Stochastic Processes and their Applications 83 (1999) 15–37

stochastic processes and their applications

www.elsevier.com/locate/spa

Differentiability of Markov semigroups for stochastic reaction–diffusion equations and applications to control

Sandra Cerrai

Dipartimento di Matematica per le Decisioni, Via Lombroso 6/17, 50134 Firenze, Italy

Received 29 April 1998; received in revised form 9 February 1999; accepted 12 February 1999

Abstract

We consider a reaction–diffusion equation in a bounded domain $\mathscr{O} \subset \mathbb{R}^d$, driven by a space–time white noise, with a drift term having polynomial growth and a diffusion term which is not boundedly invertible, in general. We are showing that the transition semigroup corresponding to the equation has a regularizing effect. More precisely, we show that it maps bounded and Borel functions defined in the Hilbert space $H = L^2(\mathscr{O})$ with values in $\mathbb{R}$ into the space of differentiable functions from H into $\mathbb{R}$. An estimate for the sup-norm of the derivative of the semigroup is given. We apply these results to the study of the corresponding Hamilton–Jacobi equation arising in stochastic control theory.

MSC: 60H15; 60J25; 93E20; 93C20

Keywords: Stochastic reaction–diffusion systems; Markov semigroups; Hamilton–Jacobi equations; Dynamic programming

1. Introduction

If $\mathscr{O}$ is a bounded domain of $\mathbb{R}^d$, we denote by H the Hilbert space $L^2(\mathscr{O})$ and by E the Banach space $C(\overline{\mathscr{O}})$. In the present paper we consider the problem

$$\mathrm{d}u(t) = [Au(t) + F(u(t))]\,\mathrm{d}t + B\,\mathrm{d}w(t), \quad u(0) = x. \tag{1.1}$$

Here $A : D(A) \subset H \to H$ is the generator of an analytic semigroup e^{tA} of class C_0, F is the Nemytskii operator associated with a function f of class C^2 having polynomial growth, with the first derivative bounded from above and satisfying other conditions, B is a non-negative bounded linear operator from H to H and $w(t)$ is a cylindrical Wiener process in H.

Under these assumptions, for any $x \in H$ the problem (1.1) has a unique solution $u(t;x)$, in a generalized sense that we will specify later on. Thus we can introduce the *Markov transition semigroup* corresponding to Eq. (1.1) by setting

$$P_t\varphi(x) = \mathbb{E}\varphi(u(t;x)), \quad x \in H,$$

for each $t \geqslant 0$ and $\varphi \in B_b(H)$ (the space of bounded and Borel functions from H into $\mathbb{R}$).

0304-4149/99/$ - see front matter
PII: S0304-4149(99)00014-9

Our aim is to show that the semigroup P_t has a smoothing effect. Namely, we want to prove that for any $t>0$

$$\varphi\in B_b(H)\Rightarrow P_t\varphi\in C_b^1(H)$$

and for any $x\in H$ the following estimate holds [1]

$$\sup_{x\in H}|\langle D(P_t\varphi)(x),h\rangle_H|\leqslant c(t\wedge 1)^{-\frac{1+\varepsilon}{2}}\sup_{x\in H}|\varphi(x)||h|_H, \tag{1.2}$$

for a suitable constant ε depending on A. When A is the realization in H of the Laplace operator with Dirichlet boundary conditions, ε may be taken equal zero if $d=1$ and strictly less than one if $d\leqslant 3$, so that the singularity at $t=0$ arising in the right-hand side of Eq. (1.2) turns out to be integrable. Moreover, we are proving that if $\varphi\in C_b(H)$, then a Bismut–Elworthy type formula holds for the derivative of $P_t\varphi$. In fact, we show that for any $x,h\in H$ and $t>0$

$$\langle D(P_t\varphi)(x),h\rangle_H=\frac{1}{t}\mathbb{E}\varphi(u(t;x))\int_0^t\langle B^{-1}v(s;x,h),\,\mathrm{d}w(s)\rangle_H, \tag{1.3}$$

where in general the process $v(t;x,h)$ is not the mean-square derivative of the solution $u(t;x)$ along the direction h, as one should expect (see Bismut (1981) and Elworthy and Li (1994) for the classical Bismut–Elworthy formula in finite dimension and Peszat and Zabczyk (1995) for its generalization to the infinite dimension). Actually, it can be proved that $v(t;x,h)$ is a *generalized* solution of the first variation equation associated with the problem (1.1). Notice that in Eq. (1.3) we have the inverse of B which is not bounded, in general. Hence, in order to give a meaning to the stochastic integral, we have first to prove that $v(s;x,h)\in D(B^{-1})$ for each $s>0$ and then

$$\mathbb{E}\int_0^t|B^{-1}v(s;x,h)|_H^2\,\mathrm{d}s<\infty.$$

The regularizing properties of transition semigroups are rather crucial in the study of the ergodicity of the system as well as in the study of the corresponding Hamilton–Jacobi equation. In the former case, if the semigroup P_t is proved to be *irreducible*, due to the Khas'minskii theorem and the Doob theorem, the *strong Feller* property of P_t implies the uniqueness and the *strongly mixing* property of any invariant measure μ (see Da Prato and Zabczyk (1996)). As far as the latter case is concerned, the differentiability of the semigroup P_t allows us to prove the existence and the uniqueness of a regular solution of the Hamilton–Jacobi equation corresponding to the problem (1.1)

$$\frac{\partial y}{\partial t}(t,x)=\frac{1}{2}\,\mathrm{Tr}[B^2D^2y(t,x)]+\langle Ax+F(x),Dy(t,x)\rangle_H-K(Dy(t,x))+g(x),$$

$$y(0,x)=\varphi(x), \tag{1.4}$$

where $g,\varphi\in C_b(H)$ and

$$K(x)=\sup_{h\in H}\{-\langle x,h\rangle_H-k(h)\},\quad x\in H,$$

[1] In general, if X is a Banach space, we will denote by $C_b^k(X)$ the Banach space of k times Fréchet differentiable functions, uniformly continuous and bounded together with their derivatives up the kth order (see Section 2.3).

for a suitable lower semicontinuous function $k : H \to]-\infty, +\infty]$. The main motivation in the study of the existence and the uniqueness of a solution for the problem (1.4) is related to the following optimal control problem: minimizing the *cost functional*

$$J(x,z) = \mathbb{E}\left(\int_0^T [g(u(t;x,z)) + k(z(t))]\,\mathrm{d}t + \varphi(u(T;x,z))\right), \quad x \in H, \tag{1.5}$$

over all controls $z \in \mathscr{H}_T(H)$.[2] Here we denote by $u(t;x,z)$ the solution of the controlled stochastic reaction–diffusion equation

$$\mathrm{d}u(t) = [Au(t) + F(u(t)) + z(t)]\,\mathrm{d}t + Bw(t), \quad u(0) = x.$$

As a matter of fact, by using the *dynamic programming* approach it is possible to prove that the *optimal cost* is given by

$$J^\star(x) = \inf\{J(x,z);\, z \in \mathscr{H}_T(H)\} = y(T,x), \quad x \in H, \tag{1.6}$$

where y is the unique solution of Eq. (1.4).

The differentiability of transition semigroups has been studied in several papers, in the linear and in the semilinear case, both with additive and with multiplicative noise (see Da Prato and Zabczyk (1996) for a comprehensive bibliography). In the semilinear case the nonlinear drift term F is always assumed to be Lipschitz continuous and differentiable. In the present paper we are dealing with a drift term F given by the Nemytskii operator corresponding to a function f having polynomial growth. Thus we override the hypothesis of Lipschitz continuity and differentiability of F (actually the Nemytskii operator F associated with any function f is never Fréchet differentiable, unless f is linear). As far as we know, Da Prato et al. (1995) is the only paper in which the case of functionals F satisfying our assumptions is considered, but it only treats the case of dimension $d = 1$ and operator B equal to identity. In such a paper it is verified that the semigroup P_t maps bounded and Borel functions into continuous functions. Here we are able to improve this result in two respects: firstly we work with operators B having an inverse not necessarily bounded, so that the case of dimension d greater than one can be covered, secondly we show that if $\varphi \in B_b(H)$ then $P_t\varphi$ is not only continuous, but it even belongs to $C_b^1(H)$.

Remark 1.1. We will denote by c (without any index) any positive constant appearing in inequalities, whose dependence on some parameters is not important. Such constants may change even in the same chain of inequalities.

2. Notations and hypotheses

Let $\mathscr{O} \subset \mathbb{R}^d$ be a bounded and regular open set. We will denote by H the Hilbert space $L^2(\mathscr{O})$, endowed with the norm $|\cdot|_H$ and the scalar product $\langle\cdot,\cdot\rangle_H$. E will be the densely and continuously embedded (as a Borel subset) Banach space $C(\overline{\mathscr{O}})$, endowed with the sup-norm $|\cdot|_E$. The duality form on $E \times E^\star$ will be represented by $\langle\cdot,\cdot\rangle_E$.

[2] For any $T > 0$, $\mathscr{H}_T(H)$ is the Hilbert space of all square integrable adapted processes, defined on $[0,T]$ and with values in H, see Section 3.

