3 The Lebesgue integral

3.1 Motivation

An excellent description of the motivation to develop the notion of the Lebesgue integral has been given by LEBESGUE himself in an article, “Development of the integral concept” (1926), which appears in [156].

We begin by recalling the definition of the Riemann integral for certain bounded functions $f : [a, b] \to \mathbb{R}$. For any partition $P : a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$ consider the numbers

$$S_P = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \quad \text{and} \quad s_P = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),$$

(3.1)

where

$$M_i = \sup \{ f(x) : x_{i-1} < x \leq x_i \}$$

and

$$m_i = \inf \{ f(x) : x_{i-1} < x \leq x_i \}.$$

Define

$$R \int_a^b f = \inf_P S_P \quad \text{and} \quad R \int_a^b f = \sup_P s_P.$$  

(3.2)

Clearly, $R \int_a^b f \geq R \int_a^b f$ and we say that $f$ is Riemann integrable if $R \int_a^b f = R \int_a^b f$. In this case, the Riemann integral of $f$ over $[a, b]$ is

$$R \int_a^b f = R \int_a^b f = R \int_a^b f.$$

Note that

$$R \int_a^b f = \inf \left\{ \int_a^b \psi : \psi \geq f, \psi = \sum_{j=1}^{n} c_j \mathbb{1}_{(x_{j-1}, x_j]} \right\},$$
where \( x_0 < x_1 < \ldots < x_n \) is a partition of \([a, b]\) and

\[
\int_a^b \psi = \sum_{j=1}^n c_j(x_j - x_{j-1}).
\]

**Example 3.1.1. A non-Riemann integrable function**

Define the function \( f : [a, b] \to \mathbb{R} \) as

\[
f(x) = \begin{cases} 
0, & \text{if } x \in [a, b] \text{ is irrational}, \\
1, & \text{if } x \in [a, b] \text{ is rational}.
\end{cases}
\]

Clearly,

\[
\int_a^b f = b - a \quad \text{and} \quad \int_a^b f = 0
\]

so that \( f \) is not Riemann integrable.

Lebesgue’s observation goes something like this. The numbers \( \overline{\int f} \) and \( \underline{\int f} \) will be close if somehow there is a lot of continuity in each interval of each partition. Example 3.1.1 shows that this will not happen if \( f \) has many discontinuities. Lebesgue’s goal was to collect approximately equal values of \( f \). He proceeded in the following way. Let \( f : [a, b] \to \mathbb{R} \) be a bounded Lebesgue measurable function, and consider the partition

\[
Q : \alpha = y_0 < y_1 < \ldots < y_n = \beta,
\]

where

\[
\alpha = \inf \{ f(x) : x \in [a, b] \} \quad \text{and} \quad \beta = \sup \{ f(x) : x \in [a, b] \},
\]

and the norm of \( Q \) is

\[
|Q| = \max \{|y_j - y_{j-1}| : j = 1, \ldots, n - 1\}.
\]

If \( A_j = \{ x : y_{j-1} < f(x) \leq y_j \}, j = 1, \ldots, n, A_0 = \{ x : f(x) = \alpha \}, \) and \( y_{-1} = \alpha \), we define

\[
S_Q = \sum_{j=0}^n y_j m(A_j) \quad \text{and} \quad s_Q = \sum_{j=0}^n y_{j-1} m(A_j),
\]

and

\[
\int_a^b f = \inf_Q S_Q \quad \text{and} \quad \int_a^b f = \sup_Q s_Q.
\]

A major initial result is the following.
### Theorem 3.1.2. Integrability of bounded measurable functions

Let \( f : [a, b] \rightarrow \mathbb{R} \) be a bounded Lebesgue measurable function. Then

\[
\int_a^b f = \int_a^b f,
\]

and the common value is denoted by \( \int_a^b f \).

**Proof.** i. We must prove that \( \sup s_Q = \inf S_Q \), where \( Q \) is a partition of \([a, \beta]\) and \( \alpha \leq f(x) \leq \beta \) for \( x \in [a, b] \). We begin by noting that \( -\infty < \sup s_Q \), and \( \inf S_Q < \infty \); in fact, \( \alpha(b - a) \leq s_Q \leq S_Q \leq \beta(b - a) \). For any \( \varepsilon > 0 \) we shall verify that \( |\sup s_Q - \inf S_Q| < \varepsilon \).

ii. If \( Q' \subseteq Q \), we note that

\[
s_Q' \leq s_Q \quad \text{and} \quad S_Q' \geq S_Q. \tag{3.3}
\]

In fact, without loss of generality, take \( Q' : \alpha = y_0 < \ldots < y_n = \beta \) and let \( Q = Q' \cup \{y\} \) where \( y_{j-1} < y < y_j \). Define \( B_j = \{x : y_{j-1} < f(x) \leq y_j\} \) and \( C_j = \{x : y_j < f(x) \leq y_{j+1}\} \) so that

\[
y_j m(B_j) + y m(C_j) \leq y_j (m(B_j) + m(C_j)) = y_j m(A_j).
\]

Consequently, \( s_Q \leq S_Q' \), with a similar calculation for \( s_Q \). (3.3) is obtained.

iii. Therefore, since \( \sup s_Q \) and \( \inf S_Q \) are finite, there are partitions \( Q_1 \) and \( Q_2 \) of \([\alpha, \beta]\) such that

\[
|\sup s_Q - s_{Q_1}| < \frac{\varepsilon}{3} \quad \text{and} \quad |\inf S_Q - S_{Q_2}| < \frac{\varepsilon}{3}.
\]

Moreover, because of (3.3), we may assume that if \( y_{i,k} = Q_k, \ k = 1, 2 \), then \( y_{i,k} - y_{i,k-1} < \varepsilon/(3(b - a)) \). Let \( \overline{Q} \) be the partition formed by the points in both \( Q_1 \) and \( Q_2 \), i.e., \( \overline{Q} = Q_1 \cup Q_2 = \{y_i\} \). Then \( |s_{\overline{Q}} - s_{\overline{Q}}| < \varepsilon \) because of the triangle inequality and the fact that

\[
0 \leq S_{\overline{Q}} - s_{\overline{Q}} = \sum_{i=0}^n (\overline{y}_{i+1} - \overline{y}_i)m(A_i) < \frac{\varepsilon}{3(b - a)} \sum_{i=0}^n m(A_i) = \frac{\varepsilon}{3}.
\]

Combining these inequalities we obtain \( |\inf s_Q - \sup S_Q| < \varepsilon \). \( \square \)

**Remark.** If \( g \) is a simple function

\[
g = \sum_{j=1}^k a_j \mathbb{1}_{A_j}
\]
defined on \([a, b], M([a, b]), m\), it is natural to define the integral \(\int_a^b g\) as
\[
\int_a^b g = \sum_{j=1}^{k} a_j m(A_j),
\]

cf., the Remark at the very beginning of Section 3.2. It is not difficult to check that this definition agrees with our definition of the Lebesgue integral in Theorem 3.1.2, see also Section 3.4. In light of Theorem 2.5.5, Theorem 3.1.2 can be restated as follows. If \(f : [a, b] \to \mathbb{R}\) is a bounded Lebesgue measurable function, then there is a sequence \(\{f_n : n = 1, \ldots\}\) of simple functions such that \(f_n \to f\) pointwise (in fact, the convergence is uniform and \(\|f_n\|_\infty \leq \|f\|_\infty\)) and
\[
\int_a^b f_n \to \int_a^b f. \tag{3.4}
\]
In this form it is interesting to compare this result with the Lebesgue dominated convergence theorem (LDC) in Section 3.3.

Because of Theorem 3.1.2 we define the Lebesgue integral of a bounded Lebesgue measurable function \(f : [a, b] \to \mathbb{R}\) to be \(\int_a^b f\).

**Example 3.1.3.** \(1_Q\) on \([a, b]\)

**a.** Take \(f\) as in Example 3.1.1. Clearly \(f\) is Lebesgue measurable. For the partition \(R : 0 = y_0 < y_1 < \ldots < y_n = 1\) we have
\[
S_R = y_1(b-a).
\]
Therefore, \(\inf_Q S_Q = \int_a^b f = 0\).

**b.** Let \(\{r_n : n = 1, \ldots\}\) be an ordering of \(\mathbb{Q} \cap [a, b]\) and define the function \(f_m : [a, b] \to \mathbb{R}\) as
\[
f_m(x) = \begin{cases} 1, & \text{if } x = r_1, \ldots, r_m, \\ 0, & \text{if otherwise.} \end{cases}
\]
Clearly \(f_m \geq 0\) and \(\{f_m : m = 1, \ldots\}\) increases pointwise to the function \(f\) defined in Example 3.1.1. Also,
\[
\forall m = 1, \ldots, \quad R \int_a^b f_m = 0. \tag{3.5}
\]
Even though (3.5) is true, \(R \int_a^b f\) does not exist as we showed in Example 3.1.1. One of the beauties of Lebesgue’s theory is that (on a finite interval \([a, b]\), say) if \(g_n \geq 0\) is measurable and the sequence \(\{g_n : n = 1, \ldots\}\) increases pointwise to a bounded function \(g\), then \(\int g\) exists and equals \(\lim_{n \to \infty} \int g_n\).

**Remark.** Example 3.1.3 is not a good example of a function which is Lebesgue integrable but not Riemann integrable. In fact \(f\) is \(0\) m-a.e. and
so there is a Riemann integrable function, viz., $g$ identically 0, in the same equivalence class as $f$, recalling that “a.e.” defines an equivalence relation. In Example 3.4.6c we shall give examples of functions $g$ whose Lebesgue integrals exist but for which there are no Riemann integrable functions $h$ with the property that $g = h$ m-a.e.

3.2 The Lebesgue integral

Let $(X, \mathcal{A}, \mu)$ be a measure space. We define the integral of a simple function

$$f = \sum_{j=1}^{n} a_{j} \mathbb{1}_{A_{j}}, \quad A_{j} \in \mathcal{A}, \ a_{j} \in \mathbb{R},$$

(3.6)

to be

$$\int f \, d\mu = \sum_{j=1}^{n} a_{j} \mu(A_{j})$$

if $A_{j} = \{x : f(x) = a_{j}\}$ and $\mu(A_{j}) < \infty$. For each such $f$ we write

$$\int_{A} f \, d\mu = \int_{X} \mathbb{1}_{A} f \, d\mu, \quad A \in \mathcal{A}.$$ 

Remark. We can write a simple function $f$ in many ways. Our canonical criterion will be (3.6) with the property that $A_{j} = \{x : f(x) = a_{j}\}$. On the other hand, the operation “$\mathbb{1}_{A}$” is a linear mapping on the vector space of simple functions which vanish outside of a set of finite measure, and so we do not have to worry if $A_{j} \cap A_{k} \neq \emptyset$. This fact can be proved in the following way.

Write a given simple function $f = \sum b_{j} \mathbb{1}_{B_{j}}$, canonically as $\sum a_{j} \mathbb{1}_{A_{j}}$; define $\int f \, d\mu$ as $\sum a_{j} \mu(A_{j})$ and check that this is well-defined; finally, calculate that $\sum a_{j} \mathbb{1}_{A_{j}} = \sum b_{j} \mathbb{1}_{B_{j}}$. All the details are routine.

Theorem 3.1.2 can be generalized in a straightforward way to the following context.

Theorem 3.2.1. Measurability criterion for bounded functions

Let $(X, \mathcal{A}, \mu)$ be a complete bounded measure space, and let $f : X \to \mathbb{R}$ be a bounded function. The function $f$ is $\mu$-measurable if and only if

$$\inf \left\{ \int_{X} h \, d\mu : f \leq h \text{ and } h \text{ is simple} \right\}$$

$$= \sup \left\{ \int_{X} g \, d\mu : f \geq g \text{ and } g \text{ is simple} \right\}.$$ 

(3.7)
The completeness hypothesis in Theorem 3.2.1 is not required to prove (3.7).

Because of Theorem 3.2.1 we define the \(\mu\)-integral of a \(\mu\)-measurable bounded function \(f : X \to \mathbb{R}, \mu(X) < \infty\), as
\[
\int_X f \, d\mu = \inf \left\{ \int_X h \, d\mu : f \leq h \text{ and } h \text{ is simple} \right\}. \tag{3.8}
\]

**Example 3.2.2. Step and regulated functions**
We say that \(f : [a, b] \to \mathbb{R}\) is a step function if \(f = \sum_{j=1}^{n} a_j \mathbf{1}_{I_j}\) on \(\bigcup_{j=1}^{n} I_j\), where \(I_j = (x_{j-1}, x_j]\) and \(x_0 = a, x_n = b\). Also, \(g : [a, b] \to \mathbb{R}\) is a regulated function if \(\lim_{y \to x_-} g(y)\) exists for each \(x \in (a, b)\), and \(\lim_{y \to a_+} g(y)\) and \(\lim_{y \to b_-} g(y)\) exist. For each step function \(f\) there is a simple function \(h\) such that \(f = h\) except for possibly finitely many points. Further, continuous functions are regulated as are functions of bounded variation, e.g., Chapter 4, and each regulated function is bounded \(m\)-a.e. A basic fact about regulated functions is the following: \(f\) is regulated if and only if \(f\) is the uniform limit of a sequence of step functions, cf., Theorem 2.5.5 and Example 10.9.7.

**Definition 3.2.3. \(\mu\)-integrable function and its integral**
Let \(f \geq 0\) be a \(\mu\)-measurable function on the measure space \((X, \mathcal{A}, \mu)\). We define
\[
\int_X f \, d\mu = \sup_{h \leq f} \int_X h \, d\mu, \tag{3.9}
\]
where \(h\) is a bounded \(\mu\)-measurable function and \(\mu(\{x : h(x) \neq 0\}) < \infty\). The function \(f\) is \(\mu\)-integrable with integral \(\int_X f \, d\mu\) if \(\int f \, d\mu < \infty\). For \(X \subseteq \mathbb{R}\), \(\mathcal{A} = \mathcal{M}(X)\), and \(\mu = m\), we write
\[
\int_X f \, dm = \int_X f(x) \, dx = \int_X f \, dx = \int_X f.
\]

This definition of a \(\mu\)-integrable function for \(f \geq 0\) is reasonable in light of Theorem 3.2.1. Moreover, because of Theorem 3.2.1, it agrees with our definition of an integral for bounded measurable functions in (3.8). On the other hand there is a slight possible pathology involved if there are not enough such functions \(h\) on the given measure space \((X, \mathcal{A}, \mu)\). This point arises explicitly in Theorem 3.3.5 and we shall tacitly assume that the measure spaces with which we deal do not have this deficiency. Also, with this definition of a \(\mu\)-integrable function, the innocent looking linearity in Theorem 3.2.5a, and a consequence of this linearity in Theorem 3.2.6a, really depends on the results in Section 3.3. Since no logical problems evolve and since aesthetically the linearity should be mentioned now, we have done so.

Next, let \(f : X \to \mathbb{R}^+\) be a \(\mu\)-measurable function on the measure space \((X, \mathcal{A}, \mu)\), and set
\[
f^+(x) = \max\{f(x), 0\} \quad \text{and} \quad f^-(x) = \max\{(-f)(x), 0\}.
\]
Since $f$ is $\mu$-measurable, both $f^+$ and $f^-$ are $\mu$-measurable and 
\[ f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^- . \]

$f$ is $\mu$-integrable if $f^+$ and $f^-$ are $\mu$-integrable, and we define the $\mu$-integral of $f$ as 
\[
\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu .
\] (3.10)

Similarly, we define $\int_X f \, d\mu$ for $f : X \to \mathbb{C}$.

Instead of considering the space of $\mu$-integrable functions it is more advantageous to employ the space $L^1_\mu(X)$ (defined below), each of whose elements is the collection of $\mu$-integrable functions which are equal $\mu$-a.e. There are two reasons that this is done: first, the operation of integration assigns the same value to two functions which are equal $\mu$-a.e.; and, second, the natural norm topology on $L^1_\mu(X)$ is Hausdorff, a notion which essentially dispenses with the topological crises inherent in the lives of identical twins. On the other hand there is no problem in computation if we deal with integrable functions instead of the corresponding elements in $L^1_\mu(X)$.

**Definition 3.2.4. $L^1_\mu(X)$**

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $L^1_\mu(X)$ be the set of all $\mu$-measurable functions $f : X \to \mathbb{C}$ such that 
\[
\int_X |f| \, d\mu < \infty .
\]

Let $\sim$ denote the equivalence relation: $f \sim g$ if $f = g$ $\mu$-a.e. We define the space $L^1_\mu(X)$ to be the collection of all equivalence classes in $L^1_\mu(X)$. Moreover, we set 
\[
\|f\|_1 = \int_X |g| \, d\mu ,
\]
where $g = f$ $\mu$-a.e.

The next theorem contains some of the basic properties of spaces $L^1_\mu(X)$.

**Theorem 3.2.5. Linearity, monotonicity, and additivity**

Let $(X, \mathcal{A}, \mu)$ be a measure space.

- **a.** $L^1_\mu(X)$ is a vector space, and 
  \[
  \int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu ,
  \]
  where $f, g \in L^1_\mu(X)$ and $\alpha$ and $\beta$ are scalars.

- **b.** If $f, g \in L^1_\mu(X)$ and $f \leq g$ $\mu$-a.e., then 
  \[
  \int_X f \, d\mu \leq \int_X g \, d\mu .
  \]
3 The Lebesgue integral

Let $A \in \mathcal{A}$ and $B \in \mathcal{A}$ be disjoint, and choose $f \in L^1_\mu(X)$. Then

$$\int_{A \cup B} f \, d\mu = \int_{A} f \, d\mu + \int_{B} f \, d\mu.$$ 

Naturally, $L^1_\mu(X)$ is a real, resp., complex, vector space if the elements of $L^1_\mu(X)$ are $\mathbb{R}$-valued, resp., $\mathbb{C}$-valued.

**Theorem 3.2.6. Elementary role of $\int_X |f| \, d\mu$**

Let $(X, \mathcal{A}, \mu)$ be a measure space.

- **a.** $f \in L^1_\mu(X) \iff |f| \in L^1_\mu(X)$ and $f$ is $\mu$-measurable.

- **b.** Assume $f \in L^1_\mu(X)$. $\int_X f \, d\mu \leq \int_X |f| \, d\mu$.

- **c.** Assume $f \in L^1_\mu(X)$. $|\int_X f \, d\mu| = \int_X |f| \, d\mu \iff$ there is $c \in \mathbb{C}$ such that $|c| = 1$ and $cf \geq 0$ $\mu$-a.e.

- **d.** Let $f$ be a $\mu$-measurable function which is non-negative $\mu$-a.e. Then $\int_X f \, d\mu = 0 \iff f = 0$ $\mu$-a.e.

**Proof.**

- **a.** Part a follows from Theorem 3.2.5 and the decomposition of $f$ in terms of $f^+$ and $f^-$.  

- **b.** There is $c \in \mathbb{C}$, for which $|c| = 1$, such that $\left| \int_X f \, d\mu \right| = \int_X fc \, d\mu$.

Thus, by the definition of the integral,

$$\left| \int_X fc \, d\mu \right| = \int_X fc \, d\mu = \int_X Re(cf) \, d\mu + i \int_X Im(cf) \, d\mu$$

$$= \int_X Re(cf) \, d\mu \leq \int_X |cf| \, d\mu,$$

where the inequality follows since $|cf| - Re(cf) \geq 0$. $Re(a)$ and $Im(a)$ are the real and imaginary parts of $a \in \mathbb{C}$.

- **c.** Because of part b we have the desired equality if and only if $Re(cf) = |cf| - Re(cf) \geq 0$. $Re(a)$ and $Im(a)$ are the real and imaginary parts of $a \in \mathbb{C}$.

- **d.** Define the sets $A_n = \{ x : f(x) > 1/n \}$ and $A = \{ x : f(x) > 0 \}$, and let $\int_X f \, d\mu = 0$. Thus, $A = \bigcup A_n$, and we need only prove that, for each $n$, $\mu(A_n) = 0$. If $\mu(A_n) > 0$ we set $g_n = (1/n)1_{A_n}$. Hence, $g_n \leq f$ and $\int_X g_n \, d\mu = (1/n)\mu(A_n)$, Thus $\int_X f \, d\mu \geq 0$, the required contradiction.

\[\square\]

**Example 3.2.7. $f$ non-measurable and $|f|$ measurable**

It is necessary to assume that $f$ is $\mu$-measurable in Theorem 3.2.6a. In fact, if we consider the measure space $([0, 1], \mathcal{M}([0, 1]), m)$ and choose $A \notin \mathcal{M}([0, 1])$, then the function

$$f(x) = \begin{cases} 1, & \text{if } x \in A, \\ -1, & \text{if } x \notin A \end{cases}$$

is not $m$-measurable whereas $|f|$ is $m$-measurable.
Remark. As we shall see, we can find functions \( f : [a, b] \to \mathbb{R} \) such that \( f' \) exists everywhere but \( f' \notin L^1_{\text{loc}}([a, b]) \). There are general integration theories with the property that the fundamental theorem of calculus holds whenever \( f' \) exists everywhere on \([a, b] \); these are due to Perron and Denjoy. Such theories are important, but, as of now, they have not achieved the general success of Lebesgue’s theory; one of the reasons for this is precisely because they do not have the property of Theorem 3.2.6a. We refer to [189] (translated from Russian) for a modern and historical approach to the Perron–Denjoy theories. Classically, there are some introductory remarks in [247], Chapter 11, and more full-fledged treatments in [138], [182], volume II, and [214], Chapters 6–8. Denjoy has written extensively on the subject. We should also mention the Kempisty integral [137] from 1925 which can be used in harmonic analysis, see [19], pages 233–234, for such an application as well as reference to its relationship to Denjoy’s work.

Example 3.2.8. A recursive integral equation

Let \( f_0 > 0 \) on \((0, 1)\) be an element of \( L^1_{\text{loc}}((0, 1)) \) and set

\[
f_{n+1}(x) = \left( \int_0^x f_n(t) \, dt \right)^{1/2}.
\]

It can be shown that \( \lim_{n \to \infty} f_n(x) = x/2 \).

