\right)\left(1+|x|_E^{\theta\vee 1}+|y|_E^{\theta\vee 1}\right)\,\frac{1}{T}.\end{aligned} \tag{7.8}$$

Next, thanks to Lemma 7.1 and Theorem 3.4, we have that the limit

$$\lim_{T\to\infty}\frac{1}{T}\int_s^{s+T}\int_E B_1(x,z)\,\mu_t^x(dz)\,dt\in\,E$$

converges to $\bar{B}_1(x)$, uniformly with respect to $s\in\mathbb{R}$ and x in any compact set $K\subset E$. Therefore, if we define

$$\alpha(T,x)=2\,\left|\frac{1}{T}\int_s^{s+T}\int_E B_1(x,z)\,\mu_t^x(dz)\,dt-\bar{B}(x)\right|_E^2,$$

we can conclude. ∎

Lemma 7.3. *Under Hypotheses* 2.1 *to* 6.1, *if α is sufficiently large and/or L_{g_2} is sufficiently small, we have that the mapping $\bar{B}:E\to E$ is locally Lipschitz continuous. Moreover, for any $x,h\in E$ and $\delta\in\mathcal{M}_h$,*

$$\left\langle\bar{B}(x+h)-\bar{B}(x),\delta\right\rangle_E\le c\,\left(1+|h|_E+|x|_E\right). \tag{7.9}$$

Proof. For any $x_1,x_2\in E$, we have

$$\bar{B}(x_1)-\bar{B}(x_2)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathbb{E}\,\left(B_1(x_1,\eta^{x_1}(t))-B_1(x_2,\eta^{x_2}(t))\right)\,dt\quad\text{ in } E.$$

By using (2.40), we have

$$\begin{aligned}&|B_1(x_1,\eta^{x_1}(t))-B_1(x_2,\eta^{x_2}(t))|_E\\ &\le c\left(1+|x_1|_E^\theta+|x_2|_E^\theta+|\eta^{x_1}(t)|_E^\theta+|\eta^{x_2}(t)|_E^\theta\right)\left(|x_1-x_2|_E+|\eta^{x_1}(t)-\eta^{x_2}(t)|_E\right),\end{aligned}$$

and then, due to (5.25), we get

$$\begin{aligned}&\sup_{t\in\mathbb{R}}|\mathbb{E}\,(B_1(x_1,\eta^{x_1}(t))-B_1(x_2,\eta^{x_2}(t)))|_E\\ &\le c\left(1+|x_1|_E^\theta+|x_2|_E^\theta\right)\left(|x_1-x_2|_E+\sup_{t\in\mathbb{R}}\left(\mathbb{E}|\eta^{x_1}(t)-\eta^{x_2}(t)|_E^2\right)^{\frac{1}{2}}\right).\end{aligned}$$

Thanks to (5.31), this implies that, for any $R>0$,

$$x_1,x_2\in\,B_E(R)\Longrightarrow|B_1(x_1)-B_1(x_2)|_E\le c_R\,|x_1-x_2|_E.$$

Concerning (7.9), if $\delta \in \mathcal{M}_h$, we have

$$\left\langle \bar{B}(x+h) - \bar{B}(x), \delta \right\rangle_E = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}\,\left\langle B_1(x+h, \eta^{x+h}(s)) - B_1(x, \eta^x(s)), \delta\right\rangle_E\, ds.$$

Now, due to (2.37), we have

$$\begin{aligned}&\left\langle B_1(x+h, \eta^{x+h}(s)) - B_1(x, \eta^x(s)), \delta\right\rangle_E\\ &\leq c\left(1 + |x|_E + |h|_E + |\eta^{x+h}(s)|_E + |\eta^x(s)|_E\right),\end{aligned}$$

and then, thanks again to (5.25), we conclude

$$\begin{aligned}&\left\langle \bar{B}(x+h) - \bar{B}(x), \delta \right\rangle_E\\ &\leq \limsup_{T\to\infty} \frac{1}{T}\int_0^T c\left(1 + |x|_E + |h|_E + \mathbb{E}|\eta^{x+h}(s)|_E + \mathbb{E}|\eta^x(s)|_E\right)\, ds\\ &\leq c\,(1 + |x|_E + |h|_E)\,.\end{aligned}$$

□

Now, we can introduce the averaged equation

$$du(t) = \left[A_1 u(t) + \bar{B}(u(t))\right]\, dt + G(u(t))\, dw^{Q_1}(t), \qquad u(0) = x \in E. \tag{7.10}$$

In view of Lemma 7.3 and [6, Theorem 5.3], for any $x \in E$, $T > 0$, and $p \geq 1$, (7.10) admits a unique mild solution $\bar{u} \in L^p(\Omega; C_b((0,T]; E))$. In the next section, we will show that the slow motion u_ϵ converges in probability to the averaged motion $\bar{u}$.

