SIAM J. MATH. ANAL.
Vol. 43, No. 6, pp. 2482–2518

AVERAGING PRINCIPLE FOR SYSTEMS OF REACTION-DIFFUSION EQUATIONS WITH POLYNOMIAL NONLINEARITIES PERTURBED BY MULTIPLICATIVE NOISE*

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Abstract. We prove the validity of an averaging principle for a class of systems of slow-fast reaction-diffusion equations with the reaction terms in both equations having polynomial growth, perturbed by a noise of multiplicative type. The models we have in mind are the stochastic Fitzhugh–Nagumo equation arising in neurophysiology and the Ginzburg–Landau equation arising in statistical mechanics.

Key words. stochastic reaction-diffusion equations, invariant measures, ergodic and strongly mixing processes, averaging principle

AMS subject classifications. 60H15, 34K33, 35K57, 37A25, 37L40

DOI. 10.1137/100806710

1. Introduction. In a series of recent papers ([8], written in collaboration with Freidlin, and [6] and [7]) we have studied the validity of an averaging principle and the normal deviations of the slow motion from the averaged motion for the following class of systems of stochastic partial differential equations of reaction-diffusion type on a bounded domain D of $\mathbb{R}^d$, with $d \geq 1$:

(1.1)
$$\left\{\begin{array}{l}
\dfrac{\partial u_\epsilon}{\partial t}(t,\xi) = \mathcal{A}_1 u_\epsilon(t,\xi) + b_1(\xi, u_\epsilon(t,\xi), v_\epsilon(t,\xi)) + g_1(\xi, u_\epsilon(t,\xi), v_\epsilon(t,\xi))\,\dfrac{\partial w^{Q_1}}{\partial t}(t,\xi),\\[2ex]
\dfrac{\partial v_\epsilon}{\partial t}(t,\xi) = \dfrac{1}{\epsilon}[(\mathcal{A}_2-\alpha)v_\epsilon(t,\xi)+b_2(\xi,u_\epsilon(t,\xi),v_\epsilon(t,\xi))]\\[2ex]
\qquad\qquad\qquad +\dfrac{1}{\sqrt{\epsilon}}\, g_2(\xi,u_\epsilon(t,\xi),v_\epsilon(t,\xi))\,\dfrac{\partial w^{Q_2}}{\partial t}(t,\xi),\\[2ex]
u_\epsilon(0,\xi) = x(\xi), \qquad v_\epsilon(0,\xi) = y(\xi), \qquad \xi \in D,\\[2ex]
\mathcal{N}_1 u_\epsilon(t,\xi) = \mathcal{N}_2 v_\epsilon(t,\xi) = 0, \qquad t \geq 0, \qquad \xi \in \partial D
\end{array}\right.$$

(here ϵ is a small positive parameter and α is a sufficiently large fixed constant). More precisely, we have proved that the slow motion u_ϵ is weakly convergent in $C([0,T];L^2(D))$, as $\epsilon \downarrow 0$, to the solution $\bar{u}$ of the so-called *averaged equation*, obtained by taking the average of the coefficients b_1 and g_1 with respect to the invariant measure of the fast equation, with frozen slow component. Moreover, we have studied the normalized difference $z_\epsilon := (u_\epsilon - \bar{u})/\sqrt{\epsilon}$ and have proved that it is weakly convergent in $C([0,T];L^2(D))$ to a process z, which is given in terms of a Gaussian processes whose covariance is explicitly described.

*Received by the editors August 27, 2010; accepted for publication July 14, 2011; published electronically November 15, 2011. This work was partially supported by NSF grant DMS-0907295, "Asymptotic Problems for SPDE's."
http://www.siam.org/journals/sima/43-6/80671.html

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In those articles, the stochastic perturbations are of multiplicative type and w^{Q_1} and w^{Q_2} are Gaussian noises, which are white in time and colored in space, in the case of space dimension $d > 1$, with covariance operators Q_1 and Q_2. $\mathcal{A}_1$ and $\mathcal{A}_2$ are second order uniformly elliptic operators endowed with the boundary conditions $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. The coefficients b_1, b_2 and g_1, g_2 are suitable real-valued functions defined on $D \times \mathbb{R}^2$ which are assumed to be Lipschitz continuous and in particular to have linear growth.

The Lipschitz-continuity assumption for the coefficients in system (1.1) has been crucial throughout the quoted papers [8], [6], and [7] for several different reasons.

First of all, the Lipschitz-continuity of coefficients implies that system (1.1) is well posed in the space $L^2(D)$, and hence we can work in the framework of square integrable functions, where, as a matter of fact, the use of stochastic analysis tools is simpler. In particular the proof of all a priori bounds for the slow and the fast motions u_ϵ and v_ϵ, which in any case requires a lot of work, is less complicated.

Second, and most importantly, if the coefficients of the slow and of the fast motion equations are Lipschitz-continuous, then under the assumption that the deterministic part of the fast motion is asymptotically stable and the noisy perturbation is not too big, it is possible to show that the averaged coefficients are Lipschitz-continuous and hence the averaged equation is well posed in $L^2(D)$.

Moreover, the Lipschitz-continuity of coefficients makes the proof of the convergence of u_ϵ to $\bar{u}$ easier, as we can apply in several situations arguments based on the use of Gronwall's lemma.

But on the other hand it is clear that the Lipschitz-continuity assumption for the coefficients in system (1.1) is too restrictive and does not allow us to study the validity of an averaging principle in many relevant situations in which, for example, the coefficients have polynomial growth. One such situation is represented by systems of reaction-diffusion equations of Fitzhugh–Nagumo or Ginzburg–Landau type, perturbed by a Gaussian noise. Such systems arise in many fields of biological and physical sciences and have attracted considerable attention. One important application in biology is the study of neurophysiology, with the Hodgkin–Huxley system, whereas in physics similar equations arise in statistical mechanics with the Ginzburg–Landau equation (see, e.g., [21] for a mathematical introduction).

Our purpose in this paper is to bridge such a gap and provide a way to study the averaging principle for a wide class of systems of two reaction-diffusion equations in a bounded domain D of $\mathbb{R}^d$, for any $d \geq 1$, perturbed by a multiplicative noise, in which the slow and the fast motions are both described by an equation having polynomially growing and monotone reaction terms b_1 and b_2 (for all details see Hypotheses 2 and 3 and section 2).

A simple example of the reaction coefficients we have in mind is given by

$$b_1(\xi, u, v) = f_1(\xi, p_1(\xi, u) + k_1(\xi, u, v))$$

and

$$b_2(\xi, u, v) = f_2(\xi, p_2(\xi, v) + k_2(\xi, u)).$$

Here $f_1, f_2 : \bar{D} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and for any $\xi \in \bar{D}$ the functions $f_1(\xi, \cdot)$ and $f_2(\xi, \cdot)$ are of class C^1, with nonnegative and uniformly bounded derivatives. The functions $p_1(\xi, \cdot)$ and $p_2(\xi, \cdot)$ are polynomials of odd degree, having the leading coefficient strictly negative, namely,

$$p_i(\xi, s) = -\gamma_i(\xi) s^{2k_i+1} + \sum_{j=1}^{2k_i} \gamma_{i,j}(\xi) s^j, \qquad i = 1, 2,$$

with

$$\inf_{\xi \in \bar{D}} \gamma_i(\xi) =: \gamma_i > 0.$$

Finally, the mappings $k_1 : \bar{D} \times \mathbb{R}^2 \to \mathbb{R}$ and $k_2 : \bar{D} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, such that $k_1(\xi, \cdot)$ and $k_2(\xi, \cdot)$ are locally Lipschitz-continuous and have linear growth, uniformly with respect to $\xi \in \bar{D}$.

Moreover, in the present paper the diffusion coefficients $g_1 : \bar{D} \times \mathbb{R} \to \mathbb{R}$ and $g_2 : \bar{D} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, with $g_1(\xi, \cdot)$ and $g_2(\xi, \cdot)$ Lipschitz-continuous, uniformly with respect to $\xi \in \bar{D}$, and, finally, α is a positive constant which has to be assumed large enough, depending on b_2 and g_2.

The key object in the analysis of averaging for system (1.1) is the fast motion equation with frozen slow component $x \in C(\bar{D})$:

$$(1.2) \qquad \begin{cases} \dfrac{\partial v^{x,y}}{\partial t}(t,\xi) = (\mathcal{A}_2 - \alpha) v^{x,y}(t,\xi) + b_2(\xi, x(\xi), v^{x,y}(t,\xi)) \\ \qquad\qquad + g_2(\xi, x(\xi), v^{x,y}(t,\xi)) \, \dfrac{\partial w^{Q_2}}{\partial t}(t,\xi), \\ v^{x,y}(0,\xi) = y(\xi), \qquad \xi \in D, \qquad \mathcal{N}_2 v^{x,y}(t,\xi) = 0, \qquad t \geq 0, \qquad \xi \in \partial D. \end{cases}$$

In section 4 we study problem (1.2) in $C(\bar{D})$ and introduce the corresponding transition semigroup P_t^x for any fixed $x \in C(\bar{D})$. We prove that, as a consequence of our assumptions, P_t^x admits an invariant measure μ^x, having all moments finite.

Once we have the existence of an invariant measure μ^x, the crucial point is proving that μ^x is the unique invariant measure, which ensures the proper convergence of $P_t^x \varphi$ to the mean of φ with respect to μ^x, as time t goes to infinity, for any φ belonging to a suitable class of functions defined on $C(\bar{D})$, with values in $\mathbb{R}$. Moreover, it is crucial to show that μ^x is nicely depending on x. To this purpose, we assume that the solution $v^{x,y}$ of problem (1.2) satisfies Hypotheses 5 and 6, and we provide the example of relevant situations in which these conditions are satisfied. In particular, we would like to stress that these results about the ergodic behavior of P_t^x are completely new, due to the lack of Lipschitz-continuity of the reaction coefficient b_2 and the presence of a noise of multiplicative type, and they can be of interest on their own, for their applications to other problems.

As we have good knowledge of the fast motion equation, the next step is introducing the averaged equation. To this purpose we notice that as the reaction coefficients do not have linear growth, system (1.1) cannot be studied in $L^2(D)$. Actually, the natural space in which to study the problem is $C(\bar{D})$, the space of continuous functions on $\bar{D}$. This means in particular that the averaged equation has to be studied in $C(\bar{D})$ and its coefficients have to be defined in $C(\bar{D})$. As b_1 is non-Lipschitz, the most delicate step is to introduce the averaged reaction coefficient. The obvious candidate is

$$(1.3) \qquad \bar{B}(x) = \int_{C(\bar{D})} B_1(x, z) \, \mu^x(dz), \qquad x \in C(\bar{D}),$$

where $B_1(x,z)(\xi) = b_1(\xi, x(\xi), z(\xi))$ for any $x, z \in C(\bar{D})$ and $\xi \in \bar{D}$. Clearly $\bar{B}$ is not Lipschitz-continuous, and for this reason we have to prove that it satisfies some monotonicity and dissipativity conditions which it inherits from b_1 and which ensure the well-posedness of the averaged equation

$$\left\{\begin{array}{l} \dfrac{\partial \bar{u}}{\partial t}(t,\xi) = \mathcal{A}_1 \bar{u}(t,\xi) + \bar{B}(\bar{u}(t))(\xi) + G(\bar{u}(t))(\xi))\,\dfrac{\partial w^{Q_1}}{\partial t}(t,\xi), \\[10pt] \bar{u}(0,\xi) = x(\xi), \qquad \xi \in D, \qquad \mathcal{N}_1 \bar{u}(t,\xi) = 0, \quad t \geq 0, \quad \xi \in \partial D, \end{array}\right. \tag{1.4}$$

in the space $C(\bar{D})$. To this purpose, it is important to stress that in this paper we show that the averaged coefficient $\bar{B}$, defined in (1.3) as a nonlocal operator, is in fact a local operator in $C(\bar{D})$, unlike the same operator defined in $L^2(D)$ (see [8] and [6]). Actually, due to the nice ergodic properties of (1.2), for any $x \in C(\bar{D})$ we obtain $\bar{B}(x)(\xi)$ at any fixed $\xi \in \bar{D}$, as the pointwise limit of suitable time averages.

Here, as in [8] and [6], our goal is proving that for any fixed $T > 0$ and $\eta > 0$ it holds that

$$\lim_{\epsilon \to 0} \mathbb{P}\left(|u_\epsilon - \bar{u}|_{C([0,T];C(\bar{D}))} > \eta\right) = 0.$$

The first step consists in proving that the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon \in (0,1]}$ is tight in $C([0,T];C(\bar{D}))$, and to this purpose we have to prove uniform bounds for u_ϵ, also in spaces of Hölder-continuous functions.

Once we have tightness, we have weak convergence in $C([0,T];C(\bar{D}))$ of a subsequence $\{u_{\epsilon_k}\}_{k \in \mathbb{N}}$ to some $\bar{u}$. Thus, we have to prove that $\bar{u}$ is the solution of the averaged equation and, as we have uniqueness for such an equation, we can conclude that the whole family $\{u_\epsilon\}$ converges to $\bar{u}$, as $\epsilon \downarrow 0$, and, as g_1 depends only on the slow motion, the convergence in fact is in probability.

In order to characterize the weak limit of the subsequence $\{u_{\epsilon_k}\}_{k \in \mathbb{N}}$ as the solution of the averaged equation, we have to prove that for any $h \in D(A_1)$

$$\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [0,T]} |R^h_\epsilon(t)|_{C([0,T];C(\bar{D}))} = 0, \tag{1.5}$$

where

$$\begin{aligned} R^h_\epsilon(t) &= \int_D u_\epsilon(t,\xi)h(\xi)\,d\xi - \int_D x(\xi)h(\xi)\,d\xi - \int_0^t \int_D u_\epsilon(s,\xi) A_1 h(\xi)\,d\xi\,ds \\ &- \int_0^t \int_D \bar{B}(u_\epsilon(s))(\xi)h(\xi)\,d\xi\,ds - \int_0^t \int_D [G_1(u_\epsilon(s)h](\xi)dw^{Q_2}(s,\xi) \\ &= \int_0^t \int_D \left(B_1(u_\epsilon(s), v_\epsilon(s))(\xi) - \bar{B}(u_\epsilon(s))(\xi)\right) h(\xi)\,d\xi\,ds. \end{aligned}$$

The proof of (1.5), as in [6], is based on the Khasminskii discretization in time, and one of the crucial steps is proving that

$$\lim_{\epsilon \to 0} \sup_{t \in [0,T]} \mathbb{E}|\hat{v}_\epsilon(t) - v_\epsilon(t)|^p_{C(\bar{D})} = 0, \tag{1.6}$$

where $\hat{v}_\epsilon$ is the solution of the problem

$$\left\{\begin{array}{l} dv(t) = \dfrac{1}{\epsilon}\left[(A_2 - \alpha)v(t) + B_2(u_\epsilon(k\delta_\epsilon), v(t))\right]\,dt + \dfrac{1}{\sqrt{\epsilon}} G_2(u_\epsilon(k\delta_\epsilon), v(t))\,dw^{Q_2}(t), \\[10pt] v(k\delta_\epsilon) = v_\epsilon(k\delta_\epsilon) \end{array}\right.$$

in each time interval $[k\delta_\epsilon, (k+1)\delta_\epsilon]$ for $k = 0, \ldots, [T/\delta_\epsilon]$ and for a suitable δ_ϵ (which has to be determined), converging to zero, as $\epsilon \downarrow 0$.

Due to the lack of Lipschitz-continuity of coefficients, the proof of (1.6) seems to be not possible, and hence we have to localize the coefficients. However, as we do not have uniform bounds with respect to $\epsilon > 0$ for the fast motion v_ϵ in $C([0,T]; C(\bar{D}))$, we cannot localize with respect to both variables u and v, but only with respect to u. This means that in the proof of (1.6) we have to use in a suitable way the dissipativity of b_2 with respect to v.

The averaging principle both for deterministically and for randomly perturbed systems, having a finite number of degrees of freedom, has been studied by many authors, under different assumptions and with different methods. The first rigorous results are due to Bogoliubov and Mitropolsky (see [2]). Further developments were obtained by Volosov, Anosov, and Neishtadt (see [19] and [23]) and by Arnold, Kozlov, and Neishtadt (see [1]). All of these references are for the deterministic case. Concerning the stochastic case, it is worth quoting the paper by Khasminskii [12], the works of Brin, Freidlin, and Wentcell (see [9], [10], [11]), Veretennikov (see [22]), and Kifer (see, for example, [13], [14], [15], [16]).

The case of systems with an infinite number of degrees of freedom is more open. Apart from a few papers dealing with averaging for infinite dimensional systems (to this purpose we refer the reader to the papers [20] by Seidler and Vrkoč and [18] by Maslowskii, Seidler, and Vrkoč, concerned with averaging for Hilbert space–valued solutions of stochastic evolution equations depending on a small parameter, and to the paper [17] by Kuksin and Piatnitski dealing with averaging for a randomly perturbed KdV equation), the behavior of solutions of infinite dimensional systems on time intervals of order ϵ^{-1} has been studied in few other papers apart from [8], [6], and [7].

Recently, Wang and Roberts in [24] studied averaging for stochastic partial differential equations (SPDEs) with non-Lipschitz coefficients. In their paper, unlike in the present paper, they have considered the particular situation of a system of two reaction-diffusion equations in an interval (so that $d = 1$), both perturbed by an additive noise, and it seems that they cannot deal with the case of white noise in space. Moreover, the only coefficient which is not Lipschitz-continuous is the reaction term in the slow equation and, due to some Sobolev embedding constraints which arise from the fact that they study the problem in L^2, its growth is dominated by a polynomial of degree 3.

