Math 410. HW 5 Solutions

1. By definition,
$$\binom{n}{0} = \frac{n!}{0!n!} = 1, \quad \binom{n}{n} = \frac{n!}{n!0!} = 1. \diamond$$

2. By definition,
$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}. \diamond$$

3.
$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!((n-k+1)+k)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}. \diamond$$

4. Let $\beta \in \mathbb{R}$ be fixed and $f(x) = (1+x)^\beta$. The Taylor polynomial of degree n around 0 is $p_n(x) = \sum_{k=0}^n \binom{\beta}{k} x^k$. The Lagrange formula for the $n-$th remainder is $R_n(x) = \binom{\beta}{n+1}(1+\xi)^{\beta-n-1}x^{n+1}$, for some $\xi \in [0, x]$. Then, to prove the Binomial Expansion for $x \in [0, 1)$, we will show that $\lim_{n\to\infty} R_n(x) = 0$. First, we find an upper bound,
$$|R_n(x)| \leq \Big|\binom{\beta}{n+1}(1+\xi)^\beta \Big(\frac{x}{1+x}\Big)^{n+1}\Big| \leq 2^\beta \Big|\binom{\beta}{n+1}\Big(\frac{x}{1+x}\Big)^{n+1}\Big| =: B_n(x).$$
If $\beta \in \mathbb{N} \cup \{0\}$, the Binomial Expansion Formula is finite, and $R_n(x) = 0$ for $n \geq \beta$. (This is the usual Binomial Formula.) If $\beta \in \mathbb{R} \setminus (\mathbb{N} \cup \{0\})$, $\binom{\beta}{k}$ is never 0, and we may use the quotient test. Keeping in mind that $x \in [0, 1)$, we have
$$\frac{B_{n+1}(x)}{B_n(x)} = \Big|\frac{\beta - n - 1}{n+1}\Big|\frac{x}{1+x} \underset{n\to\infty}{\longrightarrow} \frac{x}{1+x} < 1.$$
Thus, $\lim_{n\to\infty} B_n(x) = 0$, which implies that $\lim_{n\to\infty} R_n(x) = 0$, and the Binomial Expansion Formula holds for $x \in [0, 1)$. $\diamond$

5. We assume $\beta \notin (\mathbb{N} \cup \{0\})$, so the series of Problem 4 not finite. Let us apply the quotient test to find its radius of convergence.
$$\lim_{k\to\infty} \frac{\big|\binom{\beta}{k+1}\big|}{\big|\binom{\beta}{k}\big|} = \lim_{k\to\infty} \frac{|\beta - k|}{k+1} = 1.$$
Hence, the radius of convergence of the Binomial Expansion is 1. In particular, it does not converge for $|x| > 1$. $\diamond$