For any $p\geqslant 1$, $p\neq 2$, the usual norm in $L^p(\mathcal{O})$ will be denoted by $|\cdot|_p$. Finally, for any $s\in(0,1)$ and $p\geqslant 1$, we will denote by $W^{s,p}(\mathcal{O})$ the set of all functions $x\in L^p(\mathcal{O})$ such that

$$[x]_{s,p}=\int_{\mathcal{O}}\int_{\mathcal{O}}\frac{|x(\xi)-x(\zeta)|^p}{|\xi-\zeta|^{sp+d}}\,\mathrm{d}\xi\,\mathrm{d}\zeta<\infty.$$

$W^{s,p}(\mathcal{O})$, endowed with the norm

$$|x|_{s,p}=|x|_p+[x]_{s,p},$$

is a Banach space which is continuously embedded in E, for any s and p such that $sp>d$.

2.1. The operators A and B

In the sequel we shall assume that the operator A satisfies the following conditions.

Hypothesis 2.1. 1. *$A:D(A)\subset H\to H$ generates a negative self-adjoint analytic semigroup e^{tA} of class C_0. For any $p\geqslant 1$, e^{tA} has an extension as a semigroup on $L^p(\mathcal{O})$, that we still denote by e^{tA}.*

2. *There exists a complete orthonormal system $\{e_k\}$ on H which diagonalizes A. We denote by $\{-\alpha_k\}$ the corresponding set of eigenvalues. Concerning the eigenfunctions $\{e_k\}$, we shall assume that $e_k\in C^1(\bar{\mathcal{O}})$ and*

$$\sup_{k\in\mathbb{N}}\|e_k\|_E<\infty,\qquad \sup_{\xi\in\bar{\mathcal{O}}}|\nabla e_k(\xi)|\leqslant c\sqrt{\alpha_k}.$$

3. *If $s\in(0,1)$ and $p\geqslant 1$, e^{tA} maps $L^p(\mathcal{O})$ into $W^{s,p}(\mathcal{O})$, for any $t>0$, and there exists $r>d/2$ such that*

$$|\mathrm{e}^{tA}x|_{s,p}\leqslant ct^{-\frac{s}{r}}|x|_p, \tag{2.1}$$

for some constant c depending only on s and p.

4. *If A_E denotes the part of A in E, that is*

$$D(A_E)=\{x\in D(A)\cap E\colon Ax\in E\},\qquad A_Ex=Ax,$$

then A_E generates an analytic semigroup e^{tA_E} on E. In the sequel it will not be misleading to denote A_E and e^{tA_E} by A and e^{tA}.

We first remark that by using the semigroup law and the Sobolev embedding theorem, from the Hypothesis 2.1(3), it easily follows that e^{tA} is *ultracontractive*, that is e^{tA} is bounded from H to $L^\infty(\mathcal{O})$, for any $t>0$ and it holds

$$|\mathrm{e}^{tA}x|_\infty\leqslant ct^{-\frac{d}{2r}}|x|_H. \tag{2.2}$$

Notice that $d/2r<1$, so that the singularity arising at $t=0$ is integrable.

Moreover, since e^{tA} is self-adjoint, e^{tA} is bounded from $L^1(\mathcal{O})$ into H, for any $t>0$, and it holds

$$|\mathrm{e}^{tA}x|_H\leqslant ct^{-\frac{d}{2r}}|x|_1.$$

In particular this implies that e^{tA} is bounded from $L^1(\mathcal{O})$ into $L^\infty(\mathcal{O})$, for any $t > 0$, and

$$|e^{tA}x|_\infty \leqslant ct^{-\frac{d}{r}}|x|_1.$$

Therefore, by using the Riesz–Thorin interpolation theorem, it is easy to check that if $t > 0$ then e^{tA} is bounded from $L^p(\mathcal{O})$ into $L^q(\mathcal{O})$, for any $1 \leqslant p,\ q \leqslant \infty$, and it holds

$$|e^{tA}x|_q \leqslant ct^{-\frac{d(q-p)}{rpq}}|x|_p. \tag{2.3}$$

The *fractional powers* of the operator $-A$ are defined for any $\delta \in \mathbb{R}$ by setting

$$D((-A)^\delta) = \left\{ x \in H\colon \sum_{k=1}^{\infty} \langle x, e_k\rangle_H^2 \alpha_k^{2\delta} < \infty \right\},$$

$$(-A)^\delta x = \sum_{k=1}^{\infty} \langle x, e_k\rangle_H \alpha_k^\delta e_k, \quad x \in D((-A)^\delta).$$

We recall that since A generates an analytic semigroup, for any $t > 0$ and $\delta \in \mathbb{R}$ we have that $\mathrm{Range}(e^{tA}) \subset D((-A)^\delta)$ and

$$|(-A)^\delta e^{tA}x|_H \leqslant ct^{-\delta}|x|_H, \quad x \in H, \tag{2.4}$$

for a suitable constant c depending on δ (for more details about analytic semigroups see Lunardi (1995)).

The cylindrical Wiener process $w(t)$ is defined as

$$w(t) = \sum_{k=1}^{+\infty} e_k w_k(t), \tag{2.5}$$

where $\{e_k\}$ is the complete orthonormal system of H introduced in the Hypothesis 2.1 which diagonalizes A and $\{w_k(t)\}$ is a sequence of mutually independent real Brownian motions defined on a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and adapted to the filtration $\mathcal{F}_t$, $t \geqslant 0$. The series (2.5) does not converge in H, but it is convergent in any Hilbert space U such that the embedding $H \subset U$ is Hilbert–Schmidt (see Da Prato and Zabczyk (1992), Chapter 4).

Now, let us consider the Ornstein–Uhlenbeck equation corresponding to the problem (1.1)

$$\mathrm{d}u(t) = Au(t)\,\mathrm{d}t + B\,\mathrm{d}w(t), \quad u(0) = 0.$$

If we assume that for any $t \geqslant 0$

$$\int_0^t \mathrm{Tr}[e^{sA}B^2 e^{sA^\star}]\,\mathrm{d}s < \infty,$$

then such a problem admits a unique solution $w^A(t)$, which is the mean-square continuous Gaussian process with values in H given by

$$w^A(t) = \int_0^t e^{(t-s)A}B\,\mathrm{d}w(s). \tag{2.6}$$

In the sequel we will need $w^A(t)$ to be more regular. Thus we will assume that A and B commute and the following condition on their eigenvalues holds

Hypothesis 2.2. *$B : H \to H$ is a non-negative and self-adjoint bounded linear operator, diagonal with respect to the complete orthonormal basis $\{e_k\}$ which diagonalizes A. Moreover, if $\{\lambda_k\}$ is the corresponding set of eigenvalues, we have*

$$\sum_{k=1}^{\infty} \frac{\lambda_k^2}{\alpha_k^{1-\gamma}} < +\infty, \tag{2.7}$$

for some $\gamma \in (0,1)$.

It can be shown that, under the Hypotheses 2.1 and 2.2, $w^A(t)$ has an E-valued version with α-Hölder continuous paths, for any $\alpha \in [0, 1/4)$. Moreover, for any $T > 0$ and $p \geqslant 1$

$$\mathbb{E} \sup_{t \in [0,T]} |w^A(t)|_E^p < +\infty. \tag{2.8}$$

On the other hand, in order to get the regularizing effect of the semigroup P_t we need a sort of non degeneracy condition on B, that is we ask the range of B to be not *too small.*

Hypothesis 2.3. *There exists $\varepsilon < 1$ such that*

$$D((-A)^{-\varepsilon/2}) \subset D(B^{-1}). \tag{2.9}$$

Remark 2.4. The conditions of the Hypothesis 2.1 are all satisfied by the realization in H of the Laplace operator with Dirichlet boundary conditions

$$D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}), \quad Ax = \Delta x, \;\; x \in D(A),$$

in the case $d \leqslant 3$. Moreover, if we define for any $k \in \mathbb{N}$

$$Be_k = \frac{\beta_k}{\alpha_k^{\delta}}\, e_k,$$

(with $v < \beta_k < v^{-1}$, for some $v > 0$), then it is possible to prove that there exists $\delta \geqslant 0$ such that the Hypotheses 2.2 and 2.3 are both satisfied, if $d \leqslant 3$. Indeed it is possible to verify that Eq. (2.7) holds if and only if $\delta > (d/2 - 1)/2$. Moreover, Eq. (2.9) is satisfied by any $\delta \leqslant \varepsilon/2$, so that, if $d \leqslant 3$, it is possible to find some δ such that B verifies the Hypotheses 2.2 and 2.3 at the same time. Clearly, if we consider higher powers of the Laplace operator we can cover the case of dimension $d > 3$.

2.2. The Nemytskii operator

For any $x \in H$, the operator F is defined by

$$F(x(\xi)) = f(x(\xi)), \quad \xi \in \mathcal{O},$$

for a suitable function $f : \mathbb{R} \to \mathbb{R}$ which satisfies the following conditions.

