Haar transform

0.1. **Haar wavelets on $\mathbb{R}$.**

Definition 0.1. Let $j, k \in \mathbb{Z}$. Define the interval $I_{j,k}$ to be

$$I_{j,k} = [2^{-j}k, 2^{-j}(k+1)).$$

We say that $(j,k) \neq (j',k')$ if either $j \neq j'$ or $k \neq k'$.

Lemma 0.2. Let $j, j', k, k' \in \mathbb{Z}$ be such that $(j,k) \neq (j',k')$. Then, one of the following is true:

(i) $I_{j,k} \cap I_{j',k'} = \emptyset$, or

(ii) $I_{j,k} \subset I_{j',k'}$, or

(iii) $I_{j',k'} \subset I_{j,k}$.

Given the integers $j, k \in \mathbb{Z}$ and the interval $I_{j,k}$, we define $I^l_{j,k}$ (resp., $I^r_{j,k}$) to be the left half of $I_{j,k}$ (resp., the right half of $I_{j,k}$). Thus,

$$I^l_{j,k} = [2^{-j}k, 2^{-j}k + 2^{-j-1}),$$
$$I^r_{j,k} = [2^{-j}k + 2^{-j-1}, 2^{-j}(k+1)).$$

It is not difficult to see that:

$$I^l_{j,k} = I_{j+1,2k} \quad \text{and} \quad I^r_{j,k} = I_{j+1,2k+1}.$$

With this observation in mind, we have the following lemma.

Lemma 0.3. Let $j, j', k, k' \in \mathbb{Z}$ be such that $(j,k) \neq (j',k')$. If $I_{j,k} \subset I_{j',k'}$ then

(i) either $I_{j,k} \subset I^l_{j',k'}$

(ii) or $I_{j,k} \subset I^r_{j',k'}$.

Definition 0.4. The *system of Haar scaling functions* is defined as follows:

Let $\chi_{[0,1)}$ be the characteristic function of interval $[0,1)$, i.e.,

$$\chi_{[0,1)}(x) = \begin{cases} 1 & x \in [0,1), \\ 0 & \textit{otherwise}. \end{cases}$$

Let $p(x) = \chi_{[0,1)}(x)$, let $j, k \in \mathbb{Z}$, and let

$$p_{j,k}(x) = 2^{j/2} p(2^j x - k) = D_{2^j} T_k(p)(x),$$

where D_a is the L^2-dilation by a: $D_a(f) = \sqrt{a} f(ax)$ (this means that for all finite energy functions f, $\|f\|_2 = \|D_a(f)\|_2$), and T_b is a translation by b (T_b also has this property: $\|f\|_2 = \|T_b(f)\|_2$).

Lemma 0.5. For all $j, k \in \mathbb{Z}$,

$$p_{j,k} = \chi_{I_{j,k}}.$$

It is not difficult to observe that for all $j, k \in \mathbb{Z}$:

$$\int_{\mathbb{R}} p_{j,k}(x)\, dx = 2^{-j/2}$$

and

$$\int_{\mathbb{R}} |p_{j,k}(x)|^2\, dx = 1.$$

Definition 0.6. The *Haar wavelet system* on $\mathbb{R}$ is defined as follows:

Let $h(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x)$, let $j, k \in \mathbb{Z}$, and let

$$h_{j,k}(x) = 2^{j/2} h(2^j x - k) = D_{2^j} T_k(h)(x).$$

For each fixed $J \in \mathbb{Z}$, we refer to $\{p_{J,k}, h_{j,k} : k \in \mathbb{Z}, j \geq J\}$ as the Haar wavelet system of scale J.