8. The averaging limit. In this last section we prove that the slow motion u_ϵ converges to the averaged motion $\bar{u}$, as $\epsilon \to 0$. The proof of this averaging result is in many respects similar to the proof of [10, Theorem 61].

THEOREM 8.1. *Assume that Hypotheses* 2.1 *to* 6.1 *hold and fix* $x \in C^\theta(\bar{D})$ *for some* $\theta > 0$ *and* $y \in E$. *Then, if* α *is large enough and/or* L_{g_2} *is small enough, for any* $T > 0$ *and* $\eta > 0$, *we have*

$$\lim_{\epsilon\to 0}\, \mathbb{P}\left(\sup_{t\in\,[0,T]} |u_\epsilon(t) - \bar{u}(t)|_E > \eta\right) = 0, \tag{8.1}$$

where $\bar{u}$ *is the solution of the averaged equation* (7.10).

For any $h \in D(A_1)$, the slow motion u_ϵ satisfies the identity

$$\begin{aligned}&\int_D u_\epsilon(t,\xi) h(\xi)\, d\xi = \int_D x(\xi) h(\xi)\, d\xi + \int_0^t \int_D u_\epsilon(s,\xi) A_1 h(\xi)\, d\xi\, ds\\ &+ \int_0^t \int_D \bar{B}(u_\epsilon(s,\cdot))(\xi) h(\xi)\, d\xi\, ds + \int_0^t \int_D [G_1(u_\epsilon(s)h](\xi) dw^{Q_2}(s,\xi) + R_\epsilon(t),\end{aligned}$$

where

$$R_\epsilon(t) = \int_0^t \int_D \left(B_1(u_\epsilon(s), v_\epsilon(s))(\xi) - \bar{B}(u_\epsilon(s))(\xi)\right) h(\xi)\, d\xi\, ds.$$

Therefore, as in [9] and [10], due to the tightness of the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon\in\,(0,1]}$ in $\mathcal{P}(C([0,T]; E))$, in order to prove Theorem 8.1 it is sufficient to prove the following.

LEMMA 8.2. *Under the same hypotheses as in Theorem* 8.1*, for any* $T > 0$*, we have*

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{t\in [0,T]} |R_\epsilon(t)|_E = 0. \tag{8.2}$$

8.1. Proof of Lemma 8.2. For any $n \in \mathbb{N}$, we define

$$b_{1,n}(\xi,\sigma_1,\sigma_2) := \begin{cases} b_1(\xi,\sigma_1,\sigma_2) & \text{if } |\sigma_1| \leq n, \\ \\ b_1(\xi,\sigma_1 n/|\sigma_1|,\sigma_2) & \text{if } |\sigma_1| > n, \end{cases}$$

and

$$b_{2,n}(t,\xi,\sigma_1,\sigma_2) := \begin{cases} b_2(t,\xi,\sigma_1,\sigma_2) & \text{if } |\sigma_1| \leq n, \\ \\ b_2(t,\xi,\sigma_1 n/|\sigma_1|,\sigma_2) & \text{if } |\sigma_1| > n, \end{cases}$$

Concerning the corresponding composition operator, we have

$$x \in B_E(n) \Longrightarrow B_{1,n}(x,y) = B_1(x,y), \quad B_{2,n}(t,x,y) = B_2(t,x,y) \tag{8.3}$$

for every $t \in \mathbb{R}$ and $y \in E$. Notice that the mappings $b_{1,n}$ and $b_{2,n}$ satisfy all conditions in Hypotheses 2.6 and 2.8, respectively. For any fixed $t \in \mathbb{R}$, $\xi \in \bar{D}$, and $\sigma_2 \in \mathbb{R}$, the mappings $b_{1,n}(\xi,\cdot,\sigma_2)$, and $b_{2,n}(t,\xi,\cdot,\sigma_2)$ are Lipschitz continuous and, in view of (2.30),

$$\sup_{\substack{(t,\xi)\in \mathbb{R}\times\bar{D} \\ \sigma_2\in\mathbb{R}}} |b_{2,n}(t,\xi,\sigma_1,\sigma_2) - b_{2,n}(t,\xi,\rho_1,\sigma_2)| \leq c_n\,|\sigma_1 - \rho_1|, \qquad \sigma_1,\rho_1 \in \mathbb{R}. \tag{8.4}$$