Example 3.2.9. Improper Riemann integrability for non-Lebesgue integrable functions

a. Let \( f(x) = x^2 \sin(1/x^2) \) for \( x \in (0, 1] \), and set \( f(0) = 0 \). Then

\[
f(x) = \begin{cases} 
0, & \text{if } x = 0, \\
2x \sin \left( \frac{1}{x^2} \right) - \frac{2}{x} \cos \left( \frac{1}{x^2} \right), & \text{if } x \in (0, 1].
\end{cases}
\]

Note that, for \( x > 0 \),

\[
|f'(x)| \geq \frac{2}{x} \left| \cos \left( \frac{1}{x^2} \right) \right| - 2 \left| \sin \left( \frac{1}{x^2} \right) \right| \geq \frac{2}{x} \left| \cos \left( \frac{1}{x^2} \right) \right| - 2x.
\]

Define

\[
I_n = \left[ \left( 2n + \frac{1}{3} \right) \pi \right]^{-1/2} - \left[ \left( 2n - \frac{1}{3} \right) \pi \right]^{-1/2}.
\]

Observe that

\[
\forall x \in I_n, \quad \left| \cos \left( \frac{1}{x^2} \right) \right| \geq \frac{1}{2},
\]

in fact, for \( x = ((2n + (1/3)) \pi)^{-1/2} \),

\[
\left| \cos \left( \frac{1}{x^2} \right) \right| = \left| \cos \left( 2n \pi + \frac{\pi}{3} \right) \right| = \left| \cos \left( \frac{\pi}{3} \right) \right|.
\]
Thus, for any \( x \in I_n \), \(|f'(x)| \geq (1/x) - 2x\) and so

\[
\int_{I_n} |f'| \geq \int_{I_n} \left( \frac{1}{x} - 2x \right) \, dx = \frac{1}{2} \log \left( \frac{2n + (1/3)}{2n - (1/3)} \right) - \frac{2}{3\pi} \left( \frac{2n + (1/3)}{2n - (1/3)} \right).
\]

The fact that the sequence \( \{J_n : n = 1, \ldots\} \) is a disjoint family follows since
\[
2^{n+2} - (1/3) > 2^{n+1}(1/3) \implies [2^{n+2} - (1/3))]^{-1/2} < [(2^{(1/3)}))]^{-1/2}.
\]
Consequently, if we let \( a_N = \sum_{n=1}^{N} \frac{1}{(2n + (1/3))(2n - (1/3))} \) we have

\[
\int_{0}^{1} |f'| \geq \sum_{n=1}^{N} \int_{I_n} |f'| \geq \frac{1}{2} \sum_{n=1}^{N} \log \left( \frac{2n + (1/3)}{2n - (1/3)} \right) - \frac{2a_N}{3\pi}.
\]

Because the sequence \( \{a_N : N = 1, \ldots\} \) is convergent and

\[
\sum_{n=1}^{N} \log \left( 1 + \frac{1}{6n} \right)
\]

is divergent, noting that \( \sum_{n=1}^{N} 1/(6n) \) and therefore \( \prod_{n=1}^{N} (1 + 1/(6n)) \) diverge, we conclude that
\[
f' \notin L^1([0, 1]).
\]

b. Consider the function \( f \) of part a. From the fundamental theorem of calculus for Riemann integration, e.g., Problem 1.30, we compute

\[
R \int_{\varepsilon}^{1} f' = f(1) - f(\varepsilon) = \sin(1) - \varepsilon^2 \sin(1/\varepsilon^2)
\]

for each \( \varepsilon > 0 \), and so

\[
\lim_{\varepsilon \to 0} R \int_{\varepsilon}^{1} f' = \sin(1).
\]

We shall see in Section 3.4 that \( f' \in L^1([\varepsilon, 1]) \) since it is bounded and Riemann integrable on \([\varepsilon, 1]\); consequently,

\[
\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} f' = \sin(1).
\]

c. We again consider the function \( f \) from part a. We now observe that even though \( f' \) exists everywhere on \([0, 1]\), \( f \) is not a function of bounded variation, see Definition 4.1.1 and Theorem 4.3.2. In fact,

\[
f(1/\sqrt{k\pi}) = 0 \quad \text{and} \quad f \left( 1/\sqrt{k\pi + (\pi/2)} \right) = (-1)^k / (k\pi + (\pi/2)),
\]
3.3 The Lebesgue dominated convergence theorem (LCD)

In his thesis, cf., Section 1.3, LEBESGUE notes that his dominated convergence theorem is a generalization, with simplification in the proof, of a theorem by WILLIAM F. OSGOOD [183] (1897). OSGOOD’s result is Proposition 3.3.1 for the special case of a continuous function $f$, and Proposition 3.3.1 was originally proved by CESARE ARZELÀ [5] (1885). We shall state ARZELÀ’s result before making additional remarks.

A sequence $\{f_n : n = 1, \ldots\}$ of functions $[a;b] \to \mathbb{R}$ converges boundedly to a function $f : [a,b] \to \mathbb{R}$ if $\{f_n : n = 1, \ldots\}$ converges pointwise to $f$ and

$$\sup_{n \in \mathbb{N}} \|f_n\|_{\infty} < \infty.$$ 

Clearly, if $\{f_n : n = 1, \ldots\}$ is a sequence of bounded functions on $[a, b]$ and if $f_n \to f$ uniformly on $[a, b]$, then $f_n \to f$ boundedly.

**Proposition 3.3.1.** Let $\{f, f_n : n = 1, \ldots\}$ be a sequence of Riemann integrable functions $f, f_n : [a, b] \to \mathbb{R}$, and assume that $f_n \to f$ boundedly. Then

$$\lim_{n \to \infty} \int_{a}^{b} f_n = \int_{a}^{b} f.$$ 

Starting with the definition of the Riemann integral it is non-trivial to prove Proposition 3.3.1, whereas, starting from the axioms of measure theory it is not difficult to prove the corresponding and more general Lebesgue dominated convergence theorem (Theorem 3.3.2). The reason for this is not so mysterious. ARZELÀ’s result depends on a $\sigma$-additivity property and, starting with RIEMANN’s definition of integral, the route to proving such a property requires some effort; on the other hand, in LEBESGUE’s theory the $\sigma$-additivity is essentially built into the preliminaries. The history of the elementary (that is, without LEBESGUE’s theory) proofs of Proposition 3.3.1 has been recorded by WILHELMUS A. J. LUXEMBURG [167]; he also gives another elementary proof of his own which is basically a corrected version of an old proof due to FELIX HAUSDORFF (1927). The problem of “taking limits under the integral”
which, of course, has many forms – is one of the absolutely fundamental issues in analysis; consequently, new proofs of such results as Proposition 3.3.1 are valuable for providing insights on the matter of “switching limits”.

Arzela’s original proof depended on a complicated lemma which is, in fact, an easy corollary of our Problem 2.13b. It was in this lemma that he derived the “countable additivity property” – mentioned in the previous paragraph – that was necessary for his theorem. Mind you, Problem 2.13b is straightforward to prove when one begins with the \(-\)-additivity of Lebesgue measure. Using Problem 2.13b, we now give Arzela’s proof, properly streamlined, of Proposition 3.3.1.

**Proof.** (Proposition 3.3.1) Without loss of generality assume that \([a, b] = [0, 1] \), \( f = 0 \), and \( f_n(x) \in [-1, 1] \) for all \( x \) and \( n \). If the result is false, then \( \lim_{n \to \infty} R \int_0^1 f_n \) or \( \lim_{n \to \infty} R \int_0^1 f_n = r > 0 \). Define \( A_n = \{ x : f_n(x) \geq r/2 \} \) so that \( \lim_{n \to \infty} m(A_n) > 0 \). By Problem 2.13b we see that there is a point \( y \in [0, 1] \) such that \( f_n(y) \geq r/2 \) for infinitely many \( n \). This contradicts the hypothesis that \( f_n(y) \to 0 \).

\[ \square \]

Observe that we assumed \( f \) to be Riemann integrable in Proposition 3.3.1. The corresponding assumption will not have to be made in Theorem 3.3.2.

Recall that \( \mathbb{R}^* = \mathbb{R} \cup \{ \pm \infty \} \).

**Theorem 3.3.2. Special case of Lebesgue dominated convergence (LDC) theorem**

Let \((X, \mathcal{A}, \mu)\) be a bounded measure space and let \( \{ f_n : n = 1, \ldots \} \) be a sequence of measurable functions \( f_n : X \to \mathbb{R}^* \) for which

\[ \sup_{n \in \mathbb{N}} \| f_n \|_{\infty} = M < \infty. \]

If \( f_n \to f \) pointwise on \( X \) then \( f \in L_1^\mu(X), \| f \|_{\infty} \leq M \), and

\[ \lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0; \]

in particular,

\[ \lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu. \tag{3.11} \]

**Proof.** Clearly \( f \) is measurable by Proposition 2.5.2, and \( \| f \|_{\infty} \leq M \); thus \( f \in L_1^\mu(X) \). Take \( \varepsilon > 0 \) and choose \( N \) and \( A \in \mathcal{A} \), by Egorov’s theorem, such that \( \mu(A) < \varepsilon/(4M) \) and

\[ \forall n \geq N, \forall x \notin A, \quad |f_n(x) - f(x)| < \frac{\varepsilon}{2\mu(X)}. \]

Thus, for all \( n \geq N \),
\[ \int_X |f_n - f| \, d\mu = \int_A + \int_{X \setminus A} |f_n - f| \, d\mu \leq \frac{2M\varepsilon}{4M} + \mu(X \setminus A) \frac{\varepsilon}{2\mu(X)} \leq \varepsilon. \]

\[ \square \]

**Example 3.3.3. Elementary examples for LDC**

**a.** Obviously we can find a sequence \( \{f_n : n = 1, \ldots\} \) of simple functions on \([0, 1], M([0, 1]), m\) such that

1. \( f_n \geq 0 \),
2. \( \int_0^1 f_n(x) \, dx = 1 \),
3. \( f_n \rightarrow 0 \) pointwise,
4. \( \lim_{n \rightarrow \infty} \|f_n\| = \infty \).

In fact, take \( f_n = n(n + 1)\mathbb{1}_{[1/(n+1), 1/n]} \).

**a.’** We can also find a sequence \( \{f_n : n = 1, \ldots\} \) of simple functions on \([0, 1], M([0, 1]), m\) that satisfies

1. \( f_n \geq 0 \),
2. \( \int_0^1 f_n(x) \, dx = 1 \),
3. \( f_n \rightarrow 0 \) pointwise,
4. \( \lim_{n \rightarrow \infty} \|f_n\| = \infty \).

Take \( f_n = 2n\mathbb{1}_{[1/2 - 1/n, 1/2 + 1/n]} \).

**b.** Observe that properties ii and iii in part a cannot hold along with the condition that \( \sup_n \|f_n\| < \infty \); for if we had iii and the norm boundedness we would contradict ii by Theorem 3.3.2.

**c.** Define \( f_n(x) = x^n \) on \([0, 1]\). Then \( f_n \rightarrow f = 0 \) pointwise on \([0, 1]\) but the convergence is not uniform. On the other hand, \( |f_n| \leq 1, \int_0^1 x^n \, dx = 1/(n+1) \), and \( \int_0^1 f(x) \, dx = 0 \).

**d.** Set

\[
    f_n(x) = \begin{cases} 
    0, & \text{if } x \leq 0 \text{ and } x \geq 1/n, \\
    n^2, & \text{if } 0 < x < 1/n. 
    \end{cases}
\]

Then \( f_n \rightarrow f \) pointwise on \( \mathbb{R} \) but once again the convergence is not uniform. \( \int_\mathbb{R} f_n(x) \, dx = n \rightarrow \infty \) and \( \int_\mathbb{R} f(x) \, dx = 0 \).

**e.** Set

\[
    f_n(x) = \begin{cases} 
    1/n, & \text{if } |x| \leq n^2, \\
    0, & \text{if } |x| > n^2. 
    \end{cases}
\]

Then \( f_n \rightarrow f = 0 \) uniformly on \( \mathbb{R} \), \( \int_\mathbb{R} f_n(x) \, dx = 2n \rightarrow \infty \), and \( \int_\mathbb{R} f(x) \, dx = 0 \).

We now give Fatou’s lemma (Theorem 3.3.5) which Pierre Fatou, a friend of Lebesgue, published in his famous thesis [86]. In order to motivate this result we first establish some notation and make some remarks on Fourier series.

**Example 3.3.4. Fourier series and Fatou**

**a.** By definition, a function \( f \) is an element of \( L^1_m(T) \) if \( f \) is a 1-periodic
complex-valued \( m \)-measurable function defined on \( \mathbb{R} \) such that, for some (and therefore for all) \( r \in \mathbb{R} \),

\[
\int_r^{r+1} |f| < \infty.
\]

The associated measure space is designated by \((T, \mathcal{M}(T), m)\) and the \( L^1_m(T) \) norm of \( f \in L^1_m(T) \) is

\[
\|f\|_1 = \int_0^1 |f|.
\]

We also define \( f \in L^\infty_m(T) \) with norm \( \|f\|_\infty \) if \( f \in L^1_m(T) \) and

\[
\|f\|_\infty = \operatorname{ess sup}_{x \in \mathbb{R}} |f(x)| < \infty.
\]

If \( f \in L^1_m(T) \), then the Fourier coefficients of \( f \) are defined by

\[
\hat{f}(n) = c_n = \int_0^1 f(x) e^{-2\pi inx} \, dx, \quad n \in \mathbb{Z},
\]

and the Fourier series of \( f \) is

\[
S(f)(x) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi inx}.
\]

There are many expositions of Fourier series from [21] to Zygmund’s classic [274]. See also Appendix 11.

b. Setting

\[
u(r, x) = \sum_{n \in \mathbb{Z}} c_ne^n e^{2\pi inx}, \quad r \in (0, 1),
\]

Fatou proved

\[
\lim_{r \to 1^-} \nu(r, x) = f(x) \quad m\text{-a.e.} \quad (3.12)
\]

This result is quite important in function theory, and in order to prove it Fatou used Theorem 3.3.5. Combining (3.12) and Theorem 3.3.5 he also obtained Parseval’s equality

\[
\int_0^1 |f(x)|^2 \, dx = \sum_{n \in \mathbb{Z}} |c_n|^2
\]

for \( f \in L^2_m(T) = \{ f : f^2 \in L^1_m(T) \} \), a fact which Lebesgue initially proved only for \( f \in L^1(T) \). Shortly thereafter, F. Riesz used Fatou’s theory to prove that \( L^2_m(T) \) is a complete metric space with metric \( \rho \) defined by

\[
\rho(f, g) = \left( \int_T |f(x) - g(x)|^2 \, dx \right)^{1/2},
\]

i.e., he proved that \( L^2_m(T) \) is a Hilbert space, e.g., Appendix 10.2.
Theorem 3.3.5. Fatou lemma

Let \( \{f_n : n = 1, \ldots \} \) be a sequence of measurable functions \( X \to \mathbb{R}^* \) defined on the measure space \((X, \mathcal{A}, \mu)\). Assume that \( \{f_n : n = 1, \ldots \} \) is bounded below by some \( g \in L^1_\mu(X) \), \( f_n \to f \) \( \mu \)-a.e., and \( f \) is \( \mu \)-measurable. Then

\[
\int_X f \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu.
\]

Proof. First note that we make no claims about integrability either in the hypothesis or conclusion since our integrals may be unbounded.

Without loss of generality assume that \( g \) is identically 0, that \( f_n \to f \) pointwise everywhere, and that \( \lim \int f_n \, d\mu < \infty \). Take a bounded measurable function \( h \) such that \( 0 \leq h \leq f \) and \( h = 0 \) on a set \( Y \) of finite measure, c.f., the remark on pathology after Example 3.2.2.

Define \( h_n(x) = \min\{h(x), f_n(x)\} \). Thus \( 0 \leq h_n \leq h \) and \( h_n = 0 \) on \( Y \). Since \( f_n \to f \) pointwise and \( h \leq f \), we have \( h_n \to h \) pointwise by the definition of \( h_n \). Consequently we apply Theorem 3.3.2 to obtain

\[
\int_X h \, d\mu = \int_Y h \, d\mu = \lim_{n \to \infty} \int_Y h_n \, d\mu.
\]

Now \( f_n \geq h_n \geq 0 \) implies \( \int_X f_n \, d\mu \geq \int_Y f_n \, d\mu \geq \int_Y h_n \, d\mu \) and so

\[
\lim_{n \to \infty} \int_X f_n \, d\mu \geq \lim_{n \to \infty} \int_Y h_n \, d\mu = \lim_{n \to \infty} \int_Y h_n \, d\mu.
\]

This combined with (3.13) yields the result from the definition of \( \int_X f \, d\mu \).

The following result allows us to prove Theorem 3.2.5a. This is important since we use linearity in Theorem 3.3.7.

Theorem 3.3.6. Levi–Lebesgue theorem

Let \((X, \mathcal{A}, \mu)\) be a measure space and let \( \{f_n : n = 1, \ldots \} \) be a sequence of \( \mu \)-measurable \( \mathbb{R}^* \)-valued functions defined on \( X \). Assume that \( \{f_n : n = 1, \ldots \} \) is bounded below by some \( g \in L^1_\mu(X) \) and that \( \{f_n : n = 1, \ldots \} \) converges \( \mu \)-a.e. to a \( \mu \)-measurable function \( f \). If

\[
\forall n = 1, \ldots, \ f_n \leq f \quad \mu \text{-a.e.}
\]

then

\[
\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu.
\]

Proof. Since \( f_n \leq f \), \( \mu \)-a.e., we have \( \int f_n \, d\mu \leq \int f \, d\mu \). By Theorem 3.3.5,

\[
\int_X f \, d\mu \leq \limsup_{n \to \infty} \int_X f_n \, d\mu \leq \limsup_{n \to \infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu,
\]

and we are done.
If \( f_n \) increases to \( f \) then the conditions of Theorem 3.3.6 hold, and it was in this form that the result was proved by Beppo Levi in 1906. The fundamental criterion for taking limits under the integral sign is the following theorem.

**Theorem 3.3.7. Lebesgue dominated convergence (LDC) theorem**

Let \((X, \mathcal{A}, \mu)\) be a measure space and let \( \{f_n : n = 1, \ldots\} \) be a sequence of \( \mu \)-measurable functions each of which is complex-valued \( \mu \)-a.e. Assume that \( f_n \rightarrow f \) \( \mu \)-a.e., that \( f \) is \( \mu \)-measurable, and that there is an element \( g \in L^1_\mu(X) \) such that

\[
\forall n = 1, \ldots, \quad |f_n| \leq g \quad \mu\text{-a.e.}
\]

Then \( f \in L^1_\mu(X) \) and

\[
\lim_{n \to \infty} \int_X |f - f_n| \, d\mu = 0.
\]

In particular, \( \int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu \).

**Proof.** \( f \in L^1_\mu(X) \), since \( |f| \leq g \) \( \mu \)-a.e. Without loss of generality take \( f_n \) real-valued \( \mu \)-a.e. and so

\[
0 \leq g - f_n \quad \mu\text{-a.e.}
\]

Thus, from Theorem 3.3.5,

\[
\int_X (g - f) \, d\mu \leq \lim_{n \to \infty} \int_X (g - f_n) \, d\mu.
\]

Consequently, from properties of the \( \lim \) and \( \lim \) and since \( f, f_n \in L^1_\mu(X) \),

\[
\int_X g \, d\mu - \int_X f \, d\mu \leq \int_X g \, d\mu - \lim_{n \to \infty} \int_X f_n \, d\mu.
\]

This yields

\[
\lim_{n \to \infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu.
\]

For the opposite direction we consider \( g + f_n \) and compute

\[
\int_X f \, d\mu \leq \lim_{n \to \infty} \int_X f_n \, d\mu.
\]

This does it.

\( \square \)

**Remark.** Nets (or directed sets) [136], page 65, or filters [96] arise in non-metric convergence, e.g., Section 10.9. See [96], pages 306–307 for a brief history.

If a net of functions is indexed by a countable set then Theorem 3.3.7 holds [35], Chapter IV.3.7. We shall give an example of a convergent net.
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for which Theorem 3.3.7 fails. Let $N$ be the set of characteristic functions $\mathbb{1}_F$, where $F \subseteq [0,1]$ is a finite set. By definition (which we have not yet given!), $N$ is a net since $\mathbb{1}_{F_1 \cup F_2}$ dominates the functions $\mathbb{1}_{F_1}$ and $\mathbb{1}_{F_2}$; and the net $N$ converges to the function $\mathbb{1}_{[0,1]}$. On the other hand, Theorem 3.3.7 fails since $\int \mathbb{1}_F = 0$ and $\int \mathbb{1}_{[0,1]} = 1$. The deficiency in this example is that $\mathbb{1}_F = 0$ m-a.e.

In this context it is natural to inquire if a genuine failure of Theorem 3.3.7 for nets is possible; for this question it is well to keep in mind the fact that measurable functions are characterized in terms of sequences of measurable functions, see Corollary 2.5.15.

One means of generalizing Theorem 3.3.7 is the following, see Problem 3.7.

**Theorem 3.3.8. A general LDC**

Let $(X, A)$ be a measurable space and let $\{\mu_n : n = 1, \ldots\}$ be a sequence of measures on $A$ such that

$$\forall A \in A, \lim_{n \to \infty} \mu_n(A) = \mu(A), \tag{3.14}$$

where $\mu$ is a measure on $A$. Assume that the sequence $\{g_n : n = 1, \ldots\} \subseteq L^1_\mu(X)$ satisfies the conditions that $g_n \to g$ pointwise and

$$\lim_{n \to \infty} \int_X g_n \, d\mu_n = \int_X g \, d\mu.$$

If $\{f_n : n = 1, \ldots\}$ is a sequence of functions with the properties that $f_n \to f$ pointwise, each $f_n$ is $\mu_n$-measurable, and

$$\forall n, \quad |f_n| \leq g_n,$$

then $f \in L^1_\mu(X)$ and

$$\lim_{n \to \infty} \int_X |f - f_n| \, d\mu = 0.$$

In particular, $\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu_n$.