Clearly, we have

$$h_{j,k} = 2^{j/2} \left(\chi_{I^l_{j,k}} - \chi^r_{j,k} \right).$$

Moreover, for each $j, k \in \mathbb{Z}$, we have

$$\int_{\mathbb{R}} h_{j,k}(x)\, dx = 0$$

and

$$\int_{\mathbb{R}} |h_{j,k}(x)|^2\, dx = 1.$$

Also, because of our observations about $I^l_{j,k}$ and $I^r_{j,k}$, the Haar wavelets and scaling functions satisfy the following relations:

$$p_{l,k} = \frac{1}{\sqrt{2}}(p_{l+1,2k} + p_{l+1,2k+1})$$

and

$$h_{l,k} = \frac{1}{\sqrt{2}}(p_{l+1,2k} - p_{l+1,2k+1}).$$

Theorem 0.7. The Haar wavelet system $\{h_{j,k} : j, k \in \mathbb{Z}\}$ on $\mathbb{R}$ is an orthonormal basis for $L^2(\mathbb{R})$.

We shall only look at the proof of orthonormality of the Haar system on $\mathbb{R}$, leaving the completeness part for Advanced Calculus classes.

First, fix $j \in \mathbb{Z}$ and suppose that $k \neq k'$. We have that $h_{j,k}(x)h_{j,k'}(x) = 0$ for all $x \in R$, since the function $h_{j,k}$ assumes non-zero values only in the interval $I_{j,k}$, and for $k \neq k'$,

$$I_{j,k} \cap I_{j,k'} = \emptyset.$$

If $k = k'$, then

$$< h_{j,k}, h_{j,k} >= \int_{\mathbb{R}} |h_{j,k}(x)|^2 \, dx = 1.$$

Assume now that $j \neq j'$. Without loss of generality it is sufficient to consider the case $j > j'$. Then, it follows from the first two Lemmas that there are 3 distinct possibilities: $I_{j,k} \cap I_{j',k'} = \emptyset$, or $I_{j,k} \subset I^l_{j',k'}$, or $I_{j,k} \subset I^r_{j',k'}$.

Case $I_{j,k} \cap I_{j',k'} = \emptyset$ is elementary, as in this case $h_{j,k}(x)h_{j',k'}(x) = 0$ and so

$$< h_{j,k}, h_{j',k'} >= 0.$$

Case $I_{j,k} \subset I^l_{j',k'}$ implies that whenever $h_{j,k}$ is non-zero, $h_{j',k'}$ is constantly equal to 1. Thus,

$$< h_{j,k}, h_{j',k'} >= \int_{I_{j,k}} h_{j,k}(x)h_{j',k'}(x) \, dx = \int_{I_{j,k}} h_{j,k}(x) \, dx = 0.$$

Similarly, the case $I_{j,k} \subset I^r_{j',k'}$ implies that whenever $h_{j,k}$ is non-zero, $h_{j',k'}$ is constantly equal to -1, and so,

$$< h_{j,k}, h_{j',k'} >= \int_{I_{j,k}} h_{j,k}(x) h_{j',k'}(x)\, dx = -\int_{I_{j,k}} h_{j,k}(x)\, dx = 0.$$

This completes the proof of orthonormality of the Haar wavelet system on $\mathbb{R}$.

For the Haar wavelet systems of scale J, we have analogous result.

Theorem 0.8. The Haar wavelet systems of scale J, $\{p_{J,k}, h_{j,k} : k \in \mathbb{Z}, j \geq J\}$, on $\mathbb{R}$ is an orthonormal basis for $l^2(\mathbb{R})$.

(The proof of this result is similar to the previous one.)

0.2. **Haar wavelets on** $[0, 1]$**.** Fix an integer $J \geq 0$. The *Haar wavelet system of scale* J *on* $[0, 1]$ is defined as

$$\{p_{J,k} : 0 \leq k \leq 2^J - 1\} \cup \{h_{j,k} : j \geq J, 0 \leq k \leq 2^j - 1\}.$$

When $J = 0$ we refer to this system simply as the Haar wavelet system on $[0, 1]$.