Moreover, for any $n \in \mathbb{N}$, we define

$$g_{1,n}(\xi,\sigma_1) := \begin{cases} g_1(\xi,\sigma_1) & \text{if } |\sigma_1| \leq n, \\ \\ g_1(\xi,\sigma_1 n/|\sigma_1|) & \text{if } |\sigma_1| > n. \end{cases}$$

The corresponding composition/multiplication operator is denoted by $G_{1,n}$.

Now, for any $n \in \mathbb{N}$, we introduce the system

$$\begin{cases} du(t) = [A_1 u(t) + B_{1,n}(u(t),v(t))]\,dt + G_{1,n}(u(t))\,dw^{Q_1}(t), \\ \\ dv(t) = \dfrac{1}{\epsilon}\,[(A_2(t/\epsilon) - \alpha)v(t) + B_{2,n}(t/\epsilon,u(t),v(t))]\,dt + \dfrac{1}{\sqrt{\epsilon}}\,G_2(t/\epsilon,v(t))\,dw^{Q_2}(t) \end{cases} \tag{8.5}$$

with initial conditions $u(s) = x$ and $v(s) = y$. We denote by $z_{\epsilon,n} = (u_{\epsilon,n}, v_{\epsilon,n})$ its solution.

Next, for any $n \in \mathbb{N}$, we introduce the problem

$$dv(t) = [(A_2(t) - \alpha)v(t) + B_{2,n}(t,x,v(t))]\,dt + G_2(t,v(t))\,dw^{Q_2}(t), \quad v(s) = y, \tag{8.6}$$

whose solution will be denoted by $v_n^x(t;s,y)$. Thanks to (8.3), for any $t \geq 0$, we have

$$v_n^x(t;s,y) = \begin{cases} v^x(t;s,y) & \text{if } |x|_E \leq n, \\ \\ v^{x_n}(t;s,y) & \text{if } |x|_E > n, \end{cases} \tag{8.7}$$

where

$$x_n(\xi) := \begin{cases} x(\xi) & \text{if } |x(\xi)| < n \\ \\ n \operatorname{sign} x(\xi) & \text{if } |x(\xi)| \geq n. \end{cases}$$

This implies that, for each $n \in \mathbb{N}$ and $x \in E$, there exists an evolution of measures family $\{\mu_t^{x,n}\}_{t\in\mathbb{R}}$ for equation (8.6), and $\mu_t^{x,n}$ is given by

$$\mu_t^{x,n} = \begin{cases} \mu_t^x & \text{if } |x|_E \leq n, \\ \\ \mu_t^{x_n} & \text{if } |x|_E > n. \end{cases}$$

Moreover, due to (5.10), for any $p \geq 1$ we have

$$\mathbb{E}\,|v_n^x(t;s,y)|_E^p \leq c_{p,n}\left(1 + e^{-\delta p(t-s)}|y|_E^p\right), \qquad t > s. \tag{8.8}$$

As all coefficients in equation (8.6) satisfy the same conditions as are fulfilled by the coefficients of equation (5.2), we have that a result analogous to Lemma 7.2 holds. More precisely, if we define

$$\bar{B}_n(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_E B_{1,n}(x,y)\,\mu_t^{x,n}(dy)\,dt,$$

we have that

$$\mathbb{E}\left|\frac{1}{T}\int_s^{s+T} B_{1,n}(x, v_n^x(t;s,y))\,ds - \bar{B}_n(x)\right|_E^2 \leq \frac{c}{T}\left(1 + |x|_E^{\kappa_1} + |y|_E^{\kappa_2}\right) + \alpha(T,x) \tag{8.9}$$