Generally we shall refer to Theorem 3.3.2, Theorem 3.3.6, Theorem 3.3.7, and Theorem 3.3.8 as LDC (Lebesgue dominated convergence theorem) in the sequel. The hypothesis that $\mu$ in (3.14) is a measure raises the problem to find conditions so that (3.14) defines a measure. We shall study this question when we discuss weak convergence of measures in Chapter 6, and we shall do it in the context of trying to find necessary and sufficient conditions for the conclusion of LDC to hold. For now we refer to Problem 3.7b and the following “outline” which we shall expand on later.

**Proposition 3.3.9.** Let $(X, A, \mu)$ be a measure space and let $f \in L^1_\mu(X)$. Then

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall A \in A, \text{ for which } \mu(A) < \delta, \int_A |f| \, d\mu < \varepsilon.$$
Proof. The result is obvious if \( \|f\|_\infty < \infty \). Define
\[
f_n(x) = \begin{cases} |f|(x), & \text{if } |f(x)| \leq n, \\ n, & \text{if } |f(x)| > n. \end{cases}
\]
Then \( \|f_n\|_\infty \leq n \) and \( |f_n| \to |f| \) pointwise. From LDC
\[
3 \exists N \text{ such that } \forall n \geq N, \quad \int_X (|f| - |f_n|) \, d\mu < \varepsilon/2.
\]
Letting \( 0 < \delta < \varepsilon/(2N) \) we have
\[
\left| \int_A f \, d\mu \right| \leq \int_A |f| \, d\mu = \int_A (|f| - |f_N|) \, d\mu + \int_A |f_N| \, d\mu \\
\leq \int_X (|f| - |f_N|) \, d\mu + N \mu(A) < \varepsilon
\]
if \( \mu(A) < \delta \).

Definition 3.3.10. Absolute continuity

Let \((X, \mathcal{A}, \mu)\) be a measure space.

a. A \( \mu \)-measurable function \( f \) on \( X \) is absolutely continuous with respect to \( \mu \) if
\[
\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall A \in \mathcal{A}, \text{ for which } \mu(A) < \delta, \\
\int_A |f| \, d\mu < \varepsilon.
\]

Thus each element \( f \in L^1_\mu(X) \) is absolutely continuous with respect to \( \mu \). In Chapter 5 we shall define a “measure \( \nu \) absolutely continuous with respect to \( \mu \)” and show that such measures are actually characterized by \( L^1_\mu(X) \).

b. A collection \( \{f_n\} \subseteq L^1_\mu(X) \) is uniformly absolutely continuous if
\[
\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall A \in \mathcal{A}, \text{ for which } \mu(A) < \delta, \text{ and } \forall \alpha, \\
\int_A |f| \, d\mu < \varepsilon.
\]

c. If \( \mathcal{F} \subseteq \mathcal{P}(X) \) and \( \nu \) is a scalar-valued function on \( \mathcal{F} \), we say that \( \nu \) is Vitali continuous if for each decreasing sequence \( \{A_n : n = 1, \ldots\} \subseteq \mathcal{F} \) for which \( \bigcap A_n = \emptyset \) we can conclude that
\[
\lim_{n \to \infty} \nu(A_n) = 0.
\]

A sequence of Vitali continuous functions \( \nu_n \) on \( \mathcal{F} \) is Vitali equicontinuous if for each decreasing sequence \( \{A_n : n = 1, \ldots\} \subseteq \mathcal{F} \), for which \( \bigcap A_n = \emptyset \), we have
If $\mathcal{F}$ from part c is $\mathcal{A}$ then a sequence $\{\nu_m : m = 1, \ldots\}$ of scalar-valued functions on $\mathcal{A}$ is uniformly absolutely continuous if

\[ \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall A \in \mathcal{A}, \text{ for which } \mu(A) < \delta, \text{ and } \forall m, \]

\[ |\nu_m(A)| < \varepsilon. \]

This definition obviously generalizes the above definition of uniformly absolutely continuous sets of integrable functions.

Clearly, if $f$ from part c is $\mathcal{A}$ then a sequence $\{f_n : n = 1, \ldots\}$ of scalar-valued functions on $\mathcal{A}$ is uniformly absolutely continuous if

\[ \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathcal{A}, \text{ for which } |f(x)| < \delta, \text{ and } \forall m, \]

\[ |f_n(x)| < \varepsilon. \]

We are basically interested in finding a converse to this observation in order to obtain the best possible LDC. We make the following definition. A sequence $\{f_n : n = 1, \ldots\}$ of $\mu$-measurable functions on a measure space $(X, \mathcal{A}, \mu)$ converges in measure to a $\mu$-measurable function $f$ if (3.15) holds. Vitali made initial and deep progress in characterizing LDC with essentially the following results [259].

**Theorem 3.3.11. Vitali uniform absolute continuity theorem**

Let $(X, \mathcal{A}, \mu)$ be a bounded measure space and choose a sequence $\{f_n : n = 1, \ldots\} \subseteq L^1_\mu(X)$.

\[ \lim_{n \to \infty} \int_X |f - f_n| \, d\mu = 0 \quad (3.16) \]

for some $f \in L^1_\mu(X)$ if and only if

i. $\{f_n : n = 1, \ldots\}$ converges in measure to a $\mu$-measurable function $f$, and

ii. $\{f_n : n = 1, \ldots\}$ is uniformly absolutely continuous.

**Theorem 3.3.12. Vitali equicontinuity theorem**

Let $(X, \mathcal{A}, \mu)$ be a measure space and choose a sequence $\{f_n : n = 1, \ldots\} \subseteq L^1_\mu(X)$. (3.16) holds for some $f \in L^1_\mu(X)$ if and only if

i. $\{f_n : n = 1, \ldots\}$ converges in measure to a $\mu$-measurable function $f$, and

ii. $\{\nu_n : \forall A \in \mathcal{A}, \mu(A) = \int_A |f_n| \, d\mu\}$ is Vitali equicontinuous.

Lebesgue proved that if $f_n \to f$ $\mu$-a.e., on a measure space $(X, \mathcal{A}, \mu)$, where $f$ is $\mu$-measurable, and if $|f_n| \leq g$ $\mu$-a.e., for some $g \in L^1_\mu(X)$, then $f_n \to f$ in measure, cf., Theorem 3.3.13b and Theorem 6.1.1. Further, with these hypotheses it is straightforward to deduce part ii in Theorem 3.3.12. Thus LDC is a corollary of Theorem 3.3.12.

Part a of the following result is due to Frigyes Riesz, and part b is due to Lebesgue, see Problem 3.21d, e.
Theorem 3.3.13. F. Riesz-Lebesgue theorem
Let \((X, \mathcal{A}, \mu)\) be a measure space and let \(\{f_n : n = 1, \ldots\}\) be a sequence of measurable functions.

a. If \(f_n \to f\) in measure on \((X, \mathcal{A}, \mu)\), then there is a subsequence \(\{f_{n_k} : k = 1, \ldots\}\) which converges to \(f\) pointwise \(\mu\)-a.e.

b. Assume \((X, \mathcal{A}, \mu)\) is a bounded measure space. If \(f_n \to f\) \(\mu\)-a.e., then \(f_n \to f\) in measure.

Theorem 3.3.14. Strongly non-convergent dilations
Let \(f\) be a non-constant bounded Lebesgue measurable function defined on \(\mathbb{R}\) such that

\[ f(x + 1) = f(x). \]

Set \(f_n(x) = f(nx)\). There is no subsequence of \(\{f_n : n = 1, \ldots\}\) which converges \(m\)-a.e. on any bounded interval \([a, b]\), where \(b > a\).

Proof. i. Take any interval \([a, \beta]\). From the periodicity of \(f\),

\[
\int_{a}^{\beta} f(nx) \; dx = \frac{1}{n} \int_{a}^{\beta} f(x) \; dx
\]

\[
= \frac{1}{n} \left\{ \sum_{j=\lfloor a \rfloor}^{\lfloor \beta \rfloor} \int_{ja}^{ja+1} f(x) \; dx + \int_{\lfloor \beta \rfloor}^{\beta} f(x) \; dx - \int_{\lfloor a \rfloor}^{\lfloor a \rfloor} f(x) \; dx \right\}
\]

\[
= \frac{1}{n} \left( 1 + \lfloor n \beta \rfloor - \lfloor na \rfloor \right) \int_{0}^{1} f(x) \; dx + \frac{1}{n} \left( \int_{\lfloor \beta \rfloor}^{\beta} f(x) \; dx - \int_{\lfloor a \rfloor}^{\lfloor a \rfloor} f(x) \; dx \right),
\]

where \([x]\) is the largest integer \(k \leq x\). Clearly, the last term in the last expression is bounded by \(2\|f\|_{\infty}/n\) and so tends to 0 as \(n \to \infty\).

Now note that

\[
\frac{1}{n} \lfloor n \beta \rfloor - \lfloor na \rfloor = \frac{1}{n} (n\beta - na + (\lfloor n \beta \rfloor - n\beta) + (na - \lfloor na \rfloor))
\]

\[
= \beta - a + \frac{\lfloor n \beta \rfloor - n\beta}{n} + \frac{na - \lfloor na \rfloor}{n}.
\]

Consequently,

\[
\lim_{n \to \infty} \int_{a}^{\beta} f_n(x) \; dx = (\beta - a) \int_{0}^{1} f(x) \; dx. \tag{3.17}
\]

ii. Assume that \(f(nkx) \to g(x)\) \(m\)-a.e. on \([a, b]\) as \(k \to \infty\). Take \([\alpha, \beta] \subseteq [a, b]\) and let \(K = \int_{[a]}^{1} f(x) \; dx\). From LDC we compute

\[
\int_{a}^{\beta} g(x) \; dx = \lim_{k \to \infty} \int_{a}^{\beta} f(nkx) \; dx = (\beta - \alpha)K,
\]

and so
\[
\int_{\alpha}^{\beta} (g(x) - K) \, dx = 0.
\]
Since \([\alpha, \beta]\) is an arbitrary subinterval of \([a, b]\) we conclude that \(g = K\) \text{ m-a.e. in } [a, b].

We now use \(|f - g| = |f - K|\) and \([a, b]\) instead of \(f\) and \([\alpha, \beta]\) in (3.17). Thus,
\[
(b-a) \int_0^1 |f(x) - K| \, dx = \lim_{k \to \infty} \int_a^b |f_{n_k}(x) - K| \, dx = 0.
\]
The last equality follows by LDC, and so \(f = K\) \text{ m-a.e., on } [0, 1], a contradiction.

Example 3.3.15. Examples of strongly non-convergent dilations

Apply Theorem 3.3.14 to the following functions.

\begin{itemize}
  \item[a.] \(f(x) = x - [x]\) on \([0, 1]\) and
  \item[b.] \(f(x) = \sin(2\pi x)\) on \([0, 1]\).
\end{itemize}

\begin{remark}
Theorem 3.3.14 is interesting in light of Fejér’s theorem, e.g., Problem 3.10, which tells us, in particular, that if \(f\) is a bounded Lebesgue measurable function on \(\mathbb{R}\) with period 1 then
\[
\lim_{n \to \infty} \int_0^1 f(nx) \, dx = \int_0^1 f(x) \, dx,
\]
cf., Section 6.3.
\end{remark}

\begin{remark}
Banach suggested the following result, see [173] Problem 162: if \(f\) is Lebesgue measurable and 1-periodic function on \(\mathbb{R}\), then
\[
\lim_{n \to \infty} f(nx) = \text{ess sup}_{x \in [0, 1]} f(x)
\]
and
\[
\lim_{n \to \infty} f(nx) = \text{ess inf}_{x \in [0, 1]} f(x)
\]
for \(m\text{-a.e. } x \in \mathbb{R}\).
\end{remark}

3.4 The Riemann and Lebesgue integrals

We begin by proving a fact that we have implicitly assumed for a while now.

Proposition 3.4.1. Let \(f : [a, b] \to \mathbb{R}\) be a bounded function. If \(R \int_a^b f\) exists then \(f \in L^1_{m}([a, b])\) and
\[
R \int_a^b f = \int_a^b f(x) \, dx.
\]
Proof. Let \( g \) and \( h \) denote simple functions. Then

\[
R \int_a^b f \leq \sup_{g \leq f} \int_a^b g \leq \inf_{h \geq f} \int_a^b h \leq R \int_a^b f,
\]

since, for example, \( R \int_a^b f = \sup \{ R \int_a^b g : g = \sum a_j \mathbb{1}_{(x_{j-1},x_j]} \leq f \text{ and } a = x_0 < x_1 < \ldots < x_n = b \} \) and \( \{ g : g = \sum a_j \mathbb{1}_{(x_{j-1},x_j]} \text{ and } a = x_0 < x_1 < \ldots < x_n = b \} \) is a subfamily of the class of simple functions.

(3.18) follows because \( R \int_a^b f \) exists, by Theorem 3.2.1, and from the definition of the Lebesgue integral.

\[\square\]

**Proposition 3.4.2.** A function \( f : \mathbb{R} \to \mathbb{R} \) is continuous m-a.e. if and only if

\[
\forall V \subseteq \mathbb{R}, \text{ open, } f^{-1}(V) = U \cup A,
\]

where \( U \) is open and \( A \in \mathcal{M}(\mathbb{R}) \) has Lebesgue measure \( m(A) = 0 \).

**Proof.** (\(\Rightarrow\)) Let \( X = f^{-1}(V) \) and set \( X = X_c \cup X_d \) where \( X_c = X \cap C(f) \) and \( X_d = X \cap D(f) \). For each \( x \in X_c \) choose an open neighborhood \( U_x \) of \( x \) such that \( f(U_x) \subseteq V \); we can do this since \( f \) is continuous on \( X_c \). Clearly \( U_x \) need not be contained in \( X_c \), for example, take the ruler function. Since \( f^{-1}(V) = X \) and \( U_x \subseteq f^{-1}(V) \) we have

\[
X = X_c \cup X_d \subseteq \left( \bigcup_{x \in X_c} U_x \right) \cup X_d \subseteq X.
\]

Let \( A = X_d \) and \( U = \bigcup_{x \in X_c} U_x \). \( m(A) = 0 \) by hypothesis, and \( U \) is obviously open.

(\(\Leftarrow\)) For each \( x \in D(f) \) there is an open set \( V \) such that \( f(x) \in V \), and for each open neighborhood \( U \) of \( x \), \( f(U) \not\subseteq V \). Let \( f(x) \in N(s,r) = \{ y \in \mathbb{R} : y \in (s - r, s + r) \} \subseteq V \). Consequently, for all such sets \( U \), \( f(U) \not\subseteq N(s,r) \). By hypothesis, \( f^{-1}(N(s,r)) = U_{s,r} \cup A_{s,r} \), where \( U_{s,r} \) is open and \( m(A_{s,r}) = 0 \). Now, because of the above observations, \( x \in f^{-1}(N(s,r)) \) and \( x \notin U_{s,r} \). Thus, \( x \in A_{s,r} \) and so \( D(f) \subseteq \bigcup \{ A_{s,r} : s, r \in \mathbb{Q} \} \). Hence \( f \) is continuous m-a.e.

\[\square\]

**Example 3.4.3.** Composition of continuous m-a.e. functions

We shall show that if \( f \) and \( g \) are continuous m-a.e. on \( \mathbb{R} \), then \( f \circ g \) is not necessarily continuous m-a.e.. Define

\[
g(x) = \begin{cases} 
0, & \text{if } x \text{ is irrational,} \\
1, & \text{if } x = 0, \\
1/|q|, & \text{if } x = p/q, \text{ where } (p,q) = 1, 
\end{cases}
\]
3.4 The Riemann and Lebesgue integrals

and

\[ f(x) = \begin{cases} 1, & \text{if } x = 1/n, \ n = 1, \ldots, \\ 0, & \text{otherwise}. \end{cases} \]

Then \( f \circ g = 1_Q \) and this function is not continuous anywhere. See Problem 3.15 in this regard. Note that \( g \) is the analogue on \( \mathbb{R} \) of the ruler function.

We now sketch a “first approximation” to one direction of Lebesgue’s characterization of Riemann integrable functions. The proof is elementary and does not involve Lebesgue measure.

**Proposition 3.4.4.** Let \( f \) be a Riemann integrable function defined on \([a, b]\). Then \( f \) is continuous on a dense subset of \([a, b]\).

**Proof.** Assume that \( f \) is a real-valued function. Let \( \{P_n : n = 1, \ldots\} \) be a sequence of partitions of \([a, b]\) such that each \( P_n \) divides \([a, b]\) into \( n \) segments of equal length. Thus, \( S_{P_n} - s_{P_n} \to 0 \). Letting “\( \varepsilon = (1/2)(b - a) \)” we have

\[ \exists m \geq 4 \text{ such that } \forall n \geq m, \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < (1/2)(b - a). \]

By the definition of \( \{P_n : n = 1, \ldots\} \), and since \( n \geq 4 \),

\[ \sum_{i=1}^{n} (M_i - m_i) < (1/2)(b - a) \frac{n}{b - a} = \frac{n}{2} \leq n - 2. \]

Consequently, for each \( n \geq m \) there are at least 3 integers, \( i \), where \( 1 \leq i \leq n \), such that \( M_i - m_i < 1 \).

With these estimates we generate inductively a nested sequence of intervals \([a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]\) such that \( a_n \neq a_{n-1}, \ b_n \neq b_{n-1}, \ b_n - a_n \leq (b - a)/4^n \), and

\[ \omega(f, [a_n, b_n]) < 1/n. \]

Thus \( \bigcap [a_n, b_n] = \{x_0\} \) and \( f \) is continuous at \( x_0 \). Using the same technique we obtain continuity on a dense set.

We needed three integers above to determine continuity instead of one-sided continuity.

\[ \square \]

**Theorem 3.4.5.** Riemann integrability and continuity m-a.e.

Let \( f : [a, b] \to \mathbb{R} \) be a bounded function. \( R \int_{a}^{b} f \) exists if and only if \( f \) is continuous m-a.e.

**Proof.** \((\implies)\) Let

\[ D_k = \left\{ x : \lim_{\delta \to 0} \sup_{y, z \in [x - \delta, x + \delta]} |f(y) - f(z)| > 1/k \right\}. \]
so that \( D(f) = D_k \). We assume that \( m(D(f)) > 0 \) and that \( R \int_a^b f \) does not exist.

Since \( m(D(f)) > 0 \) there is a \( k \) for which \( m(D_k) = d_k > 0 \). Let \( P : x_0 < \ldots < x_n \) be a partition of \([a, b]\). Then \( \bigcup (x_{j-1}, x_j) \) covers all but a finite number of points of \( D_k \), and so

\[
\sum_{j \in P} (x_j - x_{j-1}) \geq d_k,
\]

where \( j \in \hat{P} \) indicates that \( (x_{j-1}, x_j) \cap D_k \neq \emptyset \). From the definition of \( D_k \), if \( j \in \hat{P} \) then

\[ M_j - m_j > 1/k. \]

Thus,

\[
S_P - s_P = \sum (M_j - m_j)(x_j - x_{j-1}) \geq \sum (M_j - m_j)(x_j - x_{j-1}) > \frac{1}{k} \sum (x_j - x_{j-1}) \geq \frac{d_k}{k} > 0. \tag{3.19}
\]

The number \( k \) is fixed and (3.19) is true for all partitions \( P \). Consequently \( R \int_a^b f \) does not exist.

\((\Leftarrow\Rightarrow)\) Let \( M = \|f\|_\infty \) and consider the functions \( R \int_a^x f \) and \( R \int_a^{x_2} f \), where \( R \int_a^x f \) is defined as \( R \int_a^y f \) if \( x \geq y \) and we similarly extend \( R \int_a^x f \). For each \( a \leq x_1 < x_2 \leq b \) we have

\[
\left| R \int_a^{x_2} f - R \int_a^{x_1} f \right| \leq M(x_2 - x_1). \tag{3.20}
\]

Let

\[ F = R \int_a^x f. \]

It is easy to check directly, e.g., Problem 1.30 and Problem 3.17, that

\[
F'(x) = f(x) \quad m\text{-a.e.} \tag{3.21}
\]

Here, of course, we use the hypothesis that \( f \) is continuous \( m\text{-a.e.} \).

Define the functions

\[ f_n(x) = n \left[ F \left( x + \frac{1}{n} \right) - F(x) \right], \quad n = 1, \ldots. \]

Each \( f_n \) is continuous on \([a, b]\) and, because of (3.20), \( \{f_n : n = 1, \ldots\} \) is a uniformly bounded sequence of functions. It is also clear, from (3.21), that \( f_n \to f \) \( m\text{-a.e.} \). Consequently, we apply LDC and obtain
Next observe that
\[
\int_a^b f_n = n \int_b^{b+(1/n)} F - n \int_a^{a+(1/n)} F = F(b) - n \int_a^{a+(1/n)} F.
\]
Also,
\[
\left| n \int_a^{a+(1/n)} F \right| \leq \sup_{x \in [a,a+(1/n)]} R \int_a^x f \int_a^x 1 \, dx \leq M/n.
\]
Hence,
\[
\lim_{n \to \infty} \int_a^b f_n = F(b) = R \int_a^b f.
\]
Combining this with (3.22) yields
\[
\int_a^b f = R \int_a^b f;
\]
and we compute similarly that
\[
\int_a^b f = R \int_a^b f.
\]
\[\square\]

**Example 3.4.6.** Lebesgue integrable non-Riemann integrable functions

**a.** Let \( E \subseteq [0,1] \) be a perfect symmetric set with \( m(E) > 0 \), and let \( f \) be its associated Volterra function defined in Example 1.3.1. Recall that the derivative of \( f \) vanishes on \( E \), and for \( x \) just to the right of \( a \), in the contiguous interval \((a,b)\), it has the value
\[
2(x-a) \sin \left( \frac{1}{x-a} \right) - \cos \left( \frac{1}{x-a} \right).
\]
From the symmetric definition of \( f \) on \((a,b)\) we therefore see that \( f' \) is bounded and Lebesgue measurable. Thus \( f' \in L^1_m([0,1]) \). Since \( f' \) is not continuous \( m\text{-a.e.} \) we can conclude from Theorem 3.4.5 that \( R \int_0^1 f' \) does not exist, see part c.ii and Example 3.4.10.