2. Setup. Let D be a bounded domain of $\mathbb{R}^d$, with $d \geq 1$, having a regular boundary. Throughout the paper, we shall denote by H the separable Hilbert space $L^2(D)$, endowed with the scalar product

$$\langle x, y \rangle_H = \int_D x(\xi) y(\xi)\, d\xi$$

and with the corresponding norm $|\cdot|_H$. We shall endow the product space $H \times H$ with the scalar product

$$\langle x, y \rangle_{H\times H} = \int_D \langle x(\xi), y(\xi) \rangle_{\mathbb{R}^2}\, d\xi = \langle x_1, y_1 \rangle_H + \langle x_2, y_2 \rangle_H$$

and the corresponding norm $|\cdot|_{H\times H}$.

Next, we shall denote by E the Banach space $C(\bar{D})$, endowed with the sup-norm

$$|x|_E = \sup_{\xi \in \bar{D}} |x(\xi)|$$

and the duality $\langle \cdot,\cdot\rangle_E$. The product space $E\times E$ shall be endowed with the norm

$$|x|_{E\times E} = \left(|x_1|_E^2 + |x_2|_E^2 \right)^{\frac{1}{2}}$$

and the corresponding duality $\langle \cdot,\cdot\rangle_{E\times E}$. Finally, for any $\theta \in (0,1)$ we shall denote by $C^\theta(\bar D)$ the subspace of θ-Hölder-continuous functions, endowed with the norm

$$|x|_{C^\theta(\bar D)} = |x|_E + [x]_\theta = |x|_E + \sup_{\substack{\xi,\eta\in \bar D\\ \xi\neq\eta}} \frac{|x(\xi)-x(\eta)|}{|\xi-\eta|^\theta}.$$

For any $p\in [1,\infty]$, with $p\neq 2$, the norm in $L^p(D)$ and $L^p(D)\times L^p(D)$ will both be denoted by $|\cdot|_p$. If $\delta>0$, we will denote by $|\cdot|_{\delta,p}$ the norm in $W^{\delta,p}(D)$:

$$|x|_{\delta,p} := |x|_p + \int_D\int_D \frac{|x(\xi)-x(\eta)|^p}{|\xi-\eta|^{\delta p+d}}\,d\xi\,d\eta. \tag{2.1}$$

Now, we introduce some notation which we will use in what follows (for all details we refer the reader to, e.g., [3, Appendix A] and [4, section 2]). Let $x\in E$ and let $\xi_x\in \bar D$ such that $|x(\xi_x)| = |x|_E$. We denote by δ_x the element of $E^\star$ defined for any $y\in E$ by

$$\langle \delta_x, y\rangle_E := \begin{cases} \dfrac{x(\xi_x)y(\xi_x)}{|x|_E} & \text{if } x\neq 0,\\[2ex] \langle \delta, y\rangle_E & \text{if } x=0,\end{cases} \tag{2.2}$$

where δ is any element of $E^\star$ of norm 1. Notice that

$$\delta_x \in \partial\, |x|_E := \{\, h\in E^\star\,;\, |h|_{E^\star}=1,\ \langle h,x\rangle_E = |x|_E\,\}$$

and for any differentiable mapping $u:[0,T]\to E$

$$\frac{d}{dt}^{-} |u(t)|_E \le \left\langle u'(t), \delta_{u(t)}\right\rangle_E. \tag{2.3}$$

Analogously, if $x\in E\times E$, we denote by δ_x the element of $(E\times E)^\star$ defined for any $y\in E\times E$ by

$$\langle \delta_x, y\rangle_{E\times E} := \begin{cases} \dfrac{x_1(\xi_{x_1})y_1(\xi_{x_1}) + x_2(\xi_{x_2})y_2(\xi_{x_2})}{|x|_{E\times E}} & \text{if } x\neq 0,\\[2ex] \langle \delta, y\rangle_{E\times E} & \text{if } x=0,\end{cases} \tag{2.4}$$

where $\xi_{x_1},\xi_{x_2}\in \bar D$ are such that $|x_i(\xi_{x_i})| = |x_i|_E$ for $i=1,2$, and δ is any element of $(E\times E)^\star$ of norm 1. As above, we have

$$\delta_x \in \partial\,|x|_{E\times E} := \left\{ h\in (E\times E)^\star\,;\, |h|_{(E\times E)^\star}=1,\ \langle h,x\rangle_{E\times E} = |x|_E \right\},$$

and (2.3) holds true, with E replaced by $E\times E$.

Now, let X be any Banach space. We shall denote by $B_b(X)$ the space of bounded Borel functions $\varphi: X\to\mathbb{R}$. $B_b(X)$ is a Banach space, endowed with the sup-norm

$$\|\varphi\|_0 := \sup_{x\in X} |\varphi(x)|.$$

$C_b(X)$ shall be the subspace of uniformly continuous mappings. Moreover, we shall denote by $\mathcal{L}(X)$ the space of bounded linear operators on X and, in the case where X is a Hilbert space, we shall denote by $\mathcal{L}_2(X)$ the subspace of Hilbert–Schmidt operators, endowed with the norm

$$\|Q\|_2 = \sqrt{\mathrm{Tr}\,[Q^\star Q]}.$$

The stochastic perturbations in the slow and in the fast motion equations (1.1) are given, respectively, by the Gaussian noises $\partial w^{Q_1}/\partial t(t,\xi)$ and $\partial w^{Q_2}/\partial t(t,\xi)$, for $t \geq 0$ and $\xi \in D$, which are assumed to be white in time and colored in space, in the case of space dimension $d > 1$. Formally, the cylindrical Wiener processes $w^{Q_i}(t,\xi)$ are defined as the infinite sums

$$w^{Q_i}(t,\xi) = \sum_{k=1}^{\infty} Q_i e_k(\xi)\,\beta_k(t), \qquad i = 1,2,$$

where $\{e_k\}_{k\in\mathbb{N}}$ is a complete orthonormal basis in H, $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is a sequence of mutually independent standard Brownian motions defined on the same complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, and Q_i is a bounded linear operator on H.

The operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are second order uniformly elliptic operators, having continuous coefficients on D, and the boundary operators $\mathcal{N}_1$ and $\mathcal{N}_2$ can be either the identity operator (Dirichlet boundary condition) or a first order operator satisfying a uniform nontangentiality condition.

The realizations A_1 and A_2 in E of the differential operators $\mathcal{A}_1$ and $\mathcal{A}_2$, endowed with the boundary conditions $\mathcal{N}_1$ and $\mathcal{N}_2$, generate two analytic semigroups e^{tA_1} and e^{tA_2}, $t \geq 0$. As in [6], we assume that the operators A_1, A_2 and Q_1, Q_2 satisfy the following conditions.

HYPOTHESIS 1. *For $i = 1,2$ there exist a complete orthonormal system $\{e_{i,k}\}_{k\in\mathbb{N}}$ of H, which is contained in $L^\infty(D)$, and two sequences of nonnegative real numbers $\{\alpha_{i,k}\}_{k\in\mathbb{N}}$ and $\{\lambda_{i,k}\}_{k\in\mathbb{N}}$ such that*

$$A_i\, e_{i,k} = -\alpha_{i,k}\, e_{i,k}, \qquad Q_i e_{i,k} = \lambda_{i,k} e_{i,k}, \quad k \geq 1,$$

and

$$\kappa_i := \sum_{k=1}^{\infty} \lambda_{i,k}^{\rho_i}\, |e_{i,k}|_\infty^2 < \infty, \qquad \zeta_i := \sum_{k=1}^{\infty} \alpha_{i,k}^{-\beta_i}\, |e_{i,k}|_\infty^2 < \infty$$

for some constants $\rho_i \in (2,+\infty]$ and $\beta_i \in (0,+\infty)$ such that

$$\frac{\beta_i(\rho_i - 2)}{\rho_i} < 1. \tag{2.5}$$

For comments and examples concerning these assumptions on the operators A_i and Q_i, we refer the reader to [6, Remark 2.1].

For any $t, \delta > 0$ and $p \geq 1$ the semigroups e^{tA_i} map $L^p(D)$ into $W^{\delta,p}(D)$ with

$$|e^{tA_i}x|_{\delta,p} \leq c_i\,(t \wedge 1)^{-\frac{\delta}{2}} |x|_p, \qquad x \in L^p(D). \tag{2.6}$$

By using the Sobolev embedding theorem and the Riesz–Thorin theorem, this implies that the semigroups e^{tA_i} map $L^p(D)$ into $L^q(D)$, for any $1 \leq p \leq q \leq \infty$, and

$$|e^{tA_i}x|_q \leq c_i\,(t \wedge 1)^{-\frac{d(q-p)}{2pq}} |x|_p, \qquad x \in L^p(D). \tag{2.7}$$

Finally, as $W^{\delta,p}(D)$ embeds into $C^\theta(\bar{D})$, for any $\theta < \delta - d/p$, we get

$$|e^{tA_i}x|_{C^\theta(\bar{D})} \leq c_i\,(t\wedge 1)^{-\frac{\theta}{2}}|x|_E. \tag{2.8}$$

As far as the reaction coefficient $b_1 : \bar{D}\times\mathbb{R}^2 \to \mathbb{R}$ in the slow equation is concerned, we assume the following condition.

Hypothesis 2.

1. *The mapping* $b_1 : \bar{D}\times\mathbb{R}^2 \to \mathbb{R}$ *is continuous and there exists* $m_1 \geq 1$ *such that*

$$\sup_{\xi\in\bar{D}} |b_1(\xi,\sigma)| \leq c\,(1 + |\sigma_1|^{m_1} + |\sigma_2|)\,,\qquad \sigma = (\sigma_1,\sigma_2)\in\mathbb{R}^2. \tag{2.9}$$

2. *There exist* $c > 0$ *and* $\theta \geq 0$ *such that*

$$\sup_{\xi\in\bar{D}} |b_1(\xi,\sigma) - b_1(\xi,\rho)| \leq c\,\left(1 + |\sigma|^\theta + |\rho|^\theta\right)|\sigma-\rho|,\qquad \sigma,\rho\in\mathbb{R}^2. \tag{2.10}$$

3. *There exists* $c > 0$ *such that for any* $\sigma, h\in\mathbb{R}^2$

$$\sup_{\xi\in\bar{D}} \left(b_1(\xi,\sigma+h) - b_1(\xi,\sigma)\right)h_1 \leq c\,|h_1|\,(1 + |\sigma| + |h|)\,. \tag{2.11}$$

Example 2.1. Let $h : \bar{D}\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that $h(\xi,\cdot) : \mathbb{R}\to\mathbb{R}$ is locally Lipschitz-continuous, uniformly with respect to $\xi\in\bar{D}$. Moreover, assume that

$$\sup_{\xi\in\bar{D}} |h(\xi,s)| \leq c\,\left(1 + |s|^m\right),\qquad s\in\mathbb{R}, \tag{2.12}$$

and

$$h(\xi,s) - h(\xi,t) = \rho(\xi,s,t)(s-t),\qquad \xi\in\bar{D},\quad s,t\in\mathbb{R}, \tag{2.13}$$

for some $\rho : \bar{D}\times\mathbb{R}^2\to\mathbb{R}$ such that

$$\sup_{\substack{\xi\in\bar{D}\\ s,t\in\mathbb{R}}} \lambda(\xi,s,t) < \infty.$$

Moreover, let $k : \bar{D}\times\mathbb{R}^2\to\mathbb{R}$ be a continuous function, such that $k(\xi,\cdot) : \mathbb{R}^2\to\mathbb{R}$ has linear growth and is locally Lipschitz-continuous, uniformly with respect to $\xi\in\bar{D}$.

Now, we fix any continuous function $f : \bar{D}\times\mathbb{R}\to\mathbb{R}$ such that $f(\xi,\cdot)$ is of class C^1, for any $\xi\in\bar{D}$, and

$$0 \leq \frac{\partial f}{\partial s}(\xi,s) \leq c,\qquad (\xi,s)\in\bar{D}\times\mathbb{R},$$

for some $c > 0$. Then, if we define

$$b(\xi,\sigma) = f(\xi, h(\xi,\sigma_1) + k(\xi,\sigma_1,\sigma_2)),$$

it is not difficult to check that conditions 1 and 3 in Hypothesis 2 are all satisfied. Moreover, if we assume that h and k are differentiable and their derivatives have polynomial growth, then condition 2 is satisfied.

Next, let γ and γ_i be continuous functions from $\bar{D}$ into $\mathbb{R}$, for $i = 1, \ldots, 2k$, with

$$\inf_{\xi \in \bar{D}} \gamma(\xi) > 0.$$

Then it is possible to check that the function

$$h(\xi, s) := -\gamma(\xi) s^{2k+1} + \sum_{i=1}^{2k} \gamma_i(\xi) s^i$$

satisfies conditions (2.12) and (2.13).

For the reaction term $b_2 : \bar{D} \times \mathbb{R}^2 \to \mathbb{R}$ in the fast equation, we assume the following conditions.

HYPOTHESIS 3.

1. *The mapping* $b_2 : \bar{D} \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and there exists* $m_2 \geq 1$ *such that*

$$\sup_{\xi \in \bar{D}} |b_2(\xi, \sigma)| \leq c \left(1 + |\sigma_1| + |\sigma_2|^{m_2}\right), \qquad \sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2. \tag{2.14}$$

2. *The mapping* $b_2(\xi, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ *is locally Lipschitz continuous, uniformly with respect to* $\xi \in \bar{D}$. *Moreover, for any* $R > 0$ *there exists* $L_R > 0$ *such that*
(2.15)
$$|\sigma_1| \leq R,\ |\rho_1| \leq R \Longrightarrow \sup_{\substack{\xi \in \bar{D} \\ \sigma_2 \in \mathbb{R}}} |b_2(\xi, \sigma_1, \sigma_2) - b_2(\xi, \rho_1, \sigma_2)| \leq L_R \, |\sigma_1 - \rho_1|.$$

3. *There exists* $c > 0$ *such that for any* $\sigma, h \in \mathbb{R}^2$

$$\sup_{\xi \in \bar{D}} \left(b_2(\xi, \sigma + h) - b_2(\xi, \sigma)\right) h_2 \leq c \, |h_2| \left(1 + |\sigma| + |h|\right). \tag{2.16}$$

4. *For any* $\sigma_1, \sigma_2, \rho_2 \in \mathbb{R}$ *we have*

$$b_2(\xi, \sigma_1, \sigma_2) - b_2(\xi, \sigma_1, \rho_2) = -\lambda(\xi, \sigma_1, \sigma_2, \rho_2)(\sigma_2 - \rho_2) \tag{2.17}$$

for some continuous function $\lambda : \bar{D} \times \mathbb{R}^3 \to [0, +\infty)$.

Example 2.2. Let $h : \bar{D} \times \mathbb{R} \to \mathbb{R}$ be as in Example 2.1. Moreover, assume that the function ρ in (2.13) satisfies

$$\sup_{\substack{\xi \in \bar{D} \\ s, t \in \mathbb{R}}} \rho(\xi, t, s) \leq 0, \tag{2.18}$$

and the mapping k satisfies

$$\sup_{\substack{\xi \in \bar{D} \\ s \in \mathbb{R}}} \left(k(\xi, s, t_1) - k(\xi, s, t_2)\right)(t_1 - t_2) \leq 0. \tag{2.19}$$

Then conditions 1, 2, and 3 in Hypothesis 3 are all satisfied by $b_2(\xi, \sigma) = f_2(\xi, h(\xi, \sigma_2) + k(\xi, \sigma))$. Notice that (2.18) holds for

$$h(\xi, s) = -\gamma(\xi) s^{2k+1} + \sum_{j=1}^{2k} \gamma_j(\xi) s^j - \lambda s$$

for λ large enough.

Concerning the diffusion coefficients g_1 and g_2 we assume they satisfy the following conditions.

HYPOTHESIS 4.

1. *The mappings* $g_1 : \bar{D} \times \mathbb{R} \to \mathbb{R}$ *and* $g_2 : \bar{D} \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous and the mappings* $g_1(\xi, \cdot) : \mathbb{R} \to \mathbb{R}$ *and* $g_2(\xi, \cdot) : \bar{D} \times \mathbb{R}^2 \to \mathbb{R}$ *are Lipschitz-continuous, uniformly with respect to* $\xi \in \bar{D}$.
2. *It holds that*

$$\sup_{\xi \in \bar{D}} |g_1(\xi, \sigma_1)| \leq c \left(1 + |\sigma_1|^{\frac{1}{m_1}}\right) \tag{2.20}$$

and

$$\sup_{\substack{\xi \in \bar{D} \\ \sigma_1 \in \mathbb{R}}} |g_2(\xi, \sigma_1, \sigma_2)| \leq c \left(1 + |\sigma_2|^{\frac{1}{m_2}}\right). \tag{2.21}$$

In what follows, for any $x, y \in E$ and for $i = 1, 2$ we shall set

$$B_i(x, y)(\xi) := b_i(\xi, x(\xi), y(\xi)), \qquad \xi \in \bar{D},$$

and

$$B := (B_1, B_2).$$

Due to Hypothesis 2, the mappings B_1 and B_2 are well defined and continuous from $E \times E$ into E, so that $B : E \times E \to E \times E$ is well defined and continuous. As the mappings b_1 and b_2 have polynomial growth, B is not well defined in $H \times H$.