for some mapping $\alpha : (0,+\infty) \times E \to [0,+\infty)$ such that

$$\sup_{T>0} \alpha(T,x) \leq c(1 + |x|_E^{m_1}), \qquad x \in E, \tag{8.10}$$

and

$$\lim_{T\to\infty} \sup_{x\in K} \alpha(T,x) = 0 \tag{8.11}$$

for every compact set $K \subset E$. Notice that

$$|x|_E \leq n \Longrightarrow \bar{B}_n(x) = \bar{B}(x).$$

Lemma 8.3. *The mapping $\bar{B}_n : E \to E$ is Lipschitz continuous.*

Proof. Due to (2.21), for every $t \in \mathbb{R}$ and $x_1, x_2 \in E$, we have

$$\begin{aligned} &|B_{1,n}(x_1, \eta_n^{x_1}(t)) - B_{1,n}(x_2, \eta_n^{x_2}(t))|_E \\ &\leq c_n\,|x_1 - x_2|_E + |B_{1,n}(x_2, \eta_n^{x_1}(t)) - B_{1,n}(x_2, \eta_n^{x_2}(t))|_E \\ &\leq c_c\,|x_1 - x_2|_E + c_n\left(1 + |\eta_n^{x_1}(t)|_E^\theta + |\eta_n^{x_2}(t)|_E^\theta\right)|\eta_n^{x_1}(t) - \eta_n^{x_2}(t)|_E. \end{aligned}$$

Due to (8.8), we have

$$\sup_{t\in\mathbb{R}} \mathbb{E}\,|\eta_n^x(t)|_E^p =: c_{p,n} < \infty, \tag{8.12}$$

and this implies

$$\left|\int_E B_{1,n}(x_1,y)\,\mu_t^{n,x_1}(dy) - \int_E B_{1,n}(x_2,y)\,\mu_t^{n,x_2}(dy)\right|_E$$

$$\leq \mathbb{E}\,|B_{1,n}(x_1,\eta_n^{x_1}(t)) - B_{1,n}(x_2,\eta_n^{x_2}(t))|_E \leq c_n\,|x_1-x_2|_E + c_n\left(\mathbb{E}\,|\eta_n^{x_1}(t)-\eta_n^{x_2}(t)|_E^2\right)^{\frac{1}{2}}.$$

Now, if we adapt the proof of Proposition 5.4 to the present situation, we can easily see that

$$\sup_{t\in\mathbb{R}}\mathbb{E}\,|\eta_n^{x_1}(t)-\eta_n^{x_2}(t)|_E^2 \leq c_n\,|x_1-x_2|_E^2,$$

and this allows us to conclude that

$$\sup_{t\in\mathbb{R}}\left|\int_E B_{1,n}(x_1,y)\,\mu_t^{n,x_1}(dy) - \int_E B_{1,n}(x_2,y)\,\mu_t^{n,x_2}(dy)\right|_E \leq c_n\,|x_1-x_2|_E,$$

which implies the Lipschitz continuity of $\bar{B}_n$. □

As in [9] and [10], we prove the validity of Lemma 8.2 by using Khasminskii's approach based on time discretization, as introduced in [25].

To this purpose, for any $\epsilon > 0$, we divide the interval $[0,T]$ in subintervals of size $\delta_\epsilon > 0$ for some constant $\delta_\epsilon > 0$ to be determined, and we introduce the auxiliary fast motion $\hat{v}_{\epsilon,n}$ defined in each time interval $[k\delta_\epsilon,(k+1)\delta_\epsilon]$ for $k = 0,1,\ldots,[T/\delta_\epsilon]$ as the solution of the problem

$$\left\{\begin{array}{l} dv(t) = \dfrac{1}{\epsilon}\left[(A_2(t/\epsilon)-\alpha)v(t) + B_{2,n}(t/\epsilon, u_{\epsilon,n}(k\delta_\epsilon), v(t))\right]dt \\ \qquad\qquad\qquad\qquad + \dfrac{1}{\sqrt{\epsilon}}\,G_2(t/\epsilon, v(t))\,dw^{Q_2}(t), \\ \\ v(k\delta_\epsilon) = v_{\epsilon,n}(k\delta_\epsilon). \end{array}\right. \tag{8.13}$$

Notice that, due to the way in which $\hat{v}_{\epsilon,n}$ has been defined, we have that an estimate analogous to (4.3) holds, that is, for any $p \geq 1$,

$$\int_0^T \mathbb{E}\,|\hat{v}_{\epsilon,n}(t)|_E^p\,dt \leq c_{p,T}\left(1+|x|_E^p+|y|_E^p\right). \tag{8.14}$$

As in [25] and [9], we want to prove the following approximation result.