**b.** For the ruler function \( r : [0,1] \to \mathbb{R} \), \( R \int_0^1 r \) exists since \( m(D(r)) = m(\mathbb{Q} \cap [0,1]) = 0 \). \( 1_{[0,1] \cap \mathbb{Q}} \) is not Riemann integrable as we saw in Example 3.1.1. For the generalized ruler function \( r_\gamma \), of which \( r \) and \( 1_{[0,1] \cap \mathbb{Q}} \) are special
cases, we observed that if $\gamma_q \to 0$ then $m(D(r_\gamma)) = m(Q \cap [0, 1]) = 0$ and so $R \int_0^1 r_\gamma = 0$, e.g., Example 1.3.8.

**c.** We have seen that $f(x) = x^{-1/2}$ and $f(x) = 1_{[0,1] \cap \mathbb{Q}}$ are not Riemann integrable on $[0, 1]$, whereas they are both Lebesgue integrable. In some sense both of these functions provide unsatisfactory examples: in the first case $f$ is unbounded, and, although the Riemann integral necessarily supposes $f$ to be bounded, $\int_0^1 x^{-1/2} \, dx$ exists in an “improper” Riemann sense; similarly $f = 1_{[0,1] \cap \mathbb{Q}} = 0$ m-a.e., so that even though $R \int_0^1 1_{[0,1] \cap \mathbb{Q}}$ does not exist the function $g$, which is identically 0 and equal to $f$ m-a.e., is Riemann integrable. We now give genuine examples of bounded Lebesgue measurable functions $f$ defined on $[0, 1]$ such that

$$\forall g = f \text{ m-a.e., } R \int_0^1 g \text{ does not exist.}$$

**c.i.** Let $E \subseteq [0, 1]$ be a perfect symmetric set with Lebesgue measure $m(E) > 0$. Setting $f = 1_E$, we obtain (3.23) since $m(D(f)) > 0$.

**c.ii.** Volterra’s example $f$ (part a) has the property that $f'$ is a bounded Lebesgue measurable function and $m(D(f')) > 0$.

A complete solution to the problem implied by Example 3.4.6b is given in the following result due to Isaac J. Schoenberg.

**Proposition 3.4.7.** The Riemann integral of the generalized ruler function $r_\gamma$ on $[0, 1]$ exists if and only if $\gamma_q \to 0$; in this case $R \int_0^1 r_\gamma = 0$.

**Proof.** Obviously we need only check that if $R \int_0^1 r_\gamma$ exists then $\gamma_q \to 0$. Assume not. We prove that $R \int_0^1 r_\gamma$ does not exist by showing that $r_\gamma$ is not continuous at any irrational and applying Theorem 3.4.5.

Given $n$, consider the $(\phi(n))$ integers $r_1 < \ldots < r_{\phi(n)} \leq n$ for which $(r_j, n) = 1$, and define the partition

$$P_n : 0 < r_1/n < \ldots < r_{\phi(n)}/n < 1$$

with norm

$$g_n = \sup \left\{ \frac{r_j}{n} - \frac{r_{j-1}}{n} : j = 1, \ldots, \phi(n) + 1, r_0 = 0, \text{ and } r_{\phi(n)+1} = n \right\}.$$

$\phi$ is the Euler function mentioned in Problem 1.25. We now make an act of faith and state that

$$\lim_{n \to \infty} g_n = 0.$$  \hspace{1cm} (3.24)

The proof of (3.24) depends on a theorem due to György (George) Pólya, e.g., [190], I, problem 188, which proves that the partitions $P_n$ are asymptotically uniformly distributed; recall Problem 1.28 and see Problem 3.24 with regard to uniform distribution.
Pick any irrational number \( y \in (0, 1) \), and for each \( n \) let \( k = k(n) \) be the integer for which
\[
\frac{r_{k-1}}{n} < y < \frac{r_k}{n}.
\]
From (3.24), \( \lim_{n \to \infty} r_{k(n)}/n = y \). On the other hand \( r_{\gamma}(y) = 0 \) and \( r_{\gamma}(r_{k(n)}/n) = \gamma_n \). Since we have assumed that \( \{\gamma_n : n = 1, \ldots\} \) does not tend to 0 we conclude that \( r_{\gamma} \) is not continuous \( m.a.e. \)

The following is left as an exercise (Problem 3.16).

**Proposition 3.4.8.** Assume that the function \( f : [a, b] \to \mathbb{R} \) has the property that
\[
\forall x \in (a, b), \quad \lim_{y \to x^\pm} f(y) \text{ exists.}
\]
Then
\[
\mathcal{R} \int_a^b f \text{ exists.}
\]

For perspective we have the following proposition.

**Proposition 3.4.9.** If \( f \in L_m^1([a, b]) \) then there is a sequence \( \{\alpha_n : n = 1, \ldots\} \) such that if \( \beta_n \in (0, \alpha_n), n = 1, \ldots \), then \( \lim_{n \to \infty} f(x + \beta_n) = f(x) \), \( m.a.e. \).

This result is proved by choosing a sequence \( \{\alpha_n : n = 1, \ldots\} \) for which
\[
\sum \|\tau_{\alpha_n} f - f\|_1 < \infty,
\]
where \( \tau_{\alpha} f(x) = f(x - \alpha) \). It should be compared with Problem 2.40; and the details of its proof are the content of Problem 3.3.

**Example 3.4.10. More on the function** \( f(x) = 1/x^{1/2} \)

\( a. \) In Problem 3.3b we show that \( f(x) = 1/x^{1/2} \in L_m^1([0, 1]) \). Using this result we now observe that if \( f \geq 0 \) is a Lebesgue measurable function defined on \( \mathbb{R} \) and if
\[
\forall a < b, \forall r \in \mathbb{R}, \quad m(\{(a, b) \cap \{x : f(x) \geq r\}\}) > 0,
\]
then it is not necessarily true that
\[
\int_{\mathbb{R}} f(x) \, dx = \infty.
\]

In fact, define
\[
f_k(x) = \begin{cases} (x - r_k)^{-1/2}, & \text{if } x \in (r_k, r_k + 1), \\ 0, & \text{otherwise,} \end{cases}
\]
and set
The Lebesgue integral

\[ f = \sum_{k=1}^{\infty} \frac{f_k}{2^k}, \]

where \( \{r_k : k = 1, \ldots\} = \mathbb{Q} \). We then compute that

\[ \int_{\mathbb{R}} f(x) \, dx = \frac{2}{3}. \]

Clearly, for any \( a < b \) and \( r > 0 \) there is \( q \in \mathbb{Q} \cap (a, b) \) and \( c \) such that \( (q, c) \subseteq (q, q + 1) \) and \( f \geq r \) on \( (q, c) \).

When in view of the function \( f(x) = 1/x^{1/2}, x \neq 0, f(0) = 0 \), defined on \([0, 1]\) we now define a function \( g \), unbounded on \([0, 1]\), such that \( g \in L_1([0, 1]) \) and the improper Riemann integral of \( g \) does not exist. Note that for \( f(x) = \sin(x)/x \) on \((0, \infty)\) the opposite phenomenon is true (Problem 3.27d). Let

\[ \sum_{k=1}^{\infty} a_k \]

be a convergent series of positive terms, and set

\[ g(x) = \sum_{k=1}^{\infty} \frac{a_k}{|x - b_k|^{1/2}}, \]

where \( \{b_k : k = 1, \ldots\} \) is a dense subset of \([0, 1]\). Of course we could even define some sort of improper Riemann integral in this case in terms of partial sums.

### 3.5 Lebesgue–Stieltjes measure and integral

In this chapter we have defined and developed a general theory of integration for arbitrary measures. For historical reasons, we would like to devote this section to a class of integrals which is closely related to the Lebesgue integral and, more importantly, which may be viewed as a nascent means of formulating the spectacular equivalence of measure theoretic integration theory and the functional analytic theory of Radon measures developed in Chapter 7.

We shall start by recalling Example 2.3.10, where we have considered the family \( \mathcal{Q} = \{(a_1, b_1] \times \ldots \times (a_d, b_d] : a_1, \ldots, a_d, b_1, \ldots, b_d \in \mathbb{R}^d\} \) of half-open parallelepipeds in \( \mathbb{R}^d \); these are half-open intervals \((a, b] \) in \( \mathbb{R} \) if \( d = 1 \). This family forms a semiring in \( \mathcal{P}(\mathbb{R}^d) \) as defined in Problem 2.22. In view of Theorem 2.3.7 and Problem 2.22, in order to find a Borel measure on \( \mathbb{R}^d \) it is enough to find a \( \sigma \)-additive non-negative set function on a \( \sigma \)-algebra that contains \( \mathcal{Q} \). We shall restrict our attention to the case \( d = 1 \) and we begin with the following observation.

**Theorem 3.5.1. \( \sigma \)-additive set functions on \( \mathcal{Q} \)**

Let \( f : \mathbb{R} \to \mathbb{R} \) be an increasing right continuous function. The set function \( \mu_f \) defined by

\[ \mu_f([a, b]) = f(b) - f(a) \]

is a non-negative, \( \sigma \)-additive set function on the semiring \( \mathcal{Q} \).
Proof. Clearly, $\mu_f$ is non-negative and finitely additive. So we only need to prove $\sigma$-additivity. Let $A = \bigcup_{j=1}^{\infty} A_j$, where $\{A = (a, b], A_j = (a_j, b_j] : j = 1, \ldots \} \subseteq Q$ and where the $A_j$ are pairwise disjoint. Since $\bigcup_{j=1}^{n} A_j \subseteq A$, it follows from the finite additivity and non-negativity of $\mu_f$ that

$$\sum_{j=1}^{n} \mu_f(A_j) \leq \mu_f(A).$$

Therefore, taking the limit as $n \to \infty$, we have

$$\sum_{j=1}^{\infty} \mu_f(A_j) \leq \mu_f(A).$$

To prove the opposite inequality fix $\varepsilon > 0$ and, using the right continuity of $f$, choose $\delta > 0$ and $\delta_j, j = 1, \ldots$, so that

$$\mu_f((a, a + \delta]) = f(a + \delta) - f(a) \leq \varepsilon$$

and

$$\mu_f((b_j, b_j + \delta_j]) = f(b_j + \delta_j) - f(b_j) \leq \frac{\varepsilon}{2^j}.$$

Then

$$[a + \delta, b] \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j + \delta_j).$$

By compactness of a closed interval there exists $k$ such that (without loss of generality)

$$[a + \delta, b] \subseteq \bigcup_{j=1}^{k} (a_j, b_j + \delta_j).$$

Therefore, the non-negativity and finite additivity of $\mu_f$ imply that

$$\mu_f((a + \delta, b]) \leq \sum_{j=1}^{k} \mu_f((a_j, b_j + \delta_j]) \leq \sum_{j=1}^{\infty} \mu_f((a_j, b_j + \delta_j])$$

$$\leq \sum_{j=1}^{\infty} \mu_f((a_j, b_j]) + \sum_{j=1}^{\infty} \mu_f((b_j, b_j + \delta_j]).$$

Considering our choice of $\delta$ and $\delta_j$ we obtain

$$\mu_f((a, b]) \leq \mu_f((a, a + \delta]) + \sum_{j=1}^{\infty} \mu_f((a_j, b_j]) + \sum_{j=1}^{\infty} \mu_f((b_j, b_j + \delta_j])$$

$$\leq \varepsilon + \sum_{j=1}^{\infty} \mu_f((a_j, b_j]) + \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j}$$

$$\leq \sum_{j=1}^{\infty} \mu_f((a_j, b_j]) + 2\varepsilon.$$
We complete the proof by letting $\varepsilon \to 0$, being technically rigorous by taking the limit of both sides.

\[ \square \]

Theorem 3.5.1 together with Problem 2.22 imply that $\mu_f$ extends to a $\sigma$-additive set function on $\mathcal{R}$, the ring generated by $\mathcal{Q}$. Following the developments in Section 2.3 that led us to Theorem 2.3.7, we note that $\mu_f$ further extends to a $\sigma$-additive set function on the $\sigma$-ring $\mathcal{A}_\mathcal{Q}$ of sets measurable with respect to $\mu_f$ and $\mathcal{Q}$. Since the elements of the ring $\mathcal{R}$ are measurable, $\mathcal{A}_\mathcal{Q}$ is in fact a $\sigma$-algebra; and this extension of $\mu_f$ is a measure. We shall also denote this new measure by $\mu_f$, and it is defined to be the Lebesgue–Stieltjes measure associated with the function $f$.

The assumption in Theorem 3.5.1 that $f$ is real-valued implies in particular that $\mu_f$ is $\sigma$-finite on $\mathcal{R}$. In view of Problem 2.20 this means that there exists a unique extension of $\mu_f$ to the smallest $\sigma$-algebra generated by $\mathcal{Q}$.

We have the following “converse” of Theorem 3.5.1.

**Proposition 3.5.2.** Let $\mu : \mathcal{Q} \to \mathbb{R}^+$ be a $\sigma$-additive set function. There exists an increasing right continuous function $f : \mathbb{R} \to \mathbb{R}$ such that
\[ \forall \ a < b, \quad \mu((a, b)) = f(b) - f(a). \]  
\((3.25)\)

**Proof.** We define
\[ f(x) = \begin{cases} 
\mu((0, x]), & \text{if } x \geq 0, \\
\mu((x, 0]), & \text{if } x < 0.
\end{cases} \]

This function satisfies (3.25) since $\mu$ is additive. Clearly this choice is not unique. $f$ is increasing since
\[ a \leq b \implies f(b) - f(a) = \mu((a, b]) \geq 0 \implies f(b) \geq f(a). \]

To show that $f$ is right continuous consider a sequence $\{a_n : n = 1, \ldots\} \subseteq \mathbb{R}$, $a_1 \leq \ldots \leq a_n \ldots$, $a_n \to b$. We may write
\[ [a_1, b] = \bigcup_{j=1}^{\infty} (a_j, a_{j+1}], \]
and use the $\sigma$-additivity of $\mu$ to obtain
\[ f(b) - f(a_1) = \mu((a_1, b]) = \sum_{j=1}^{\infty} \mu((a_j, a_{j+1}]) \]
\[ = \sum_{j=1}^{\infty} (f(a_j) - f(a_{j+1})) \]
\[ = \lim_{j \to \infty} f(a_j) - f(a_1). \]  
\[ \square \]
Corollary 3.5.3. Let $\mu : B(\mathbb{R}) \to \mathbb{R}^+ \cup \{\infty\}$ be a measure that is finite on all bounded intervals. There exists an increasing right continuous function $f : \mathbb{R} \to \mathbb{R}$ such that (3.25) holds.

The integral of Section 3.2 associated with the measure $\mu_f$ is called the Lebesgue–Stieltjes integral, and the Lebesgue–Stieltjes integral of $h$ is naturally denoted by

$$\int_\mathbb{R} h \, d\mu_f.$$  

Clearly, Lebesgue measure and the Lebesgue integral are a special example of the Lebesgue–Stieltjes theory for the case that $f(x) = x$ and $\mu_f = m$.

In view of the results of Section 3.4, we shall now define the Riemann–Stieltjes integral with respect to an increasing right continuous function and we shall give its relationship with the Lebesgue–Stieltjes integral.

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing right continuous function. Let $h : [a, b] \to \mathbb{R}$ be a bounded function. Analogous to Section 3.1, for any partition $P : a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$ we let

$$S^f_P(h) = \sum_{i=1}^n M_i (f(x_i) - f(x_{i-1}))$$

and

$$s^f_P(h) = \sum_{i=1}^n m_i (f(x_i) - f(x_{i-1})),$$

where

$$M_i = \sup \{h(x) : x_{i-1} < x \leq x_i\}$$

and

$$m_i = \inf \{h(x) : x_{i-1} < x \leq x_i\},$$

for $i = 1, \ldots, n$. Define

$$\int_a^b h \, df = \inf_P S^f_P(h) \quad \text{and} \quad \int_a^b h \, df = \sup_P s^f_P(h). \quad (3.26)$$

As in the case for the Riemann integral, we have

$$\int_a^b h \, df \geq \int_a^b h \, df.$$  

If $\int_a^b h \, df = \int_a^b h \, df$, we say that $h$ is Riemann–Stieltjes integrable with respect to $f$. In this case, the Riemann–Stieltjes integral of $f$ over $[a, b]$ is

$$\int_a^b h \, df = \int_a^b h \, df = \int_a^b h \, df. \quad (3.27)$$
The Riemann–Stieltjes integral played a fundamental role historically in the functional analytic formulation of integration theory exemplified by the Riesz representation theorem, see Theorem 7.1.1.

In analogy with Theorem 3.4.5 we have the following result.

**Theorem 3.5.4. Riemann–Stieltjes integrability and continuity $\mu_f$-a.e.**

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing right continuous function and let $h : [a, b] \to \mathbb{R}$ be a bounded function. $h$ is Riemann–Stieltjes integrable with respect to $f$ if and only if $h$ is continuous $\mu_f$-a.e.

The proof of Theorem 3.5.4 is left as an exercise (Problem 3.18).

We shall now use Theorem 3.5.4 to prove that Lebesgue–Stieltjes integration generalizes the Riemann–Stieltjes idea.

**Theorem 3.5.5. Lebesgue–Stieltjes integrals generalize Riemann–Stieltjes integrals**

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing right continuous function and let $h : [a, b] \to \mathbb{R}$ be a bounded function. If $h$ is Riemann–Stieltjes integrable, then it is integrable with respect to the measure $\mu_f$, and

$$
\int_a^b h \, df = \int_a^b h \, d\mu_f.
$$

(3.28)

**Proof.** Reformulating the definition of Riemann–Stieltjes integrability it is not difficult to see that there are increasing sequences $\{P_k\}$ and $\{Q_k\}$ of partitions with norms $|P_k|, |Q_k| \to 0$. Associated with these partitions there are sequences $\{h_k, k = 1, \ldots\}$ and $\{H_k, k = 1, \ldots\}$ of simple functions, such that $h_k \leq h_{k+1} \leq h \leq H_{k+1} \leq H_k$ on $[a, b]$, and for which

$$
\int_a^b h \, df = \lim_{k \to \infty} \int_a^b h_k \, d\mu_f
$$

and

$$
\int_a^b h \, df = \lim_{k \to \infty} \int_a^b H_k \, d\mu_f.
$$

As such, we define the functions

$$
m(x) = \lim_{k \to \infty} h_k(x)
$$

and

$$
M(x) = \lim_{k \to \infty} H_k(x).
$$

Clearly, $m(x) \leq h(x) \leq M(x)$ on $[a, b]$.

Further, we verify that if $h$ is continuous at $x \in [a, b]$ then $m(x) = h(x) = M(x)$. Indeed, if $h$ is continuous at $x$, then for every $\varepsilon > 0$ there is $\delta > 0$ such
that $|x - y| < \delta$ implies $|h(x) - h(y)| < \varepsilon$. Because we have assumed that the norms of partitions converge to 0, we obtain $m(x) = h(x) = M(x)$ by letting $\varepsilon \to 0$.

On the other hand, a converse statement is true for all but countably many $x \in [a, b]$. To see this, let $x \notin \bigcup_{k=1}^{\infty} (P_k \cup Q_k)$ and suppose $m(x) = h(x) = M(x)$. Then, there exists a decreasing sequence of partition intervals $(x_k, y_k)$ containing $x$ in their interiors such that $m_k = \inf \{h(y) : y \in (x_k, y_k]\} \to h(x)$ and $M_k = \sup \{h(y) : y \in (x_k, y_k]\} \to h(x)$. This implies the continuity of $h$ at $x$.

The functions $h_k$ and $H_k$ are $\mu_f$-measurable. Therefore $m$ and $M$ are $\mu_f$-measurable as limits of $\mu_f$-measurable functions.

By Theorem 3.5.4 Riemann–Stieltjes integrability of $h$ with respect to $f$ implies that $h$ is continuous $\mu_f$-a.e. In view of our previous observation this means that $m(x) = h(x) = M(x)$ $\mu_f$-a.e. Since both $m$ and $M$ are $\mu_f$-measurable, and since, in view of Problem 2.26, the Lebesgue–Stieltjes measure is complete, it follows from Theorem 2.5.4 that $f$ is measurable with respect to the measure $\mu_f$ on the $\sigma$-algebra of measurable sets generated by $Q$. Moreover, by LDC,

$$
\int_a^b h \, df = \int_a^b m \, d\mu_f = \int_a^b M \, d\mu_f = \int_a^b h \, d\mu_f
$$

and so (3.28) is obtained.

The above discussion extends easily to $\mathbb{R}^d$ in the following way. Let $Q$ be the family of all half-open parallelepipeds in $\mathbb{R}^d$ and let $f : \mathbb{R} \to \mathbb{R}$ be an increasing right continuous function. Then there exists a non-negative finitely additive set function $\mu_f^d$ on $Q$ such that

$$
\mu_f^d((a_1, b_1] \times \cdots \times (a_d, b_d]) = (f(b_1) - f(a_1)) \cdot \ldots \cdot (f(b_d) - f(a_d)).
$$

We leave the verification of this assertion for the reader, as well as proving the fact that such a function $\mu_f^d$ extends to a measure on $\mathcal{B}(\mathbb{R}^d)$, see Problem 3.26. With this definition of $\mu_f^d$ all of the results in this section can be extended to $\mathbb{R}^d$, cf., [7]. For a more general approach to Lebesgue–Stieltjes measures on $\mathbb{R}^d$, see [6].

### 3.6 Some fundamental applications

The following three results are variants of LDC, and they are left as exercises (Problem 3.19).

**Theorem 3.6.1. The interchange of limits and integration**

Let $(X, \mathcal{A}, \mu)$ be a measure space and choose a subset $S \subseteq \mathbb{R}$. Pick $t_0 \in S$. Define the function $f(x, t)$ on $X \times S$ assuming that
\textbf{The Lebesgue integral}

i. \( \forall t \in S, \ f(\cdot, t) \) is a \( \mu \)-measurable function,

ii. \( \lim_{t \to t_0} f(x, t) = f_0(x) \) exists \( \mu \)-a.e.

and

iii. \( \exists g \in L_\mu^1(X) \) such that \( \forall t \in S, \ |f(x, t)| \leq g(x) \) \( \mu \)-a.e.