In view of (2.9) and (2.14), for any $x, y \in E$ we have

$$|B_1(x, y)|_E \leq c\,(1 + |x|_E^{m_1} + |y|_E), \quad |B_2(x, y)|_E \leq c\,(1 + |x|_E + |y|_E^{m_2}), \tag{2.22}$$

so that, in particular,

$$|B(x, y)|_{E \times E} \leq c\,(1 + |x|_E^{m_1} + |y|_E^{m_2}), \quad x, y \in E. \tag{2.23}$$

As a consequence of (2.11) and (2.16), it is immediate to check that for any $x, y, h, k \in E$

$$\langle B_i(x + h, y + k) - B_i(x, y), \delta_h \rangle_E \leq c\,(1 + |h|_E + |k|_E + |x|_E + |y|_E), \qquad i = 1, 2, \tag{2.24}$$

so that for any $(x, y), (h, k) \in E \times E$

$$\left\langle B(x + h, y + k) - B(x, y), \delta_{(h,k)} \right\rangle_{E \times E} \leq c\,(1 + |(h, k)|_{E \times E} + |(x, y)|_{E \times E}). \tag{2.25}$$

Moreover, from (2.17) we have

$$\langle B_2(x, y + k) - B_2(x, y), \delta_k \rangle_E \leq 0. \tag{2.26}$$

Finally, in view of (2.10) we have

$$|B_1(x_1, y_1) - B_1(x_2, y_2)|_E \leq c\,\left(1 + |(x_1, y_1)|^{\theta}_{E \times E} + |(x_2, y_2)|^{\theta}_{E \times E}\right)(|x_1 - x_2|_E + |y_1 - y_2|_E). \tag{2.27}$$

Next, for any $x, y, z \in E$ we define

$$[G_1(x)z](\xi) = g_1(\xi, x(\xi))z(\xi), \qquad [G_2(x, y)z](\xi) := g_2(\xi, x(\xi), y(\xi))z(\xi), \qquad \xi \in \bar{D}.$$

Due to Hypothesis 4, we have that the mappings

$$x \in E \mapsto G_1(x) \in \mathcal{L}(E)$$

and

$$(x,y) \in E \times E \mapsto G_2(x,y) \in \mathcal{L}(E)$$

are Lipschitz-continuous for $i = 1, 2$, so that the same is true for the mapping $G = (G_1, G_2)$ defined on $E \times E$ with values in $\mathcal{L}(E \times E)$.

3. Solvability and a priori bounds for the slow-fast system. With the notation introduced in section 2, system (1.1) can be rewritten in the following abstract form:

$$\left\{\begin{array}{l} du_\epsilon(t) = [A_1 u_\epsilon(t) + B_1(u_\epsilon(t), v_\epsilon(t))]\, dt + G_1(u_\epsilon(t))\, dw^{Q_1}(t), \\[10pt] dv_\epsilon(t) = \dfrac{1}{\epsilon}\,[(A_2 - \alpha)v_\epsilon(t) + B_2(u_\epsilon(t), v_\epsilon(t))]\, dt + \dfrac{1}{\sqrt{\epsilon}}\, G_2(u_\epsilon(t), v_\epsilon(t))\, dw^{Q_2}(t), \end{array}\right. \tag{3.1}$$

with initial conditions $u_\epsilon(0) = x \in E$ and $v_\epsilon(0) = y \in E$. We are here concerned with the solvability of the system above in $L^p(\Omega; C((0,T]; E \times E) \cap L^\infty(0,T; E \times E))$ and with uniform bounds for its solution.

As proved in [4, Theorem 5.3], under Hypotheses 1 to 4 for any $\epsilon > 0$ and $x, y \in E$ there exists a unique adapted mild solution to problem (3.1) in $L^p(\Omega; C((0,T]; E \times E) \cap L^\infty(0,T; E \times E))$, with $T > 0$ and $p \geq 1$. This means that there exist two adapted processes u_ϵ and v_ϵ in $L^p(\Omega; C((0,T]; E) \cap L^\infty(0,T; E))$ such that

$$u_\epsilon(t) = e^{tA_1}x + \int_0^t e^{(t-s)A_1} B_1(u_\epsilon(s), v_\epsilon(s))\, ds + \int_0^t e^{(t-s)A_1} G_1(u_\epsilon(s))\, dw^{Q_1}(s)$$

and

$$v_\epsilon(t) = e^{t\frac{(A_2-\alpha)}{\epsilon}} y + \frac{1}{\epsilon}\int_0^t e^{(t-s)\frac{(A_2-\alpha)}{\epsilon}} B_2(u_\epsilon(s), v_\epsilon(s))\, ds$$

$$+ \frac{1}{\sqrt{\epsilon}}\int_0^t e^{(t-s)\frac{(A_2-\alpha)}{\epsilon}} G_2(u_\epsilon(s), v_\epsilon(s))\, dw^{Q_2}(s),$$

and such processes are unique.

LEMMA 3.1. *Under Hypotheses 1 to 4, for any $p \geq 1$ and $T > 0$ there exists a constant $c_{p,T} > 0$ such that for any $x, y \in E$ and $\epsilon \in (0,1]$*

$$\mathbb{E} \sup_{t \in [0,T]} |u_\epsilon(t)|_E^p \leq c_{p,T}\, (1 + |x|_E^p + |y|_E^p) \tag{3.2}$$

and

$$\mathbb{E} \int_0^T |v_\epsilon(t)|_E^p\, dt \leq c_{p,T}\, (1 + |x|_E^p + |y|_E^p)\,. \tag{3.3}$$

Proof. Let $\epsilon \in (0,1]$ be fixed. We denote

$$\Gamma_{1,\epsilon}(t) := \int_0^t e^{(t-s)A_1} G_1(u_\epsilon(s))\, dw^{Q_1}(s), \qquad t \in [0,T],$$

and we set $\Lambda_{1,\epsilon}(t) := u_\epsilon(t) - \Gamma_{1,\epsilon}(t)$. We have

$$\frac{d}{dt}\Lambda_{1,\epsilon}(t) = A_1\Lambda_{1,\epsilon}(t) + B_1(\Lambda_{1,\epsilon}(t) + \Gamma_{1,\epsilon}(t), v_\epsilon(t)), \qquad \Lambda_{1,\epsilon}(0) = x,$$

and then, according to (2.22) and (2.24),

$$\begin{aligned}
&\frac{d}{dt}^{-} |\Lambda_{1,\epsilon}(t)|_E \leq \left\langle A_1\Lambda_{1,\epsilon}(t), \delta_{\Lambda_{1,\epsilon}(t)}\right\rangle_E \\
&+ \left\langle B_1(\Lambda_{1,\epsilon}(t)+\Gamma_{1,\epsilon}(t), v_\epsilon(t)) - B_1(\Gamma_{1,\epsilon}(t), v_\epsilon(t)), \delta_{\Lambda_{1,\epsilon}(t)}\right\rangle_E + \left\langle B_1(\Gamma_{1,\epsilon}(t), v_\epsilon(t)), \delta_{\Lambda_{1,\epsilon}(t)}\right\rangle_E \\
&\leq c\,|\Lambda_{1,\epsilon}(t)|_E + c\,(1 + |\Gamma_{1,\epsilon}(t)|_E + |v_\epsilon(t)|_E) + c\,|B_1(\Gamma_{1,\epsilon}(t), v_\epsilon(t))|_E \\
&\leq c\,|\Lambda_{1,\epsilon}(t)|_E + c\,(1 + |\Gamma_{1,\epsilon}(t)|_E^{m_1} + |v_\epsilon(t)|_E).
\end{aligned}$$

By comparison, this implies

$$\begin{aligned}
&|u_\epsilon(t)|_E \leq |\Lambda_{1,\epsilon}(t)|_E + |\Gamma_{1,\epsilon}(t)|_E \\
&\leq c(t)\,|x|_E + c(t) \sup_{s\in[0,t]} |\Gamma_{1,\epsilon}(s)|_E^{m_1} + c(t)\int_0^t (1 + |v_\epsilon(s)|_E)\, ds,
\end{aligned}$$

so that for any $p \geq 1$ we obtain

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} |u_\epsilon(s)|_E^p \\
&\leq c_p(t)\,(1 + |x|_E^p) + c_p(t)\,\mathbb{E} \sup_{s\in[0,t]} |\Gamma_{1,\epsilon}(s)|_E^{pm_1} + c_p(t)\int_0^t \mathbb{E}\,|v_\epsilon(r)|_E^p\, ds.
\end{aligned}$$

As we are assuming (2.20), by proceeding as in [4, proof of Theorem 4.2, and Remark 4.3] and as in [6, Lemma 4.1] it is possible to show that for any T and $q \geq 1$

$$\mathbb{E} \sup_{t\in[0,T]} |\Gamma_{1,\epsilon}(t)|_E^q \leq c_q(T)\left(1 + \int_0^T \mathbb{E}\,|u_\epsilon(t)|_E^{\frac{q}{m_1}}\, dt\right),$$

for some continuous increasing function $c_q(t)$ which is clearly independent of ϵ and vanishes at $t = 0$. Hence, we have

$$\mathbb{E} \sup_{s\in[0,t]} |u_\epsilon(s)|_E^p \leq c_p(t)\left(1 + |x|_E^p + \int_0^t \mathbb{E}\,|v_\epsilon(s)|_E^p\, ds\right) + c_p(t)\int_0^t \mathbb{E}\,|u_\epsilon(s)|_E^p\, ds. \tag{3.4}$$

Thus, in order to conclude the proof of (3.2) we have to estimate

$$\int_0^t \mathbb{E}\,|v_\epsilon(s)|_E^p\, ds.$$

As before, if we define

$$\Gamma_{2,\epsilon}(t) = \frac{1}{\sqrt{\epsilon}}\int_0^t e^{(t-s)\frac{(A_2-\alpha)}{\epsilon}} G_2(u_\epsilon(s), v_\epsilon(s))\, dw^{Q_2}(s)$$

and set $\Lambda_{2,\epsilon}(t) := v_\epsilon(t) - \Gamma_{2,\epsilon}(t)$, it follows that

$$\frac{d}{dt}\Lambda_{2,\epsilon}(t) = \frac{1}{\epsilon}\left[(A_2 - \alpha)\Lambda_{2,\epsilon}(t) + B_2(u_\epsilon(t), \Lambda_{2,\epsilon}(t) + \Gamma_{2,\epsilon}(t))\right], \qquad \Lambda_{2,\epsilon}(0) = y.$$

Thanks to (2.22) and (2.26), for any $p \geq 1$ we have

$$\frac{1}{p}\frac{d}{dt}^{-}|\Lambda_{2,\epsilon}(t)|_E^p \leq \frac{1}{\epsilon}\left\langle (A_2-\alpha)\Lambda_{2,\epsilon}(t), \delta_{\Lambda_{2,\epsilon}(t)}\right\rangle_E |\Lambda_{2,\epsilon}(t)|_E^{p-1}$$

$$+\frac{1}{\epsilon}\left\langle B_2(u_\epsilon(t), \Lambda_{2,\epsilon}(t)+\Gamma_{2,\epsilon}(t)) - B_2(u_\epsilon(t),\Gamma_{2,\epsilon}(t)), \delta_{\Lambda_{2,\epsilon}(t)}\right\rangle_E |\Lambda_{2,\epsilon}(t)|_E^{p-1}$$

$$+\frac{1}{\epsilon}\left\langle B_2(u_\epsilon(t),\Gamma_{2,\epsilon}(t)), \delta_{\Lambda_{2,\epsilon}(t)}\right\rangle_E |\Lambda_{2,\epsilon}(t)|_E^{p-1}$$

$$\leq -\frac{\alpha}{2\epsilon}|\Lambda_{2,\epsilon}(t)|_E^p + \frac{c_p}{\epsilon}\left(1+|\Gamma_{2,\epsilon}(t)|_E^{m_2 p} + |u_\epsilon(t)|_E^p\right).$$

Integrating both sides in time, it follows that

$$(3.5)\qquad \begin{aligned} |\Lambda_{2,\epsilon}(t)|_E^p &\leq |y|_E^p - \frac{c_p(t)}{\epsilon}\int_0^t |\Lambda_{2,\epsilon}(s)|_E^p\,ds + \frac{c_p(t)}{\epsilon}\left(1+\sup_{s\in[0,t]}|u_\epsilon(s)|_E^p\right)\\ &+\frac{c_p}{\epsilon}\int_0^t |\Gamma_{2,\epsilon}(s)|_E^{m_2 p}\,ds. \end{aligned}$$

Thus, by comparison this yields

$$\int_0^t |\Lambda_{2,\epsilon}(s)|_E^p\,ds \leq c_p(t)\,|y|_E^p + c_p(t)\left(1+\sup_{s\in[0,t]}|u_\epsilon(s)|_E^p\right)$$

$$+\,c_p\int_0^t |\Gamma_{2,\epsilon}(s)|_E^{m_2 p}\,ds,$$

so that

$$\int_0^t \mathbb{E}\,|v_\epsilon(s)|_E^p\,ds \leq c_p\int_0^t \mathbb{E}\,|\Lambda_{2,\epsilon}(s)|_E^p\,ds + c_p\int_0^t \mathbb{E}|\Gamma_{2,\epsilon}(s)|_E^p\,ds$$

$$\leq c_p(t)\,|y|_E^p + c_p(t)\left(1+\mathbb{E}\sup_{s\in[0,t]}|u_\epsilon(s)|_E^p\right) + c_p\int_0^t \mathbb{E}\,|\Gamma_{2,\epsilon}(s)|_E^{m_2 p}\,ds.$$

By using a factorization argument and by proceeding as in [6, proof of Proposition 4.2], due to (2.5) and (2.21) it is possible to prove that for any $k \geq 1$

$$\sup_{\epsilon\in(0,1]}\int_0^t \mathbb{E}|\Gamma_{2,\epsilon}(s)|_E^k\,ds \leq c_k(t)\int_0^t\left(1+\mathbb{E}\,|v_\epsilon(s)|_E^{\frac{k}{m_2}}\right)ds$$

for some continuous increasing function $c_k(t)$ such that $c_k(0)=0$. Then we have

$$\int_0^t \mathbb{E}\,|v_\epsilon(s)|_E^p\,ds \leq c_p(t)\,\left(|y|_E^p+1\right) + c_p(t)\,\mathbb{E}\sup_{s\in[0,t]}|u_\epsilon(s)|_E^p + c_{m_2 p}(t)\int_0^t \mathbb{E}\,|v_\epsilon(s)|_E^p\,ds.$$

As $c_{m_2p}(0)=0$, we can fix $t_0>0$ such that $c_{m_2p}(t)\leq 1/2$, for any $t\leq t_0$, so that

$$(3.6)\qquad \int_0^t \mathbb{E}\,|v_\epsilon(s)|_E^p\,ds \leq 2\,c_p(t)\,\left(|y|_E^p+1\right) + 2\,c_p(t)\,\mathbb{E}\sup_{s\in[0,t]}|u_\epsilon(s)|_E^p,\quad t\leq t_0.$$

If we plug the inequality above into (3.4), for any $\epsilon \in (0,1]$ we easily get

$$\mathbb{E} \sup_{s\in [0,t]} |u_\epsilon(s)|_E^p \leq c_p(t)\,(1+|x|_E^p+|y|_E^p) + c_p(t)\,\mathbb{E} \sup_{s\in [0,t]} |u_\epsilon(t)|_E^p, \qquad t \leq t_0.$$

Now, from the computations above, it is immediate to check that the function $c_p(t)$ is continuous and vanishes at $t=0$, so that we can fix $0 < t_1 \leq t_0$ such that $c_p(t_1) \leq 1/2$. This implies

$$\mathbb{E} \sup_{s\in [0,t_1]} |u_\epsilon(s)|_E^p \leq 2\,c_p(t_1)\,(1+|x|_E^p+|y|_E^p)\,.$$

As the same argument can be repeated in the intervals $[t_1, 2t_1]$, $[2t_1, 3t_1]$, etc., we can conclude that (3.2) is true.

Finally, if we plug (3.2) into (3.6), we immediately get (3.3). ∎

Next, we show that the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon\in (0,1]}$ is tight in $C([0,T];E)$. To this purpose, in order to use the Ascoli–Arzelà theorem, we need a uniform bound for u_ϵ in $L^\infty(0,T;C^\theta(\bar{D}))$, for some $\theta > 0$, and uniform bounds for the increments of the mapping $t \in [0,T] \mapsto u_\epsilon(t) \in E$.