Hypothesis 2.5. 1. *$f \in C^2(\mathbb{R})$ and there exists $m \geqslant 1$ such that*

$$\sup_{t \in \mathbb{R}} \frac{|f^{(j)}(t)|}{1+|t|^{2m+1-j}} < +\infty, \quad j=0,1,2.$$

2. *There exists $c \in \mathbb{R}$ such that for any $t \in \mathbb{R}$*

$$f'(t) \leqslant c.$$

Since f is not assumed to have linear growth, it easily follows that F is not well defined from H into itself. Nevertheless, it is well defined and continuous from E into E and it holds

$$|F(x)|_E \leqslant c(1+|x|_E^{2m+1}), \quad x \in E.$$

Moreover, since we assumed f to be twice differentiable, F is twice Fréchet differentiable in E and for any $x, h_1, h_2 \in E$ and $\xi \in \mathcal{O}$ we have

$$\begin{aligned} &DF(x)h_1(\xi) = f'(x(\xi))h_1(\xi), \\ &DF^2(x)(h_1,h_2)(\xi) = f''(x(\xi))h_1(\xi)h_2(\xi). \end{aligned} \tag{2.10}$$

This implies that for $j=1,2$

$$|DF^j(x)|_{\mathscr{L}^j(E)} \leqslant c(1+|x|_E^{2m+1-j}), \quad x \in E, \tag{2.11}$$

where $\mathscr{L}^1(E)=\mathscr{L}(E;E)$ and $\mathscr{L}^2(E)=\mathscr{L}(E;\mathscr{L}(E;E))$. In particular F and its derivative are locally Lipschitz continuous on E. Furthermore, from the Hypothesis 2.5(2) for any $x, h \in E$

$$\langle DF(x)h,h\rangle_H = \int_{\mathcal{O}} f'(x(\xi))h^2(\xi)\,\mathrm{d}\xi \leqslant c|h|_H^2. \tag{2.12}$$

In particular, for any $x, y \in E$ we have

$$\langle F(x)-F(y), x-y\rangle_H \leqslant c|x-y|_E^2. \tag{2.13}$$

The following stronger dissipativity condition will be assumed in the sequel.

Hypothesis 2.6. *There exist $a>0$ and $b, c \in \mathbb{R}$ such that for any $\sigma>0$ and $\rho \in \mathbb{R}$*

$$f(\sigma+\rho)-f(\rho) \leqslant -a\sigma^{2m+1}+b|\rho|^{2m+1}+c. \tag{2.14}$$

Remark 2.7. Let f_0 be a function of class C^2 such that

$$|f_0''(\sigma)| \leqslant c(1+|\sigma|^{2m-1-\varepsilon}), \quad \sigma \in \mathbb{R},$$

for some $\varepsilon > 0$. Then it is easy to show that the Hypotheses 2.5 and 2.6 are both satisfied by any function f given by

$$f(\sigma) = -a\sigma^{2m+1} + f_0(\sigma),$$

for a constant $a>0$ sufficiently large.

2.3. Functional spaces

If X is a Banach space with norm $|\cdot|_X$, we denote by $B_b(X)$ the Banach space of bounded and Borel functions $\varphi : X \to \mathbb{R}$, endowed with the *sup-norm*

$$||\varphi||_0^X = \sup_{x\in X} |\varphi(x)|.$$

$C_b(X)$ is the subspace of all uniformly continuous functions. The subspace of Lipschitz continuous functions is denoted by $C_b^{0,1}(X)$ and is endowed with the norm

$$||\varphi||_{0,1}^X = ||\varphi||_0^X + \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|_X}.$$

For each $k\geqslant 1$, $C_b^k(X)$ is the space of all functions $\varphi : X \to \mathbb{R}$ which are k times Fréchet differentiable, with bounded and uniformly continuous derivatives up the kth order. If we define

$$[\varphi]_h^X = \sup_{x\in X} |D^h\varphi(x)|, \quad h=1,\dots,k,$$

then $C_b^k(X)$ is a Banach space endowed with the norm

$$||\varphi||_k^X = ||\varphi||_0^X + \sum_{h=1}^k [\varphi]_h^X.$$

Clearly, if a Banach space X_1 is continuously embedded in another Banach space X_2, then $C_b^k(X_2)$ is continuously embedded in $C_b^k(X_1)$, for any $k\geqslant 0$. On the other hand, in the case $X_1=E$ and $X_2=H$, for any $\varphi\in C_b(E)$ there exists a sequence $\{\varphi_n\}\subset C_b(H)$ such that

$$\begin{aligned}&\lim_{n\to+\infty} \varphi_n(x)=\varphi(x), \quad x\in E,\\ &\sup_{n\in\mathbb{N}} ||\varphi_n||_0^H \leqslant ||\varphi||_0^E.\end{aligned} \tag{2.15}$$

Indeed, by standard arguments of reflection, it is possible to prove that there exists a bounded linear extension operator $P : H \to L^2(\mathbb{R}^d)$ such that for any $x\in E$ the function $Px\in E$. Thus, by setting

$$x_n(\xi) = n^d \int_{\xi_d}^{\xi_d+1/n} \cdots \int_{\xi_1}^{\xi_1+1/n} Px(\zeta)\,d\zeta_1\cdots d\zeta_d, \quad \xi\in\mathcal{O},$$

it can be proved that $x_n\in E$, $\varphi_n\in C_b(H)$ and Eq. (2.15) holds (for a more detailed proof see Cerrai (1999), Proposition 2.7).

3. The transition semigroup

Let X be a Banach space. For any $T>0$ we denote by $\mathscr{H}_T(X)$ the space of all adapted measurable processes $u : [0,T]\times\Omega\to X$ such that

$$||u||^2_{\mathscr{H}_T(X)} = \int_0^T \mathbb{E}|u(t)|_X^2\,dt < \infty.$$

$\mathscr{H}_T(X)$ is a Banach space endowed with the norm $\|\cdot\|_{\mathscr{H}_T(X)}$. Notice that if X is a Hilbert space, then $\mathscr{H}_T(X)$ is a Hilbert space, as well.

We denote by $\mathscr{K}_T(X)$ the subspace of $\mathscr{H}_T(X)$ consisting of all processes u such that

$$\|u\|^2_{\mathscr{K}_T(X)} = \sup_{t\in[0,T]} \mathbb{E}|u(t)|^2_X < \infty.$$

$\mathscr{K}_T(X)$ endowed with the norm $\|\cdot\|_{\mathscr{K}_T(X)}$ is a Banach space continuously embedded in $\mathscr{H}_T(X)$.

In this section we want to establish some properties of the solution of the following problem

$$\mathrm{d}u(t) = [Au(t) + F(u(t))]\,\mathrm{d}t + B\,\mathrm{d}w(t), \quad u(0) = x, \tag{3.1}$$

in order to introduce and describe the associated transition semigroup.

Definition 3.1. 1. A process $u(t;x)$ is a *mild* solution of Eq. (3.1) if

$$u(t;x) = \mathrm{e}^{tA}x + \int_0^t \mathrm{e}^{(t-s)A}F(u(s;x))\,\mathrm{d}s + w^A(t), \tag{3.2}$$

where $w^A(t)$ is the process given by Eq. (2.6).

2. A process $u(t;x)$ is a *generalized* solution of the equation (3.1) if for an arbitrary sequence $\{x_n\}\subset E$ converging to x in H, the corresponding sequence of mild solutions $\{u(t;x_n)\}$ converges to $u(t;x)$ in $C([0,T];H)$, $\mathbb{P}$-a.s. for any $T>0$.

In Da Prato and Zabczyk (1992), Theorem 7.13, it is proved that under the Hypotheses 2.1, 2.2 and 2.5 for any $x\in E$ Eq. (3.1) has a unique mild solution $u(x)$ in $L^2(\Omega; C(]0,+\infty[;E)\cap L^\infty_{\mathrm{loc}}(0,+\infty;E))$ and in Cerrai (1999) the following estimate is established

$$|u(t;x)|_E \leqslant \mathrm{e}^{ct}|x|_E + h(t), \quad \mathbb{P}\text{-a.s.} \tag{3.3}$$

where h is the process defined by

$$h(t) = c\mathrm{e}^{ct}\int_0^t (1 + |w^A(s)|_E^{2m+1})\,\mathrm{d}s + \sup_{s\in[0,t]} |w^A(s)|_E, \quad t\geqslant 0. \tag{3.4}$$

Moreover, if the Hypothesis 2.6 holds, as well, then for any $t>0$ we have

$$\sup_{x\in E} |u(t;x)|_E \leqslant k(t)t^{-\frac{1}{2m}}, \quad \mathbb{P}\text{-a.s.} \tag{3.5}$$

where the process k is defined by

$$k(t) = c\left(1 + \sup_{s\in[0,t]} |w^A(s)|_E\right), \quad t\geqslant 0. \tag{3.6}$$

Proposition 3.2. *Under the Hypotheses* 2.1, 2.2 *and* 2.5, *for any* $x,y\in E$ *and* $t\geqslant 0$ *we have*

$$|u(t;x) - u(t;y)|_H \leqslant \mathrm{e}^{ct}|x-y|_H, \quad \mathbb{P}\text{-}a.s. \tag{3.7}$$

In particular, for any $x \in H$ *Eq.* (3.1) *has a unique generalized solution* $u(t;x)$ *and it holds*

$$|u(t;x)|_H \leqslant e^{ct}|x|_H + h(t), \quad \mathbb{P}\text{-}a.s. \tag{3.8}$$

where h is defined by (3.4). *Moreover, if also the Hypothesis* 2.6 *is satisfied, then for any* $t > 0$ *we get*

$$\sup_{x \in H} |u(t;x)|_H \leqslant k(t) t^{-\frac{1}{2m}}, \quad \mathbb{P}\text{-}a.s. \tag{3.9}$$

where k is defined by (3.6).