Then \( f_0 \in L_\mu^1(X) \) and

\[
\lim_{t \to t_0} \int_X f(x, t) \, d\mu(x) = \int_X f_0 \, d\mu.
\]

Theorem 3.6.1 is true for the case that \( S \) is a metric space, e.g., Appendix 10.1.

\textbf{Theorem 3.6.2. The interchange of summation and integration}

Let \((X, \mathcal{A}, \mu)\) be a measure space and assume that \( \{f_n : n = 1, \ldots\} \) is a sequence of \( \mu \)-measurable functions for which

\[
\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty.
\]

Then \( \sum_{n=1}^{\infty} f_n \in L_\mu^1(X) \) and

\[
\int_X \left( \sum_{n=1}^{\infty} f_n \right) \, d\mu = \sum_{n=1}^{\infty} \left( \int_X f_n \, d\mu \right).
\]

\textbf{Theorem 3.6.3. The interchange of differentiation and integration}

Let \((X, \mathcal{A}, \mu)\) be a \( \sigma \)-finite measure space and choose an open interval \( S \subseteq \mathbb{R} \). Define the function \( f(x, t) \) on \( X \times S \) assuming that:

i. \( f(\cdot, t) \) is a \( \mu \)-measurable function for \( m \)-a.e. \( t \in S \), and there exists \( t_0 \in S \) such that \( f(\cdot, t_0) \) is integrable,

ii. \( \frac{d}{dt}f(x, t) \) exists and is continuous on \( S \) \( \mu \)-a.e.,

and

iii. \( \exists h \in L_\mu^1(X) \) such that \( \forall t \in S, \ |\frac{d}{dt}f(x, t)| \leq h(x) \) \( \mu \)-a.e.

Then

\[
\frac{d}{dt} \int_X f(x, t) \, d\mu(x) = \int_X \frac{d}{dt}f(x, t) \, d\mu(x).
\]

Theorems 3.6.1, 3.6.2, and 3.6.3 generalize classical criteria which one proves with a uniform convergence hypothesis, e.g., [4], [211].

The \textit{Riemann–Lebesgue lemma}, which we now prove, is a fundamental result in Fourier series; we indicate another proof of it in Problem 3.10d.

\textbf{Theorem 3.6.4. Riemann–Lebesgue lemma}

For each \( f \in L_\mu^1(T) \)

\[
\lim_{|n| \to \infty} \hat{f}(n) = 0.
\]
Proof. If \((a, b) \subseteq [0, 1),\)

\[
\lim_{|n| \to \infty} \left| \int_a^b e^{-2\pi inx} \, dx \right| = \lim_{|n| \to \infty} \left| \frac{1}{n} \left( e^{-2\pi inb} - e^{-2\pi ina} \right) \right| = 0.
\]

Consequently, the result is true for characteristic functions of intervals; and the expected application of our approximation theorems yields the result for arbitrary elements of \(L^1_m(\mathbb{T}).\)

\[\Box\]

**Example 3.6.5. Weak convergence that is not strong convergence**

There is a sequence \(\{f_n : n = 1, \ldots\} \subseteq L^1_m(\mathbb{T})\) such that

\[
\forall A \in \mathcal{M}(\mathbb{T}), \quad \lim_{n \to \infty} \int_A f_n(x) \, dx = 0
\]

and

\[
\lim_{n \to \infty} \int_0^1 |f_n(x)| \, dx > 0.
\]

In fact, \(\lim_{n \to \infty} \int_A \sin(2\pi nx) \, dx = 0\) by Theorem 3.6.4, whereas

\[
\int_0^1 |\sin(2\pi nx)| \, dx = \frac{2}{\pi}.
\]

From the point of view of functional analysis this will serve as an example of weak convergence which is not strong convergence. We shall discuss this phenomenon more fully in Chapter 6.

**Example 3.6.6. Unboundedness associated with a generalized sinc function on \(\mathbb{Z}\)**

We shall construct an open set \(S \subseteq [0, 1)\) such that

\[
\lim_{n \to \infty} |n \mathbb{1}_S(n)| = \infty; \quad (3.29)
\]

note that \(\lim_{n \to \infty} \mathbb{1}_S(n) = 0\) because of Theorem 3.6.4. In particular \(\mathbb{1}_S\) is not a function of bounded variation, e.g., Problem 4.4d, a notion we study systematically in the next chapter. If \(S = [-T, T], 0 < T < 1/2,\) then

\[
\mathbb{1}_S(n) = \frac{1}{\pi n} \sin(2\pi Tn),
\]

and so it is natural to seek sets \(S\) for which (3.29) holds, see [170], [39], and [105].

For each \(j\) and each \(k = 1, \ldots, 2j\) set

\[
S_{j,k} = ((k - 1/2)/(2j)!, k/(2j)!).
\]

These \(2j\) intervals are obviously disjoint. We define
The Lebesgue integral

\[ S_j = \bigcup_{k=1}^{2j} S_{j,k} \quad \text{and} \quad S = \bigcup_{j=1}^{\infty} S_j, \]

noting that the sequence \( \{S_j : j = 1, \ldots\} \) forms a disjoint family. We shall prove that

\[ \lim_{m \to \infty} (2m)! \mathbb{1}_S(-(2m)!)) = \infty. \]

We first calculate

\[
\int_{S_{j,k}} e^{2\pi i (2m)!x} \, dx = \frac{e^{2\pi i k(2m)!/(2j)!} - e^{2\pi i (k-1/2)(2m)!/(2j)!}}{2\pi i (2m)!}. \tag{3.30}
\]

Observe that the numerator in (3.30) vanishes if \( j < m \) and takes the value 2 if \( j = m \). Therefore

\[
\mathbb{1}_S(-(2m)!)) = \sum_{j=m}^{\infty} \int_{S_j} e^{2\pi i (2m)!x} \, dx = \sum_{j=m}^{\infty} \int_{S_j} e^{2\pi i (2m)!x} \, dx + \frac{2m}{\pi i (2m)!}.
\]

Further, \( \bigcup_{j=m+1}^{\infty} S_j \subseteq (0, 1/(2m+1)!)) \) and so

\[
\left| \sum_{j=m+1}^{\infty} \int_{S_j} e^{2\pi i (2m)!x} \, dx \right| \leq \frac{1}{(2m+1)!}.
\]

Consequently,

\[
|(2m)! \mathbb{1}_S(-(2m)!))| \geq \frac{2m}{\pi} - \frac{1}{2m+1} \to \infty.
\]

3.7 Fubini and Tonelli theorem

We now present the Fubini and Tonelli theorems. These are among the most important results in analysis. Let \((X, \mathcal{A}_1, \mu)\) and \((Y, \mathcal{A}_2, \nu)\) be \(\sigma\)-finite measure spaces. The Fubini–Tonelli theorems give conditions so that

\[
\iint_{X \times Y} f \, d(\mu \times \nu) = \int_X \left( \int_Y f(x,y) \, d\nu(y) \right) \, d\mu(x) = \int_Y \left( \int_X f(x,y) \, d\mu(x) \right) \, d\nu(y). \tag{3.31}
\]

Obviously, just the formal expressions in (3.31) require explanations. We shall start by constructing a \textit{product measure} on the subsets of the direct product \(X \times Y\) of the spaces \(X\) and \(Y\).
3.7 Fubini and Tonelli theorem

Let $A \in \mathcal{A}_1$ and $B \in \mathcal{A}_2$, $A \times B$ is a measurable rectangle. The collection $\mathcal{R}$ of all such rectangles is a semialgebra as defined in Problem 2.22, i.e. $X \times Y \in \mathcal{R}$ it is closed under finite intersections,

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2),$$

and a complement of a rectangle is a finite union of rectangles,

$$(A \times B)^\complement = (A^\complement \times B) \cup (A \times B^\complement) \cup (A^\complement \times B^\complement).$$

Let $\mathcal{A}$ be the collection of all finite unions of disjoint elements of $\mathcal{R}$. Clearly $\mathcal{A}$ is an algebra, e.g., Section 2.3.

For any rectangle $A \times B \in \mathcal{R}$ we define

$$\omega(A \times B) = \mu(A) \cdot \nu(B),$$

with the convention that $0 \cdot \infty = 0$. This convention is consistent with the assumption that the measure spaces $(X, \mathcal{A}_1, \mu)$ and $(Y, \mathcal{A}_2, \nu)$ are $\sigma$-finite.

Indeed, if e.g., $\mu(A) = 0$ and $\nu(B) = \infty$, then there exist sets $\{B_n : n = 1, \ldots\}$ such that $B \subseteq \bigcup B_n$ and $\nu(B_n) < \infty$, $n = 1, \ldots$. Hence, if we want $\omega$ to be extended to a $\sigma$-additive set function, we must put $\omega(A \times B) = 0$, because $\sum_1^N \omega(A \times B_i) = 0$.

For any $\bigcup_1^N (A_i \times B_i) \in \mathcal{A}$, a disjoint sum, we let

$$\omega \left( \bigcup_1^N (A_i \times B_i) \right) = \sum_1^N \omega(A_i \times B_i).$$

Clearly, $\omega : \mathcal{A} \to \mathbb{R}^+$ is a $\sigma$-additive set function on the algebra $\mathcal{A}$. Thus, following our construction of a measure in Section 2.3, or using Carathéodory’s theorem (Problem 2.20), there exists an extension of $\omega$ to a $\sigma$-additive set function on the $\sigma$-algebra generated by $\mathcal{A}$. We denote this $\sigma$-algebra by $\mathcal{A}_1 \times \mathcal{A}_2 \subseteq \mathcal{P}(X \times Y)$. Moreover, since we have assumed that $(X, \mathcal{A}_1, \mu)$ and $(Y, \mathcal{A}_2, \nu)$ are $\sigma$-finite, such an extension is unique. We call this extension the product measure on the measurable space $(X \times Y, \mathcal{A}_1 \times \mathcal{A}_2)$, and $(X \times Y, \mathcal{A}_1 \times \mathcal{A}_2, \mu \times \nu)$ is the associated product measure space where $\mu \times \nu$ is the unique extension of $\omega$.

We combine a few simple properties of product measures in the next proposition.

**Proposition 3.7.1.** Let $(X, \mathcal{A}_1, \mu)$ and $(Y, \mathcal{A}_2, \nu)$ be measure spaces.

a. If $(X, \mathcal{A}_1, \mu)$ and $(Y, \mathcal{A}_2, \nu)$ are bounded measure spaces, then $(X \times Y, \mathcal{A}_1 \times \mathcal{A}_2, \mu \times \nu)$ is a bounded measure space.

b. If $(X, \mathcal{A}_1, \mu)$ and $(Y, \mathcal{A}_2, \nu)$ are $\sigma$-finite measure spaces, then $(X \times Y, \mathcal{A}_1 \times \mathcal{A}_2, \mu \times \nu)$ is a $\sigma$-finite measure space.

c. If $(X, \mathcal{A}_1, \mu)$ and $(Y, \mathcal{A}_2, \nu)$ are complete $\sigma$-finite measure spaces, then $(X \times Y, \mathcal{A}_1 \times \mathcal{A}_2, \mu \times \nu)$ is not necessarily complete.
For a set $E \subseteq X \times Y$, we define its section $E_x \subseteq Y$ as

$$E_x = \{ y \in Y : (x, y) \in E \}.$$ 

Similarly, we define the section $E_y \subseteq X$ of $E$ to be

$$E^y = \{ x \in X : (x, y) \in E \}.$$ 

**Proposition 3.7.2.** Let $(X, \mathcal{A}_1, \mu)$ and $(Y, \mathcal{A}_2, \nu)$ be sigma-finite measure spaces, and let $E \in \mathcal{A}_1 \times \mathcal{A}_2$. Then

$$\forall x \in X, \ E_x \in \mathcal{A}_2, \quad \text{and} \quad \forall y \in Y, \ E^y \in \mathcal{A}_1.$$ 

**Proof.** Since the situation is symmetric for $x$ and $y$, we shall only prove the first part. Thus, it is enough to show that the collection $\mathcal{E}$ of all sets $E \in \mathcal{A}_1 \times \mathcal{A}_2$ for which $E_x \in \mathcal{A}_2$, for all $x \in X$, is a $\sigma$-algebra which contains $\mathcal{A}$.

Clearly, if $E = A \times B$ is a measurable rectangle, then for all $x \in X$ $E_x \in \mathcal{A}_2$, since either $E_x = B$ or $E_x = \emptyset$. In particular, $X \times Y \in \mathcal{E}$. Also, if $E \in \mathcal{E}$, then

$$\forall x \in X, \ (E \setminus E_x)_x = Y \setminus E_x \in \mathcal{A}_2,$$

because $\mathcal{A}_2$ is an algebra. If $\{E_n : n = 1, \ldots\} \subseteq \mathcal{E}$ is a pairwise disjoint collection of elements of $\mathcal{E}$, then from the fact that $\mathcal{A}_2$ is a $\sigma$-algebra we have that

$$\left( \bigcup_{n=1}^{\infty} E_n \right)_x = \bigcup_{n=1}^{\infty} (E_n)_x$$

is also an element of $\mathcal{E}$. □

For a function $f : X \times Y \to \mathbb{R}^*$ we define sections $f_x$ and $f^y$ of $f$ by

$$f_x(y) = f^y(x) = f(x, y).$$

**Proposition 3.7.3.** Let $(X, \mathcal{A}_1 \times \mathcal{A}_2, \mu \times \nu)$ be the $\sigma$-finite product measure space associated with the $\sigma$-finite measure spaces $(X, \mathcal{A}_1, \mu)$ and $(Y, \mathcal{A}_2, \nu)$. Let $f : X \times Y \to \mathbb{R}^*$ be a $(\mu \times \nu)$-measurable function. For all $x \in X$, $f_x$ is $\nu$-measurable, and for all $y \in Y$, $f^y$ is $\mu$-measurable.

**Proof.** The proof is again symmetric for $x$ and $y$, and so we shall only prove the first part. For a set $I \subseteq \mathbb{R}^*$, we have

$$f_x^{-1}(I) = \left( f^{-1}(I) \right)_x.$$ 

Thus the $\nu$-measurability of $f_x$ follows from Proposition 3.7.2 and Proposition 2.4.11. □
Theorem 3.7.4. Measurability of sections

Let \( (X, \mathcal{A}_1, \mu) \) and \( (Y, \mathcal{A}_2, \nu) \) be \( \sigma \)-finite measure spaces. For every \( (\mu \times \nu) \)-measurable set \( E \in \mathcal{A}_1 \times \mathcal{A}_2 \), the functions \( \nu(E_x) \) and \( \mu(E^y) \) are measurable, e.g., \( \nu(E_x) \) is a \( \mu \)-measurable function \( X \to \mathbb{R}^+ \), and

\[
\int_X \nu(E_x) \, d\mu(x) = \int_Y \mu(E^y) \, d\nu(y) = (\mu \times \nu)(E). \tag{3.32}
\]

Proof. The proof is symmetric for \( \nu(E_x) \) and \( \mu(E^y) \). We shall provide details for \( \nu(E_x) \).

Assume \( \mu(X), \nu(Y) < \infty \). If \( E = A \times B \) then \( \nu(E_x) = \nu(B) \cdot 1_A(x) \). Thus,

\[
\int_X \nu(E_x) \, d\mu(x) = \int_X \nu(B) \cdot 1_A(x) \, d\mu(x) = \mu(A) \cdot \nu(B).
\]

For any \( E = \bigcup_{n=1}^N E_n = \bigcup_{n=1}^N (A_n \times B_n) \in \mathcal{A} \), a disjoint sum, we have by the additivity of \( \nu \) that

\[
\int_X \nu(E_x) \, d\mu(x) = \int_X \nu \left( \bigcup_{n=1}^N E_n \right) \, d\mu(x)
= \int_X \nu \left( \bigcup_{n=1}^N (E_n)_x \right) \, d\mu(x)
= \int_X \sum_{n=1}^N \nu((E_n)_x) \, d\mu(x)
= \sum_{n=1}^N \int_X \nu(B_n) \cdot 1_{A_n}(x) \, d\mu(x)
= \sum_{n=1}^N \mu(A_n) \cdot \nu(B_n).
\]

Therefore, for all \( E \in \mathcal{A} \), \( \nu(E_x) \) is \( \mu \)-measurable and

\[
\int_X \nu(E_x) \, d\mu(x) = \mu \times \nu(E).
\]

Let \( \mathcal{E} \) denote the family of all sets \( E \in \mathcal{A}_1 \times \mathcal{A}_2 \) for which our hypothesis is true. We have just showed that \( \mathcal{A} \subseteq \mathcal{E} \).

In view of Problem 2.17, in order to show that \( \mathcal{E} \) contains \( \mathcal{A}_1 \times \mathcal{A}_2 \), the \( \sigma \)-algebra generated by \( \mathcal{A} \), it is enough to prove that \( \mathcal{E} \) is closed under increasing countable unions and decreasing countable intersections.

If \( E = \bigcup_{n=1}^\infty E_n \), \( E_n \subseteq E_{n+1} \), \( n = 1, \ldots \), then the functions \( \nu((E_n)_x) \) are measurable and they increase pointwise to the function \( \nu(E_x) \). Thus, \( \nu(E_x) \) is also measurable. From Theorem 3.3.6 it follows that
The Lebesgue integral

\[
\int_X \nu(E_x) \, d\mu(x) = \lim_{n \to \infty} \int_X \sum_{n=1}^{N} \nu ((E_n)_x) \, d\mu(x)
\]

\[
= \lim_{n \to \infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E),
\]

where the last equality holds in view of Problem 2.42.

The proof that \( \mathcal{E} \) is closed under countable decreasing intersections is similar, and it uses Theorem 3.3.7 and the fact that the measures are bounded.

If we now assume that \( \mu \) and \( \nu \) are \( \sigma \)-finite, then

\[
X \times Y = \bigcup_{n=1}^{\infty} X_n \times Y_n,
\]

where \( \mu \times \nu(X_n \times Y_n) < \infty \) and \( X_n \times Y_n \subseteq X_{n+1} \times Y_{n+1}, \ n = 1, \ldots \). For \( E \in \mathcal{A}_1 \times \mathcal{A}_2 \) and for all \( n, \nu((E \cap (X_n \times Y_n))_x) \) is \( \mu \)-measurable and \( \mu((E \cap (X_n \times Y_n))_y) \) is \( \nu \)-measurable, and, hence, respectively so are \( \nu(E_x) \) and \( \mu(E_y) \). Equation (3.32) now follows from another application of Theorem 3.3.6:

\[
\int_X \nu(E_x) \, d\mu(x) = \int_X \lim_{n \to \infty} \nu((E \cap (X_n \times Y_n))_x) \, d\mu(x)
\]

\[
= \int_X \lim_{n \to \infty} 1_{X_n}(x) \nu(E_x \cap Y_n) \, d\mu(x)
\]

\[
= \lim_{n \to \infty} \int_{X_n} \nu(E_x \cap Y_n) \, d\mu(x)
\]

\[
= \lim_{n \to \infty} (\mu \times \nu)(E \cap (X_n \times Y_n)) = (\mu \times \nu)(E).
\]

A similar argument works for \( \mu(E_y) \).

\( \square \)

Theorem 3.7.5. Fubini theorem for integrable functions

Let \((X, \mathcal{A}_1, \mu)\) and \((Y, \mathcal{A}_2, \nu)\) be \( \sigma \)-finite measure spaces, and let \( h : X \times Y \to \mathbb{R}^* \) be an element of \( L^1_{\mu \times \nu}(X \times Y) \).

a. For \( \mu \)-a.e. \( x \in X \), \( h_x \in L^1_\mu(Y) \), and \( \int_Y h_x \, d\nu \in L^1_\mu(X) \).

b. Similarly, for \( \nu \)-a.e. \( y \in Y \), \( h^y \in L^1_\nu(X) \), and \( \int_X h^y \, d\mu \in L^1_\nu(Y) \).

c. Moreover,

\[
\iint_{X \times Y} h(x, y) \, d(\mu \times \nu)(x, y) = \int_X \left( \int_Y h(x, y) \, d\nu(y) \right) \, d\mu(x)
\]

\[
= \int_Y \left( \int_X h(x, y) \, d\mu(x) \right) \, d\nu(y).
\]

(3.33)

Proof. It follows from Theorem 3.7.4 that the statement of the theorem is true for characteristic functions \( 1_E \) of \( \mu \times \nu \)-measurable sets \( E \subseteq X \times Y \). Thus it also holds for any measurable simple function \( \sum_{j=1}^{\infty} a_j 1_{E_j} \).
3.7 Fubini and Tonelli theorem

Let \( h \in L^1_{\mu\times\nu}(X \times Y) \). Write \( h = h^+ - h^- \), with \( h^+, h^- \geq 0 \). There exist monotone sequences of simple functions \( \{h_n^+: n = 1, \ldots\} \) and \( \{h_n^- : n = 1, \ldots\} \), such that \( h^+ = \lim_{n \to \infty} h_n^+ \) and \( h^- = \lim_{n \to \infty} h_n^- \), pointwise. This implies, in particular, that \( h_n^\pm = \lim_{n \to \infty} (h_n^\pm)_x \), and the convergence is monotone. Thus, the functions \( h_n^\pm \) are \( \nu \)-measurable for all \( x \), and it follows from Theorem 3.3.6 that

\[
\int_Y h_n^\pm(y) \, d\nu(y) = \lim_{n \to \infty} \int_Y (h_n^\pm)_x(y) \, d\nu(y).
\]

Consequently, \( \int_Y h_n^\pm(y) \, d\nu(y) \) is \( \mu \)-measurable.