LEMMA 8.4. *Assume Hypotheses* 2.1 *to* 2.10 *and fix* $x \in C^\theta(\bar{D})$ *and* $y \in E$. *Then, there exists a constant* $\kappa > 0$ *such that if*

$$\delta_\epsilon = \epsilon\,\log\epsilon^{-\kappa},$$

then, for any fixed $n \in \mathbb{N}$,

$$\lim_{\epsilon\to 0}\sup_{t\in[0,T]}\mathbb{E}|\hat{v}_{\epsilon,n}(t)-v_{\epsilon,n}(t)|_H^2 = 0. \tag{8.15}$$

Proof. Let $\epsilon > 0$ and $n \in \mathbb{N}$ be fixed. For $k = 0,\ldots,[T/\delta_\epsilon]$ and $t \in [k\delta_\epsilon,(k+1)\delta_\epsilon]$, let $\Lambda_{\epsilon,n}(t)$ be the solution of the problem

$$d\Lambda_{\epsilon,n}(t) = \frac{1}{\epsilon}\,(A_2-\alpha)\Lambda_{\epsilon,n}(t)\,dt + \frac{1}{\sqrt{\epsilon}}\,K_{\epsilon,n}(t)\,dw^{Q_2}, \qquad \Lambda_{\epsilon,n}(k\delta_\epsilon) = 0,$$

where

$$K_{\epsilon,n}(t) := G_{2,n}(t/\epsilon, \hat{v}_{\epsilon,n}(t)) - G_{2,n}(t/\epsilon, v_{\epsilon,n}(t)).$$

Notice that, with the notations of section 2, we can write

$$\Lambda_{\epsilon,n}(t) = \psi_{\alpha,\epsilon}(\Lambda_{\epsilon,n}; k\delta_\epsilon)(t) + \Gamma_{\epsilon,n}(t), \qquad t \in [k\delta_\epsilon, (k+1)\delta_\epsilon], \tag{8.16}$$

where

$$\Gamma_{\epsilon,n}(t) = \frac{1}{\sqrt{\epsilon}} \int_{k\delta_\epsilon}^t U_{\alpha,\epsilon}(t,r) K_{\epsilon,n}(r)\, dw^{Q_2}(r).$$

If we define $\rho_{\epsilon,n}(t) := \hat{v}_{\epsilon,n}(t) - v_{\epsilon,n}(t)$ and $z_{\epsilon,n}(t) := \rho_{\epsilon,n}(t) - \Lambda_{\epsilon,n}(t)$, we have

$$dz_{\epsilon,n}(t) = \frac{1}{\epsilon}\left[(A_2(t/\epsilon) - \alpha) z_{\epsilon,n}(t) + H_{\epsilon,n}(t)\right] dt, \qquad z_{\epsilon,n}(k\delta_\epsilon) = 0,$$

where, in view of (2.31),

$$\begin{aligned} H_{\epsilon,n}(t) &:= B_{2,n}(t/\epsilon, u_{\epsilon,n}(k\delta_\epsilon), \hat{v}_{\epsilon,n}(t)) - B_{2,n}(t/\epsilon, u_{\epsilon,n}(t), v_{\epsilon,n}(t)) \\ &= B_{2,n}(t/\epsilon, u_{\epsilon,n}(k\delta_\epsilon), \hat{v}_{\epsilon,n}(t)) - B_{2,n}(t/\epsilon, u_{\epsilon,n}(t), \hat{v}_{\epsilon,n}(t)) \\ &\quad - \lambda(t/\epsilon, \cdot, u_{\epsilon,n}(t), \hat{v}_{\epsilon,n}(t), v_{\epsilon,n}(t))(z_{\epsilon,n}(t) + \Lambda_{\epsilon,n}(t)). \end{aligned}$$

By proceeding as in the proof of [10, Lemma 6.2], we have

$$\begin{aligned} |z_{\epsilon,n}(t)|_E &\leq \frac{c_n}{\epsilon} \int_{k\delta_\epsilon}^t e^{-\frac{\alpha}{\epsilon}(t-s)} |u_{\epsilon,n}(k\delta_\epsilon) - u_{\epsilon,n}(s)|_E \, ds \\ &\quad + \frac{1}{\epsilon} \int_{k\delta_\epsilon}^t \exp\left(-\frac{1}{\epsilon}\int_s^t \lambda_{\epsilon,n}(r)\, dr\right) \lambda_{\epsilon,n}(s)\, |\Lambda_{\epsilon,n}(s)|_E \, ds \end{aligned}$$

where

$$\lambda_{\epsilon,n}(t) := \lambda(t/\epsilon, \xi_{\epsilon,n}(t), u_{\epsilon,n}(t, \xi_{\epsilon,n}(t)), \hat{v}_{\epsilon,n}(t, \xi_{\epsilon,n}(t)), v_{\epsilon,n}(t, \xi_{\epsilon,n}(t)))$$

and $\xi_{\epsilon,n}(t)$ is a point in $\bar{D}$ such that

$$|z_{\epsilon,n}(t, \xi_{\epsilon,n}(t))| = |z_{\epsilon,n}(t)|_E.$$

Now, it is not difficult to check that an estimate analogous to (4.5) is also valid for $u_{\epsilon,n}$. Therefore, we get