Proof. Let us fix $x, y \in E$. If we define

$$v(t) = u(t;x) - u(t;y), \quad t \geqslant 0,$$

then v is the unique mild solution of the problem

$$\frac{d}{dt} v(t) = Av(t) + F(u(t;x)) - F(u(t;y)), \quad v(0) = x - y. \tag{3.10}$$

We can assume that v is a strict solution of Eq. (3.10). If that is not the case, we can approximate v by means of a more regular sequence in $C^1([0,T];H) \cap C([0,T];D(A))$ which converges to v in $C([0,T];H)$, $\mathbb{P}$-a.s. Indeed for any $\lambda \in \mathbb{N}$ we introduce the problem

$$\frac{d}{dt} v^\lambda(t) = Av^\lambda(t) + F(u(t;x)) - F(u(t;y)), \quad v^\lambda(0) = \lambda(\lambda - I)^{-1}(x - y)$$

which has a unique solution $v^\lambda(t;x,y)$ in $C([0,+\infty);H)$. For any $T > 0$ we have that $F(u(\cdot;x)) - F(u(\cdot;y)) \in C([0,T];H)$ and $v^\lambda(0) \in D(A)$. Then, as shown in Lunardi (1995) Proposition 4.1.8, v^λ is a strong solution, that is there exists a sequence $\{v^{\lambda,n}\} \subset C^1([0,T];H) \cap C([0,T];D(A))$ such that

$$v^{\lambda,n} \to v^\lambda, \quad \frac{d}{dt} v^{\lambda,n} - Av^{\lambda,n} \to F(u(\cdot;x)) - F(u(\cdot;y)) \quad \text{in } C([0,T];H),$$

as $n \to +\infty$. Then, if v is a strict solution of Eq. (3.10), from Eq. (2.13) we have

$$\frac{1}{2}\frac{d}{dt}|v(t)|_H^2 = \langle Av(t), v(t)\rangle_H + \langle F(u(t;x)) - F(u(t;y)), v(t)\rangle_H \leqslant c|v(t)|_H^2,$$

and by the Gronwall lemma this yields (3.7). Now, the existence and the uniqueness of a generalized solution of Eq. (3.1) follows from Eq. (3.7). Indeed, if $\{x_n\} \subset E$ is a sequence converging to x in H, due to Eq. (3.7) the corresponding sequence of mild solutions $\{u(t;x_n)\}$ is a Cauchy sequence in $C([0,T];H)$, $\mathbb{P}$-a.s. Hence it admits a limit in $C([0,T];H)$ as $n \to +\infty$, $\mathbb{P}$-a.s. which is the unique generalized solution $u(t;x)$. For any $x \in H$ we have

$$|u(t;x)|_H \leqslant |u(t;x) - u(t;0)|_H + |u(t;0)|_H \leqslant |u(t;x) - u(t;0)|_H + c|u(t;0)|_E,$$

and then Eq. (3.8) is a consequence of Eqs. (3.3) and (3.7). Finally, if $\{x_n\}$ is a sequence in E which converges to x in H as $n \to +\infty$, we have that

$$|u(t;x)|_H = \lim_{n \to +\infty} |u(t;x_n)|_H \leqslant c \limsup_{n \to +\infty} |u(t;x_n)|_E,$$

so that from Eq. (3.5) we get Eq. (3.9). □

Due to the previous proposition, we can introduce the *transition semigroup* associated with Eq. (3.1). For any $\varphi \in B_b(H)$ and $x \in H$ we define

$$P_t\varphi(x) = \mathbb{E}\varphi(u(t;x)), \quad t \geqslant 0,$$

where $u(t;x)$ is the unique generalized solution of Eq. (3.1). From Eq. (3.7), due to the definition of generalized solution, it easily follows that for any $t \geqslant 0$ and $x, y \in H$

$$|u(t;x) - u(t;y)|_H \leqslant e^{ct}|x-y|_H, \quad \mathbb{P}\text{-a.s.} \tag{3.11}$$

and then P_t is a *Feller* semigroup, that is for any $\varphi \in C_b(H)$ we have that $P_t\varphi \in C_b(H)$. Moreover,

$$||P_t\varphi||_0^H = \sup_{x \in H} |\mathbb{E}\varphi(u(t;x))| \leqslant ||\varphi||_0^H,$$

so that P_t is a contraction on $C_b(H)$.

Proposition 3.3. *For any $\varphi \in C_b(H)$ the family of functions $\{P_t\varphi;\ t \in [0,T]\}$ is equi-uniformly continuous, for any $T > 0$. Moreover, for any $x \in H$ the mapping*

$$[0,+\infty) \to \mathbb{R}, \quad t \mapsto P_t\varphi(x)$$

is continuous. In particular, for any $\varphi \in C_b(H)$ the mapping

$$[0,+\infty) \times H \to \mathbb{R}, \quad (t,x) \mapsto P_t\varphi(x)$$

is continuous.

Proof. The equi-uniform continuity of the family of functions $\{P_t\varphi;\ t \in [0,T]\}$ follows directly from Eq. (3.11). The continuity of the mapping

$$[0,+\infty) \to \mathbb{R}, \quad t \mapsto P_t\varphi(x)$$

follows from the dominated convergence theorem, as $u(.;x) \in C([0,T];H)$ for any fixed $x \in H$ and $T > 0$, $\mathbb{P}$-a.s. □

In Da Prato et al. (1995) it has been proved that, if $d = 1$ and $B = I$, the semigroup P_t enjoys the *strong Feller* property, that is

$$\varphi \in B_b(H) \Rightarrow P_t\varphi \in C_b(H), \quad t > 0. \tag{3.12}$$

Actually, it is proved that $P_t\varphi$ is Lipschitz continuous, for any $t > 0$ and

$$||P_t\varphi||_{0,1}^H \leqslant c(t \wedge 1)^{-1/2}||\varphi||_0^H.$$

In the present paper we want to show that, under the more general Hypotheses 2.1–2.3, $P_t\varphi \in C_b^1(H)$, for any $t > 0$ and $\varphi \in B_b(H)$, and

$$||P_t\varphi||_1^H \leqslant c(t \wedge 1)^{-\frac{1+\varepsilon}{2}}||\varphi||_0^H.$$

In Da Prato and Zabczyk (1992), Theorem 7.13, it is proved that for any $x \in E$ Eq. (3.1) has a unique solution $u(t;x)$ which is an E-valued process. Then for any $\varphi \in B_b(E)$ it is possible to define

$$P_t^E\varphi(x) = \mathbb{E}\varphi(u(t;x)), \quad t \geqslant 0.$$

P_t^E is the *transition semigroup* associated with Eq. (3.1), regarded as a stochastic differential equation on the Banach space E. As proved in Cerrai (1999), such a semigroup has a smoothing effect. Namely, if $f\in C^{k+1}(\mathbb{R})$ then

$$\varphi\in B_b(E)\Rightarrow P_t^E\varphi(x)\in C_b^k(E),\quad t>0. \tag{3.13}$$

Notice that since $B_b(H)$ is continuously embedded in $B_b(E)$, for any $\varphi\in B_b(H)$ and $x\in E$ it holds

$$P_t^E\varphi(x)=P_t\varphi(x),\quad t\geqslant 0. \tag{3.14}$$

4. Some approximation results

In Cerrai (1999) it is proved that if $f\in C^{k+1}(\mathbb{R})$, then the unique mild solution of Eq. (3.1) is k-times mean-square differentiable in E with respect to the initial datum $x\in E$. This means that for any fixed t the mapping

$$E\to L^2(\Omega;E),\quad x\mapsto u(t;x),$$

is k-times Fréchet differentiable. Besides, if $D_xu(t;x)h$ denotes the first order mean-square derivative of $u(t;x)$, at the point x and along the direction h, $D_xu(t;x)h$ is the unique mild solution of the following deterministic problem with random coefficients

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t)=Av(t)+DF(u(t;x))v(t),\quad v(0)=h. \tag{4.1}$$

We are proving now that

$$\sup_{x\in E}|D_xu(t;x)h|_H\leqslant \mathrm{e}^{ct}|h|_H,\quad \mathbb{P}\text{-a.s.} \tag{4.2}$$

As in the proof of the Proposition 3.2, we can assume that $D_xu(t;x)h$ is a strict solution of Eq. (4.1). Then we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|D_xu(t;x)h|_H^2$$
$$=\langle AD_xu(t;x)h,D_xu(t;x)h\rangle_H+\langle DF(u(t;x))D_xu(t;x)h,D_xu(t;x)h\rangle_H$$

and from Eq. (2.12) it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|D_xu(t;x)h|_H^2\leqslant c|D_xu(t;x)h|_H^2,$$

which easily implies Eq. (4.2).

Now, recalling Eq. (2.10) and the estimates satisfied by f and f', for any $x,y\in E$ we have

$$|DF(x)y|_H^2=\int_{\mathcal{O}}|f'(x(\xi))y(\xi)|^2\,\mathrm{d}\xi\leqslant c\int_{\mathcal{O}}(1+|x(\xi)|^{4m})|y(\xi)|^2\,\mathrm{d}\xi,$$

so that

$$|DF(x)y|_H\leqslant c|y|_E(1+|x|_E^{2m-1}|x|_H). \tag{4.3}$$

Then, since $D_x u(t;x)h$ is the mild solution of Eq. (4.1), from the *ultracontractivity* property of e^{tA} (see the Hypothesis 2.1(5)), we have

$$\begin{aligned}&|D_x u(t;x)h|_E\\ &\quad\leqslant ct^{-\frac{d}{2r}}|h|_H + c\int_0^t (t-s)^{-\frac{d}{2r}}|DF(u(s;x))D_x u(s)(x)h|_H\,ds\\ &\quad\leqslant ct^{-\frac{d}{2r}}|h|_H + c\int_0^t (t-s)^{-\frac{d}{2r}}(1+|u(s;x)|_E^{2m-1}|u(s;x)|_H)|D_x u(s;x)h|_E\,ds.\end{aligned}$$

By using Eqs. (3.8) and (3.5) this yields

$$\begin{aligned}|D_x u(t;x)h|_E &\leqslant ct^{-\frac{d}{2r}}|h|_H + c\int_0^t (t-s)^{-\frac{d}{2r}}(1+k(s)^{2m-1}s^{-1+\frac{1}{2m}}\\ &\qquad\times(e^{cs}|x|_H + h(s)))|D_x u(s;x)h|_E\,ds\end{aligned}$$

and due to a generalization of the Gronwall lemma, this implies that for any $x,h\in E$

$$|D_x u(t;x)h|_E \leqslant \theta^{|x|_H}(t)t^{-\frac{d}{2r}}|h|_H, \quad \mathbb{P}\text{-a.s.} \tag{4.4}$$

for a suitable positive process $\theta^{|x|_H}(t)$ increasing with respect to t and finite $\mathbb{P}$-a.s.