Another application of Theorem 3.3.6 yields:

\[
\int_X \left( \int_Y h_n^\pm(y) \, d\nu(y) \right) \, d\mu(x) = \int_X \lim_{n \to \infty} \left( \int_Y (h_n^\pm)_x(y) \, d\nu(y) \right) \, d\mu(x)
\]

\[
= \lim_{n \to \infty} \int_X \left( \int_Y (h_n^\pm)_x(y) \, d\nu(y) \right) \, d\mu(x)
\]

\[
= \lim_{n \to \infty} \iint_{X \times Y} h_n^\pm(x,y) \, d(\mu \times \nu)(x,y)
\]

\[
= \iint_{X \times Y} h^\pm(x,y) \, d(\mu \times \nu)(x,y) < \infty.
\]

Therefore \( \int_Y h_n^\pm(y) \, d\nu(y) \in L^1_{\mu}(X) \) and, consequently, \( h_n^\pm \in L^1_{\mu}(Y) \) for \( \mu \)-a.e. \( x \). Analogous arguments hold for \( (h^\pm)^y \).

\[ \square \]

**Example 3.7.6. Intuitive proof of Fubini theorem**

**a.** We outline the following alternative and intuitive proof of Theorem 3.7.5. Let \((X, \mathcal{A}_1, \mu) \) and \((Y, \mathcal{A}_2, \nu) \) be \( \sigma \)-finite complete measure spaces and let \( \mu \times \nu \) be an associated complete product measure.

As a first step, if \( Q \subseteq X \times Y \) and \( Q_x \in \mathcal{A}_2 \) we set

\[
F(x) = \nu(Q_x).
\]

Consequently, as in the case of ordinary integration, we should have

\[
(\mu \times \nu)(Q) = \int_X F(x) \, d\mu(x),
\]

cf., our remarks on Peano at the beginning of Section 2.1. Precisely, let \( \mathcal{R}_\sigma \) denote the collection of countable unions of elements of the ring \( \mathcal{R} \) of measurable rectangles in \( X \times Y \), and let \( \mathcal{R}_{\sigma\delta} \) denote the collection of countable intersections of elements of \( \mathcal{R}_\sigma \). Take \( Q \in \mathcal{R}_{\sigma\delta} \) with \( (\mu \times \nu)(Q) < \infty \). Then \( F \) is \( \mu \)-measurable and (3.34) is valid.

The next step extends the last statement to any \( Q \in \mathcal{A}_1 \times \mathcal{A}_2 \) satisfying \( \mu \times \nu(Q) < \infty \). In order to prove this we must use the fact, which is easy
to show, that if \( Q \in A_1 \times A_2 \) then there is \( P \in R_{\sigma} \) such that \( Q \subseteq P \) and \( \mu \times \nu(P) = \mu \times \nu(Q) \), cf., Proposition 2.2.3. Note that this step clinches (3.33) for the case of 

\[ f(x, y) = \mathbb{1}_Q(x, y). \]

The final step in the proof of Theorem 3.7.5 for “arbitrary” functions \( f(x, y) \) involves standard approximation techniques.

b. We note that, in order to use the above argument for an arbitrary \( Q \in A_1 \times A_2 \ R_{\sigma} \) cannot approximate \( Q \) from within by an element \( P \in R_{\sigma} \).

Let \( (X, A_1, \mu) \) and \( (Y, A_2, \nu) \) be \( ([0, 1], M([0, 1]), m) \), and choose a perfect totally disconnected set \( E \subseteq [0, 1] \) satisfying \( m(E) > 0 \). Set 

\[ Q = \{(x, y) : x - y \in E, x \in X, y \in Y\}, \]

so that \( (m \times m)(Q) > 0 \), cf., Problem 3.4. Now if we take any \( A \times B \in R \) then \( A \setminus B \subseteq E \) if \( A \times B \subseteq Q \). Note that is \( A \) and \( B \) are both of positive measure then \( A \setminus B \) contains an interval, e.g., Problem 3.4b. This contradicts the fact that \( A \setminus B \subseteq E \). Thus \( m \times m(A \times B) = 0 \).

In general, to apply Fubini’s theorem, we must know that the function \( h \) is integrable with respect to the product measure \( \mu \times \nu \). It is a condition that is not always easy to check. However, the following result provides us with a large class of functions for which we do not have to worry about this assumption.

**Theorem 3.7.7. Fubini theorem for non-negative functions**

Let \( (X, A_1, \mu) \) and \( (Y, A_2, \nu) \) be \( \sigma \)-finite measure spaces, and let \( h : X \times Y \rightarrow \mathbb{R}^+ \) be a \( (\mu \times \nu) \)-measurable function. Then \( \int_Y h_x \mathrm{d} \nu \) is \( \mu \)-measurable, and \( \int_X h^y \mathrm{d} \mu \) is \( \nu \)-measurable. Moreover, the equations (3.33) are valid.

**Theorem 3.7.8. Tonelli theorem**

Let \( (X, A_1, \mu) \) and \( (Y, A_2, \nu) \) be \( \sigma \)-finite measure spaces, and let \( h : X \times Y \rightarrow \mathbb{R}^* \) be a \( (\mu \times \nu) \)-measurable function such that

\[ \int_X \int_Y |h(x, y)| \mathrm{d} \nu(y) \mathrm{d} \mu(x) < \infty \quad \text{or} \quad \int_Y \int_X |h(x, y)| \mathrm{d} \mu(x) \mathrm{d} \nu(y) < \infty. \]

Then \( h \in L^1_{\mu \times \nu}(X \times Y) \) and the equations (3.33) are valid.

**Proof.** It is enough to apply Theorem 3.7.7 to the function \( |h(x, y)| \).

The proof of the following result is Problem 3.33.

**Theorem 3.7.9. Fubini–Tonelli theorem for complete \( \sigma \)-finite measure spaces**

Let \( (X, A_1, \mu) \) and \( (Y, A_2, \nu) \) be complete \( \sigma \)-finite measure spaces, and let
3.7 Fubini and Tonelli theorem

Let $(X \times Y, \mathcal{A}_1 \times \mathcal{A}_2, \mu \times \nu)$ be the completion of the product measure space $(X \times Y, \mathcal{A}_1 \times \mathcal{A}_2, \mu \times \nu)$. Let $h : X \times Y \to \mathbb{C}$ be $(\mu \times \nu)$-measurable.

**a.** If $h \in L^1_{\mu \times \nu}(X \times Y)$ then $h_x \in L^1_\mu(X)$ for $\mu$-a.e. $x \in X$, $h^y \in L^1_\nu(Y)$ for $\nu$-a.e. $y \in Y$, $\int_X h^y \, d\mu \in L^1_\mu(X)$, and the equations (3.33) are valid.

**b.** If $h \geq 0$ then $h_x$ is $\nu$-measurable for $\mu$-a.e. $x \in X$, $h^y$ is $\mu$-measurable for $\nu$-a.e. $y \in Y$, $\int_X h^y \, d\mu$ is $\mu$-measurable, $\int_Y h_x \, d\nu$ is $\mu$-measurable, and the equations (3.33) are valid.

Guido Fubini proved his theorem in 1907, [95]. Two years later, in 1909, Leonida Tonelli [248] observed that in case of $\sigma$-finite product measures it is enough to assume that a function is nonnegative to obtain the conclusions of Fubini’s result.

In the context of the Fubini-Tonelli theorem it is natural to ask if a function which is continuous in each variable must necessarily be measurable. This question was studied by Lebesgue in his first published paper, see [212] for more on this matter.

**Theorem 3.7.10. Lebesgue’s first theorem**

Let $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be continuous in $x$ for each $y \in \mathbb{R}$ and let it be continuous in $y$ for each $x \in \mathbb{R}$. Then $f$ is Lebesgue measurable.

**Proof.** Let $X_n = \{(k/n, y) : k \in \mathbb{Z}, y \in \mathbb{R}\}$, for $n = 1, \ldots$. Define $f_n$ to be equal to $f$ on $X_n$, and for $(x, y)$ such that $k/n < x < k + 1/n$ for some $k \in \mathbb{Z}$, let $f_n(x, y)$ be the linear interpolation between $f(k/n, y)$ and $f((k + 1)/n, y)$. For each $n$, $f_n$ is continuous on $\mathbb{R}^2$, because $f$ is continuous in $y$ for each $x$. On the other hand, since $f$ is continuous in $x$ for each fixed $y$, we have

$$f(x, y) = \lim_{n \to \infty} f_n(x, y).$$

Thus, $f$ is Lebesgue measurable because the limit of measurable functions is measurable.

We close this section with a complete elementary proof of the Fubini-Tonelli theorem for a special case. We need the following standard result from advanced calculus, which itself is easy to prove.

**Lemma 3.7.11.** Let $g : [a, b] \to \mathbb{C}$ be bounded and assume $\lim_{n \to \infty} S_{P_n}$ and $\lim_{n \to \infty} s_{P_n}$ exist and are equal for every sequence $\{P_n : n = 1, \ldots\}$ of partitions for which $\text{card} \, P_n = n + 1$ and the norms $|P_n| \to 0$. Then $\int_a^b g$ exists.

**Theorem 3.7.12. Fubini–Tonelli theorem for Lebesgue and Riemann integrable functions**

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \times [a, b] \to \mathbb{C}$. Assume
The Lebesgue integral

\[ \int_a^b f(x, y) \, dy \] exists \( \mu \)-a.e., \( f \) is \( \mu \)-measurable for all \( y \in [a,b] \), and that there is \( F \in L^1(X) \) for which

\[ \forall (x,y) \in X \times [a,b], \quad |f(x,y)| \leq F(x). \]

Then \( \int_X f(x,t) \, d\mu(x) \) is Riemann integrable and

\[ \int_X \left( \int_a^b f(x,y) \, dy \right) \, d\mu(x) = \int_a^b \left( \int_X f(x,y) \, d\mu(x) \right) \, dy. \]

Proof. Let \( \{P_n : n = 1, \ldots \} \) be a sequence of partitions of \([a,b]\) for which the norms \( |P_n| \to 0 \) and card \( P_n = n + 1 \). Form

\[ S_{P_n}(x) = \sum_{j=1}^n f(x, M_j)(y_j - y_{j-1}). \]

Then

\[ |S_{P_n}(x)| \leq (b-a)F(x) \quad (3.35) \]

and

\[ \lim_{n \to \infty} S_{P_n}(x) = R \int_a^b f(x,y) \, dy \quad \mu\text{-a.e.} \]

Now

\[ \int_X S_{P_n}(x) \, d\mu(x) = \sum_{j=1}^n \left( \int_X f(x, M_j) \, d\mu(x) \right)(y_j - y_{j-1}). \]

By Theorem 3.3.7, which we can use by (3.35),

\[ \lim_{n \to \infty} \sum_{j=1}^n \left( \int_X f(x, M_j) \, d\mu(x) \right)(y_j - y_{j-1}) = \lim_{n \to \infty} \int_X S_{P_n}(x) \, d\mu(x) \]

\[ = \int_X \left( R \int_a^b f(x,y) \, dy \right) \, d\mu(x). \quad (3.36) \]

This is true for all such \( \{P_n : n = 1, \ldots \} \) and for \( s_{P_n}(x) \). Consequently, by Lemma 3.7.11,

\[ R \int_a^b \left( \int_X f(x,y) \, d\mu(x) \right) \, dy \]

exists, and (3.36) completes the proof.

If a function does not satisfy the hypotheses of the Fubini–Tonelli theorem as stated in this section (noting that a measure space can always be completed), there is probably no other useful version of Fubini–Tonelli which will help to switch the order of integration - counterexamples abound; and if it is possible to switch the order at all, it will have to be done by ad hoc, and probably ingenious, methods.
3.8 Measure zero and sets of uniqueness

3.8.1 B. Riemann

Bernhard Riemann’s life (September 17, 1826–July 20, 1866) has been documented by his friend Dedekind. Our interest for the sequel focuses on his Habilitationsschrift [195]; here he begins with an important historical note on Fourier series, defines the Riemann integral to provide a broader setting for an analytically precise theory of Fourier series, and develops the Riemann localization principle which is a key technique in the study of $U$-sets. $E \subseteq [0, 1) = \mathbb{T}$ is a $U$-set if

$$\lim_{N \to \infty} \sum_{|n| \leq N} c_n e^{2\pi i n x} = 0 \text{ off } E \iff \forall n \in \mathbb{Z}, \; c_n = 0.$$

The problem to determine $U$-sets is important since one would like to know if the representation of a function by a trigonometric series is unique or not. The first explicit results in this direction were given by Cantor, e.g., Section 3.8.2, although the following fundamental theorem for uniqueness questions, first proved by Cantor, was apparently known by Riemann [154], page 110, [195]:

$$\lim_{N \to \infty} \sum_{|n| \leq N} c_n e^{2\pi i n x} = 0 \text{ on } [a, b] \implies \lim_{|n| \to \infty} c_n = 0 \quad (3.37)$$

e.g., Problem 3.29e.

It is interesting to observe the overlap between Dini and Riemann. Riemann convalesced and toured Italy during the winter of 1862; and then came to Pisa during 1863. He became quite friendly with Enrico Betti and Eugenio Beltrami. Betti was director of the Scuola Normale Superiore in Pisa from 1865 to 1892; and there is a Betti–Riemann correspondence which has yet to be studied as far as we know. Dini graduated from the Scuola Normale in 1864 when he was 19 years old; he then spent a year in Paris with Bertrand, and returned to the Scuola Normale where he spent the next 52 years. Dini became one of the 19th-century giants in real variable and Fourier analysis, and includes Volterra and Vitali among his students. Riemann returned to Germany for the winter of 1864–65, but then came back to Pisa. He died and was buried at Biganzolo in the northern part of Verbania, the Italian resort town on the western banks of Lago Maggiore just 15 miles south of the Swiss border.

3.8.2 G. Cantor

Georg Cantor (March 3, 1845–January 6, 1918) wrote several important papers on $U$-sets during the early 1870s (Crelle’s Journal, volumes 72 and 73, and Math. Ann., volume 5). In the first, he proved (3.37) (see Section
The Lebesgue integral

3.8.1) and, using this fact, proved (in the second) that $\emptyset$ is a $U$-set. The subsequent papers gave simplifications of proof and extensions of the basic result, showing finally that certain countable infinite sets are $U$-sets. The study of special types of infinite sets in this work influenced his later research activity, and it was in 1874 that he gave his famous and controversial proof that there are only countably many algebraic numbers, cf., Problem 1.24.

The remainder of his life was devoted to the study of the infinite, not only in a mathematical milieu, but often delving into various philosophical notions of infinity due to the Greeks, the scholastic philosophers, and his contemporaries. An interesting letter in this latter regard was sent by CANTOR, in London at the time, to BERTRAND RUSSELL (at Trinity College, Cambridge); he writes: “... and I am quite an adversary of Old KANT, who, in my eyes has done much harm and mischief to philosophy, even to mankind; as you easily see by the most perverted development of metaphysics in Germany in all that followed him, as in Fichte, Schilling, HEGEL, HERBART, Schopenhauer, Hartman, Nietzsche, etc. etc. on to this very day. I never could understand why ... reasonable ... peoples ... could follow yonder sophistical philistine, who was so bad a mathematician.”

DEDEKIND was also involved in work realted to CANTOR’s, and many of his set theoretic contributions are found in their 27-year correspondence (edited by JEAN CAVAILLÈS and EMMY NOETHER and published by Hermann of Paris in 1937). FEDOR A. MEDVEDEV (1966) has studied these particular results. The marvelous DEDEKIND, by the way, taught at the Gymnasium in Brunswick for fifty years beginning in 1862, cf., Section 6.6.

CANTOR’s $U$-set papers were preceded by H. HEINE’s uniqueness theorem (Crelle’s Journal, volume 71) in 1870 which assumed that the given trigonometric series were uniformly convergent off arbitrary neighborhoods of a finite number of points. HEINRICH E. HEINE was at the University of Halle with CANTOR and attributes this approach to CANTOR.

CANTOR, of course, tried to prove that all countable sets are $U$-sets; and this was finally achieved by F. BERNSTEIN (1908) and W.H. YOUNG (1909). Actually, BERNSTEIN proved somewhat more when applied to compact Abelian groups, showing that $E$ is a $U$-set if it does not contain any non-empty closed sets without isolated points.

3.8.3 D. Menshov

DMITRI MENSHOV (April 18, 1892–November 25, 1988) proved a key result on $U$-sets in 1916 by finding a non-$U$-set $E$ with Lebesgue measure $m(E) = 0$. He did this just after graduating from Moscow University, where he wrote his thesis under NIKOLAI N. LUZIN. His example stimulated a great deal of study about sets of measure zero; and research about specific sets of measure zero now forms a significant part of modern Fourier analysis and potential theory. Actually, on the basis of MENSHOV’s example, LUZIN and BARI defined the notion “$U$-set” as such. Earlier, DE LA VALLÉE-POUSSIN had proved that if
a trigonometric series converges to $f \in L^1_m(T)$ off a countable set $E$ then
the series is the Fourier series of $f$; and it was generally felt that the same
would be true if $m(E) = 0$. Consequently, Menshov’s example had a certain
amount of shock value to say the least.

Since we shall be discussing Carleson’s solution to Luzin’s problem in
Section 4.7.3, it is interesting to note that Menshov solved the analogue
for measurable functions in 1940–41. Luzin, in 1915, had noted that if $f$ is
measurable on $T$ and finite $m$-a.e. then there is a trigonometric series which
converges to $f$ by both Riemann and Abel summation. The problem was to see
if such a series exists which converges pointwise $m$-a.e. to $f$; Menshov
showed precisely this. Thus with the Carleson–Hunt theorem and Kol-
mogorov’s example of $f \in L^1_m(T)$ with Fourier series diverging everywhere
(1926), “all that remains” (in the broad sense) of Luzin’s problem is an in-
vestigation of the analogous situation for $f$ measurable but taking infinite
values on a set of positive measure. Actually Menshov has an affirmative
answer on this latter problem for the case of convergence in measure instead
of convergence $m$-a.e..

3.8.4 N. Bari and A. Rajchman

What with Menshov’s example, Aleksander Rajchman (November 13,
1890 – Sachsenhausen, 1940) “seems to have been the first to realize that
for sets of measure zero that occur in the theory of trigonometric series it
is not so much the metric as the arithmetic properties that matter” [218]
(from Zygmund’s biography of Salem). Rajchman [194] (1922) proved the
existence of closed uncountable $U$-sets. He was motivated by some work of
Hardy and Littlewood (Acta Math. 37 (1914)), and later (1920) Stein-
haus (who was Rajchman’s adviser), on diophantine approximation to in-
troduce “$H$-sets” and proved that such sets are $U$-sets. Rajchman, in a
letter to Luzin, thought that any $U$-set is contained in a countable union
of $H$-sets, and it was only in 1952 that Pyatetskii–Shapiro proved this
conjecture false. The Cantor set is $H$ and therefore $U$. It is easy to verify
that if $m(E) > 0$ then $E$ is not $U$.

Actually, Nina Bari had proved the existence of closed uncountable $U$-
sets in 1921 and presented her results at Luzin’s seminar at University of
Moscow; they were unpublished at the time Rajchman’s paper, although
they were communicated to him in [194] (1923). This does not minimize
the importance of Rajchman’s results since he established a large class of
uncountable $U$-sets and illustrated the need for diophantine properties in
constructing such sets.

Nina Bari (November 19, 1901–July 15, 1961) established her first results
on $U$-sets as an undergraduate, and throughout her life, although she engaged
in several other research areas, was an outstanding expositor and contributor
on the tricky business of uniqueness. One of her major results is that the
countable union of closed $U$-sets is a $U$-set; although the problem is open for
the finite union of arbitrary $U$-sets. Another of her results, which she proved in 1936–37 and which has an interesting sequel, e.g., Section 7.63, shows that if $\alpha$ is rational and $E$ is the perfect symmetric set determined by $\xi_k = \alpha$, then $E$ is a $U$-set if and only if $1/\alpha$ is an integer. Her theorem depends heavily on Diophantine considerations. See [176] and [145] for recent developments.

### 3.9 Potpourri and titillation

1. “Sensational” is the proper level of sober language to describe the discovery in 1899 of a palimpsest containing a Greek Orthodox ritual, but with an underlying Archimedes text, theretofore thought lost to civilization. The text, from the 10th century, contained Archimedes’ Method ($\hat{\text{M}e\theta\omicron\delta\omicron\kappa}$), see [?], [?] for more on this riveting saga in scholarship.

Archimedes (c. 287 B.C. - 212 B.C.) began the Method with a letter to Eratosthenes saying that he first writes down what “became clear to us by the mechanical method” and then he formulates these results in the geometrical rigour du jour. The latter, of course, with its systematic sequence of logical steps, often has the capability of obfuscating ideas, especially as mathematics weaned itself from Euclidean geometrical technology through the centuries. So, the Method really illustrates how Archimedes divined some of his greatest creations. In his letter he writes that “there will be some among [...] future generations who by means of the method here explained will be enabled to find other theorems which have not yet fallen to our share”.

As an illustration, Archimedes tells about the 1 : 2 : 3 theorem first stated by Democritus (c. 460 B.C. - c. 370 B.C.) and first proved by Eudoxus (c. 408 B.C. - c. 355 B.C.).

To state this theorem, let $T$ be an isosceles triangle whose base is twice its height. We inscribe $T$ in a semi-circle $C$ which, in turn, we inscribe in a rectangle $R$, see Figure ???. We generate the corresponding cone, hemisphere, and cylinder by rotating $T, C,$ and $R$ about the segment $AO$. The respective volumes of these solids are denoted by $V_1, V_2,$ and $V_3$. Archimedes 1 : 2 : 3 theorem is

$$V_1/V - 2 = 1/2 \quad \text{and} \quad V_2/V_3 = 2/3.$$  

Archimedes proved this result geometrically in On the sphere and cylinder. Elaborating on the contributions of Democritus and Eudoxus, Archimedes says that Democritus discovered the fact that $V_1/V_3 = 1/3$, but that the part of the proof depending on the method of exhaustion was due to Eudoxus. This proof is found in Euclid’s Book 12, Proposition 10. Archimedes completed the proof of the 1 : 2 : 3 theorem by proving that $V_1/V_2 = 1/2$.

Archimedes obviously considered the 1 : 2 : 3 theorem to be a remarkable result since he asked that his tombstone have the carving of a sphere inscribed in a cylinder. This request was followed, and Cicero (106 B.C.}
43 B.C.), the Roman orator, found the tombstone when he was quaestor of Sicily. At that time the tombstone had been neglected, and Cicero was responsible for its restoration (the tombstone was again forgotten and rediscovered in 1965). Unfortunately, Cicero more than balanced this action with the remark: “Among the Greeks nothing was more glorious than mathematics. But we [the Romans] have limited the usefulness of this art to measuring and calculating”.