(8.17)

$$\begin{aligned} \mathbb{E}\, |\hat{v}_{\epsilon,n}(t) - v_{\epsilon,n}(t)|_E^2 &\leq c_p\, \mathbb{E}\, |\Lambda_{\epsilon,n}(t)|_E^2 + c_n \left(1 + |x|^{2m_2}_{C^\theta(\bar{D})} + |y|_E^2\right) \delta_\epsilon^{\gamma(\theta)2} \\ &\quad + c\, \mathbb{E} \sup_{s \in [k\delta_\epsilon, t]} |\Lambda_{\epsilon,n}(s)|_E^2 \\ &\qquad \left(\frac{1}{\epsilon}\int_{k\delta_\epsilon}^t \exp\left(-\frac{1}{\epsilon}\int_s^t \lambda_{\epsilon,n}(r)\, dr\right) \lambda_{\epsilon,n}(s)\, ds\right)^2 \\ &\leq c_n \left(1 + |x|^{2m_2}_{C^\theta(\bar{D})} + |y|_E^2\right) \delta_\epsilon^{\gamma(\theta)2} + c\, \mathbb{E} \sup_{s \in [k\delta_\epsilon, t]} |\Lambda_{\epsilon,n}(s)|_E^2. \end{aligned}$$

Since, for any $\alpha \geq 0$ and $\epsilon > 0$, we have

$$U_{\alpha,\epsilon}(t,s) = U_{\alpha,\epsilon}(t,r)U_{\alpha,\epsilon}(r,s), \qquad s < r < t,$$

the usual factorization argument used in the autonomous case can also be used here, so that, for $s \in [k\delta_\epsilon, (k+1)\delta_\epsilon]$ and $\eta \in (0,1)$, we have

$$\Gamma_{\epsilon,n}(s) = \frac{\sin \pi\eta}{\pi}\frac{1}{\sqrt{\epsilon}}\int_{k\delta_\epsilon}^s (s-r)^{\eta-1}U_{\alpha,\epsilon}(s,r)\,Y_{\eta,\epsilon,n}(r)\,dr,$$

where

$$Y_{\eta,\epsilon,n}(r) = \int_{k\delta_\epsilon}^r (r-\rho)^{-\eta}U_{\alpha,\epsilon}(r,\rho)\,K_{\epsilon,n}(\rho)\,dw^{Q_2}(\rho).$$

Therefore, by proceeding as in the proof of [10, Lemma 6.2], we have

$$\mathbb{E}\sup_{s\in[k\delta_\epsilon,t]}|\Gamma_{\epsilon,n}(s)|_E^2 \leq c_\eta\frac{1}{\epsilon}\int_{k\delta_\epsilon}^t \mathbb{E}\,|\hat{v}_{\epsilon,n}(s) - v_{\epsilon,n}(s)|_E^2\,ds. \tag{8.18}$$

Thanks to (8.16) and (8.18), this implies

$$\mathbb{E}\sup_{s\in[k\delta_\epsilon,t]}|\Lambda_{\epsilon,n}(s)|_E^2 \leq c_\eta\frac{1}{\epsilon}\int_{k\delta_\epsilon}^t \mathbb{E}\,|\hat{v}_{\epsilon,n}(s) - v_{\epsilon,n}(s)|_E^2\,ds$$

so that, thanks to (8.17), for $t \in [k\delta_\epsilon, (k+1)\delta_\epsilon]$,

$$\mathbb{E}\,|\hat{v}_{\epsilon,n}(t) - v_{\epsilon,n}(t)|_E^2 \leq c_\eta\left(1 + |x|_{C^\theta(\bar{D})}^{2m_2} + |y|_E^2\right)\,\delta_\epsilon^{\gamma(\theta)2} + \frac{c}{\epsilon}\int_{k\delta_\epsilon}^t \mathbb{E}\,|\hat{v}_{\epsilon,n}(s) - v_{\epsilon,n}(s)|_E^2\,ds.$$