Proposition 3.2. *Under Hypotheses* 1 *to* 4, *there exists* $\bar{\theta} > 0$ *such that for any* $\theta \in [0,\bar{\theta})$, $x \in C^\theta(\bar{D})$, $y \in E$, *and* $T > 0$

$$\sup_{\epsilon\in (0,1]} \mathbb{E}\,|u_\epsilon|_{L^\infty(0,T;C^\theta(\bar{\theta}))} \leq c_T\left(1+|x|_{C^\theta(\bar{D})}+|y|_E\right). \tag{3.7}$$

Proof. By proceeding as in [4, Proposition 4.5 and Remark 4.6] we have that there exist $\theta_1 > 0$ and $\bar{p} \geq 1$ such that for any $T > 0$ and $p \geq \bar{p}$

$$\mathbb{E}\,|\Gamma_{1,\epsilon}|^p_{L^\infty(0,T;C^{\theta_1}(\bar{D}))} \leq c_p(T)\left(1+\mathbb{E}\,|u_\epsilon|^p_{L^\infty(0,T;E)}\right). \tag{3.8}$$

Next, according to (2.8) and (2.22), if $\theta < 1$, we have

$$\left|\int_0^t e^{(t-s)A_1} B_1(u_\epsilon(s),v_\epsilon(s))\,ds\right|_{C^\theta(\bar{D})} \leq c\int_0^t (t-s)^{-\frac{\theta}{2}}\,(1+|u_\epsilon(s)|_E^{m_1}+|v_\epsilon(s)|_E)\,ds$$
$$\leq c(t)\left(1+\sup_{t\in [0,T]} |u_\epsilon(s)|_E^{m_1}\right)+\left(\int_0^t s^{-\theta}\,ds\right)^{\frac{1}{2}}\left(\int_0^t |v_\epsilon(s)|_E^2\,ds\right)^{\frac{1}{2}},$$

and, according to (3.2) and (3.3), this implies that for any $p \geq 2$

$$\mathbb{E} \sup_{t\in [0,T]} \left|\int_0^t e^{(t-s)A_1} B_1(u_\epsilon(s),v_\epsilon(s))\,ds\right|^p_{C^\theta(\bar{D})}$$
$$\leq c_p(T)\left(1+\mathbb{E} \sup_{t\in [0,T]} |u_\epsilon(s)|_E^{pm_1}\right)+c_p(T)\int_0^T \mathbb{E}\,|v_\epsilon(s)|_E^p\,ds \leq c_p(T)\,(1+|x|_E^{pm_1}+|y|_E^p)\,. \tag{3.9}$$

Moreover, if we assume that $x \in C^\theta(\bar{D})$, we have

$$|e^{tA_1}x|_{C^\theta(\bar{D})} \leq c\,|x|_{C^\theta(\bar{D})}.$$

Thus, thanks to (3.8) and (3.9), for any $\theta \leq \theta_1 \wedge 1$ and $\xi,\eta \in \bar{D}$ we get

$$\sup_{\epsilon\in (0,1]} \mathbb{E} \sup_{t\in [0,T]} |u_\epsilon(t,\xi)-u_\epsilon(t,\eta)|^p \leq c_p(T)\left(1+|x|^{pm_1}_{C^\theta(\bar{D})}+|y|_E^p\right)|\xi-\eta|^{p\theta},$$

so that, due to the characterization of the space $W^{\delta,p}(D)$ given in (2.1), we have that for any $\delta < \theta$

$$\sup_{\epsilon\in\,(0,1]} \mathbb{E} \sup_{t\in\,[0,T]} |u_\epsilon(t)|_{\delta,p} \leq c_p(T) \left(1 + |x|^{m_1}_{C^\theta(\bar{D})} + |y|_E\right).$$

As $W^{\delta,p}(D)$ embeds into $C^{\bar{\theta}}(\bar{D})$, for p large enough, with $\bar{\theta} = \theta_1 \wedge 1$, we get (3.7). □

Proposition 3.3. *Assume Hypotheses* 1 *to* 4. *Then for any $\theta > 0$ there exists $\gamma(\theta) > 0$ such that for any $T > 0$, $p \geq 2$, $x \in C^\theta(\bar{D})$, $y \in E$, and $t, s \in [0,T]$ we have*

$$\sup_{\epsilon\in\,(0,1]} \mathbb{E}\,|u_\epsilon(t) - u_\epsilon(s)|^p_E \leq c_p(T) \left(1 + |x|^{pm_1}_{C^\theta(\bar{D})} + |y|^p_E\right) |t-s|^{\gamma(\theta)p}. \tag{3.10}$$

In view of estimates (3.2) and (3.3), the proof of the proposition above turns out to be analogous to the proof of Proposition 4.4 in [6], and we do not repeat it.

Now, according to Proposition 3.2 and 3.3, thanks to the Ascoli–Arzelà theorem and the Garcia–Rademich–Rumsey theorem we can conclude that the following result holds true (for more details, see, e.g., [6, Corollary 4.5]).

Theorem 3.4. *Under Hypotheses* 1 *to* 4, *for any $x \in C^\theta(\bar{D})$, with $\theta > 0$ and $y \in E$, the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon\in\,(0,1]}$ is tight in $C([0,T];E)$.*

4. The fast equation. For any frozen slow component $x \in E$ and any initial datum $y \in E$, we introduce the problem

$$dv(t) = [(A_2 - \alpha)v(t) + B_2(x, v(t))]\,dt + G_2(x, v(t))\,dw^{Q_2}(t), \qquad v(0) = y. \tag{4.1}$$

According to Hypotheses 1, 3, and 4, equation (4.1) is well defined in E. That is, for any $x, y \in E$ there exists a unique mild solution $v^{x,y} \in L^p(\Omega; C((0,T];E) \cap L^\infty(0,T;E))$, with $p \geq 1$ and $T > 0$ (for a proof, see, e.g., [4]). This allows us to introduce the transition semigroup P^x_t associated with (4.1), which is defined by

$$P^x_t \varphi(y) = \mathbb{E}\,\varphi(v^{x,y}(t)), \qquad t \geq 0, \quad y \in E,$$

for any $\varphi \in B_b(E)$.

For any $\lambda > 0$, (4.1) can be rewritten as

$$dv(t) = [(A_2 - \lambda)v(t) + B_{2,\lambda}(x, v(t))]\,dt + G_2(x, v(t))\,dw^{Q_2}(t), \qquad v(0) = y,$$

where

$$B_{2,\lambda}(x,y) = B_2(x,y) + (\lambda - \alpha)\,y.$$

In what follows, for any $x \in E$ and $u \in L^p(\Omega; C_b((0,T];E))$ we shall set

$$\Gamma^x_\lambda(u)(t) = \int_0^t e^{(t-s)(A_2-\lambda)} G_2(x, u(s))\,dw^{Q_2}(s) \tag{4.2}$$

and

$$\Gamma^x(u)(t) = \int_0^t e^{(t-s)A_2} G_2(x, u(s))\,dw^{Q_2}(s).$$

As proved in [5, Lemma 3.1], there exists $\bar{p} > 1$ such that for any $t > 0$, $p \geq \bar{p}$, and $0 < \delta < \lambda$ and for any $u, v \in L^p(\Omega; C_b((0,T]; E))$

$$\sup_{s\in[0,t]} e^{\delta s p}\mathbb{E}\,|\Gamma^x_\lambda(u)(s) - \Gamma^x_\lambda(v)(s)|^p_E \leq c_{p,1}\frac{L^p_{g_2}}{(\lambda-\delta)^{c_{p,2}}}\sup_{s\in[0,t]} e^{\delta s p}\,\mathbb{E}\,|u(s)-v(s)|^p_E, \tag{4.3}$$

where L_{g_2} is the Lipschitz constant of g_2 and $c_{p,1}, c_{p,2}$ are two suitable positive constants. Moreover, according to (2.21), we have

$$\sup_{s\in[0,t]} e^{\delta s p}\mathbb{E}\,|\Gamma^x_\lambda(u)(s)|^p_E \leq c_{p,1}\frac{M^p_{g_2}}{(\lambda-\delta)^{c_{p,2}}}\sup_{s\in[0,t]} e^{\delta s p}\left(1+\mathbb{E}\,|u(s)|_E^{\frac{p}{m_2}}\right), \tag{4.4}$$

where

$$M_{g_2} = \sup_{\xi\in\bar{D},\ \sigma\in\mathbb{R}^2}\frac{|g_2(\xi,\sigma)|}{1+|\sigma_2|^{\frac{1}{m_2}}}$$

(see [5, Remark 3.2]).

PROPOSITION 4.1. *Assume Hypotheses* 1, 3, *and* 4. *Then there exists* $\delta > 0$ *such that for any* $x, y \in E$ *and* $p \geq 1$

$$\mathbb{E}\,|v^{x,y}(t)|^p_E \leq c_p\left(1+e^{-\delta p t}\,|y|^p_E + |x|^p_E\right), \qquad t \geq 0. \tag{4.5}$$

Proof. If we set $z_\lambda(t) := v^{x,y}(t) - \Gamma^x_\lambda(t)$, where $\Gamma^x_\lambda(t) = \Gamma^x_\lambda(v^{x,y})(t)$ is the stochastic convolution defined in (4.2), and $\lambda > \alpha$, thanks to (2.26) we have

$$\frac{d^-}{dt}|z_\lambda(t)|_E \leq \left\langle (A_2-\lambda)z_\lambda(t), \delta_{z_\lambda(t)}\right\rangle_E + \left\langle B_{2,\lambda}(x, z_\lambda(t)+\Gamma^x_\lambda(t)) - B_{2,\lambda}(x,\Gamma^x_\lambda(t)), \delta_{z_\lambda(t)}\right\rangle_E$$

$$+\left\langle B_{2,\lambda}(x,\Gamma^x_\lambda(t)), \delta_{z_\lambda(t)}\right\rangle_E \leq -\alpha\,|z_\lambda(t)|_E + c\,(1+|x|_E+|\Gamma^x_\lambda(t)|^{m_2}_E) + (\lambda-\alpha)\,|\Gamma^x_\lambda(t)|_E$$

$$\leq -\alpha\,|z_\lambda(t)|_E + c\,(1+|x|_E+|\Gamma^x_\lambda(t)|^{m_2}_E) + (\lambda-\alpha)^{\frac{m_2}{m_2-1}},$$

the last estimate following from the Young inequality. By comparison we get

$$|z_\lambda(t)|_E \leq e^{-\alpha t}|y|_E + c\left(1+|x|_E+(\lambda-\alpha)^{\frac{m_2}{m_2-1}}\right) + c\int_0^t e^{-\alpha(t-s)}|\Gamma^x_\lambda(s)|^{m_2}_E\,ds,$$

so that for any $p \geq 1$

$$|v^{x,y}(t)|^p_E \leq c_p|\Gamma^x_\lambda(t)|^p_E + c_p\,e^{-\alpha p t}|y|^p_E$$

$$+\,c_p\left(1+|x|^p_E+(\lambda-\alpha)^{\frac{pm_2}{m_2-1}}\right) + c_p\left(\int_0^t e^{-\alpha(t-s)}|\Gamma^x_\lambda(s)|^{m_2}_E\,ds\right)^p.$$

Next, we fix $0 < \delta < \alpha/2$. Then, thanks to (4.4), we get
(4.6)

$$
\begin{aligned}
&e^{2\delta pt}\mathbb{E}\,|v^{x,y}(t)|_E^p \leq c_p e^{2\delta pt}\,\mathbb{E}\,|\Gamma_\lambda^x(t)|_E^p + c_p\,|y|_E^p + c_p e^{2\delta pt}\,\left(1+|x|_E^p+(\lambda-\alpha)^{\frac{pm_2}{m_2-1}}\right)\\
&+c_p\left(\frac{p-1}{(\alpha-2\delta)p}\right)^{p-1}\int_0^t e^{2\delta ps}\,\mathbb{E}\,|\Gamma_\lambda^x(s)|_E^{pm_2}\,ds\\
&\leq c_p\frac{M_{g_2}^p}{(\lambda-2\delta)^{c_{p,2}}}\sup_{s\in[0,t]}e^{2\delta sp}\,\left(1+\mathbb{E}\,|v^{x,y}(s)|_E^{\frac{p}{m_2}}\right)\\
&+c_p\,|y|_E^p + c_p e^{2\delta pt}\,\left(1+|x|_E^p+(\lambda-\alpha)^{\frac{pm_2}{m_2-1}}\right)\\
&+c_p\left(\frac{p-1}{(\alpha-2\delta)p}\right)^{p-1}\frac{M_{g_2}^p}{(\lambda-2\delta)^{c_{p,2}}}\int_0^t\sup_{r\leq s}e^{2\delta pr}\,\left(1+\mathbb{E}\,|v^{x,y}(r)|_E^p\right)\,ds.
\end{aligned}
$$

Now, if we take

$$\lambda_1 = \left(2c_pM_{g_2}^p\right)^{\frac{1}{c_{p,2}}}+2\delta,$$

we have

$$c_p\frac{M_{g_2}^p}{(\lambda-2\delta)^{c_{p,2}}}\leq\frac{1}{2},\qquad \lambda\geq\lambda_1.$$

Thus, if we take $\lambda_2 \geq \lambda_1$ such that

$$c_p\left(\frac{p-1}{(\alpha-2\delta)p}\right)^{p-1}\frac{M_{g_2}^p}{(\lambda-2\delta)^{c_{p,2}}}\leq\delta p,\qquad\lambda\geq\lambda_2,$$

from (4.6) we have

$$
\begin{aligned}
&\sup_{s\leq t}e^{2\delta ps}\mathbb{E}\,|v^{x,y}(s)|_E^p\\
&\leq 2\,c_p\,|y|_E^p + c_p e^{2\delta pt}\,\left(1+|x|_E^p+(\lambda_2-\alpha)^{\frac{pm_2}{m_2-1}}\right)+\delta p\int_0^t\sup_{r\leq s}e^{2\delta pr}\,\mathbb{E}\,|v^{x,y}(r)|_E^p\,ds.
\end{aligned}
$$

Due to the Gronwall lemma this yields

$$
\begin{aligned}
&\sup_{s\leq t}e^{2\delta ps}\mathbb{E}\,|v^{x,y}(s)|_E^p\leq 2\,c_p\,|y|_E^p + c_p e^{2\delta pt}\,\left(1+|x|_E^p+(\lambda_2-\alpha)^{\frac{pm_2}{m_2-1}}\right)\\
&+\delta p\int_0^t e^{\delta p(t-s)}\left(2\,c_p\,|y|_E^p+c_pe^{2\delta ps}\,\left(1+|x|_E^p+\lambda_2^{\frac{pm_2}{m_2-1}}\right)\right)ds,\qquad t\geq 0,
\end{aligned}
$$

and this immediately implies (4.5). $\Box$

As proved in [4, Theorem 6.2], there exists some $\theta > 0$ such that for any $a > 0$ and $x, y \in E$

$$\sup_{t\geq a}\mathbb{E}\,|v^{x,y}(t)|_{C^\theta(\bar{D})}\leq c_a\,(1+|x|_E+|y|_E). \tag{4.7}$$

This implies that if $x, y \in E$, then for any $a > 0$ the family $\{\mathcal{L}(v^{x,y}(t))\}_{t \geq a}$ is tight in $\mathcal{P}(E, \mathcal{B}(E))$, and hence there exists an invariant measure μ^x for the semigroup P_t^x associated with (4.1) in E.

In view of (4.5), the invariant measure has all moments finite.

PROPOSITION 4.2. *For any $\rho \geq 1$ there exists $c_\rho > 0$ such that*

$$\int_E |z|_E^\rho \, \mu^x(dz) \leq c_\rho \, (1 + |x|_E^\rho) . \tag{4.8}$$

Proof. Due to (4.5), for any $t \geq 0$ we have

$$\int_E |z|_E^\rho \, \mu^x(dz) = \int_E P_t^x |z|_E^\rho \, \mu^x(dz) = \int_E \mathbb{E} \, |v^{x,z}(t)|_E^\rho \, \mu^x(dz)$$
$$\leq c_\rho \, (1 + |x|_E^\rho) + c_\rho \, e^{-\delta \rho t} \int_E |z|_E^\rho \, \mu^x(dz).$$

Therefore, if we take $\bar{t} > 0$ such that

$$c_\rho \, e^{-\delta \rho \bar{t}} < \frac{1}{2},$$

we get immediately (4.8). ∎

In what follows, we shall assume that the solution of problem (4.1) satisfies the following conditions.

HYPOTHESIS 5. *There exists a function $\beta : [0, \infty)^2 \to [0, \infty)$ such that*

$$\sup_{x \in E} \mathbb{E} \, |v^{x,y_1}(t) - v^{x,y_2}(t)|_E^2 \leq \beta(t, |y_1 - y_2|_E), \qquad t \geq 0, \tag{4.9}$$

for any $y_1, y_2 \in E$. Moreover, there exists some $\kappa \geq 0$ such that

$$\sup_{t,s \geq 0} \frac{\beta(t,s)}{s^\kappa} < \infty$$

and

$$\lim_{t \to \infty} \sup_{s \geq 0} \frac{\beta(t,s)}{s^\kappa} = 0. \tag{4.10}$$

Conditions (4.9) and (4.10) have important consequences regarding the ergodic behavior of system (4.1). Actually, as shown in the next lemma, (4.9) implies that μ^x is the unique invariant measure for P_t^x and it is also strongly mixing.