Proposition 4.1. *Assume the Hypotheses* 2.1, 2.2, 2.5 *and* 2.6. *Let* $x,h\in H$ *and let* $\{x_n\}$ *and* $\{h_n\}$ *be any two sequences in E converging, respectively, to x and h in H. Then the sequence* $\{D_x u(\cdot;x_n)h_n\}$ *converges in* $C([0,T];H)$, $\mathbb{P}$*-a.s to a process* $v(\cdot;x,h)$ *which will be called the* generalized solution of the first variation equation *relative to the problem* (3.1).

Proof. If we prove that for any $x,y,h,k\in E$ and $t\geqslant 0$

$$|D_x u(t;x)h - D_x u(t;y)k|_H \leqslant \theta^{|x|_H}(t)(|x-y|_H|k|_H + |h-k|_H), \quad \mathbb{P}\text{-a.s.} \tag{4.5}$$

then $\{D_x u(\cdot;x_n)h_n\}$ is a Cauchy sequence in $C([0,T];H)$, $\mathbb{P}$-a.s. and it converges to a process v which is independent on the particular choice of the sequences $\{x_n\}$ and $\{h_n\}$.

We set

$$z(t;x) = D_x u(t;x)h - D_x u(t;y)k \tag{4.6}$$

and from Eq. (4.1) we have

$$\begin{aligned}z(t;x) &= e^{tA}(h-k)\\ &\quad + \int_0^t e^{(t-s)A}(DF(u(s;x))D_x u(s;x)h - DF(u(s;y))D_x u(s;y)k)\,ds.\end{aligned}$$

Therefore, by using Eq. (2.3) for any $\theta\in(1,2]$ it follows

$$\begin{aligned}|z(t;x)|_H &\leqslant c|h-k|_H + c\int_0^t (t-s)^{-\frac{d(2-\theta)}{2r\theta}}|DF(u(s;x))z(s;x)|_\theta\,ds\\ &\quad + c\int_0^t (t-s)^{-\frac{d(2-\theta)}{2r\theta}}|(DF(u(s;x)) - DF(u(s;y)))D_x u(s;y)k|_\theta\,ds\\ &= c|h-k|_H + J_1(t) + J_2(t).\end{aligned}$$

Now, let us estimate $J_1(t)$ and $J_2(t)$, one by one. We first remark that if $x\in E$ and $y\in H$ and if $\theta\in(1,2]$ we have

$$|DF(x)y|_\theta^\theta\leqslant c\int_{\mathcal{O}}(1+|x(\xi)|^{2m})^\theta|y(\xi)|^\theta\,\mathrm{d}\xi\leqslant c|y|_H^\theta\left(1+\left(\int_{\mathcal{O}}|x(\xi)|^{\frac{4m\theta}{2-\theta}}\,\mathrm{d}\xi\right)^{\frac{2-\theta}{2}}\right),$$

so that by easy calculations

$$|DF(x)y|_\theta\leqslant c|y|_H\left(1+|x|_E^{2m-\frac{2-\theta}{\theta}}|x|_H^{\frac{2-\theta}{\theta}}\right). \tag{4.7}$$

Hence we get

$$\begin{aligned}J_1(t) &\leqslant c\int_0^t(t-s)^{-\frac{d(2-\theta)}{2r\theta}}\left(1+|u(s;x)|_E^{2m-\frac{2-\theta}{\theta}}|u(s;x)|_H^{\frac{2-\theta}{\theta}}\right)|z(s;x)|_H\,\mathrm{d}s\\ &\leqslant c\int_0^t(t-s)^{-\frac{d(2-\theta)}{2r\theta}}\left(1+k(s)^{2m-\frac{2-\theta}{\theta}}s^{-1+\frac{2-\theta}{2m\theta}}\right.\\ &\quad\left.\times(\mathrm{e}^{cs}|x|_H+h(s))^{\frac{2-\theta}{\theta}}\right)|z(s;x)|_H\,\mathrm{d}s,\end{aligned}$$

last inequality following from Eqs. (3.8) and (3.5).

Concerning the second term $J_2(t)$, for any $x,y\in E$, $z\in H$ and $\theta\in(1,2)$, we have

$$\begin{aligned}&|(DF(x)-DF(y))z|_\theta^\theta\\ &\quad=\int_{\mathcal{O}}\left|\int_0^1 f''(\rho x(\xi)+(1-\rho)y(\xi))\,\mathrm{d}\rho\right|^\theta|x(\xi)-y(\xi)|^\theta|z(\xi)|^\theta\,\mathrm{d}\xi\\ &\quad\leqslant c\int_{\mathcal{O}}(1+|x(\xi)|^{(2m-1)\theta}+|y(\xi)|^{(2m-1)\theta})|x(\xi)-y(\xi)|^\theta|z(\xi)|^\theta\,\mathrm{d}\xi\\ &\quad\leqslant c\int_{\mathcal{O}}(1+|x(\xi)|^{(2m-1)\theta}+|y(\xi)|^{(2m-1)\theta})|x(\xi)-y(\xi)|^{2-\theta}\\ &\qquad\times(|x(\xi)|^{2(\theta-1)}+|y(\xi)|^{2(\theta-1)})|z(\xi)|^\theta\,\mathrm{d}\xi\end{aligned}$$

and then it follows

$$|DF(x)z-DF(y)z|_\theta\leqslant c\left(1+|x|_E^{2m+1-\frac{2}{\theta}}+|y|_E^{2m+1-\frac{2}{\theta}}\right)|x-y|_H^{\frac{2-\theta}{\theta}}|z|_H. \tag{4.8}$$

Therefore, we have

$$\begin{aligned}J_2(t) &\leqslant c\int_0^t(t-s)^{-\frac{d(2-\theta)}{2r\theta}}\left(1+|u(s;x)|_E^{2m-\frac{2-\theta}{\theta}}+|u(s;y)|_E^{2m-\frac{2-\theta}{\theta}}\right)\\ &\quad\times|u(s;x)-u(s;y)|_H^{\frac{2-\theta}{\theta}}|D_xu(s;y)k|_H\,\mathrm{d}s,\end{aligned}$$

and from Eqs. (3.7), (3.5) and (4.2) we have

$$J_2(t) \leqslant c \int_0^t (t-s)^{-\frac{d(2-\theta)}{2r\theta}} \left(1 + k(s)^{2m - \frac{2-\theta}{\theta}} s^{-1+\frac{2-\theta}{2m\theta}}\right) e^{\frac{2c}{\theta}s} \, ds |x-y|_H^{\frac{2-\theta}{\theta}} |k|_H.$$

If we define

$$L(t,s) = c(t-s)^{-\frac{d(2-\theta)}{2r\theta}} \left(1 + k(s)^{2m - \frac{2-\theta}{\theta}} s^{-1+\frac{2-\theta}{2m\theta}} (e^{cs}|x|_H + h_s)^{\frac{2-\theta}{\theta}}\right)$$

$$M(t,s) = c(t-s)^{-\frac{d(2-\theta)}{2r\theta}} \left(1 + k(s)^{2m - \frac{2-\theta}{\theta}} s^{-1+\frac{2-\theta}{2m\theta}}\right) e^{\frac{2c}{\theta}s},$$

we have that $L(t,s)$ and $M(t,s)$ are both integrable in $[0,t]$ with respect to s and it holds

$$|z(t;x)|_H \leqslant |h-k|_H + |x-y|_H |k|_H \int_0^t M(t,s)\, ds + \int_0^t L(t,s)|z(s;x)|_H \, ds.$$

By using a modification of the Gronwall lemma, this implies Eq. (4.5). □

Before proving next lemma, we recall that, according to Eq. (2.4) and the Hypothesis 2.3 the operator $\Gamma(t) = B^{-1}e^{tA}$ is bounded for any $t > 0$ and

$$|\Gamma(t)x|_H \leqslant ct^{-\varepsilon/2}|x|_H, \quad x \in H. \tag{4.9}$$

Moreover, since $\Gamma(t) = \Gamma(t/2)S(t/2)$, from Eqs. (2.3) and (4.9) $\Gamma(t)$ maps $L^\theta(\mathcal{O})$ into H, for any $t > 0$ and $\theta \in (1,2]$, and it holds

$$|\Gamma(t)x|_H \leqslant ct^{-\frac{\varepsilon}{2}} |S(t/2)x|_H \leqslant ct^{-\frac{\varepsilon}{2} - \frac{d(2-\theta)}{2r\theta}} |x|_\theta, \quad x \in L^\theta(\mathcal{O}). \tag{4.10}$$

We recall that in Cerrai (1999) we proved that if also the Hypothesis 2.3 holds, then the process $D_x u(t;x)h$ belongs to $D(B^{-1})$ for any $t > 0$ and

$$\mathbb{E} \sup_{x \in E} \int_0^t |B^{-1} D_x u(s;x)h|_H^2 \, ds \leqslant c(t) t^{1-\varepsilon} |h|_H^2, \quad h \in E, \tag{4.11}$$

where $c(t) > 0$ is a continuous function, increasing in t.