From the Gronwall lemma, this gives

$$\mathbb{E}\,|\hat{v}_{\epsilon,n}(t) - v_{\epsilon,n}(t)|_E^2 \leq c_\eta\left(1 + |x|_{C^\theta(\bar{D})}^{2m_2} + |y|_E^2\right)\,\delta_\epsilon^{\gamma(\theta)2}\exp\left(\frac{c\,\delta_\epsilon}{\epsilon}\right).$$

Now, since

$$\exp\left(\frac{c\,\delta_\epsilon}{\epsilon}\right) = \exp\left(c\,\log\epsilon^{-\kappa}\right) = \epsilon^{-c\,\kappa},$$

we have

$$\delta_\epsilon^{\gamma(\theta)2}\exp\left(\frac{c\,\delta_\epsilon}{\epsilon}\right) = \delta_\epsilon^{\gamma(\theta)2}\epsilon^{-c\,\kappa} = \epsilon^{-c\,\kappa+2\gamma(\theta)}\left(\log\epsilon^{-\kappa}\right)^{2\gamma(\theta)}.$$

Hence, if we take $\kappa < 2\,\gamma(\theta)/c$, we have (8.15). ∎

Finally, we can prove (8.2). As in [10], we can show that, for any $n \in \mathbb{N}$,

$$\mathbb{E}\sup_{t\in[0,T]}|R_\epsilon(t)| \leq \mathbb{E}\left(\sup_{t\in[0,T]}|R_{\epsilon,n}(t)|\right) + \frac{c_T}{n}\left(1 + |x|_E^{2m_1} + |y|_E^2\right)|h|_E.$$

Therefore, due to the arbitrariness of $n \in \mathbb{N}$, (8.2) follows once we prove that, for any fixed $n \in \mathbb{N}$,

$$\lim_{\epsilon\to 0}\mathbb{E}\left(\sup_{t\in[0,T]}|R_{\epsilon,n}(t)|\right) = 0. \tag{8.19}$$

We have

(8.20)
$$\begin{aligned}
&\limsup_{\epsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \left\langle B_{1,n}(u_{\epsilon,n}(s),v_{\epsilon,n}(s))-\bar{B}_n(u_{\epsilon,n}(s)),h\right\rangle_H\,ds\right|\\
&\leq \limsup_{\epsilon\to 0}\mathbb{E}\int_0^T\left|\langle B_{1,n}(u_{\epsilon,n}(s),v_{\epsilon,n}(s))-B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s)),h\rangle_H\right|\,ds\\
&\quad+\limsup_{\epsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s))-\bar{B}_n(u_{\epsilon,n}(s)),h\right\rangle_H\,ds\right|.
\end{aligned}$$

As in the proofs of [9, Lemma 6.3] and [10, Lemma 6.2], we have

$$\begin{aligned}
&\mathbb{E}\int_0^T\left|\langle B_{1,n}(u_{\epsilon,n}(s),v_{\epsilon,n}(s))-B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s)),h\rangle_H\right|\,ds\\
&\leq c_{T,n}\,|h|_H\left(1+|x|^{(2\vee\theta)m_1}_{C^\theta(\bar{D})}+|y|^{2\vee\theta}_E\right)\left(\delta_\epsilon^{\gamma(\theta)}+\sup_{t\in[0,T]}\left(\mathbb{E}\,|v_{\epsilon,n}(t)-\hat{v}_{\epsilon,n}(t)|^2_E\right)^{\frac{1}{2}}\right).
\end{aligned}$$

Therefore, in view of Lemma 8.4, from (8.20),

$$\begin{aligned}
&\limsup_{\epsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \left\langle B_{1,n}(u_{\epsilon,n}(s),v_{\epsilon,n}(s))-\bar{B}_n(u_{\epsilon,n}(s)),h\right\rangle_H\,ds\right|\\
&=\limsup_{\epsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s))-\bar{B}_n(u_{\epsilon,n}(s)),h\right\rangle_H\,ds\right|.
\end{aligned}$$