LEMMA 4.3. *Let $\varphi : E \to \mathbb{R}$ such that*

$$|\varphi(y) - \varphi(z)| \leq c_\varphi \, |y - z|_E \left(1 + |y|_E^\theta + |z|_E^\theta\right), \qquad y, z \in E,$$

for some $\theta \geq 0$. Then there exists $\kappa_\theta \geq 0$ such that for any $x, y \in E$ and $t \geq 0$

$$\left| P_t^x \varphi(y) - \int_E \varphi(z) \, \mu^x(dz) \right| \leq c_\varphi \, (1 + |y|_E^{\kappa_\theta} + |x|_E^{\kappa_\theta}) \, \beta(t) \tag{4.11}$$

for a function $\beta \in L^\infty(0, \infty)$ such that

$$\lim_{t \to \infty} \beta(t) = 0.$$

Proof. We have

$$\left|P_t^x\varphi(y)-\int_E\varphi(z)\,\mu^x(dz)\right|\leq\int_E\left|P_t^x\varphi(y)-P_t^x\varphi(z)\right|\,\mu^x(dz)$$

$$\leq\int_E\mathbb{E}\left|\varphi(v^{x,y}(t))-\varphi(v^{x,z}(t))\right|\mu^x(dz)$$

$$\leq c_\varphi\int_E\left(\mathbb{E}\left|v^{x,y}(t)-v^{x,z}(t)\right|_E^2\right)^{\frac12}\left(1+\mathbb{E}\left|v^{x,y}(t)\right|^{2\theta}+\mathbb{E}\left|v^{x,z}(t)\right|^{2\theta}\right)^{\frac12}\,\mu^x(dz).$$

Then, according to (4.5), (4.8), and Hypothesis 5, we have

$$\left|P_t^x\varphi(y)-\int_E\varphi(z)\,\mu^x(dz)\right|\leq c_\varphi\int_E\sqrt{\beta(t,|y-z|_E)}\left(1+|x|_E^\theta+|y|_E^\theta+|z|_E^\theta\right)\,\mu^x(dz)$$

$$\leq c_\varphi\sup_{s\geq0}\sqrt{\frac{\beta(t,s)}{s^\kappa}}\int_E|y-z|_E^{\frac{\kappa}{2}}\left(1+|x|_E^\theta+|y|_E^\theta+|z|_E^\theta\right)\,\mu^x(dz)$$

$$\leq c_\varphi\sup_{s\geq0}\sqrt{\frac{\beta(t,s)}{s^\kappa}}\left(1+|x|_E^{\theta\vee\frac{\kappa}{2}}+|y|_E^{\theta\vee\frac{\kappa}{2}}\right).$$

Therefore, if we set

$$\beta(t):=\sqrt{\frac{\beta(t,s)}{s^\kappa}},\qquad\kappa_\theta:=\theta\vee\frac{\kappa}{2},$$

we conclude the proof of the lemma. ∎

Now, we describe a couple of situations in which condition (4.9) is satisfied.

Example 4.4. Let us consider the following reaction term for the fast equation:

$$b_2(\xi,\sigma_1,\sigma_2)=f(\xi,p(\xi,\sigma_2)+h(\xi,\sigma_1)),\tag{4.12}$$

where

$$p(\xi,s)=-\gamma(\xi)s^{2k+1}+\sum_{j=1}^{2k}\gamma_j(\xi)s^j-\lambda s,\tag{4.13}$$

with $\lambda>0$ and

$$\inf_{\xi\in\bar D}\gamma(\xi)>0,$$

and $h:\bar D\times\mathbb{R}\to\mathbb{R}$ is continuous with $h(\xi,\cdot)$ locally Lipschitz-continuous, with linear growth, uniformly with respect to $\xi\in\bar D$. Assume that $f:\bar D\times\mathbb{R}\to\mathbb{R}$ is continuous, and $f(\xi,\cdot)\in C^1(\mathbb{R})$ with

$$0\leq\frac{\partial f}{\partial s}(\xi,s)\leq c,\qquad\xi\in\bar D,\quad s\in\mathbb{R}.\tag{4.14}$$

We want to prove that condition (4.9) is satisfied by the solution of problem (4.1), when b_2 is given by (4.12).

To this purpose, if we define $\rho(t) := v^{x,y_1}(t) - v^{x,y_2}(t)$, then we have that ρ solves the linear equation

$$d\rho(t) = [(A_2 - \alpha)\rho(t) - J(t)\rho(t)]\, dt + K(t)\rho(t)dw^{Q_2}(t), \qquad \rho(0) = y_1 - y_2, \tag{4.15}$$

where for any $(t,\xi) \in [0,+\infty) \times \bar{D}$

$$J(t,\xi) := H(t,\xi)\left(\frac{p(\xi, v^{x,y_1}(t,\xi)) - p(\xi, v^{x,y_2}(t,\xi))}{v^{x,y_2}(t,\xi) - v^{x,y_1}(t,\xi)}\right)$$

with

$$H(t,\xi) := \int_0^1 \frac{\partial f}{\partial s}\left(\xi, \theta\, p(\xi, v^{x,y_1}(t,\xi)) + (1-\theta)\, p(\xi, v^{x,y_2}(t,\xi)) + h(\xi,x)\right)\, d\theta$$

and

$$K(t,\xi) = \frac{g_2(\xi, x(\xi), v^{x,y_1}(t,\xi)) - g_2(\xi, x(\xi), v^{x,y_2}(t,\xi))}{v^{x,y_1}(t,\xi) - v^{x,y_2}(t,\xi)}.$$

Notice that, due to (4.14), for λ large enough we have

$$J(t,\xi) \geq 0, \quad (t,\xi) \in [0,\infty) \times \bar{D}, \qquad \mathbb{P}\text{-a.s.} \tag{4.16}$$

and, due to the Lipschitz continuity of $g_2(\xi,\cdot) : \mathbb{R}^2 \to \mathbb{R}$, uniform with respect to $\xi \in \bar{D}$,

$$\sup_{(t,\xi)\in[0,\infty)\times\bar{D}} |K(t,\xi)| \leq \sup_{\substack{\xi\in\bar{D}\\ h\in\mathbb{R}}} [g_2(\xi,h,\cdot)]_{\text{Lip}} < \infty, \qquad \mathbb{P}\text{-a.s.} \tag{4.17}$$

In addition to problem (4.23), we introduce the other linear problem

$$d\rho'(t) = (A_2 - \alpha)\rho'(t)\, dt + K(t)\rho'(t)dw^{Q_2}(t), \qquad \rho'(0) = y_1 - y_2, \tag{4.18}$$

By adapting the arguments used in [5] to (4.18), we have that if we assume

$$\sup_{\substack{\xi\in\bar{D}\\ h\in\mathbb{R}}} [g_2(\xi,h,\cdot)]_{\text{Lip}} < \alpha,$$

then there exists some $\delta > 0$ such that for any $p \geq 1$

$$\mathbb{E}\,|\rho'(t)|_E^p \leq c_p\, e^{-\delta p t}|y_1 - y_2|_E^p, \qquad t \geq 0. \tag{4.19}$$

Next, by a comparison argument, it is possible to prove that

$$y_1 \geq y_2 \Longrightarrow \mathbb{P}\text{-a.s.} \;\; \rho'(t,\xi) \geq 0, \quad t \geq 0, \quad \xi \in D.$$

Moreover, we have that the same is true for ρ, that is,

$$y_1 \geq y_2 \Longrightarrow \mathbb{P}\text{-a.s.} \;\; \rho(t,\xi) \geq 0, \quad t \geq 0, \quad \xi \in D.$$

Therefore, since in view of the sign condition (4.16) we have

$$y_1 \geq y_2 \Longrightarrow \mathbb{P}\text{-a.s.} \;\; \rho(t,\xi) \leq \rho'(t,\xi), \quad t \geq 0, \quad \xi \in D, \tag{4.20}$$

from (4.19) we can conclude that

$$\mathbb{E}\,|v^{x,y_1}(t)-v^{x,y_2}(t)|_E^p=\mathbb{E}\,|\rho(t)|_E^p\leq c_p\,e^{-\delta pt}|y_1-y_2|_E^p,\qquad t\geq 0,\quad y_1\geq y_2.$$

Finally, in the general case $y_1,y_2\in E$ we have

$$\mathbb{E}\,|v^{x,y_1}(t)-v^{x,y_2}(t)|_E^p\leq 2^p\,\mathbb{E}\,|v^{x,y_1}(t)-v^{x,y_1\wedge y_2}(t)|_E^p+2^p\,\mathbb{E}\,|v^{x,y_1\wedge y_2}(t)-v^{x,y_2}(t)|_E^p$$

$$\leq 2^p\,c_pe^{-\delta pt}\left(|y_1-y_1\wedge y_2|_E^p+|y_1\wedge y_2-y_2|_E^p\right)\leq c_p\,e^{-\delta pt}|y_1-y_2|_E^p,$$

and this yields (4.9), with

$$\beta(t,s)=c_2\,e^{-2\delta t}s.$$

Example 4.5. Assume that the diffusion coefficient g_2 in the fast equation does not depend on the fast motion, that is,

$$g_2(\xi,\sigma_1,\sigma_2)=g_2(\xi,\sigma_1),\qquad \xi\in\bar{D},\quad(\sigma_1,\sigma_2)\in\mathbb{R}^2.$$

Then, if we define as in the previous example $\rho(t):=v^{x,y_1}(t)-v^{x,y_2}(t)$, we have

$$\frac{d}{dt}\rho(t)=A_2\rho(t)+B_2(x,v^{x,y_1}(t))-B_2(x,v^{x,y_2}(t)),\qquad \rho(0)=y_1-y_2.$$

This yields

$$\frac{d}{dt}^{-}|\rho(t)|_E\leq\left\langle B_2(x,v^{x,y_1}(t))-B_2(x,v^{x,y_2}(t)),\delta_{\rho(t)}\right\rangle_E. \tag{4.21}$$

Now, if we assume

$$\sup_{\xi\in\bar{D}}\left(b_2(\xi,\sigma_1,\sigma_2+h_2)-b_2(\xi,\sigma_1,\sigma_2)\right)h_2\leq -a\,|h_2|^{m_2+1},$$

we have

$$\langle B_2(x,y+k)-B_2(x,y),\delta_k\rangle_E\leq -a\,|k|_E^{m_2},$$

and then from (4.21) we get

$$\frac{d}{dt}^{-}|\rho(t)|_E\leq -a\,|\rho(t)|_E^{m_2}.$$

If $m_2>1$, by comparison this implies

$$|v^{x,y_1}(t)-v^{x,y_2}(t)|_E\leq|y_1-y_2|_E\left(1+a(m_2-1)|y_1-y_2|_E^{m_2-1}\,t\right)^{-\frac{1}{m_2-1}},$$

so that (4.9) follows for

$$\beta(t,s)=s^2\left(1+a(m_2-1)s^{m_2-1}\,t\right)^{-\frac{2}{m_2-1}}.$$

Notice that in this case

$$\lim_{t\to\infty}\sup_{s\geq 0}\beta(t,s)=0.$$

In the case $m_2 = 1$, instead of polynomial we have exponential decay to zero, as in the previous example, and so we get (4.9) with

$$\beta(t,s) = c_2 e^{-2\alpha t}\, s.$$

In order to prove the averaging result of this paper we must also require that the solution of (4.1) satisfy the following condition. (In what follows, for any $R > 0$ we denote by $B_E(R)$ the ball in E of radius R.)

HYPOTHESIS 6. *For any $R > 0$ there exists $K_R > 0$ such that*

$$x_1, x_2 \in B_E(R) \Longrightarrow \limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}\,|v^{x_1,0}(t) - v^{x_2,0}(t)|_E^2\, dt \le K_R\, |x_1 - x_2|_E^2. \tag{4.22}$$

As for Hypothesis 5, we describe relevant situations in which the condition described in Hypothesis 6 is satisfied.

Example 4.6. We consider the same situation described in Example 4.4, that is,

$$b_2(\xi,\sigma_1,\sigma_2) = f(\xi, p(\xi,\sigma_2) + h(\xi,\sigma_1)),$$

where p is the polynomial introduced in (4.13) and $h : \bar{D} \times \mathbb{R} \to \mathbb{R}$ is a continuous function which is locally Lipschitz-continuous and has linear growth in the variable σ_1, uniformly with respect to $\xi \in \bar{D}$.

In this case, if we define $\rho(t) := v^{x_1,y}(t) - v^{x_2,y}(t)$, we have that ρ solves the linear equation

$$d\rho(t) = [(A_2 - \alpha)\rho(t) - J(t)\rho(t) + I]\; dt + K(t)\, dw^{Q_2}, \qquad \rho(0) = 0, \tag{4.23}$$

where

$$I(\xi) = H(t,\xi)\,(h(\xi, x_1(\xi)) - h(\xi, x_2(\xi)))$$

and

$$J(t,\xi) := H(t,\xi)\left(\frac{p(\xi, v^{x_1,y}(t,\xi)) - p(\xi, v^{x_2,y}(t,\xi))}{v^{x_2,y}(t,\xi) - v^{x_1,y}(t,\xi)}\right),$$

with $H(t,\xi)$ defined by

$$\int_0^1 \frac{\partial f}{\partial s}\,(\xi, \theta\,[p(\xi, v^{x_1,y}(t,\xi)) + h(\xi, x_1(\xi))] + (1-\theta)\,[p(\xi, v^{x_2,y}(t,\xi)) + h(\xi, x_2(\xi))])\; d\theta$$

and

$$K(t,\xi) = g_2(\xi, x_1(\xi), v^{x_1,y}(t,\xi)) - g_2(\xi, x_2(\xi), v^{x_2,y}(t,\xi)).$$

As $h(\xi,\cdot)$ is locally Lipschitz-continuous, uniformly with respect to $\xi \in \bar{D}$, for any $R > 0$ there exists $c_R > 0$ such that

$$x_1, x_2 \in B_E(R) \Longrightarrow |I|_E \le c_R\, |x_1 - x_2|_E. \tag{4.24}$$

Now, we denote by $\Gamma(t)$ the solution of the problem

$$d\Gamma(t) = (A_2 - \alpha)\Gamma(t)\, dt + K(t)\, dw^{Q_2}(t), \qquad \Gamma(0) = 0.$$

By proceeding as in [4, Theorem 4.2 and Proposition 4.5], due to the Lipschitz-continuity of g_2 it is possible to prove that for any $t > 0$ and $p \geq 1$

$$\mathbb{E} \sup_{s\in[0,t]} |\Gamma(s)|_E^p \leq c_{\alpha,p}(t) \sup_{\xi\in\bar{D}} [g_2(\xi,\cdot)]_{\text{Lip}}^p \left(|x_1 - x_2|_E^p + \mathbb{E} \sup_{s\in[0,t]} |\rho(s)|_E^p \right) \tag{4.25}$$

for some function $c_{\alpha,p} : [0,\infty) \to [0,\infty)$ such that

$$\lim_{\alpha\to\infty} \sup_{t\geq 0} c_{\alpha,p}(t) = 0. \tag{4.26}$$

Next, if we define $z(t) := \rho(t) - \Gamma(t)$, we have that z solves the problem

$$dz(t) = (A_2 - \alpha)z(t) - J(t)z(t) + I - J(t)\Gamma(t), \qquad z(0) = 0.$$

For any $t \geq 0$ we denote by ξ_t the point in $\bar{D}$ such that

$$|z(t, \xi_t)| = |z(t)|_E$$

and

$$\left\langle h, \delta_{z(t)} \right\rangle_E = \frac{1}{|z(t)|_E} h(\xi_t) z(t, \xi_t)$$

for any $h \in E$. Since, in view of (4.16), we have that for λ large enough $J(t, \xi_t) \geq 0$, $\mathbb{P}$-a.s., we thus get

$$\frac{d}{dt}^- |z(t)|_E \leq \left\langle (A_2 - \alpha)z(t), \delta_{z(t)} \right\rangle_E - \left\langle J(t)z(t), \delta_{z(t)} \right\rangle_E + \left\langle I, \delta_{z(t)} \right\rangle_E - \left\langle J(t)\Gamma(t), \delta_{z(t)} \right\rangle_E$$

$$\leq -(\alpha + J(t, \xi_t))|z(t)|_E + |I|_E + J(t, \xi_t)|\Gamma(t)|_E.$$

By comparison this yields

$$|z(t)|_E \leq \frac{1}{\alpha}|I|_E + \int_0^t \exp\left(-\int_s^t (\alpha + J(r, \xi_r))\, dr \right) J(s, \xi_s)|\Gamma(s)|_E\, ds,$$

and then

$$|z(t)|_E \leq \frac{1}{\alpha}|I|_E + \sup_{s\leq t} |\Gamma(s)|_E \int_0^t \exp\left(-\int_s^t J(r, \xi_r)\, dr \right) J(s, \xi_s)\, ds$$

$$= \frac{1}{\alpha}|I|_E + \sup_{s\leq t} |\Gamma(s)|_E \left(1 - \exp\left(-\int_0^t J(r, \xi_r)\, dr \right) \right) \leq \frac{1}{\alpha}|I|_E + \sup_{s\leq t} |\Gamma(s)|_E.$$

This yields

$$\sup_{s\leq t} |\rho(t)|_E^2 \leq \frac{2}{\alpha^2}|I|_E^2 + 8 \sup_{s\leq t} |\Gamma(s)|_E^2,$$

so that, thanks to (4.24) and (4.25), we have that for any $R > 0$ and $x_1, x_2 \in B_E(R)$

$$\mathbb{E} \sup_{s\leq t} |\rho(t)|_E^2 \leq \left(\frac{2\, c_R^2}{\alpha^2} + c_{\alpha,2}(t) \right) |x_1 - x_2|_E^2 + 8\, c_{\alpha,2}(t) \sup_{\xi\in\bar{D}} [g_2(\xi,\cdot)]_{\text{Lip}}^2\, \mathbb{E} \sup_{s\in[0,t]} |\rho(s)|_E^2$$

$$\leq \left(\frac{2c_R^2}{\alpha^2} + |c_{\alpha,2}|_\infty^2 \right) |x_1 - x_2|_E^2 + 8\, |c_{\alpha,2}|_\infty^2 \sup_{\xi\in\bar{D}} [g_2(\xi,\cdot)]_{\text{Lip}}^2\, \mathbb{E} \sup_{s\in[0,t]} |\rho(s)|_E^2.$$

Therefore, due to (4.26) we can fix $\bar{\alpha}$ large enough such that

$$8\,|c_{\bar{\alpha},2}|_\infty^2 \sup_{\xi\in\bar{D}}[g_2(\xi,\cdot)]_{\text{Lip}}^2 \leq \frac{1}{2},$$

so that

$$\mathbb{E}\sup_{s\leq t}|\rho(t)|_E^2 \leq 2\left(\frac{2c_R^2}{\bar{\alpha}^2}+|c_{\bar{\lambda},2}|_\infty^2\right)|x_1-x_2|_E^2 =: K_R\,|x_1-x_2|_E^2.$$

This clearly implies the validity of Hypothesis 6.