Now, we prove that if in addition to the hypotheses of the previous proposition, the Hypothesis 2.3 is also assumed, then the following estimate holds.

Lemma 4.2. *For any $R > 0$ and $t \geqslant 0$ there exists a random variable $c^R(t)$ which is finite, $\mathbb{P}$-a.s., such that for any $x, y, h, k \in E$, with $x, y \in \{z \in H: |z|_H \leqslant R\}$, it holds*

$$\int_0^t |B^{-1}(D_x u(s;x)h - D_x u(s;y)k)|_H^2 \, ds \leqslant c^R(t)(|x-y|_H^\beta |k|_H + |h-k|_H)^2, \tag{4.12}$$

$\mathbb{P}$-a.s., for any $\beta < r(1-\varepsilon)/d$.

Proof. If we define $z(t) = D_x u(t;x)h - D_x u(t;y)k$, we have

$$B^{-1}z(t) = \Gamma(t)(h-k) + \int_0^t \Gamma(t-s) DF(u(s;x)) z(s)\, ds$$
$$+ \int_0^t \Gamma(t-s)(DF(u(s;x)) - DF(u(s;y))) D_x u(t;y)k \, ds.$$

We remark that if

$$\theta^\star = \frac{2d}{r(1-\varepsilon)+d}, \tag{4.13}$$

then for any $\theta > \theta^\star$ we have

$$\varepsilon + \frac{\mathrm{d}(2-\theta)}{r\theta} < 1. \tag{4.14}$$

Hence, by using Eqs. (4.9) and (4.10), for any $\theta^\star < \theta < 2$ we get

$$\begin{aligned}|B^{-1}z(t)|_H &\leqslant ct^{-\frac{\varepsilon}{2}}|h-k|_H + c\int_0^t (t-s)^{-\frac{\varepsilon}{2}-\frac{d(2-\theta)}{2r\theta}}|DF(u(s;x))z(s)|_\theta\,\mathrm{d}s\\ &\quad + c\int_0^t (t-s)^{-\frac{\varepsilon}{2}-\frac{d(2-\theta)}{2r\theta}}|(DF(u(s;x))-DF(u(s;y)))D_xu(t;y)k|_\theta\,\mathrm{d}s\\ &= ct^{-\frac{\varepsilon}{2}}|h-k|_H + I_1(t)+I_2(t).\end{aligned}$$

By using Eqs. (3.8), (3.5) and (3.7) and the Proposition 4.1, we have

$$\begin{aligned}I_1(t) &\leqslant c(|x-y|_H|k|_H + |h-k|_H)\int_0^t (t-s)^{-\frac{\varepsilon}{2}-\frac{d(2-\theta)}{2r\theta}}\theta^{|x|_H}(s)\\ &\quad\times\left(1+k(s)^{2m-\frac{2-\theta}{\theta}}s^{-1+\frac{2-\theta}{2m\theta}}(\mathrm{e}^{\gamma s}|x|_H + h(s))^{\frac{2-\theta}{\theta}}\right)\mathrm{d}s\\ &\leqslant (|x-y|_H|k|_H + |h-k|_H)\theta^{|x|_H}(t)t^{-\frac{\varepsilon}{2}-\frac{d(2-\theta)}{2r\theta}}, \quad \mathbb{P}\text{-a.s.}\end{aligned} \tag{4.15}$$

Concerning $I_2(t)$, by using Eqs. (3.7), (3.5) and (4.2) and according to Eq. (4.8) we get

$$\begin{aligned}I_2(t) &\leqslant c\int_0^t (t-s)^{-\frac{\varepsilon}{2}-\frac{d(2-\theta)}{2r\theta}}\left(1+k(s)^{2m-\frac{2-\theta}{\theta}}s^{-1+\frac{2-\theta}{2\theta m}}\right)\mathrm{e}^{\frac{2c}{\theta}s}\,\mathrm{d}s|x-y|_H^{\frac{2-\theta}{\theta}}|k|_H\\ &\leqslant \theta(t)t^{-\frac{\varepsilon}{2}-\frac{d(2-\theta)}{2r\theta}}|x-y|_H^{\frac{2-\theta}{\theta}}|k|_H, \quad \mathbb{P}\text{-a.s.}\end{aligned} \tag{4.16}$$

Then, from Eqs. (4.15) and (4.16) we get that

$$\begin{aligned}&|B^{-1}(D_xu(s;x)h - D_xu(s;y)k)|_H\\ &\quad\leqslant \theta^{|x|_H}(t)t^{-\frac{\varepsilon}{2}-\frac{d(2-\theta)}{2r\theta}}\left(|x-y|_H|k|_H + |x-y|_H^{\frac{2-\theta}{\theta}}|k|_H + |h-k|_H\right)\\ &\quad\leqslant \theta^{|x|_H,|y|_H}(t)t^{-\frac{\varepsilon}{2}-\frac{d(2-\theta)}{2r\theta}}\left(|x-y|_H^{\frac{2-\theta}{\theta}}|k|_H + |h-k|_H\right) \quad \mathbb{P}\text{-a.s.}\end{aligned} \tag{4.17}$$

This implies that

$$\int_0^t |B^{-1}(D_x u(s;x)h - D_x u(s;y)k)|_H^2\, \mathrm{d}s$$

$$\leqslant \int_0^t |\theta^{\,|x|_H,|y|_H}(s)|^2 s^{-\varepsilon-\frac{d(2-\theta)}{r\theta}}\, \mathrm{d}s \left(|x-y|_H^{\frac{2-\theta}{\theta}}|k|_H + |h-k|_H\right)^2$$

and hence, by setting

$$c^R(t) = \int_0^t |\theta^{R,R}(s)|^2 s^{-\varepsilon-\frac{d(2-\theta)}{r\theta}}\, \mathrm{d}s,$$

Eq. (4.12) follows. Now, since $\theta > \theta^\star$ we get that

$$\frac{2-\theta}{\theta} = \frac{2}{\theta} - 1 < \frac{2}{\theta^\star} - 1 = \frac{(1-\varepsilon)r}{\mathrm{d}}.$$

Proposition 4.3. *Let $x, h \in H$ and let $\{x_n\}$ and $\{h_n\}$ be any two sequences in E converging respectively to x and h in H. Then, under the Hypotheses* 2.1–2.3, 2.5 *and* 2.6 *we have*

$$\lim_{n\to+\infty} \mathbb{E} \int_0^t |B^{-1}(D_x u(s;x_n)h_n - v(s;x,h))|_H^2\, \mathrm{d}s = 0 \tag{4.18}$$

and it holds

$$\sup_{x\in H} \mathbb{E} \int_0^t |B^{-1}v(s;x,h)|_H^2\, \mathrm{d}s \leqslant c(t)t^{1-\varepsilon}|h|_H^2. \tag{4.19}$$

Proof. From Eq. (4.12) we have that the sequence $\{B^{-1}D_x u(x_n)h_n\}$ is a Cauchy sequence in $L^2([0,T];H)$, for any $T>0$. Then, since B^{-1} is closed, Eq. (4.18) follows from the Proposition 4.1 by the dominated convergence theorem. Finally, we get Eq. (4.19) from Eq. (4.11). □

5. Smoothing property of the transition semigroup

If $f \in C^{k+1}(\mathbb{R})$, then $P_t^E \varphi \in C_b^k(E)$, for any $\varphi \in B_b(E)$ and $t>0$. Moreover for any $\varphi \in C_b(E)$ and $x, h \in E$ the following Bismut–Elworthy type formula holds

$$\langle D(P_t^E\varphi)(x), h\rangle_E = \frac{1}{t}\mathbb{E}\varphi(u(t;x)) \int_0^t \langle B^{-1}D_x u(s;x)h, \mathrm{d}w(s)\rangle_H \tag{5.1}$$

(for a proof see Cerrai (1999)). Notice that, since B is not assumed to have a bounded inverse, in order to make the formula above meaningful we have preliminarily to prove that the estimate (4.11) holds.

Now, by using the approximation results proved in the previous section, we can state the main result of this work.