Here, the choice of k's and the assumption about $J \geq 0$ are necessary so that the system we have created is a collection of functions which are non-zero only in the interval $[0, 1]$.

Remark. Note that each and every Haar system on $[0, 1]$ consists of **both** Haar wavelet functions and Haar scaling functions. This is to compensate the fact that we have restricted the set of possible parameters j, k.

Theorem 0.9. The Haar wavelet system of scale J on $[0, 1]$ is an orthonormal basis on $[0, 1]$.

0.3. **Discrete Haar transform.** Fix $N > 0$ and let $c_0(k) =< f, p_{N,k} >$, $k = 0, \ldots, 2^N - 1$. This is our starting finite sequence of length 2^N. One may think of this sequence as of a finite approximation to a given signal f of length 1. We fix $J > 0$ and for each $1 \leq j \leq J$ we define the coefficients:

$$c_j(k) =< f, p_{N-j,k} >$$

and

$$d_j(k) = < f, h_{N-j,k} > .$$

Because of the relationship that holds together Haar wavelets and Haar scaling functions (stated in these notes before Theorem 0.7), the following formulas hold for the coefficients:

$$c_j(k) = \frac{1}{\sqrt{2}}(c_{j-1}(2k) + c_{j-1}(2k+1))$$

and

$$d_j(k) = \frac{1}{\sqrt{2}}(c_{j-1}(2k) - c_{j-1}(2k+1)).$$

These equations can be rewritten equivalently in the matrix-vector form:

$$\begin{pmatrix} c_j(k) \\ d_j(k) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_{j-1}(2k) \\ c_{j-1}(2k+1) \end{pmatrix}.$$

By taking the inverse of the above matrix (which has a non-zero determinant, and so, is invertible), we can re-write the matrix-vector equation above as follows:

$$\begin{pmatrix} c_{j-1}(2k) \\ c_{j-1}(2k+1) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_j(k) \\ d_j(k) \end{pmatrix}.$$

Definition 0.10. Given $J < N$ and a finite sequence $c_0 = (c_0(0), \ldots, c_0(2^N - 1))$, the Discrete Haar transform DHT_J of scale J of c_0 is defined to be the finite sequence of coefficients:

$$\{d_j(k) : 1 \le j \le J; 0 \le k \le 2^{N-j} - 1\} \cup \{c_J(k) : 0 \le k \le 2^{N-j} - 1\},$$

where we use the following formulas to compute c_j's and d_j's:

$$c_j(k) = \frac{1}{\sqrt{2}}(c_{j-1}(2k) + c_{j-1}(2k+1))$$

and

$$d_j(k) = \frac{1}{\sqrt{2}}(c_{j-1}(2k) - c_{j-1}(2k+1)).$$

Define the following auxiliary $l/2 \times L$ matrices:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 0 & \dots \\ \dots & & & & & \\ 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix}$$

and

$$G = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & -1 & 0 & \dots \\ \dots & & & & & \\ 0 & \dots & \dots & 0 & 1 & -1 \end{pmatrix}$$

In the matrix-vector notation, the DHT_J produces the vector

$$(d_1,\ d_2,\ \dots\ d_J,\ c_J)$$

where each d_j is a vector of length 2^{N-j}, $1 \le j \le J$, and c_J is a vector of length 2^{N-J}, and where

$$\begin{pmatrix} c_j \\ d_j \end{pmatrix} = \begin{pmatrix} H \\ G \end{pmatrix} c_{j-1}.$$

Here, H and G are $2^{N-j-2} \times 2^{N-j-1}$ matrices, and

In the extreme case when $J = N - 1$, d_J and c_J are both vectors of length 2.

The inverse of DHT_J is computed by applying the formula:

$$c_{j-1} = H^*(c_j) + G^*(d_j),$$

up to $j = 1$. This is related to the fact that the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is its own inverse; hence, the inverse of

$$\begin{pmatrix} H \\ G \end{pmatrix}$$

is (H^*, G^*).