Example 4.7. If we consider the case covered in Example 4.5, then with arguments similar to those used throughout Example 4.5 itself, it is possible to prove that Hypothesis 6 is satisfied.

5. The averaged equation. In this section we introduce the averaged equation. The key point is constructing the coefficients and describing some of its properties.

For any fixed $x\in E$, we introduce the mapping

$$y\in E\mapsto B_1(x,y)\in E.$$

Due to our assumptions, the mapping $B_1(x,\cdot):E\to E$ is continuous and

$$|B_1(x,y)|_E \leq c\,(1+|x|_E^{m_1}+|y|_E)\,. \tag{5.1}$$

Now, we define

$$\bar{B}(x) := \int_E B_1(x,y)\,\mu^x(dy),\qquad x\in E.$$

Due to the estimates above and to (4.8), the mapping $\bar{B}:E\to E$ is well defined and

$$|\bar{B}(x)|_E \leq c\,(1+|x|_E^{m_1})\,. \tag{5.2}$$

Actually, in view of (5.1) we have

$$|\bar{B}(x)|_E \leq \int_E |B_1(x,y)|_E\,\mu^x(dy) \leq c\,\left(1+|x|_E^{m_1}\right)+c\int_E |y|_E\,\mu^x(dy),$$

and then, thanks to (4.8), we have

$$|\bar{B}(x)|_E \leq c\,\left(1+|x|_E^{m_1}\right)+c_1\,\left(1+|x|_E\right),$$

which implies (5.2).

As a consequence of (4.11), we can give the following characterization of $\bar{B}$.

LEMMA 5.1. *Assume Hypotheses* 1 *to* 5 *hold. Then there exist some constants* $\kappa_1,\kappa_2\geq 0$ *such that for any* $T>0$, $t\geq 0$, *and* $x,y\in E$ *and for any* $\Lambda\in E^\star$

$$\mathbb{E}\left|\frac{1}{T}\int_t^{t+T}\langle B_1(x,v^{x,y}(s)),\Lambda\rangle_E\,ds-\left\langle\bar{B}(x),\Lambda\right\rangle_E\right| \leq \alpha(T)\,(1+|x|_E^{\kappa_1}+|y|_E^{\kappa_2})\,|\Lambda|_{E^\star} \tag{5.3}$$

for some mapping $\alpha:[0,\infty)\to[0,\infty)$ *such that*

$$\lim_{T\to\infty}\alpha(T)=0.$$

Proof. For any fixed $\Lambda \in E^\star$ and $x \in E$, we denote by $\Pi^x_\Lambda B_1$ the mapping

$$z \in E \mapsto \Pi^x_\Lambda B_1(z) := \left\langle B_1(x,z) - \bar{B}(x), \Lambda \right\rangle_E \in \mathbb{R}.$$

By proceeding as in the proof of [6, Lemma 2.3], due to Markovianity of $v^{x,y}(t)$ we have

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{T}\int_t^{t+T} \langle B_1(x, v^{x,y}(s)), \Lambda\rangle_E \, ds - \left\langle \bar{B}(x), \Lambda\right\rangle_E\right)^2 \\
&= \frac{2}{T^2}\int_t^{t+T}\int_r^{t+T} \mathbb{E}\left[\Pi^x_\Lambda B_1(v^{x,y}(r))\, P^x_{s-r}\Pi^x_\Lambda B_1(v^{x,y}(r))\right] ds\, dr \\
&\leq \frac{2}{T^2}\int_t^{t+T}\int_r^{t+T} \left(\mathbb{E}\,|\Pi^x_\Lambda B_1(v^{x,y}(r))|^2\right)^{\frac{1}{2}} \left(\mathbb{E}\,|P^x_{s-r}\Pi^x_\Lambda B_1(v^{x,y}(r))|^2\right)^{\frac{1}{2}} ds\, dr.
\end{aligned} \tag{5.4}$$

Due to (2.22), (4.5), and (5.2), we have

$$\begin{aligned}
&\mathbb{E}\,|\Pi^x_\Lambda B_1(v^{x,y}(r))|^2 \leq c\left(1 + |x|_E^{2m_1} + \mathbb{E}\,|v^{x,y}(r)|_E^2\right)|\Lambda|^2_{E^\star} \\
&\leq c\left(1 + |x|_E^{2m_1} + e^{-2\delta r}|y|_E^2\right)|\Lambda|^2_{E^\star}.
\end{aligned} \tag{5.5}$$

Moreover, as due to (2.27) we have

$$|\langle B_1(x,y), \Lambda\rangle_E - \langle B_1(x,z), \Lambda\rangle_E| \leq c\,|y-z|_E\left(1 + |x|_E^\theta + |y|_E^\theta + |z|_E^\theta\right)|\Lambda|_{E^\star},$$

thanks to (4.11) we have

$$\mathbb{E}\,|P^x_{s-r}\Pi^x_\Lambda B_1(v^{x,y}(r))|^2 \leq c\left(1 + |x|_E^{2(\theta\vee\kappa_\theta)} + |y|_E^{2(\theta\vee\kappa_\theta)}\right)|\Lambda|^2_{E^\star}\beta^2(s-r).$$

So, if we plug the above estimate and estimate (5.5) into (5.4), we get

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{T}\int_t^{t+T} \langle B_1(x, v^{x,y}(s)), \Lambda\rangle_E \, ds - \left\langle \bar{B}(x), \Lambda\right\rangle_E\right)^2 \\
&\leq c\,\left(1 + |x|_E^{m_1} + |y|_E\right)\left(1 + |x|_E^{\theta\vee\kappa_\theta} + |y|_E^{\theta\vee\kappa_\theta}\right)|\Lambda|^2_{E^\star}\frac{1}{T^2}\int_t^{t+T}\int_r^{t+T}\beta(s-r)\, ds\, dr,
\end{aligned}$$

so that, if we define

$$\alpha(T) = \left(\sup_{t\geq 0}\frac{1}{T^2}\int_t^{t+T}\int_r^{t+T}\beta(s-r)ds\, dr\right)^{\frac{1}{2}},$$

we have

$$\begin{aligned}
&\mathbb{E}\left|\frac{1}{T}\int_t^{t+T} \langle B_1(x, v^{x,y}(s)), \Lambda\rangle_E \, ds - \left\langle \bar{B}(x), \Lambda\right\rangle_E\right| \\
&\leq c\,\left(1 + |x|_E^{\kappa_1} + |y|_E^{\kappa_2}\right)|\Lambda|_{E^\star}\alpha(T)
\end{aligned}$$

for some constants $\kappa_1, \kappa_2 \geq 0$. Therefore, in order to conclude the proof of the lemma, we have to show that $\lim_{T\to\infty} \alpha(T) = 0$.

For any $\epsilon > 0$ we can fix $T_\epsilon > 0$ such that $\beta(s) \leq \epsilon$ for $s \geq T_\epsilon$. Thus, for any $T > T_\epsilon$ we have

$$\frac{1}{T^2}\int_t^{t+T}\int_r^{t+T}\beta(s-r)ds\,dr = \frac{1}{T^2}\int_t^{t+T}\int_0^{t+T-r}\beta(s)ds\,dr$$

$$= \frac{1}{T^2}\int_t^{t+T-T_\epsilon}\int_0^{t+T-r}\beta(s)ds\,dr + \frac{1}{T^2}\int_{t+T-T_\epsilon}^{t+T}\int_0^{t+T-r}\beta(s)ds\,dr$$

$$=: I_{1,\epsilon}(t,T) + I_{2,\epsilon}(t,T).$$

For the first term $I_{1,\epsilon}$ we have

(5.6)
$$I_{1,\epsilon}(t,T) = \frac{1}{T^2}\int_t^{t+T-T_\epsilon}\left[\int_0^{T_\epsilon}\beta(s)ds + \int_{T_\epsilon}^{t+T-r}\beta(s)\,ds\right]dr$$

$$\leq \frac{(T-T_\epsilon)T_\epsilon}{T^2}|\beta|_\infty + \frac{\epsilon}{T^2}\int_t^{t+T-T_\epsilon}(t+T-T_\epsilon-r)\,dr = \frac{(T-T_\epsilon)T_\epsilon}{T^2}|\beta|_\infty + \frac{\epsilon(T-T_\epsilon)^2}{2T^2}.$$

For the second term $I_{2,\epsilon}$ we have

$$I_{2,\epsilon}(t,T) \leq \frac{1}{T^2}\int_{t+T-T_\epsilon}^{t+T}|\beta|_\infty T_\epsilon\,dr = \frac{|\beta|_\infty T_\epsilon^2}{T^2},$$

and then, combining this estimate with (5.6), we obtain

$$\frac{1}{T^2}\int_t^{t+T}\int_r^{t+T}\beta(s-r)ds\,dr \leq \frac{T_\epsilon}{T}|\beta|_\infty + \frac{\epsilon}{2} + \frac{|\beta|_\infty T_\epsilon^2}{T^2}, \qquad T>0, \quad t\geq 0,$$

so that, due to the arbitrariness of ϵ, we conclude that

$$\lim_{T\to\infty}\alpha(T) = 0. \qquad \square$$

In view of the previous lemma, we have that for any $x \in E$ and $T > 0$

$$\left|\left\langle \frac{1}{T}\int_0^T \mathbb{E}\,B_1(x, v^{x,0}(s))\,ds - \bar{B}(x), \Lambda\right\rangle_E\right|$$

$$\leq \mathbb{E}\left|\frac{1}{T}\int_0^T \left\langle B_1(x, v^{x,0}(s)), \Lambda\right\rangle_E\,ds - \left\langle \bar{B}(x), \Lambda\right\rangle_E\right| \leq \alpha(T)\,(1+|x|^{\kappa_1})\,|\Lambda|_{E^\star}.$$

This implies

$$\lim_{T\to\infty}\left|\frac{1}{T}\int_0^T \mathbb{E}\,B_1(x, v^{x,0}(s))\,ds - \bar{B}(x)\right|_E = 0, \tag{5.7}$$

and in particular we get

$$\bar{B}(x)(\xi) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{E}\,b_1(\xi, x(\xi), v^{x,0}(t,\xi))\,dt, \quad \xi \in \bar{D}.$$

Moreover, we can prove that $\bar{B}$ fulfills the following properties.

LEMMA 5.2. *Under Hypotheses* 1 *to* 6, *we have that* $\bar{B} : E \to E$ *is locally Lipschitz-continuous. Moreover, for any* $x, h \in E$

$$\left\langle \bar{B}(x+h) - \bar{B}(x), \delta_h \right\rangle_E \leq c\,(1 + |h|_E + |x|_E)\,. \tag{5.8}$$

Proof. According to Lemma 5.1 and to (5.7), for any $x_1, x_2 \in E$ we have

$$\bar{B}(x_1) - \bar{B}(x_2) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}\,\left(B_1(x_1, v^{x_1,0}(s)) - B_1(x_2, v^{x_2,0}(s))\right)\,ds \quad \text{in } E. \tag{5.9}$$

Now, by using (2.27) we have

$$\begin{aligned}&\left|B_1(x_1, v^{x_1,0}(s)) - B_1(x_2, v^{x_2,0}(s))\right|_E\\ &\leq c\left(1 + |x_1|_E^\theta + |x_2|_E^\theta + |v^{x_1,0}(s)|_E^\theta + |v^{x_2,0}(s)|_E^\theta\right)\left(|x_1 - x_2|_E + |v^{x_1,0}(s) - v^{x_2,0}(s)|_E\right),\end{aligned}$$

and then, due to (4.5),

$$\begin{aligned}&\left|\mathbb{E}\left(B_1(x_1, v^{x_1,0}(s)) - B_1(x_2, v^{x_2,0}(s))\right)\right|_E\\ &\leq c\left(1 + |x_1|_E^\theta + |x_2|_E^\theta\right)\left(|x_1 - x_2|_E + \left(\mathbb{E}|v^{x_1,0}(s) - v^{x_2,0}(s)|_E^2\right)^{\frac{1}{2}}\right).\end{aligned}$$

Thanks to (4.22), this implies that for any $T > 0$ and $R > 0$, if $x_1, x_2 \in B_E(R)$, then

$$\begin{aligned}&\left|\frac{1}{T}\int_0^T \mathbb{E}\left(B_1(x_1, v^{x_1,0}(s)) - B_1(x_2, v^{x_2,0}(s))\right)\,ds\right|_E\\ &\leq c\left(1 + |x_1|_E^\theta + |x_2|_E^\theta\right)\left(|x_1 - x_2|_E + \frac{1}{T}\int_0^T \left(\mathbb{E}|v^{x_1,0}(s) - v^{x_2,0}(s)|_E^2\right)^{\frac{1}{2}}\,ds\right)\\ &\leq c\,K_R\,\left(1 + |x_1|_E^\theta + |x_2|_E^\theta\right)|x_1 - x_2|_E \leq C_R\,|x_1 - x_2|_E,\end{aligned}$$

so that, due to (5.9), the local Lipschitz-continuity of $\bar{B}$ follows.

Concerning (5.8), we have

$$\left\langle \bar{B}(x+h) - \bar{B}(x), \delta_h\right\rangle_E = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{E}\,\left\langle B_1(x+h, v^{x+h,0}(s)) - B_1(x, v^{x,0}(s)), \delta_h\right\rangle_E\,ds.$$

Now, due to (2.24) we have

$$\begin{aligned}&\left\langle B_1(x+h, v^{x+h,0}(s)) - B_1(x, v^{x,0}(s)), \delta_h\right\rangle_E\\ &\leq c\left(1 + |x|_E + |h|_E + |v^{x+h,0}(s)|_E + |v^{x,0}(s)|_E\right),\end{aligned}$$

and then, thanks to (4.5),

$$\begin{aligned}&\left\langle \bar{B}(x+h) - \bar{B}(x), \delta_h\right\rangle_E\\ &\leq \limsup_{T\to\infty}\frac{1}{T}\int_0^T c\left(1 + |x|_E + |h|_E + \mathbb{E}|v^{x+h,0}(s)|_E + \mathbb{E}|v^{x,0}(s)|_E\right)\,ds\\ &\leq c\,(1 + |x|_E + |h|_E)\,. \qquad \square\end{aligned}$$

Now, we can introduce the averaged equation

$$du(t) = \left[A_1 u(t) + \bar{B}(u(t))\right]\, dt + G(u(t))\, dw(t), \qquad u(0) = x \in E. \tag{5.10}$$

In view of Lemma 5.2 and [4, Theorem 5.3], for any $x \in E$, $T > 0$, and $p \geq 1$ equation (5.10) admits a unique mild solution $\bar{u} \in L^p(\Omega; C((0,T]; E) \cap L^\infty(0,T;E))$. In the next section we will show that the slow motion u_ϵ converges in probability to the averaged motion $\bar{u}$.

6. The averaging limit. In this last section we prove the following averaging result.

THEOREM 6.1. *Assume that Hypotheses* 1 *to* 6 *hold and fix* $x \in C^\theta(\bar{D})$ *for some* $\theta > 0$ *and* $y \in E$. *Then, for any* $T > 0$ *and* $\eta > 0$,

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\sup_{t \in [0,T]} |u_\epsilon - \bar{u}(t)|_E > \eta\right) = 0, \tag{6.1}$$

where $\bar{u}$ *is the solution of the averaged equation* (5.10).

Proof. For any $h \in D(A_1)$, the slow motion u_ϵ satisfies the identity

$$\int_D u_\epsilon(t,\xi) h(\xi)\, d\xi = \int_D x(\xi) h(\xi)\, d\xi + \int_0^t \int_D u_\epsilon(s,\xi) A_1 h(\xi)\, d\xi\, ds$$

$$+ \int_0^t \int_D \bar{B}(u_\epsilon(s,\cdot))(\xi) h(\xi)\, d\xi\, ds + \int_0^t \int_D [G_1(u_\epsilon(s)h](\xi) dw^{Q_2}(s,\xi) + R_\epsilon(t),$$

where

$$R_\epsilon(t) = \int_0^t \int_D \left(B_1(u_\epsilon(s), v_\epsilon(s))(\xi) - \bar{B}(u_\epsilon(s))(\xi)\right) h(\xi)\, d\xi\, ds.$$

In order to prove Lemma 6.1, we shall need the following result, whose proof is postponed.

LEMMA 6.2. *Under the same hypotheses of Theorem* 6.1, *for any* $T > 0$ *we have*

$$\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [0,T]} |R_\epsilon(t)|_E = 0. \tag{6.2}$$

Once we have the key result of Lemma 6.2, we proceed exactly as in [6, proof of Theorem 6.4]. For the reader's convenience, we repeat the main steps.