Again, as in the proofs of [9, Lemma 6.3] and [10, Lemma 6.2], we have

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s))-\bar{B}_n(u_{\epsilon,n}(s)),h\right\rangle_H\,ds\right|\\
&\leq\sum_{k=0}^{[T/\delta_\epsilon]}\mathbb{E}\left|\int_{k\delta_\epsilon}^{(k+1)\delta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s))-\bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|\\
&\quad+\,c_{T,n}\,|h|_H\left(1+|x|^{m_1}_{C^\theta(\bar{D})}+|y|_E\right)[T/\delta_\epsilon]\,\delta_\epsilon^{\gamma(\theta)+1}
\end{aligned}$$

so that we have to show that

(8.21)
$$\lim_{\epsilon\to 0}\sum_{k=0}^{[T/\delta_\epsilon]}\mathbb{E}\left|\int_{k\delta_\epsilon}^{(k+1)\delta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s))-\bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|=0.$$

If we set $\zeta_\epsilon := \delta_\epsilon/\epsilon$, we have

$$\begin{aligned}
&\mathbb{E}\left|\int_{k\delta_\epsilon}^{(k+1)\delta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s))-\bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|\\
&=\mathbb{E}\left|\int_0^{\delta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(k\delta_\epsilon s))-\bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|\\
&=\mathbb{E}\left|\int_0^{\delta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\tilde{v}_n^{u_{\epsilon,n}(k\delta_\epsilon),v_{\epsilon,n}(k\delta_\epsilon)}(s/\epsilon)-\bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|\\
&=\delta_\epsilon\mathbb{E}\left|\frac{1}{\zeta_\epsilon}\int_0^{\zeta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\tilde{v}_n^{u_{\epsilon,n}(k\delta_\epsilon),v_{\epsilon,n}(k\delta_\epsilon)}(s)-\bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|,
\end{aligned}$$

where $\tilde{v}_n^{u_{\epsilon,n}(k\delta_\epsilon),v_{\epsilon,n}(k\delta_\epsilon)}(s)$ is the solution of the fast motion equation (5.1) with frozen slow component $u_{\epsilon,n}(k\delta_\epsilon)$ and initial datum $v_{\epsilon,n}(k\delta_\epsilon)$, and noise $\tilde{w}^{Q_2}$ independent of both of them. According to (8.9), (4.2), and (4.3), this yields

$$\begin{aligned}
&\mathbb{E}\left|\int_{k\delta_\epsilon}^{(k+1)\delta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s))-\bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|\\
&\leq\delta_\epsilon\frac{c}{\zeta_\epsilon}\left(1+\mathbb{E}\,|u_{\epsilon,n}(k\delta_\epsilon)|_E^{\kappa_1}+\mathbb{E}\,|v_{\epsilon,n}(k\delta_\epsilon)|_E^{\kappa_2}\right)|h|_1+\mathbb{E}\,\alpha(\zeta_\epsilon,u_{\epsilon,n}(k\delta_\epsilon)).
\end{aligned}$$

Now, the family

$$\{u_{\epsilon,n}(k\delta_\epsilon)\,:\,\epsilon>0,\ n\in\mathbb{N},\ k=0,\dots,[T/\delta_\epsilon]\}$$

is tight. Then, for any $\eta>0$, there exists a compact set $K_\eta\subset E$ such that

$$\mathbb{P}\left(u_{\epsilon,n}(k\delta_\epsilon)\in K_\eta^c\right)\leq\eta.$$

Therefore, due to (8.10), we have

$$\begin{aligned}
&\mathbb{E}\alpha(\zeta_\epsilon,u_{\epsilon,n}(k\delta_\epsilon))\\
&=\mathbb{E}\left(\alpha(\zeta_\epsilon,u_{\epsilon,n}(k\delta_\epsilon))\,;\,u_{\epsilon,n}(k\delta_\epsilon)\in K_\eta\right)+\mathbb{E}\left(c\,(1+|u_{\epsilon,n}(k\delta_\epsilon)|_E^{m_1})\,;\,u_{\epsilon,n}(k\delta_\epsilon)\in K_\eta^c\right)\\
&\leq\sup_{x\in K_\eta}\alpha(\zeta_\epsilon,x)+\sqrt{\eta}\,c\left(1+\left(\mathbb{E}\,|u_{\epsilon,n}(k\delta_\epsilon)|_E^{2m_1}\right)^{\frac{1}{2}}\right).
\end{aligned}$$

Thanks to (8.11), we can conclude that

$$\limsup_{\epsilon\to0}\mathbb{E}\,\alpha(\zeta_\epsilon,u_{\epsilon,n}(k\delta_\epsilon))\leq c\,(1+|x|_E^\kappa+|y|_E^\kappa)\,\sqrt{\eta},$$

for some $\kappa>0$, and, due to the arbitrariness of η, this implies (8.21).

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