We shall verify that \( V_1/V_2 = 1/2 \). We shall give the mechanical proof that is in Archimedes’ Method. This proof uses the Democritus-Eudoxus result that \( V_3 = 3V_1 \). We begin by describing the law of the lever: the lever in Figure ?? is in equilibrium if

\[
aA = bB,
\]

where \( A \) and \( B \) are weights having distance \( a \) and \( b \), respectively, from the fulcrum.

Figure ?? is the figure we shall use to prove \( V_1/V_2 = 1/2 \). Let \( S \) be a sphere of radius \( r \) and center \( 0 \); we shall work on a plane \( P \) through \( 0 \). Take perpendicular diameters \( AB \) and \( CD \) of \( S \) (in \( P \)).

When we rotate the triangle \( ACD \) about the segment \( AO \) we obtain a cone incribed in the hemisphere obtained by rotating the arc \( CAD \) about \( AO \). We also form the rectangle \( EFGH \) determined by the lines parallel to the circle at \( A \) and \( B \), and by the points \( F \) and \( H \) which are one the lines \( AC \) and \( AD \). Let \( XY \) be a line intersecting \( AO \) at \( Z \) and parallel to \( CD \). Also, \( XY \) intersects the circle at \( M \) and \( N \) and the triangle \( ACD \) at \( U \) and \( V \). Finally, we extend \( AB \) to the point \( I \) so that the lengths \( |IA| \) and \( |IB| \) satisfy \( |IA| = 2r \) and \( |TB| = 4r \), respectively.

Note that

\[
|FB| = |BH| = 2r.
\]

In fact, \( AOC \) and \( ABF \) are similar triangles, and so (3.39) follows since \( |AB| = 2r \), \( |AO| = r \), and \( |OC| = r \). Next set \( |UZ| = a \) and \( |MZ| = b \), and note that \( |AZ| = a \) since \( AUZ \) is a right triangle and the angle \( \angle ZAU \) is \( \pi/4 \) radians. Consequently, the right triangle \( AMZ \) has the property that

\[
|AM|^2 = a^2 + b^2.
\]

Observe that \( AMB \) is a right triangle with right angle at \( M \) since \( AB \) is a diameter. Thus,

\[
|AM|^2 = (2r)^2 - |MB|^2
\]

From the right triangle \( ZMB \) we obtain,

\[
b^2 = |MB|^2 - (2r - a)^2.
\]

(3.40) and (3.42) yield

\[
|AM|^2 = a^2 + |MB|^2 - (2r)^2 + 4ra - a^2;
\]
so that by adding (3.41) and (3.43) we obtain
\[ |AM|^2 = 2ra. \quad (3.44) \]

Also we combine (3.40) and (3.44), and have
\[ 2ra = a^2 + b^2; \]
we write this as
\[ 2r \pi (a^2 + b^2) = a \pi (2r)^2. \quad (3.45) \]

We now rotate our figure about \( IB \) so that the circle generates \( S \), the triangle \( AFH \) generates a cone whose base is a circle of radius \( |BF| = 2r \) and whose height is \( |AB| = 2r \), and the rectangle \( EFGH \) generates a cylinder whose base has radius \( |BF| = 2r \) and whose height is \( |AB| = 2r \). In this rotation, \( XY \) determines a plane which intersects the cone in a circle \( C_C \) of radius \( a \), the sphere in a circle \( C_S \) of radius \( 2r \). With this notation and the fact that \( |AI| = 2r \), (3.45) becomes
\[ |AI| (A(C_C) + A(C_S)) = |AZ| A(C_C); \quad (3.46) \]
where \( A(X) \) represents the area of \( X \).

(3.46) should be compared with (3.38); in fact, we suppose \( IZ \) is a lever with fulcrum at \( A \). Heuristically, then, we consider circular discs with weights proportional to their areas so that (3.46) expresses the law of the lever. Consequently, if we consider \( n \) such cuts \( XY \) equidistributed by the points \( A = Z_0, Z_1, \ldots, Z_n = 0 \) along \( AO \), then we can think of the sum of \( n \) areas \( A(\cdot) \) as approximating the volume \( V \) of the corresponding solid. Thus, since \( |AI| = 2r \), the left-hand side of (3.46) becomes
\[ 2r(V_1 + V_2). \quad (3.47) \]

For the right hand side we see from the equidistribution that
\[
\sum_{j=1}^{n} |AZ_j| A(C_C) = \frac{1}{n} V_3 \sum_{j=1}^{n} |AZ_j| = \frac{1}{n} V_3 r \left( \frac{1}{n} + \frac{2}{n} + \ldots + \frac{n}{n} \right)
= \frac{r V_3 n(n+1)}{2n^2},
\]
where \( V_3 = \pi (2r)^2 r \) is the volume of the cylinder whose base has radius \( |AE| = 2r \) and whose height is \( |AO| = r \). Since \( V_3 = \pi r^3 \), we have \( V_3 = 4V_3 \). Therefore, equating (3.47) and (3.48), we compute
\[ V_1 + V_2 = V_3 \quad (3.49) \]
since \( \lim_{n \to \infty} (n+1)/n = 1 \). We now use the DEMOCRITUS-EUDOXUS result, \( V_3 = 3V - 1 \), to obtain ARCHIMEDES’ theorem, \( V_2 = 2V_1 \).
The point of this illustration of Archimedes’ Method, especially the previous paragraph, is to relate Archimedes’ analysis with the definition of the Riemann integral, see his prescient comment above about “future generations”.

2. Let \((X, \mathcal{A}, p)\) be a probability space and let \(f : X \to \mathbb{R}\) be a random variable as in Section 2.6. If \(f \in L^1_p(X)\) then

\[
E(f) = \int_X f \, dp
\]

is the \textit{mean value} or \textit{expected value} of \(f\). The map

\[
p_f : \mathcal{B}(\mathbb{R}) \to \mathbb{R}, \quad B \mapsto p(f^{-1}(B))
\]

is a probability measure on \(\mathbb{R}\) (the range of \(f\)) called the \textit{distribution} of \(f\), not to be confused with the Schwartz distributions of Section 7.5.

If \(g : \mathbb{R} \to \mathbb{R}\) is \(\mathcal{B}(\mathbb{R})\)-measurable and \(f \in L^1_p(X)\) then it is not difficult to prove that

\[
E(g \circ f) = \int_{\mathbb{R}} g \, dp_f
\]

and so

\[
E(f) = \int_{\mathbb{R}} t \, dp_f(t).
\]

Also, in probabilistic language, if \(\{f_n : n = 1, \ldots\}\) and \(f\) are random variables on \(X\), then one says that \(\{f_n\}\) \textit{converges in probability to} \(f\) if \(\{f_n\}\) converges in measure to \(f\).

3.10 Problems

Some of the more elementary problems in this set include Problems 3.1, 3.2, 3.3, 3.7, 3.8, 3.9, 3.13, 3.14, 3.15, 3.17, 3.21, 3.22, 3.28.

3.1. Let \(f_n(x) = (\pi n)^{-1/2} e^{-x^2/n}\) on \(\mathbb{R}\) and consider \(f(x) = \lim_{n \to \infty} f_n(x)\). Compute \(f\), and determine whether (and why) \(f\) is measurable and integrable on \(\mathbb{R}\).

[\textit{Hint.} In order to prove that

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi},
\]

Pierre-Simon Laplace considered the product

\[
\left( \int_0^{\infty} e^{-x^2} \, dx \right) \left( \int_0^{\infty} e^{-y^2} \, dy \right)
\]

and then used polar coordinates. Brilliant!]
3.2. Let \( f \in L^1_{m}(\mathbb{R}) \), \( f(0) = 0 \), and assume \( f'(0) \) exists. Prove that
\[
\int_{-\infty}^{\infty} \frac{f(x)}{x} \, dx
\]
equals
exists.

3.3. \( a \). Let \( f \geq 0 \) on \([a, b]\) be Lebesgue measurable. Prove that
\[
\int_{a}^{b} f = \lim_{\varepsilon \to 0^+} \int_{a}^{b} f \mathbb{1}_{[a, b+\varepsilon)},
\]
where \( \int_{a}^{b} f \) may be infinite.

\( b \). Prove that \( \int_{0}^{1} x^{a-1} \, dx = 1/\alpha \) if \( \alpha > 0 \).

\( c \). With respect to part \( a \) and Problem 1.28a, find a function \( f : (0,1] \to \mathbb{R} \) such that
\[
\forall \varepsilon > 0, \quad f \in L^1_{m}((\varepsilon,1])
\]
and \( \lim_{n \to \infty} (1/n) \sum_{k \leq n} f(k/n) \) exists, but \( \lim_{\varepsilon \to 0^+} \int f \mathbb{1}_{[\varepsilon,1]} \) does not exist.

3.4. \( a \). For real-valued functions \( f \in L^\infty_{m}((0, \infty)) \) define
\[
f \circ f(x) = \int_{0}^{x} f(t) f(x-t) \, dt.
\]
It is not unreasonable to expect that \( f \circ f \geq 0 \) on some small interval \((0, \varepsilon)\).
Show that this is not necessarily the case.

[Hint. Take \( f(t) = \sin(1/t) \).]

\( b \). The \textit{convolution} of the functions \( f \) and \( g \), where \( f, g \in L^1_{m}(\mathbb{R}) \), is
\[
f * g(x) = \int_{-\infty}^{\infty} f(t) g(x-t) \, dt.
\]
As such \( L^1_{m}(\mathbb{R}) \) becomes a Banach algebra, a most elegant gateway to harmonic analysis, see Appendix 11. For this problem prove \textsc{Steinhaus}' result, mentioned in Problem 1.13 and in Problem 2.23:

\( If X \in \mathcal{M}(\mathbb{R}) \) and \( m(X) > 0 \), then \( X - X = \{x - y : x, y \in X\} \) is a neighborhood of 0.

\textsc{Steinhaus}' result can be used to characterize absolute continuity as we observe in the remark after Proposition 5.4.4 at the end of Section 5.4.

[Hint. Let \( f(x) = 1_X(x) \) and note that \( f \circ f \) is continuous. We can also prove the result set theoretically.]

3.5. \( a \). Let \( X \subseteq \mathbb{R} \) have the property that finite linear combinations of elements from \( X \) with rational coefficients generate an interval, \( I \). Prove that \( X \) contains a Hamel basis, see Problem 2.24 for the definition and see Problem 3.12. Thus, \textit{the Cantor set} \( C \) contains a Hamel basis.
[Hint. Let $0 \in I$; and so
\[
\forall x \in \mathbb{R}, \exists r \in \mathbb{Q} \text{ such that } (1/r)x \in I.
\]
Hence, every real number is a finite rational linear combination from $X$. Let $\mathcal{F}$ be the family of all subsets $Y \subseteq X$ with the same property. Apply Zorn’s lemma.]

**b.** Take $A \in \mathcal{M}(\mathbb{R})$ for which $0 < m(A) < \infty$; prove that $A$ contains a Hamel basis.

[Hint. Use part $a$ and the method of proof from Problem 3.4b. This is simpler to prove than Problem 2.24e since we do not have to deal with the likes of (2.25); of course, it is also a weaker result.]

3.6. **a.** Consider a sequence of functions $f, f_n \in L^1_m([a,b]), n = 1, \ldots,$ such that $f = \lim_{n \to \infty} f_n$ m-a.e. Prove that
\[
\lim_{n \to \infty} \int_a^b |f_n - f| = 0 \iff \lim_{n \to \infty} \int_a^b |f_n| = \int_a^b |f|,
\]
cf., Theorem 6.5.3.

**b.** Find an example of a sequence of $m$-integrable functions $\{f_n\}$ on $[a,b]$ which converges m-a.e. to an $m$-integrable function $f$, such that
\[
\lim_{n \to \infty} \int_a^b f_n = \int_a^b f, \text{ but } \lim_{n \to \infty} \int_a^b |f_n| \neq \int_a^b |f|.
\]

[Hint. Take $f_n(x) = \begin{cases} n, & \text{if } x \in [0, 1/(2n)), \\ -n, & \text{if } x \in [1/(2n), 1/n), \\ 0, & \text{otherwise.} \end{cases}$]

3.7. **a.** Prove Theorem 3.3.8.

[Hint. Generalize Theorem 3.3.5 appropriately.]

**b.** Show that there can be a strict inequality in Theorem 3.3.5.

[Hint. Let $f_n = 1_{[n,n+1)}.$]

**c.** Prove that Theorem 3.3.6 can fail if we replace “$f_n \leq f$ $\mu$-a.e.” by the hypothesis that
\[
\forall x \text{ and } \forall n, \ f_{n+1}(x) \leq f_n(x).
\]

[Hint. Let $f_n = 1_{[n,\infty)}.$]

3.8. Find a function $f : [0,1] \to \mathbb{R}$ such that $f'$ exists on $[0,1]$, $f''(0)$ exists, and $R \int_0^b f'$ does not exist for any $b \in (0,1].$

[Hint. Let $g$ be the Volterra function defined in Example 1.3.1 and set $f(x) = x^2 g(x).$]
3.9. Let \( f : \mathbb{R} \to \mathbb{R} \) be a bounded Lebesgue measurable function with the following property:

\[
\exists C > 0 \text{ such that } \forall \varepsilon > 0, \, m(\{x \in \mathbb{R} : |f(x)| > \varepsilon\}) \leq \frac{C}{\sqrt{\varepsilon}}.
\]

Prove that \( f \in L^1_m(\mathbb{R}) \).

3.10. \( a \). Prove that for each \( f \in L^1_m(\mathbb{R}) \) and \( \varepsilon > 0 \) there is a continuous function \( g \) such that

\[
m(\{x : g(x) \neq 0\}) < \infty
\]

and

\[
\int_{\mathbb{R}} |f(x) - g(x)| \, dx < \varepsilon.
\]

[Hint. Use LDC and the observation on this matter given after Example 2.5.11.]

\( b \). Prove:

\[
\forall f \in L^1_m(\mathbb{R}), \quad \lim_{h \to 0} \int_{\mathbb{R}} |f(x + h) - f(x)| \, dx = 0. \quad (3.50)
\]

[Hint. Use part \( a \) and a fact about uniformly continuous functions.]

We shall later characterize functions of bounded variation, resp., constants, on \([a, b]\) as those elements in \( L^1_m([a, b]) \) which satisfy:

\[
\int_a^b |f(x + h) - f(x)| \, dx = O(|h|), \quad \text{resp., } o(|h|), \quad h \to 0.
\]

The “\( O \)” and “\( o \)” notation is standard in classical analysis. \( F(x) = O(G(x)) \), resp., \( o(G(x)) \), as \( x \to r \), means that

\[
\exists M > 0 \text{ and } \exists y \text{ such that } \forall x \in [r - y, r + y], \quad |F(x)| \leq MG(x), \quad \text{resp., } \lim_{x \to r} F(x)/G(x) = 0,
\]

where \( G \) is positive. For \( r = \pm \infty \) there is an obvious adjustment in the definition.

\( c \). A trigonometric polynomial on \( \mathbb{T} \) has the form

\[
t_N(x) = \sum_{|n| \leq N} a_N e^{2\pi i nx}.
\]

Using part \( a \) prove that

\[
\forall \varepsilon > 0 \text{ and } \forall f \in L^1_m(\mathbb{T}), \exists t_N \text{ such that } \int_{\mathbb{T}} |f - t_N| < \varepsilon.
\]
3.10 Problems

\[\text{Hint.} \text{ Find a function } g \in C^2(\mathbb{R}) \text{ in terms of indefinite integrals such that } g \text{ has period 1 and } \|f - g\|_1 < \varepsilon/2; \text{ then find } N \text{ for which } \|g(x) - \sum_{|n|\leq N} \hat{g}(n) e^{2\pi i nx}\|_\infty < \varepsilon, \text{ where } \hat{g}(n) = \int_T g(x) e^{-2\pi i nx} \, dx.\]

\[d. \text{ We proved the Riemann–Lebesgue lemma in Theorem 3.6.4. Complete the following proof which uses (3.5):}\]

\[|\hat{f}(n)| = (1/2) \left| (f - \tau_{-1/(2n)} f) (n) \right| \leq \frac{1}{2} \int_0^1 |f(x) - f(x + 1/2n)| \, dx,\]

where \(\tau_y f(x) = f(x - y).\)

\[e. \text{ Prove Fejér's lemma: If } f \in L^1_m(T) \text{ and } g \in L^\infty_m(T) \text{ then}\]

\[\lim_{|n| \to \infty} \int_0^1 f(x) g(nx) \, dx = \int_0^1 f(x) \, dx \int_0^1 g(x) \, dx.\]

This result reduces to the Riemann–Lebesgue lemma for the case of \(g(x) = e^{2\pi i x}.\)

3.11. Let \(\{f_n : n = 1, \ldots\}\) be a sequence of non-negative functions defined on \([0, 1].\) Assume that \(\sum_{n=1}^\infty \int_0^1 f_n < \infty\) and prove that \(f_n \to 0 \text{ m-a.e.}\) This result is an often used technical device, e.g., Proposition 3.4.9.

\[\text{Hint. } \int \sum f_n = \sum \int f_n < \infty \text{ and so } \sum f_n(x) \text{ is finite m-a.e.}\]

3.12. Let \(H \subseteq \mathbb{R}\) be a Hamel basis, see Problem 2.24 for the definition. Prove that

\[\forall a \in \mathbb{R}, \ a \neq 1, \ \exists h \in H \text{ such that } ah \notin H.\]

\[\text{Hint. Take such an } a \text{ and assume } aH \subseteq H; \text{ for each } x = \sum r_\alpha h_\alpha \text{ define } f(x) = \sum r_\alpha, \text{ and note that } f(ax) = f(x). \text{ Compute } f(x), \text{ where } x = h/(a-1) \text{ and } h \in H.\]

3.13. Let \(f \in L^1_m(0, \infty)\) and define \(I_r\) to be a subset of \((0, \infty)\) parametrized by \(r > 0.\)

\[a. \text{ Prove that } \lim_{r \to \infty} \int_{I_r} f(x) \cos(rx) \, dx = 0 \text{ if } I_r \text{ is an interval.}\]

\[b. \text{ Show that part } a \text{ is not generally true if } I_r \text{ is taken to be a finite union of intervals.}\]

These results are interesting in light of the Riemann–Lebesgue lemma.

3.14. Compute \(\lim_{n \to \infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} \, dx.\)

\[\text{Hint. Transform the integral to } \int_0^n \text{ and note, comparing like terms of the binomial expansions of } (1 + t/n)^n \text{ and } (1 + t/(n+1))^{n+1}, \text{ that } (1 + t/n)^n e^{-t/2} \text{ increases to } e^{t/2}; \text{ then use LDC.}\]

3.15. Prove that if \(f : \mathbb{R} \to \mathbb{R}\) is a continuous function and \(g : \mathbb{R} \to \mathbb{R}\) is continuous \(m\text{-a.e.}, \) then \(f \circ g\) is continuous \(m\text{-a.e.}, \) cf., Problem 4.46.
3.16. Prove Proposition 3.4.8.
[Hint. The characterization of regulated functions in Example 3.2.2 gives us a means of verifying Proposition 3.4.8. In fact, if \( f \) is a regulated function then there is a function \( F \) such that \( F' = f \) except for possibly a countable set.]

3.17. Prove that if \( f \) is bounded on \([a, b]\) and continuous \( m\text{-a.e.} \), then \( \mathcal{F} = f \) \( m\text{-a.e.} \), where
\[
\mathcal{F}(x) = \mathcal{R}\int_{a}^{x} f.
\]

3.18. Prove Theorem 3.5.4.

3.19. Verify Theorems 3.6.1, 3.6.2 and 3.6.3.

3.20. Let \((X, \mathcal{A}, \mu)\) be a measure space, where \( X \) is a locally compact space (Appendix 10.1) and \( \mu \) is a positive measure on \( X \). Choose an open interval \( S \subseteq \mathbb{R} \) and let \( f(x, t) \) be a function on \( X \times S \). Assume that
i. \( \forall t \in S, f(\cdot, t) \) is a \( \mu \)-measurable function,
ii. \( \exists g \in L^1_\mu(X) \), non-negative, such that \( \forall t \in S, |f(x, t)| \leq g(x) \) \( \mu\text{-a.e.} \),
iii. \( \exists t_0 \in S \) such that \( f(x, \cdot) \) is continuous at \( t_0 \) \( \mu\text{-a.e.} \),
iv. \( f(x, \cdot) \) is finite and differentiable on \( S \) \( \mu\text{-a.e.} \),
v. \( \exists h \in L^1_\mu(X) \), non-negative, such that \( \forall t \in S, |d/dtf(x, t)| \leq h(x) \) \( \mu\text{-a.e.} \).

Prove that \( \int_{X} f(x, t) \, d\mu(x) \) is differentiable on \( S \) and
\[
\frac{d}{dt} \int_{X} f(x, t) \, d\mu(x) = \int_{X} \frac{d}{dt} f(x, t) \, d\mu(x).
\]

3.21. a. Let \((X, \mathcal{A}, \mu)\) be a bounded measure space. Let \( Y \) be the space of real-valued \( \mu \)-measurable functions defined on \( X \); and let \( \hat{Y} \) be the space of equivalence classes of such functions where \( f, g : X \to \mathbb{R} \) are defined to be equivalent if \( f = g \) \( \mu\text{-a.e.} \). Define
\[
\forall f, g \in Y, \quad \rho(f, g) = \int_{X} \frac{|f - g|}{1 + |f - g|} \, d\mu.
\]

Prove that \( \rho \) is a metric on \( \hat{Y} \times \hat{Y} \) (Appendix 10.1).
[Hint. If \( a \) and \( b \) are real, then \( |a + b|/(1 + |a + b|) \leq |a|/(1 + |a|) + |b|/(1 + |b|) \).]

b. Consider the setting of part a. Prove that \( \rho(f_n, f) \to 0 \) if and only if \( f_n \to f \) in measure.
[Hint. Let \( A_n(\varepsilon) = \{x : |f_n(x) - f(x)| \geq \varepsilon\} \) and prove that
\[
\varepsilon \int \frac{\mu(A_n(\varepsilon))}{1 + \varepsilon} \leq \rho(f_n, f) \leq \mu(A_n(\varepsilon)) + \varepsilon \mu(X).
\]

c. Find an example of a sequence \( \{f_n : n = 1, \ldots\} \) of \( m \)-measurable functions (on the measure space \([0, 1], \mathcal{M}([0, 1]), m\)) such that \( f_n \to 0 \) in
Problem 3.10

measure, whereas \( \{f_n : n = 1, \ldots \} \) does not converge pointwise at any point. Actually, the example can be constructed so that \( \lim_{n \to \infty} \|f_n\|_1 = 0 \) [35], Chapitre IV, page 236.

d. Prove Theorem 3.3.13 (F. Riesz–Lebesgue).
[Hint. Use Egorov’s theorem for part b.]

e. Show by example that the hypothesis, \( \mu(X) < \infty \), is necessary generally in Theorem 3.3.13b.