In view of Theorem 3.4, the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon \in (0,1]}$ is tight in $\mathcal{P}(C([0,T];E))$. Hence, for any two sequences $\{\epsilon_n\}_{n \in \mathbb{N}}$ and $\{\epsilon_m\}_{m \in \mathbb{N}}$ converging to zero, we can find two subsequences $\{\epsilon_{n(k)}\}_{k \in \mathbb{N}}$ and $\{\epsilon_{m(k)}\}_{k \in \mathbb{N}}$ and a sequence

$$\{X_k\}_{k \in \mathbb{N}} = \{(u_{1,k}, u_{2,k}, \hat{w}_k)\} \subset \mathcal{C} := [C([0,T];E)]^2 \times C([0,T]; \mathcal{D}'(D)),$$

defined on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, which converges $\hat{\mathbb{P}}$-a.s. to some $X = (u_1, u_2, \hat{w}) \in \mathcal{C}$ and such that

$$\mathcal{L}(X_k) = (u_{\epsilon_{n(k)}}, u_{\epsilon_{m(k)}}, w^{Q_1}), \qquad k \in \mathbb{N}. \tag{6.3}$$

Next, for $k \in \mathbb{N}$ and $i = 1, 2$, we define

(6.4)
$$R_{i,k}(t) := \int_D u_{i,k}(t,\xi)h(\xi)\,d\xi - \int_D x(\xi)h(\xi)\,d\xi - \int_0^t \int_D u_{i,k}(s,\xi)A_1 h(\xi)\,d\xi\,ds$$
$$- \int_0^t \int_D \bar{B}(u_{i,k}(s,\cdot))(\xi)h(\xi)\,d\xi\,ds - \int_0^t \int_D [G_1(u_{i,k}(s)h](\xi)dw^{Q_2}(s,\xi).$$

Due to (6.3), we have that $\mathcal{L}(R_{1,k}) = \mathcal{L}(R_{\epsilon_{n(k)}})$ and $\mathcal{L}(R_{2,k}) = \mathcal{L}(R_{\epsilon_{m(k)}})$ in $C([0,T];E)$, so that

$$\lim_{k\to\infty} \hat{\mathbb{E}} \sup_{t\in[0,T]} |R_{i,k}|_E = 0.$$

Moreover, it is possible to prove that for $i = 1, 2$ the right-hand side in (6.4) converges $\hat{\mathbb{P}}$-a.s. to

$$\int_D u_i(t,\xi)h(\xi)\,d\xi - \int_D x(\xi)h(\xi)\,d\xi - \int_0^t \int_D u_i(s,\xi)A_1 h(\xi)\,d\xi\,ds$$
$$- \int_0^t \int_D \bar{B}(u_i(s,\cdot))(\xi)h(\xi)\,d\xi\,ds - \int_0^t \int_D [G_1(u_i(s)h](\xi)dw^{Q_2}(s,\xi),$$

so that both u_1 and u_2 are mild solutions of the averaged equation (5.10). By uniqueness, this implies $u_1 = u_2$ and this means that the two sequences $\mathcal{L}(u_{\epsilon_{n(k)}})$ and $\mathcal{L}(u_{\epsilon_{m(k)}})$ are both weakly convergent to the same limit $\bar{u}$, solution of (5.10). As shown in [6], due to a general argument by Gyöngy and Krylov, this implies that u_ϵ converges in probability to $\bar{u}$. ∎

6.1. Proof of Lemma 6.2. For any $n \in \mathbb{N}$, we define

$$b_{i,n}(\xi,\sigma_1,\sigma_2) := \begin{cases} b_i(\xi,\sigma_1,\sigma_2) & \text{if } |\sigma_1| \le n, \\ \\ b_i(\xi,\sigma_1 n/|\sigma_1|,\sigma_2) & \text{if } |\sigma_1| > n, \end{cases} \qquad i = 1, 2.$$

In correspondence to each $b_{i,n}$, we denote by $B_{i,n}$ the corresponding composition operator and obtain

$$|x|_E \le n \Longrightarrow B_{i,n}(x,y) = B_i(x,y), \qquad y \in E. \tag{6.5}$$

Notice that the mappings $b_{1,n}$ and $b_{2,n}$ satisfy all conditions in Hypotheses 2 and 3, respectively. For any fixed $\xi \in \bar{D}$ and $\sigma_2 \in \mathbb{R}$, the mappings $b_{i,n}(\xi,\cdot,\sigma_2) : \mathbb{R} \to \mathbb{R}$ are Lipschitz-continuous and, in view of (2.15),

$$\sup_{\substack{\xi\in\bar{D} \\ \sigma_2\in\mathbb{R}}} |b_{2,n}(\xi,\sigma_1,\sigma_2) - b_{2,n}(\xi,\rho_1,\sigma_2)| \le c_n\,|\sigma_1 - \rho_1|, \qquad \sigma_1, \rho_1 \in \mathbb{R}. \tag{6.6}$$

Moreover, for any $n \in \mathbb{N}$ we define

$$g_{1,n}(\xi,\sigma_1) := \begin{cases} g_1(\xi,\sigma_1) & \text{if } |\sigma_1| \le n, \\ \\ g_1(\xi,\sigma_1 n/|\sigma_1|) & \text{if } |\sigma_1| > n \end{cases}$$

and

$$g_{2,n}(\xi,\sigma_1,\sigma_2) := \begin{cases} g_2(\xi,\sigma_1,\sigma_2) & \text{if } |\sigma_1| \leq n, \\ \\ g_2(\xi,\sigma_1 n/|\sigma_1|,\sigma_2) & \text{if } |\sigma_1| > n. \end{cases}$$

The corresponding composition/multiplication operators are denoted by $G_{1,n}$ and $G_{2,n}$.

Now, for any $n \in \mathbb{N}$ we introduce the system

$$\text{(6.7)} \qquad \begin{cases} du(t) = [A_1 u(t) + B_{1,n}(u(t),v(t))]\, dt + G_{1,n}(u(t))\, dw^{Q_1}(t), \\ \\ dv(t) = \dfrac{1}{\epsilon}\, [A_2 v(t) + B_{2,n}(u(t),v(t))]\, dt + \dfrac{1}{\sqrt{\epsilon}}\, G_{2,n}(u(t),v(t))\, dw^{Q_2}(t), \end{cases}$$

with initial conditions $u(0) = x$ and $v(0) = y$. We denote its solution by $(u_{\epsilon,n}, v_{\epsilon,n})$. It is important to stress that, as $b_{1,n}$ satisfies Hypothesis 2 and $b_{2,n}$ satisfies Hypothesis 3, then $u_{\epsilon,n}$ and $v_{\epsilon,n}$ satisfy estimates (3.2), (3.3), (3.7), and (3.10).

Next, for any $n \in \mathbb{N}$ we introduce the problem

$$\text{(6.8)} \qquad dv(t) = [A_2 v(t) + B_{2,n}(x,v(t))]\, dt + G_{2,n}(x,v(t))\, dw^{Q_2}(t), \qquad v(0) = y,$$

whose solution will be denoted by $v_n^{x,y}$. Thanks to (6.5), for any $t \geq 0$ we have

$$\text{(6.9)} \qquad v_n^{x,y}(t) = \begin{cases} v^{x,y}(t) & \text{if } |x|_E \leq n, \\ \\ v^{xn/|x|_E,y}(t) & \text{if } |x|_E > n. \end{cases}$$

This implies that $v_n^{x,y}$ satisfies Hypothesis 5 and a stronger version of Hypothesis 6, namely,

$$\text{(6.10)} \qquad \limsup_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}\, |v_n^{x_1,0}(t) - v_n^{x_2,0}(t)|_E\, dt \leq c_n\, |x_1 - x_2|_E, \qquad x_1, x_2 \in E.$$

In particular, for each $x \in E$ there exists a unique invariant measure μ_n^x for (6.8), and μ_n^x is given by

$$\mu_n^x = \begin{cases} \mu^x & \text{if } |x|_E \leq n, \\ \\ \mu^{xn/|x|_E} & \text{if } |x|_E > n. \end{cases}$$

Moreover, due to (4.5), for any $T > 0$ and $p \geq 1$ we have

$$\text{(6.11)} \qquad \mathbb{E}\, |v_n^{x,y}(t)|_E^p \leq c_{p,n} \left(1 + e^{-\delta p t} |y|_E^p\right).$$

As $v_n^{x,y}$ satisfies both Hypothesis 5 and Hypothesis 6, we have that a result analogous to Lemma 5.1 holds. More precisely, if we define

$$\bar{B}_n(x) = \int_E B_{1,n}(x,y)\, \mu_n^x(dy),$$

we have that
(6.12)
$$\mathbb{E}\left|\frac{1}{T}\int_t^{t+T}\left\langle B_{1,n}(x,v_n^{x,y}(s)),\Lambda\right\rangle_E\,ds-\left\langle \bar{B}_n(x),\Lambda\right\rangle_E\right|\leq \alpha(T)\,(1+|x|_E^{\kappa_1}+|y|_E^{\kappa_2})\,|\Lambda|_{E^\star}.$$

Notice that

$$|x|_E\leq n\Longrightarrow \bar{B}_n(x)=\bar{B}(x).$$

Moreover, as $\bar{B}_n$ satisfies (6.12), thanks to (6.6), (6.10), and (6.11) it is immediate to check that $\bar{B}_n : E\to E$ is globally Lipschitz-continuous.

As in [6], we prove the validity of Lemma 6.2 by using the Khasminskii approach based on time discretization introduced in [12].

To this purpose, for any $\epsilon>0$ we divide the interval $[0,T]$ into subintervals of size $\delta_\epsilon>0$ for some constant δ_ϵ to be determined, and we introduce the auxiliary fast motion $\hat{v}_{\epsilon,n}$ defined in each time interval $[k\delta_\epsilon,(k+1)\delta_\epsilon]$ for $k=0,1,\ldots,[T/\delta_\epsilon]$ as the solution of the problem
(6.13)
$$\left\{\begin{array}{l} dv(t)=\dfrac{1}{\epsilon}\left[(A_2-\alpha)v(t)+B_{2,n}(u_{\epsilon,n}(k\delta_\epsilon),v(t))\right]\,dt+\dfrac{1}{\sqrt{\epsilon}}\,G_{2,n}(u_{\epsilon,n}(k\delta_\epsilon),v(t))\,dw^{Q_2}(t),\\[10pt] v(k\delta_\epsilon)=v_{\epsilon,n}(k\delta_\epsilon).\end{array}\right.$$

Notice that, due to the way $\hat{v}_{\epsilon,n}$ has been defined, we have that an estimate analogous to (3.3) holds, that is, for any $p\geq 1$

$$\sup_{t\in[0,T]}\mathbb{E}\,|\hat{v}_{\epsilon,n}(t)|_E^p\leq c_{p,T}\,(1+|x|_E^p+|y|_E^p)\,. \tag{6.14}$$

As in [12] and [6], we want to prove the following approximation result.

Lemma 6.3. *Assume Hypotheses* 1 *to* 4 *hold and fix $x\in C^\theta(\bar{D})$ and $y\in E$. Then there exists a constant $\kappa>0$ such that if*

$$\delta_\epsilon=\epsilon\,\log\epsilon^{-\kappa},$$

then for any fixed $n\in\mathbb{N}$ and $p\geq 1$

$$\lim_{\epsilon\to 0}\sup_{t\in[0,T]}\mathbb{E}|\hat{v}_{\epsilon,n}(t)-v_{\epsilon,n}(t)|_H^p=0. \tag{6.15}$$

Proof. Let $\epsilon>0$ and $n\in\mathbb{N}$ be fixed. For $k=0,\ldots,[T/\delta_\epsilon]$ and $t\in[k\delta_\epsilon,(k+1)\delta_\epsilon]$, let $\Gamma_{\epsilon,n}(t)$ be the solution of the problem

$$d\Gamma_{\epsilon,n}(t)=\frac{1}{\epsilon}\,(A_2-\alpha)\Gamma_{\epsilon,n}(t)\,dt+\frac{1}{\sqrt{\epsilon}}\,K_{\epsilon,n}(t)\,dw^{Q_2},\qquad \Gamma_{\epsilon,n}(k\delta_\epsilon)=0,$$

where

$$K_{\epsilon,n}(t):=G_{2,n}(u_{\epsilon,n}(k\delta_\epsilon),\hat{v}_{\epsilon,n}(t))-G_{2,n}(u_{\epsilon,n}(t),v_{\epsilon,n}(t)).$$

If we define $\rho_{\epsilon,n}(t):=\hat{v}_{\epsilon,n}(t)-v_{\epsilon,n}(t)$ and $z_{\epsilon,n}(t):=\rho_{\epsilon,n}(t)-\Gamma_{\epsilon,n}(t)$, we have

$$dz_{\epsilon,n}(t)=\frac{1}{\epsilon}\left[(A_2-\alpha)z_{\epsilon,n}(t)+H_{\epsilon,n}(t)\right]\,dt,\qquad z_{\epsilon,n}(k\delta_\epsilon)=0,$$

where

$$H_{\epsilon,n}(t) := B_{2,n}(u_{\epsilon,n}(k\delta_\epsilon), \hat{v}_{\epsilon,n}(t)) - B_{2,n}(u_{\epsilon,n}(t), v_{\epsilon,n}(t)).$$

Notice that in view of (2.17) we have

$$\begin{aligned} H_{\epsilon,n}(t) = {} & B_{2,n}(u_{\epsilon,n}(k\delta_\epsilon), \hat{v}_{\epsilon,n}(t)) - B_{2,n}(u_{\epsilon,n}(t), \hat{v}_{\epsilon,n}(t)) \\ & - \lambda(\cdot, u_{\epsilon,n}(t), \hat{v}_{\epsilon,n}(t), v_{\epsilon,n}(t))\, \rho_{\epsilon,n}(t). \end{aligned}$$

Thanks to (6.6), this yields

$$\begin{aligned} \frac{d}{dt}^{-} |z_{\epsilon,n}(t)|_E \leq {} & \frac{1}{\epsilon} \left\langle (A_2 - \alpha) z_{\epsilon,n}(t), \delta_{z_{\epsilon,n}(t)} \right\rangle_E \\ & + \frac{1}{\epsilon} \left\langle B_{2,n}(u_{\epsilon,n}(k\delta_\epsilon), \hat{v}_{\epsilon,n}(t)) - B_{2,n}(u_{\epsilon,n}(t), \hat{v}_{\epsilon,n}(t)), \delta_{z_{\epsilon,n}(t)} \right\rangle_E \\ & - \frac{1}{\epsilon} \left\langle \lambda(\cdot, u_{\epsilon,n}(t), \hat{v}_{\epsilon,n}(t), v_{\epsilon,n}(t)) z_{\epsilon,n}(t), \delta_{z_{\epsilon,n}(t)} \right\rangle_E \\ & - \frac{1}{\epsilon} \left\langle \lambda(\cdot, u_{\epsilon,n}(t), \hat{v}_{\epsilon,n}(t), v_{\epsilon,n}(t)) \Gamma_{\epsilon,n}(t), \delta_{z_{\epsilon,n}(t)} \right\rangle_E \\ \leq {} & -\frac{\alpha}{\epsilon} |z_{\epsilon,n}(t)|_E + \frac{c_n}{\epsilon} |u_{\epsilon,n}(k\delta_\epsilon) - u_{\epsilon,n}(t)|_E - \frac{\lambda_{\epsilon,n}(t)}{\epsilon} |z_{\epsilon,n}(t)|_E + \frac{\lambda_{\epsilon,n}(t)}{\epsilon} |\Gamma_{\epsilon,n}(t)|_E, \end{aligned}$$

where

$$\lambda_{\epsilon,n}(t) := \lambda(\xi_{\epsilon,n}(t), u_{\epsilon,n}(t, \xi_{\epsilon,n}(t)), \hat{v}_{\epsilon,n}(t, \xi_{\epsilon,n}(t)), v_{\epsilon,n}(t, \xi_{\epsilon,n}(t))),$$

and $\xi_{\epsilon,n}(t)$ is the point in $\bar{D}$ such that

$$|z_{\epsilon,n}(t, \xi_{\epsilon,n}(t))| = |z_{\epsilon,n}(t)|_E.$$

Therefore, by comparison

$$\begin{aligned} |z_{\epsilon,n}(t)|_E \leq {} & \frac{c_n}{\epsilon} \int_{k\delta_\epsilon}^t e^{-\frac{\alpha}{\epsilon}(t-s)} |u_{\epsilon,n}(k\delta_\epsilon) - u_{\epsilon,n}(s)|_E \, ds \\ & + \frac{1}{\epsilon} \int_{k\delta_\epsilon}^t \exp\left(-\frac{1}{\epsilon} \int_s^t \lambda_{\epsilon,n}(r)\, dr \right) \lambda_{\epsilon,n}(s)\, |\Gamma_{\epsilon,n}(s)|_E \, ds, \end{aligned}$$

and, due to (3.10), this yields for any $p \geq 1$

$$\begin{aligned} \mathbb{E}\, |\hat{v}_{\epsilon,n}(t) - v_{\epsilon,n}(t)|_E^p \leq {} & c_p\, \mathbb{E}\, |\Gamma_{\epsilon,n}(t)|_E^p + c_{p,n} \left(1 + |x|_{C^\theta(\bar{D})}^{pm_2} + |y|_E^p \right) \delta_\epsilon^{\gamma(\theta)p} \\ & + c_p\, \mathbb{E} \sup_{s \in [k\delta_\epsilon, t]} |\Gamma_{\epsilon,n}(s)|_E^p \left(\frac{1}{\epsilon} \int_{k\delta_\epsilon}^t \exp\left(-\frac{1}{\epsilon} \int_s^t \lambda_{\epsilon,n}(r)\, dr \right) \lambda_{\epsilon,n}(s)\, ds \right)^p \\ \leq {} & c_{p,n} \left(1 + |x|_{C^\theta(\bar{D})}^{pm_2} + |y|_E^p \right) \delta_\epsilon^{\gamma(\theta)p} + c_p\, \mathbb{E} \sup_{s \in [k\delta_\epsilon, t]} |\Gamma_{\epsilon,n}(s)|_E^p. \end{aligned} \tag{6.16}$$