Theorem 5.1. *For any $\varphi \in B_b(H)$ and $t>0$ we have that $P_t\varphi \in C_b^1(H)$. Moreover, if $\varphi \in C_b(H)$, for any $x, h \in H$ it holds*

$$\langle D(P_t\varphi)(x), h\rangle_H = \frac{1}{t}\mathbb{E}\varphi(u(t;x)) \int_0^t \langle B^{-1}v(s;x,h), \mathrm{d}w(s)\rangle_H, \tag{5.2}$$

where the process $v(t;x,h)$ *is the one introduced in Proposition* 4.1. *In particular this implies that*

$$||P_t\varphi||_1^H \leqslant c(t\wedge 1)^{-\frac{1+\varepsilon}{2}}||\varphi||_0^H. \tag{5.3}$$

Proof. Let us fix $x,h\in H$ and let $\{x_n\}$ and $\{h_n\}$ be two sequences in E, converging, respectively, to x and h in H. Since $C_b(H)$ is continuously embedded in $C_b(E)$ and $P_t^E\varphi$ is differentiable, for any $n\in\mathbb{N}$ and $\varphi\in C_b(H)$ we have

$$\begin{aligned}&P_t^E\varphi(x_n+h_n)-P_t^E\varphi(x_n)\\&\quad=\langle D(P_t^E\varphi)(x_n),h_n\rangle_E+\int_0^1\langle D(P_t^E\varphi)(x_n+\rho h_h)-D(P_t^E\varphi)(x_n),h_n\rangle_E\,\mathrm{d}\rho.\end{aligned}$$

Recalling that $u(t;x+h)$ and $u(t;x)$ are generalized solutions of Eq. (3.1) with initial data $x+h$ and x, we have that $u(t;x_n+h_n)\to u(t;x+h)$ and $u(t;x_n)\to u(t;x)$ in $C([0,T];H)$, $\mathbb{P}$-a.s. as $n\to+\infty$. Then, since $\varphi\in C_b(H)$ from the dominated convergence theorem and from Eq. (3.14) it follows

$$\lim_{n\to+\infty}P_t^E\varphi(x_n+h_n)-P_t^E\varphi(x_n)=P_t\varphi(x+h)-P_t\varphi(x).$$

Besides, by using these arguments and the Proposition 4.3 it is easy to check that for any $t>0$

$$\lim_{n\to+\infty}\langle D(P_t^E\varphi)(x_n),h_n\rangle_E=\frac{1}{t}\mathbb{E}\varphi(u(t;x))\int_0^t\langle B^{-1}v(s;x,h),\mathrm{d}w(s)\rangle_H.$$

Thus, if we define

$$R_t(x,h)=\lim_{n\to+\infty}\int_0^1\langle D(P_t^E\varphi)(x_n+\rho h_h)-D(P_t^E\varphi)(x_n),h_n\rangle_E\,\mathrm{d}\rho$$

and if we show that

$$\lim_{|h|_H\to 0}\frac{|R_t(x,h)|}{|h|_H}=0, \tag{5.4}$$

it follows that $P_t\varphi$ is differentiable and

$$\langle D(P_t\varphi)(x),h\rangle_H=\frac{1}{t}\mathbb{E}\varphi(u(t;x))\int_0^t\langle B^{-1}v(s;x,h),\mathrm{d}w(s)\rangle_H. \tag{5.5}$$

According to Eq. (4.19), for any $\varphi\in C_b(H)$ we have that

$$\begin{aligned}|\langle D(P_t\varphi)(x),h\rangle_H|&\leqslant\frac{1}{t}||\varphi||_0^H\left(\mathbb{E}\int_0^t|B^{-1}v(s;x,h)|_H^2\,\mathrm{d}s\right)^{1/2}\\&\leqslant c(t)t^{-\frac{1+\varepsilon}{2}}||\varphi||_0^H|h|_H.\end{aligned}$$

Then, since P_t is a contraction semigroup and $P_t\varphi=P_1(P_{t-1}\varphi)$ for any $t\geqslant 1$, we get

$$\sup_{x\in H}|\langle D(P_t\varphi)(x),h\rangle_H|\leqslant c(t\wedge 1)^{-\frac{1+\varepsilon}{2}}||\varphi||_0^H|h|_H,\quad h\in H. \tag{5.6}$$

In particular, for any $t>0$ and $x\in H$, $D(P_t\varphi)(x):H\to\mathbb{R}$ is continuous and bounded, $P_t\varphi$ is Lipschitz continuous and it holds

$$||P_t\varphi||_{0,1}^H\leqslant c(t\wedge 1)^{-\frac{1+\varepsilon}{2}}||\varphi||_0^H. \tag{5.7}$$

From the semigroup law, for any $x, y, h \in E$ we have

$$\langle D(P_t^E\varphi)(x) - D(P_t^E\varphi)(y), h\rangle_E$$
$$= \frac{2}{t}\mathbb{E}(P_{t/2}^E\varphi(u(t/2;x)) - P_{t/2}^E\varphi(u(t/2;y)))\int_0^{t/2}\langle B^{-1}D_xu(s;x)h, \mathrm{d}w(s)\rangle_H$$
$$+ \frac{2}{t}\mathbb{E}(P_{t/2}^E\varphi)(u(t/2;y))\int_0^{t/2}\langle B^{-1}(D_xu(s;x)h - D_xu(s;y)h), \mathrm{d}w(s)\rangle_H$$

and then by easy computations from Eq. (5.7) we have

$$|\langle D(P_t^E\varphi)(x) - D(P_t^E\varphi)(y), h\rangle_E|$$
$$\leqslant \frac{c}{t}(t\wedge 1)^{-\frac{1+\varepsilon}{2}}||\varphi||_0^H$$
$$\times(\mathbb{E}|u(t/2;x) - u(t/2;y)|_H^2)^{1/2}\left(\mathbb{E}\int_0^{t/2}|B^{-1}D_xu(s;x)h|_H^2\,\mathrm{d}s\right)^{1/2}$$
$$+\frac{2}{t}||\varphi||_0^H\left(\mathbb{E}\int_0^{t/2}|B^{-1}(D_xu(s;x)h - D_xu(s;y)h)|_H^2\,\mathrm{d}s\right)^{1/2}.$$

Hence, by using Eq. (4.11) we have

$$\left|\int_0^1\langle D(P_t^E\varphi)(x_n+\rho h_h) - D(P_t^E\varphi)(x_n), h_n\rangle_E\,\mathrm{d}\rho\right|$$
$$\leqslant c(t\wedge 1)^{-(1+\varepsilon)}||\varphi||_0^H|h_n|_H^2 + \frac{2}{t}\|\varphi\|_0^H$$
$$\times\int_0^1\left(\mathbb{E}\int_0^{t/2}|B^{-1}(D_xu(s;x_n+ph_n)h_n - D_xu(s;x_n)h_n)|_H^2\mathrm{d}s\right)^{1/2}\mathrm{d}\rho.$$

Thus, if

$$\lim_{|h|_H\to 0}\frac{1}{|h|_H^2}\lim_{n\to+\infty}\mathbb{E}\int_0^{t/2}|B^{-1}(D_xu(s;x_n+ph_n)h_n - D_xu(s;x_n)h_n)|_H^2\mathrm{d}s = 0, \quad (5.8)$$

then we get Eq. (5.4) and the differentiability of $P_t\varphi$ follows for any $\varphi\in C_b(H)$. But (5.8) easily follows from (4.12) and from the dominated convergence theorem. Finally, the continuity of $D(P_t\varphi)$ is a consequence of Eq. (5.8), recalling that for any $x, h\in H$

$$\langle D(P_t\varphi)(x), h\rangle_H = \lim_{n\to+\infty}\langle D(P_t^E\varphi)(x_n), h_n\rangle_E.$$

Now, in order to prove that $P_t\varphi\in C_b^1(H)$, for any $\varphi\in B_b(H)$, we remark that from Eq. (5.7) we get

$$\mathrm{Var}(P_t(x,\cdot) - P_t(y,\cdot))\leqslant c(t\wedge 1)^{-\frac{1+\varepsilon}{2}}|x-y|_H,$$

where $P_t(x,\cdot)$ is the law of $u(t;x)$. This implies that for any $\varphi\in B_b(H)$ and $t>0$ $P_t\varphi\in C_b(H)$. Therefore, since $P_t\varphi = P_{t/2}(P_{t/2}\varphi)$, it follows that $P_t\varphi\in C_b^1(H)$. □

Remark 5.2. P_t is the restriction of P_t^E to $B_b(H)$, that is for any $\varphi\in B_b(H)$ $P_t^E\varphi$ extends to a Borel and bounded function defined in the whole space H and $P_t^E\varphi = P_t\varphi$.

Indeed, let $x \in H$ and let $\{x_n\} \subset E$ be a sequence converging to x in H. For any $n \in \mathbb{N}$ and $\varphi \in B_b(H)$ we have that $P_t^E \varphi(x_n) = P_t\varphi(x_n)$. Now, due to the previous theorem $P_t\varphi \in C_b(H)$ for any $t > 0$ and then for any $t > 0$

$$\exists \lim_{n\to+\infty} P_t^E \varphi(x_n) = P_t\varphi(x).$$

Thus, if we define

$$P_t^E \varphi(x) = \lim_{n\to+\infty} P_t^E \varphi(x_n), \quad x \in H, \tag{5.9}$$

we have that $P_t^E \varphi \in B_b(H)$ and $P_t^E \varphi = P_t\varphi$.

6. An application: the Hamilton–Jacobi equation

We are here concerned with the problem

$$\begin{aligned} &\frac{\partial y}{\partial t}(t,x) = \frac{1}{2}\mathrm{Tr}[B^2D^2 y(t,x)] + \langle Ax + F(x), Dy(t,x)\rangle_H - K(Dy(t,x)) + g(x) \\ &y(0,x) = \varphi(x), \end{aligned} \tag{6.1}$$

where $g, \varphi \in C_b(H)$ and $K : H \to \mathbb{R}$ is Lipschitz continuous. The problem (6.1) can be rewritten in the *mild* form

$$y(t,x) = P_t\varphi(x) + \int_0^t P_{t-s}(-K(Dy(s,\cdot)) + g)(x)\,\mathrm{d}s, \quad t \geqslant 0. \tag{6.2}$$

We are looking for a solution of Eq. (6.2) in the space Z_T consisting of all bounded continuous functions $y : [0,T] \times H \to \mathbb{R}$ such that $y(t,\cdot) \in C_b^1(H)$, for all $t > 0$ and the mapping

$$(0,T] \times H \to H, \quad (t,x) \mapsto t^{\frac{1+\varepsilon}{2}} Dy(t,x)$$

is measurable and bounded. Notice that ε is the constant strictly less than one appearing in Hypothesis 2.3. It is easy to prove that Z_T endowed with the norm

$$||y||_{Z_T} = \sup_{t\in[0,T]} ||y(t,\cdot)||_0^H + \sup_{t\in(0,T]} t^{\frac{1+\varepsilon}{2}} ||Dy(t,\cdot)||_0^H,$$

is a Banach space.

Theorem 6.1. *Under the same hypotheses as Theorem* 5.1, *Eq.* (6.2) *has a unique mild solution* $y \in Z_T$, *for any* $T > 0$.

Proof. We write Eq. (6.2) as

$$y = \gamma(y) + y_0,$$

where for any $(t,x) \in [0,T] \times H$

$$\gamma(y)(t,x) = -\int_0^t P_{t-s}(K(Dy(s,\cdot)))(x)\,\mathrm{d}s,$$

$$y_0(t,x) = P_t\varphi(x) + \int_0^t P_{t-s}g(x)\,\mathrm{d}s.$$

Clearly, we have

$$||P_t\varphi||_0^H \leqslant ||\varphi||_0^H \quad \text{and} \quad \left|\left| \int_0^t P_{t-s}g(\cdot)\,\mathrm{d}s \right|\right|_0^H \leqslant T||g||_0^H$$

and due to the Proposition 3.3 it follows that the mapping

$$y_0 : [0,T] \times H \to \mathbb{R}$$

is continuous and bounded. Moreover, by Theorem 5.1 we have

$$\sup_{x\in H} |D(P_t\varphi)(x)| \leqslant c(t\wedge 1)^{-\frac{1+\varepsilon}{2}} ||\varphi||_0^H,$$

$$\sup_{x\in H} \left| D\left(\int_0^t P_{t-s}g(\cdot)\,\mathrm{d}s \right)(x) \right| \leqslant c\int_0^t ((t-s)\wedge 1)^{-\frac{1+\varepsilon}{2}}\,\mathrm{d}s||g||_0^H \leqslant c||g||_0^H.$$

The measurability and the boundedness of the mapping

$$(0,T]\times H \to \mathbb{R}, \quad (t,x) \mapsto t^{\frac{1+\varepsilon}{2}} |Dy_0(t,x)|,$$

follows from the formula for the derivative of the semigroup P_t, so that $y_0 \in Z_T$. Notice that here and in the sequel it is crucial that $\varepsilon < 1$, so that $(1+\varepsilon)/2 < 1$ and the singularities arising for the derivatives are integrable.

Concerning the operator γ, it maps Z_T into itself. Indeed, for any $y \in Z_T$ we have

$$||\gamma(y)(t,\cdot)||_0^H \leqslant c\int_0^t ||K(Dy(s,\cdot))||_0^H\,\mathrm{d}s.$$

Since K is assumed to be Lipschitz continuous we have

$$|K(Dy(s,x))| \leqslant c|Dy(s,x)| + |K(0)| \leqslant cs^{-\frac{1+\varepsilon}{2}} ||y||_{Z_T} + |K(0)|$$

and then

$$||\gamma(y)(t,\cdot)||_0^H \leqslant cT^{\frac{1-\varepsilon}{2}} ||y||_{Z_T} + T|K(0)|. \tag{6.3}$$

Moreover, due to the Theorem 5.1, for any $s < t$ we have that $P_{t-s}(K(Dy(s,\cdot)))$ belongs to $C_b^1(H)$ and

$$\sup_{x\in H} |D[P_{t-s}(K(Dy(s,\cdot)))](x)|_H \leqslant c((t-s)\wedge 1)^{-\frac{1+\varepsilon}{2}} \sup_{x\in H} |K(Dy(s,x))|.$$

Therefore, since $\varepsilon < 1$, from Eq. (6.3) it follows that

$$\begin{aligned} \sup_{x\in H} |D(\gamma(y)(t,\cdot))(x)| &\leqslant c\int_0^t ((t-s)\wedge 1)^{-\frac{1+\varepsilon}{2}} s^{-\frac{1+\varepsilon}{2}}\,\mathrm{d}s||y||_{Z_T} \\ &\quad + c\int_0^t ((t-s)\wedge 1)^{-\frac{1+\varepsilon}{2}}\,\mathrm{d}s|K(0)|, \end{aligned} \tag{6.4}$$

so that $\gamma(y) \in Z_T$.

Now, if we prove that for T_0 sufficiently small γ is a contraction on Z_{T_0}, it follows that there exists a unique solution of Eq. (6.2) on Z_{T_0}. Let $y,z \in Z_T$. Proceeding as before, for any $t \geqslant 0$ we have

$$||\gamma(y)(t,\cdot) - \gamma(z)(t,\cdot)||_0^H \leqslant c \int_0^t ||K(Dy(s,\cdot)) - K(Dz(s,\cdot))||_0^H \, \mathrm{d}s,$$

$$||K(Dy(s,\cdot)) - K(Dz(s,\cdot))||_0^H \leqslant c||Dy(s,\cdot) - Dz(s,\cdot)||_0^H.$$

Then we get

$$||\gamma(y)(t,\cdot) - \gamma(z)(t,\cdot)||_0^H \leqslant c \int_0^t s^{-\frac{1+\varepsilon}{2}} \, \mathrm{d}s ||y - z||_{Z_T} = cT^{\frac{1-\varepsilon}{2}} ||y - z||_{Z_T}.$$

Furthermore, by arguing as in Eq. (6.4) we have

$$\sup_{x \in H} |D(\gamma(y)(t,\cdot) - \gamma(z)(t,\cdot))(x)|$$

$$\leqslant c \int_0^t (t-s)^{-\frac{1+\varepsilon}{2}} s^{-\frac{1+\varepsilon}{2}} \, \mathrm{d}s ||y - z||_{Z_T} \leqslant cT^{\frac{1-\varepsilon}{2}} ||y - z||_{Z_T},$$

which yields

$$||\gamma(y) - \gamma(z)||_{Z_T} \leqslant cT^{\frac{1-\varepsilon}{2}} |y - z||_{Z_T}. \tag{6.5}$$

Thus there exists T_0 small enough such that γ is a contraction on Z_{T_0} and existence and uniqueness of a mild solution on Z_{T_0} follow. We can extend such a solution in Z_T in a standard way, by repeating the same arguments in the intervals $[T_0, 2T_0]$, $[2T_0, 3T_0]$ and so on. □

7. For further reading

The following references are also of interest to the reader: Cannarsa and Da Prato (1991), Cerrai (1998a,b), Davies (1989), Gross (1967), Krylov (1995), Kusuoka and Stroock (1985) and Stroock (1981).

References

Bismut, J.M., 1981. Martingales, the Malliavin calculus and hypoellipticity general Hörmander's conditions. Z. Wahrscheinlichkeitstheorie verw. Gebiete 56, 469–505.

Cannarsa, P., Da Prato, G., 1991. Second-order Hamilton–Jacobi equations in infinite dimensions. SIAM J. Control Optim. 29(2), 474–492.

Cerrai, S., 1998a. Differentiability with respect to initial datum for solutions of SPDE'S with no Fréchet differentiable drift term. Comm. Appl. Anal. 2 (2), 249–270.

Cerrai, S., 1998b. Kolmogorov equations in Hilbert spaces with non-smooth coefficients. Comm. Appl. Anal. 2 (2), 271–297.

Cerrai, S., 1999. Smoothing properties of transition semigroups relative to SDE's with values in Banach spaces. Probab. Theory Rel. Fields 113(1), 85–114.

Da Prato, G., Zabczyk, J., 1992. Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge.

Da Prato, G., Zabczyk, J., 1996. Ergodicity for Infinite Dimensional Systems. Cambridge University Press, Cambridge.

Da Prato, G., Elworthy, K.D., Zabczyk, J., 1995. Strong Feller property for stochastic semilinear equations. Stochastic Anal. Appl. 13 (1), 35–45.

Davies, E.B., 1989. Heat Kernels and Spectral Theory. Cambridge University Press, Cambridge.

Elworthy, K.D., Li, X.M., 1994. Formulae for the derivatives of heat semigroups. J. Funct. Anal. 125, 252–286.

Gross, L., 1967. Potential theory in Hilbert spaces. J. Funct. Anal. 1, 123–189.

Krylov, N.V., 1995. Introduction to the Theory of Diffusions Processes. American Mathematical Society, Providence, RI.

Kusuoka, S., Stroock, D.W., 1985. Some boundedness properties of certain stationary diffusion semigroups. J. Funct. Anal. 60, 243–264.

Lunardi, A., 1995. Analytic Semigroups and Optimal Regularity in Parabolic Problems. Birkhäuser, Basel, 1995.

Peszat, S., Zabczyk, J., 1995. Strong Feller property and irreducibility for diffusion processes on Hilbert spaces. Ann. Probab. 23 (1), 157–172.

Stroock, D.W., 1981. The Malliavin calculus, a functional analytic approach. J. Funct. Anal. 40, 212–257.