Now, by using a factorization argument, for $s \in [k\delta_\epsilon, (k+1)\delta_\epsilon]$ and $\eta \in (0,1)$ we have

$$\Gamma_{\epsilon,n}(s) = \frac{\sin \pi\eta}{\pi}\frac{1}{\sqrt{\epsilon}}\int_{k\delta_\epsilon}^s (s-r)^{\eta-1} e^{(s-r)\frac{(A_2-\alpha)}{\epsilon}} Y_{\eta,\epsilon,n}(r)\,dr,$$

where

$$Y_{\eta,\epsilon,n}(r) = \int_{k\delta_\epsilon}^r (r-\rho)^{-\eta} e^{(r-\rho)\frac{(A_2-\alpha)}{\epsilon}} K_{\epsilon,n}(\rho)\,dw^{Q_2}(\rho).$$

Hence, for $\delta > d/p$, that is, $\eta > (d+2)/2p$, due to (2.6) we have

(6.17)

$$\begin{aligned}
&|\Gamma_{\epsilon,n}(s)|_E^p \leq c_p\,|\Gamma_{\epsilon,n}(s)|_{\delta,p}^p \\
&\leq \frac{c_\eta}{\epsilon^{\frac{p}{2}}}\left(\int_{k\delta_\epsilon}^s \left[(s-r)^{\eta-1}\left(\frac{s-r}{\epsilon}\right)^{-\frac{\delta}{2}} e^{-\frac{\alpha(s-r)}{\epsilon}}\right]^{\frac{p}{p-1}} dr\right)^{p-1} \int_{k\delta_\epsilon}^s |Y_{\eta,\epsilon,n}(r)|_p^p\,dr \\
&\leq c_\eta\,\epsilon^{-\frac{p}{2}}\epsilon^{p-1}\epsilon^{(\eta-1)p}\int_{k\delta_\epsilon}^s |Y_{\eta,\epsilon,n}(r)|_p^p\,dr = c_\eta\,\epsilon^{-\frac{p}{2}+\eta p-1}\int_{k\delta_\epsilon}^s |Y_{\eta,\epsilon,n}(r)|_p^p\,dr.
\end{aligned}$$

Now, by proceeding as in [4, proof of Theorem 4.2] and [6, proof of Proposition 4.2], we have

$$\begin{aligned}
\mathbb{E}\,|Y_{\eta,\epsilon,n}(r)|_p^p \leq c_p\mathbb{E}&\left(\int_{k\delta_\epsilon}^r (r-\rho)^{-2\eta}\left(\frac{r-\rho}{\epsilon}\right)^{-\frac{\beta_2(\rho_2-2)}{\rho_2}} e^{-\alpha\frac{\rho_2+2}{\rho_2}\frac{(r-\rho)}{\epsilon}}\right. \\
&\times\left.\left[|u_{\epsilon,n}(k\delta_\epsilon) - u_{\epsilon,n}(\rho)|_E^2 + |\hat{v}_{\epsilon,n}(\rho) - v_{\epsilon,n}(\rho)|_E^2\right]\,dr\right)^{\frac{p}{2}}.
\end{aligned}$$

In view of (2.5), there exists p large enough such that there exists $\eta \in (0,1)$ with

$$\frac{d+2}{p} < 2\eta < 1 - \frac{\beta_2(\rho_2-2)}{\rho_2}.$$

Then

$$\begin{aligned}
&\mathbb{E}\int_{k\delta_\epsilon}^s |Y_{\eta,\epsilon,n}(r)|_p^p\,dr \\
&\leq c_p\,\epsilon^{-\left(\eta p+\frac{p}{2}\right)}\int_{k\delta_\epsilon}^s \left(\mathbb{E}\,|u_{\epsilon,n}(k\delta_\epsilon) - u_{\epsilon,n}(r)|_E^p + \mathbb{E}\,|\hat{v}_{\epsilon,n}(r) - v_{\epsilon,n}(r)|_E^p\right)\,dr.
\end{aligned}$$

In view of (6.17) and (3.10), this yields

$$\begin{aligned}
\mathbb{E}\sup_{s\in[k\delta_\epsilon,t]} |\Gamma_{\epsilon,n}(s)|_E^p \leq c_{\eta,p}&\left(1 + |x|_{C^\theta(\bar{D})}^{pm_2} + |y|_E^p\right)\delta_\epsilon^{\gamma(\theta)p}\zeta_\epsilon \\
&+ c_{\eta,p}\frac{1}{\epsilon}\int_{k\delta_\epsilon}^t \mathbb{E}\,|\hat{v}_{\epsilon,n}(s) - v_{\epsilon,n}(s)|_E^p\,ds,
\end{aligned}$$

so that, thanks to (6.16), for $t \in [k\delta_\epsilon, (k+1)\delta_\epsilon]$

$$\mathbb{E}\,|\hat{v}_{\epsilon,n}(t) - v_{\epsilon,n}(t)|_E^p \leq c_{\eta,p}\left(1 + |x|^{pm_2}_{C^\theta(\bar{D})} + |y|_E^p\right)\delta_\epsilon^{\gamma(\theta)p}\,(1+\zeta_\epsilon)$$
$$+\frac{c_p}{\epsilon}\int_{k\delta_\epsilon}^t \mathbb{E}\,|\hat{v}_{\epsilon,n}(s) - v_{\epsilon,n}(s)|_E^p\,ds.$$

From the Gronwall lemma, this gives

$$\mathbb{E}\,|\hat{v}_{\epsilon,n}(t) - v_{\epsilon,n}(t)|_E^p \leq c_{\eta,p}\left(1 + |x|^{pm_2}_{C^\theta(\bar{D})} + |y|_E^p\right)\delta_\epsilon^{\gamma(\theta)p}\,(1+\zeta_\epsilon)\,e^{\zeta_\epsilon}.$$

Now, if

$$e^{\zeta_\epsilon} = \exp\left(\log\frac{1}{\epsilon^\kappa}\right) = \frac{1}{\epsilon^\kappa},$$

we have

$$\delta_\epsilon^{\gamma(\theta)p}\,(1+\zeta_\epsilon)\,e^{\zeta_\epsilon} = \frac{1}{\epsilon^\kappa}\delta_\epsilon^{\gamma(\theta)p}\left(1+\log\frac{1}{\epsilon^\kappa}\right) = \left(\log\frac{1}{\epsilon^\kappa}\right)^\kappa\left(1+\log\frac{1}{\epsilon^\kappa}\right)\delta_\epsilon^{\gamma(\theta)p-\kappa}.$$

Hence, if we take $\kappa < \gamma(\theta)$, we have (6.15) for any $p \geq 1$. ∎

Finally, we can prove (6.2). For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}|R_\epsilon(t)| = \mathbb{E}\left(\sup_{t\in[0,T]}|R_\epsilon(t)|;\ \sup_{t\in[0,T]}|u_\epsilon(t)|_E \leq n\right)\\
&\quad+\mathbb{E}\left(\sup_{t\in[0,T]}|R_\epsilon(t)|;\ \sup_{t\in[0,T]}|u_\epsilon(t)|_E > n\right)\\
&\leq \mathbb{E}\left(\sup_{t\in[0,T]}|R_{\epsilon,n}(t)|\right) + \mathbb{E}\left(\sup_{t\in[0,T]}|R_\epsilon(t)|_E^2\right)^{\frac{1}{2}}\mathbb{P}\left(\sup_{t\in[0,T]}|u_\epsilon(t)|_E > n\right)^{\frac{1}{2}},
\end{aligned}\tag{6.18}$$

where

$$R_{\epsilon,n}(t) = \int_0^t\int_D\left(B_{1,n}(u_{\epsilon,n}(s), v_{\epsilon,n}(s))(\xi) - \bar{B}_n(u_{\epsilon,n}(s))(\xi)\right)h(\xi)\,d\xi\,ds.$$

Now, according to (2.22) and (5.2), we have

$$|R_\epsilon(t)| \leq c\,|h|_E\int_0^t\left(1 + |u_\epsilon(s)|_E^{m_1} + |v_\epsilon(s)|_E\right)ds,$$

so that, thanks to (3.2) and (3.3), we obtain

$$\mathbb{E}\sup_{t\in[0,T]}|R_\epsilon(t)|^2 \leq c_T\left(1 + |x|_E^{2m_1} + |y|_E^2\right)|h|_E^2.$$

By using again estimate (3.2), due to (6.18) this yields

$$\mathbb{E}\sup_{t\in[0,T]}|R_\epsilon(t)| \leq \mathbb{E}\left(\sup_{t\in[0,T]}|R_{\epsilon,n}(t)|\right) + \frac{c_T}{n}\left(1 + |x|_E^{2m_1} + |y|_E^2\right)|h|_E.$$

Therefore, due to the arbitrariness on $n \in \mathbb{N}$, (6.2) follows once we prove that for any fixed $n \in \mathbb{N}$

$$\lim_{\epsilon\to 0} \mathbb{E}\left(\sup_{t\in[0,T]} |R_{\epsilon,n}(t)|\right) = 0. \tag{6.19}$$

In view of the key Lemma 6.3, of estimates (3.2) and (3.3), and of the Lipschitz-continuity of $B_{1,n}(\xi,\cdot,y)$, for any fixed $\xi \in \bar{D}$ and $y \in E$, and of $\bar{B}_n$, the proof of (6.19) is analogous to [6, proof of Lemma 6.3]. Here we repeat it for the reader's convenience.

We have

(6.20)

$$\begin{aligned}
&\limsup_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} \left|\int_0^t \left\langle B_{1,n}(u_{\epsilon,n}(s), v_{\epsilon,n}(s)) - \bar{B}_n(u_{\epsilon,n}(s)), h\right\rangle_H \, ds\right| \\
&\leq \limsup_{\epsilon\to 0} \mathbb{E} \int_0^T \left|\langle B_{1,n}(u_{\epsilon,n}(s), v_{\epsilon,n}(s)) - B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon), \hat{v}_{\epsilon,n}(s)), h\rangle_H\right| \, ds \\
&\quad + \limsup_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} \left|\int_0^t \left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon), \hat{v}_{\epsilon,n}(s)) - \bar{B}_n(u_{\epsilon,n}(s)), h\right\rangle_H \, ds\right|.
\end{aligned}$$

Now, due to (2.27) and (6.6) we have

$$\begin{aligned}
&\mathbb{E} \int_0^T \left|\langle B_{1,n}(u_{\epsilon,n}(s), v_{\epsilon,n}(s)) - B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon), \hat{v}_{\epsilon,n}(s)), h\rangle_H\right| \, ds \\
&\leq c_n \, |h|_H \int_0^T \mathbb{E} \, |u_{\epsilon,n}(s) - u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon)|_E \, ds \\
&\quad + c \, |h|_H \int_0^T \mathbb{E} \, |v_{\epsilon,n}(s) - \hat{v}_{\epsilon,n}(s)|_E \left(1 + |u_{\epsilon,n}(s)|_E^\theta + |v_{\epsilon,n}(s)|_E^\theta + |\hat{v}_{\epsilon,n}(s)|_E^\theta\right) ds \\
&\leq c_n \, |h|_H \int_0^T \mathbb{E} \, |u_{\epsilon,n}(s) - u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon)|_E \, ds + c \, |h|_H \sup_{t\in[0,T]} \left(\mathbb{E} \, |v_{\epsilon,n}(t) - \hat{v}_{\epsilon,n}(t)|_E^2\right)^{\frac{1}{2}} \\
&\quad \times \int_0^T \left(\mathbb{E}\left(1 + |u_{\epsilon,n}(s)|_E^{2\theta} + |v_{\epsilon,n}(s)|_E^{2\theta} + |\hat{v}_{\epsilon,n}(s)|_E^{2\theta}\right)\right)^{\frac{1}{2}} \, ds.
\end{aligned}$$

Then, thanks to (3.2), (3.3), (3.10), and (6.14), we conclude

$$\begin{aligned}
&\mathbb{E} \int_0^T \left|\langle B_{1,n}(u_{\epsilon,n}(s), v_{\epsilon,n}(s)) - B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon), \hat{v}_{\epsilon,n}(s)), h\rangle_H\right| \, ds \\
&\leq c_{T,n} \, |h|_H \left(1 + |x|_{C^\theta(\bar{D})}^{(2\vee\theta)m_1} + |y|_E^{2\vee\theta}\right) \left(\delta_\epsilon^{\gamma(\theta)} + \sup_{t\in[0,T]} \left(\mathbb{E} \, |v_{\epsilon,n}(t) - \hat{v}_{\epsilon,n}(t)|_E^2\right)^{\frac{1}{2}}\right).
\end{aligned}$$

Therefore, in view of Lemma 6.3, from (6.20)

$$\begin{aligned}
&\limsup_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} \left|\int_0^t \left\langle B_{1,n}(u_{\epsilon,n}(s), v_{\epsilon,n}(s)) - \bar{B}_n(u_{\epsilon,n}(s)), h\right\rangle_H \, ds\right| \\
&= \limsup_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} \left|\int_0^t \left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon), \hat{v}_{\epsilon,n}(s)) - \bar{B}_n(u_{\epsilon,n}(s)), h\right\rangle_H \, ds\right|.
\end{aligned}$$

Now, we have

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s)) - \bar{B}_n(u_{\epsilon,n}(s)),h\right\rangle_H\,ds\right|\\
&\leq \sum_{k=0}^{[T/\delta_\epsilon]}\mathbb{E}\left|\int_{k\delta_\epsilon}^{(k+1)\delta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s)) - \bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|\\
&\quad+\sum_{k=0}^{[T/\delta_\epsilon]}\int_{k\delta_\epsilon}^{(k+1)\delta_\epsilon}\mathbb{E}\left|\left\langle \bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)) - \bar{B}_n(u_{\epsilon,n}(s),h\right\rangle_H\right|\,ds,
\end{aligned}$$

and then, due to (3.10) and to the global Lipschitz-continuity of $\bar{B}_n$,

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s)) - \bar{B}_n(u_{\epsilon,n}(s)),h\right\rangle_H\,ds\right|\\
&\leq \sum_{k=0}^{[T/\delta_\epsilon]}\mathbb{E}\left|\int_{k\delta_\epsilon}^{(k+1)\delta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s)) - \bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|\\
&\quad+c_{T,n}\,|h|_H\left(1+|x|^{m_1}_{C^\theta(\bar{D})}+|y|_E\right)[T/\delta_\epsilon]\,\delta_\epsilon^{\gamma(\theta)+1}.
\end{aligned}$$

This means that in order to obtain (6.19) and conclude the proof of Lemma 6.2, it remains to show that

(6.21)

$$\lim_{\epsilon\to 0}\sum_{k=0}^{[T/\delta_\epsilon]}\mathbb{E}\left|\int_{k\delta_\epsilon}^{(k+1)\delta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s)) - \bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right| = 0.$$

We have

$$\begin{aligned}
&\mathbb{E}\left|\int_{k\delta_\epsilon}^{(k+1)\delta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s)) - \bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|\\
&=\mathbb{E}\left|\int_0^{\delta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(k\delta_\epsilon s)) - \bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|\\
&=\mathbb{E}\left|\int_0^{\delta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\tilde{v}_n^{u_{\epsilon,n}(k\delta_\epsilon),v_{\epsilon,n}(k\delta_\epsilon)}(s/\epsilon) - \bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|\\
&=\delta_\epsilon\mathbb{E}\left|\frac{1}{\zeta_\epsilon}\int_0^{\zeta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\tilde{v}_n^{u_{\epsilon,n}(k\delta_\epsilon),v_{\epsilon,n}(k\delta_\epsilon)}(s) - \bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|,
\end{aligned}$$

where $\tilde{v}_n^{u_{\epsilon,n}(k\delta_\epsilon),v_{\epsilon,n}(k\delta_\epsilon)}(s)$ is the solution of the fast motion equation (4.1), with frozen slow component $u_{\epsilon,n}(k\delta_\epsilon)$ and initial datum $v_{\epsilon,n}(k\delta_\epsilon)$ and noise $\tilde{w}^{Q_2}$ independent of both of them. According to (6.12), (3.2), and (3.3), this yields

$$\begin{aligned}
&\mathbb{E}\left|\int_{k\delta_\epsilon}^{(k+1)\delta_\epsilon}\left\langle B_{1,n}(u_{\epsilon,n}([s/\delta_\epsilon]\delta_\epsilon),\hat{v}_{\epsilon,n}(s)) - \bar{B}_n(u_{\epsilon,n}(k\delta_\epsilon)),h\right\rangle_H\,ds\right|\\
&\leq \delta_\epsilon\alpha(\zeta_\epsilon)\left(1+\mathbb{E}\,|u_{\epsilon,n}(k\delta_\epsilon)|_E^{\kappa_1}+\mathbb{E}\,|v_{\epsilon,n}(k\delta_\epsilon)|_E^{\kappa_2}\right)|h|_1\\
&\leq d_\epsilon\alpha(\zeta_\epsilon)\left(1+|x|_E^{\kappa_1}+|y|_E^{\kappa_2}\right)|h|_1,
\end{aligned}$$

and (6.21) follows.

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