3.22. a. Show that if \( f \in L^1_c(\mathbb{Z}) \), where \( c \) is counting measure defined in Example 2.4.2, then

\[
\lim_{|n| \to \infty} f(n) = 0.
\]

b. Prove that there are functions \( f \in L^1_m(\mathbb{R}) \) such that

\[
\lim_{|x| \to \infty} |f(x)| = \infty.
\]

3.23. The Fejér kernel \( \{F_N : N = 1, \ldots \} \) is defined by

\[
\forall x \in \mathbb{R} \text{ and } \forall N \geq 0, \quad F_N(x) = \sum_{|n| \leq N} \left( 1 - \frac{|n|}{N + 1} \right) e^{2\pi i nx}.
\]

a. Prove that if \( x \in \mathbb{Z} \) then \( F_N(x) = N + 1 \), and that if \( x \) is any other real number then

\[
F_N(x) = \frac{1}{N + 1} \frac{\sin^2((N + 1)x/2)}{\sin^2(x/2)}.
\]

b. Clearly \( F_N \geq 0 \). Prove that

\[
\int_0^1 F_N = 1,
\]

and that

\[
\forall \delta \in (0, 1/2), \quad \lim_{N \to \infty} \int_0^{1-\delta} |F_N| = 0.
\]

c. An approximate identity in \( L^1_m(\mathbb{T}) \) is a sequence \( \{K_N : N = 1, \ldots \} \subseteq L^1_m(\mathbb{T}) \) which satisfies

\[
\sup N \int_0^1 |K_N| < \infty,
\]

\[
\lim_{N \to \infty} \int_0^1 K_N = 1,
\]

and

\[
\forall \delta \in (0, 1/2), \quad \lim_{N \to \infty} \int_0^{1-\delta} |K_N| = 0.
\]
Thus \( \{F_N : N = 1, \ldots \} \) is an approximate identity. We defined convolution in \( L^1_m(\mathbb{R}) \) in Problem 3.4b; the same definition makes sense on \( \mathbb{T} \) as long as we define our functions periodically. Hence, if \( f, g \in L^1_m(\mathbb{T}) \) we have

\[
 f \ast g(x) = \int_0^1 f(x-y)g(y) \, dy.
\]

Prove that if \( f \in C(\mathbb{R}) \) has period 1 and \( \{K_N : N = 1, \ldots \} \) is an approximate identity in \( L^1_m(\mathbb{T}) \), then

\[
 \lim_{N \to \infty} \|K_N \ast f - f\|_\infty = 0,
\]

where \( \| \cdot \|_\infty \) designates the \( L^\infty(\mathbb{T}) \)-norm.

**[Hint.]**

\[
 \|K_N \ast f - f\|_\infty \leq \int_0^1 |K_N(y)||\tau_y f - f\|_\infty \, dy = I
\]

Take \( \varepsilon > 0 \) and fix \( 0 < \delta < 1/2 \) such that \( |\tau_y f - f| < \varepsilon \) if \( y \in [0, \delta] \cup [1-\delta, 1] \).

Compute

\[
 I \leq M \varepsilon + 2\|f\|_\infty \int_0^{1-\delta} |K_N|,
\]

where \( M \) is independent of \( N \).

In particular, we have proved that

\[
 f(0) = \lim_{N \to \infty} \frac{1}{N+1} \int_0^1 f(x) \frac{\sin^2((N+1)x/2)}{\sin^2(x/2)} \, dx.
\]

**d.** Let \( f : [a, b] \times [a, b] \to \mathbb{R} \) be a function which is continuous in each variable separately. Prove that

\[
 \forall (x, y) \in [a, b] \times [a, b], \quad \lim_{N \to \infty} f_N(x, y) = f(x, y),
\]

where \( \{F_N : N = 1, \ldots \} \) is a sequence of continuous functions on \( [a, b] \times [a, b] \).

**[Hint.]** Assume without loss of generality that \( [a, b] = [0, 1] \), that \( \|f\|_\infty < \infty \), and that for each \( y, f(0, y) = f(1, y) \). Set

\[
 f_N(x, y) = \sum_{|n| \leq N} \left( 1 - \frac{|n|}{N+1} \right) c_n(y)e^{2\pi inx},
\]

where

\[
 c_n(y) = \int_0^1 f(x, y)e^{-2\pi inx} \, dx.
\]

From part c,

\[
 \forall y \in \mathbb{T}, \quad \lim_{N \to \infty} \sup_{x \in \mathbb{T}} |f_N(x, y) - f(x, y)| = 0.
\]

In particular, we have the desired pointwise convergence. To prove that \( f_N \) is continuous we need only to check that each \( c_n \) is continuous, and this is clear from LDC.
3.24. We have mentioned the concept of uniform distribution in Problem 1.28 and Proposition 3.4.7. We shall now define it. A sequence \( \{x_n : n = 1, \ldots\} \subseteq \mathbb{R} \) is uniformly distributed modulo \( a, a > 0 \), if, when \( y_n \in [0, a) \) and \( (x_n - y_n)/a \in \mathbb{Z} \), we can conclude that

\[
\forall I \subseteq [0, a), \quad \lim_{N \to \infty} \frac{N_I}{N} = \frac{1}{a} m(I),
\]

where \( I \subseteq [0, a) \) is an arbitrary interval and \( N_I = \text{card} \{y_1, \ldots, y_N\} \cap I \). This concept was introduced by Hermann Weyl in 1914. Weyl proved that a sequence \( \{x_n : n = 1, \ldots\} \subseteq \mathbb{R} \) is uniformly distributed modulo \( a \) if and only if

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(y_n) = R \int_0^a f
\]

for every Riemann integrable function \( f \) on \([0, a] \), e.g., [148]. Here \( \{y_n\} \subseteq [0, a) \) corresponds to \( \{x_n\} \) as in the definition.

\textbf{a.} Let \( x \in \mathbb{R} \setminus \mathbb{Q} \). Prove that for each Riemann integrable function \( g \) defined on \([0, 1) \),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} g(nx \mod 1) = \hat{g}(0);
\]

[\textit{Hint.} First verify that

\[
\left| \sum_{n=1}^{N} e^{2\pi i m nx} \right| \leq \frac{1}{|\sin(\pi mx)|};
\]

and then use the hypothesis that \( x \) is irrational to prove that if \( m \neq 0 \) then

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m nx} = 0.
\]

The rest of the demonstration follows from standard approximation results.]

\textbf{b.} Let \( x \in \mathbb{R} \setminus \mathbb{Q} \). Prove that the sequence \( \{nx : n = 1, \ldots\} \) is uniformly distributed modulo 1.

[\textit{Hint.} Part \( b \) is clear from part \( a \) by taking the appropriate function \( g \).]

3.25. For \( a > 0 \) and a sequence \( \{x_n : n = 1, \ldots\} \subseteq \mathbb{R} \) define \( y_n \in \mathbb{R} \) as in Problem 3.24 by the condition that \( y_n \in [0, a) \) and \( (x_n - y_n)/a \in \mathbb{Z} \). The antithesis of uniform distribution for a sequence \( \{x_n : n = 1, \ldots\} \subseteq \mathbb{R} \) occurs when \( y_n \to 0 \). Assume that the sequence \( \{x_n : n = 1, \ldots\} \subseteq \mathbb{R} \) does not tend to 0 and let \( A = \{a > 0 : y_n \to 0\} \). Prove that \( m(A) = 0 \).

[\textit{Hint.} Without loss of generality we may assume that \( \lim_{n \to \infty} |x_n| = \infty \). If \( a \in A \) then \( \lim_{n \to \infty} e^{2\pi i x_n/a} = 1 \). Consequently, \( \lim_{n \to \infty} e^{2\pi i x_r} = 1 \) for \( r \in A^{-1} = \{a^{-1} : a \in A\} \). Thus,
\[ \forall b > 0, \quad \lim_{n \to \infty} \int_{A_b^{-1}} e^{2\pi i\alpha r} \, dr = m(A_b^{-1}), \]

where \( A_b^{-1} = A^{-1} \cap (0, b) \), by LDC. On the other hand, from the Riemann–Lebesgue lemma

\[ \lim_{n \to \infty} \int_{A_b^{-1}} e^{2\pi i\alpha r} \, dr = 0. \]

Therefore, \( m(A^{-1}) = m(A) = 0 \).

Note that \( A \) can be uncountable.

3.26. Let \( Q \) be the family of all half-open parallelepipeds in \( \mathbb{R}^d \) and let \( f : \mathbb{R} \to \mathbb{R} \) be an increasing right continuous function. Prove that there exists a non-negative finitely additive set function \( \mu_f^q \) on \( Q \) such that

\[ \mu_f^q((a_1, b_1] \times \ldots \times (a_d, b_d]) = (f(b_1) - f(a_1)) \ldots (f(b_d) - f(a_d)). \]

3.27. Given the open interval \((a, b)\), where \( b \) can be \( \infty \), and a function \( f : (a, b) \to \mathbb{R} \). Assume that \( \int_a^r f \) exists for each \( r \in (a, b) \). The Cauchy–Riemann integral of \( f \) on \((a, b)\) is

\[ CR \int_a^b f = \lim_{r \to b^-} R \int_a^r f \]

when the right-hand side exists. Naturally we can define the “Cauchy–Lebesgue” integral if we replace \( \int_a^r f \) by \( \int_a^r f \).

a. Give an example of an unbounded function on \((a, \infty)\) whose Cauchy–Riemann integral exists, e.g., Problem 3.22.

b. Let \( f \) be a non-negative function on the interval \((a, b)\) and assume that \( \int_a^r f \) exists for all \( r > a \). Prove that \( CR \int_a^\infty f \) exists if and only if \( f \in L^1_{\text{loc}}((a, \infty)) \).

c. Prove that if \( f \in L^1_{\text{loc}}((a, \infty)) \) and \( \int_a^r f \) exists for all \( r > a \) then \( \int_a^\infty f = CR \int_a^\infty f \).

d. Let \( f(x) = \sin(x)/x \). Show that \( CR \int_0^\infty f \) exists but that \( f \notin L^1_{\text{loc}}((0, \infty)) \).

[Hint.]

\[ \int_0^\infty \left| \frac{\sin(x)}{x} \right| \, dx = \sum_{k=0}^{\infty} \int_{k\pi}^{(k+1)\pi} \left| \frac{\sin(x)}{x} \right| \, dx \]

\[ \leq \sum_{k=0}^{\infty} \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin(x)| \, dx. \]

Thus \( f \) can be integrated whereas \(|f|\) cannot be integrated; this phenomenon is referred to as conditional convergence.

e. Show that the Cauchy–Riemann integral does not integrate every bounded Lebesgue measurable function on \([a, b], b < \infty\).
3.10 Problems

\[ f \] Find a Cauchy–Riemann integrable function \( f \) on \([a, b], b < \infty\), such that \( f \notin L^1_{\text{loc}}([a, b]) \).

[Hint. Example 3.2.9a.]

\[ g \] Let \( f(x) = (1/x) \sin(\pi/x) \) on \((0, 1)\). Is \( f' \) Cauchy–Riemann integrable on \((0, 1)\)? Does \( f' \in L^1_{\text{loc}}([0, 1]) \)?

\[ h \] In light of Problem 3.22 we would like to know conditions on a function \( f \) so that we could conclude that

\[ \lim_{x \to 1} f(x) = 0 \quad (3.51) \]

when \( \text{CR} f = \int_a^\infty f \) exists. In fact, the following is true.

Let \( f \) be continuous on \([a, b), b < \infty\), and assume that \( \text{CR} f = \int_a^\infty f \) exists; then (3.51) is valid if and only if \( f \) is uniformly continuous.

Prove this assertion.

[Hint. Integrate \( f f'' \) by parts and use part \( h \).]

\[ i \] Let \( f \) be a twice continuously differentiable real-valued function defined on \((a, b), a, b < \infty\), and assume that \( \text{CR} f_1 = \int_a^\infty f' \) and \( \text{CR} f_2 = \int_a^\infty (f'')^2 \) exist. Prove that (3.51) is valid.

[Hint. Integrate \( ff'' \) by parts and use part \( h \).]

3.28. \[ a \] In light of Example 3.6.5, does there exist a sequence \( \{f_n : n = 1, \ldots\} \subseteq L^1_{\text{loc}}([0, 1]) \) such that \( f_n \to 0 \) pointwise and

\[ \forall A \in \mathcal{M}([0, 1]), \quad \lim_{n \to \infty} \int_0^1 f_n(x) 1_A(x) \, dx = 0, \]

whereas \( \lim_{n \to \infty} \int |f_n(x)| \, dx > 0 \)?

\[ b \] Construct a sequence \( \{f_n : n = 1, \ldots\} \subseteq L^1_{\text{loc}}(\mathbb{R}) \) such that

i. \( f_n \) is continuous and \( \lim_{|x| \to \infty} f_n(x) = 0 \),

ii. \( \sup_{n} \int_{\mathbb{R}} |f_n(x)| \, dx < \infty \),

iii. \( f_n \to f \) uniformly on \( \mathbb{R} \),

but such that \( \{f_n : n = 1, \ldots\} \) contains no subsequence \( \{f_{n_k} : k = 1, \ldots\} \) for which

\[ \int_{\mathbb{R}} |f_{n_k}(x) - f(x)| \, dx \to 0. \]

[Hint. Start with \( g_n = (1/n) 1_{[n, 2n]} \).]

3.29. Let \( \{f_n : n = 1, \ldots\} \) be a sequence of functions on \([a, b]\). The general problem for this exercise is to investigate pointwise convergence of subsequences on certain subsets of \([a, b]\).

\[ a \] Let \( \{n_k : k = 1, \ldots\} \subseteq \mathbb{N} \) be a sequence converging to infinity. We let \( S_{n_k} \) denote the set \( \{x : \lim_{k \to \infty} \sin(2\pi n_k x) \text{ exists} \} \subseteq \mathbb{R} \). Show that even though \( m(S_{n_k}) = 0 \), it is possible to find subsequences \( \{n_k : k = 1, \ldots\} \subseteq \mathbb{N} \) for which card \( S_{n_k} = \text{card} \mathbb{R} \).

\[ b \] Does there exist \( \delta \in (0, 1] \) and \( X \subseteq [0, 1) \) such that \( m(X) > 0 \) and

\[ \text{card} \{n : \sup_{x \in X} \sin(2\pi n x) \geq \delta \} < \infty? \]
c. Solve part b for any non-constant periodic function \( f \in L^\infty_m(\mathbb{R}) \), where \( \sin(2\pi nx) \) is replaced by \( f(nx) \).

d. Let \( A_{n_k} = S_{n_k} \cap [0,1) \). Give examples to show whether \( A_{n_k} \) can be closed and infinite, countable, or finite. Find \( A_{n!} \). See the second remark below.

e. Prove the Cantor–Lebesgue theorem: if \( \sum_{|n| \leq N} c_n e^{2\pi nx} \to 0 \) pointwise on a set \( X \subseteq [0,1) \), as \( N \to \infty \), and \( m(X) > 0 \) then \( \lim_{|n| \to \infty} c_n = 0 \).

[Hint. Without loss of generality consider the series \( \sum_{n=0}^{\infty} r_n \cos(2\pi(nx+a_n)) \), and assume that there is a \( \delta > 0 \) and a subsequence \( \{n_k : k = 1,\ldots\} \subseteq \mathbb{N} \) such that for each \( k \), \( r_{n_k} > \delta > 0 \). Thus, \( \lim_{k \to \infty} \cos(2\pi(n_kx + a_{n_k})) = 0 \).

Now use LDC and the fact that

\[
\int_X \cos^2(2\pi(kx + a)) \, dx = \frac{1}{2} m(X) + \frac{1}{2} \int_X \cos(4\pi(kx + a)) \, dx. \tag{3.52}
\]

Obtain the contradiction by employing the Riemann–Lebesgue lemma.]

See the remarks in Sections 3.8.1 and 3.8.2.

f. Suppose that \( \sum_{n=1}^{\infty} a_n \) is an absolutely convergent series. By way of considering the converse situation to part e, prove the following: if \( \sum_{n=1}^{\infty} (a_n/n) \cos(2\pi nx) \) is not constant on \([0,1)\), then \( \sum_{n=1}^{\infty} a_n \sin(2\pi nx) \) cannot converge pointwise \( m\text{-a.e.} \) to 0.

Remark. Using (3.52) and LDC we have another proof, besides Theorem 3.3.14, that \( m(A_{n_k}) = 0 \).

Remark. S. Mazurkiewicz [174] proved that if \( \{f_n : n = 1,\ldots\} \) is a uniformly bounded sequence of continuous functions on \([a,b]\), then there is a closed uncountable set \( F \subseteq [a,b] \) (without isolated points) and integers \( n_1 < n_2 < \ldots \) such that \( \{f_{n_k} : k = 1,\ldots\} \) converges uniformly on \( F \). Using the continuum hypothesis, Sierpiński [234] proved the following result.

There is a sequence of non-measurable functions on \([a,b]\) which is uniformly bounded but such that no subsequence converges on an uncountable set.

Note with regard to Problem 3.29d, that if \( \{f_n : n = 1,\ldots\} \) is any sequence of functions and \( D \subseteq [a,b] \) is countable then there is a subsequence \( \{f_{n_k} : k = 1,\ldots\} \) which converges on \( D \).

Remark. Some interesting positive results on the general problem stated at the beginning of this exercise have been given by K. Schrader [222], [223]. For example, he proved the following theorem. Let \( f_n : [a,b] \to \mathbb{R} \) be continuous; and assume that for some \( N \geq 0 \) the sequence \( \{f_n : n = 1,\ldots\} \) has the following property:

if \( f_k = f_j \) for more than \( N \) values of \( x \in [a,b] \) then \( f_k \) is identical to \( f_j \) on \([a,b]\).

Then there is a subsequence \( \{f_{n_k} : k = 1,\ldots\} \) which converges pointwise, possibly with infinite value, on \([a,b]\).
3.10 Problems

3.30. a. Prove that \( \sum_{n=1}^{\infty} e^{2\pi inx}/n \) converges on \([0,1)\) if and only if \( x \neq 0 \). [Hint. Use partial summation obtaining a bound on the geometric series \( \sum_{n=1}^{\infty} e^{2\pi inx} \).]

b. Let \( \{a_n : n = 1, \ldots\} \) be a sequence decreasing to 0 and define \( g(x) = \sum_{n=1}^{\infty} a_n \sin(2\pi nx) \). By an argument similar to that of part a, the series converges for each \( x \neq 0 \). Prove that \( g \in L^1_m(\mathbb{T}) \) if and only if \( \sum_{n=1}^{\infty} a_n/n < \infty \), e.g., [80], volume I, page 115. Note that \( \sum_{n=1}^{\infty} e^{2\pi inx}/n \in L^2_m(\mathbb{T}) \) by the Parseval equality.

3.31. Let \( f \in L^1_m([0,1]) \) and assume that

\[
\int_0^1 f \phi = 0
\]

for all functions \( \phi \in C([0,1]) \) such that \( \phi(0) = \phi(1) = 0 \). Prove that \( f = 0 \) m-a.e.

[Hint. Assume that the result is not true. Then \( \int_0^1 |f| = \alpha > 0 \). For any \( \varepsilon > 0 \) consider a set \( G_\varepsilon \subseteq [0,1] \) such that \( \int_{[0,1] \setminus G_\varepsilon} |f| < \varepsilon \). Using Theorem 10.1.2, construct a continuous function \( \phi_\varepsilon \) on \([0,1]\) such that \( |\phi_\varepsilon| \leq 1, \phi_\varepsilon(0) = \phi_\varepsilon(1) = 0, \) and \( f\phi_\varepsilon = |f| \) on \( G_\varepsilon \), a compact subset of \( G_\varepsilon \) such that \( \int_{G_\varepsilon \setminus C_\varepsilon} |f| < \varepsilon \) and \( 0,1 \notin C_\varepsilon \). Note that there exists \( \delta > 0 \) such that, for all \( \varepsilon > 0 \), \( \int_{C_\varepsilon} |f| > \delta \), which yields a contradiction with the fact that \( \int_0^1 f \phi = 0 \).]

3.32. Let \( f : [0,1] \to \mathbb{R} \) be an increasing right continuous function with the property that

\[
\forall g \in C([0,1]), \quad \int_0^1 g \, df = 0.
\]

Prove that \( f \) is a constant function.

3.33. Prove Theorem 3.7.9.

3.34. Let \( X \) be an uncountable set and let

\[
\mathcal{A} = \{ A \subseteq X : A \text{ is countable or } A^c \text{ is countable} \}.
\]

Define \( \mu(A) = 1 \) is \( \text{card } A^c \leq \aleph_0 \) and \( \mu(A) = 0 \) if \( \text{card } A \leq \aleph_0 \). prove that \((X,\mathcal{A},\mu)\) is a measure space.

In this case, and with the discrete topology on \( X \), we have a measure space with open covering \( \{U_\alpha\} \), such that \( \mu(X \cap U_\alpha) = 0 \) for each \( \alpha \), whereas \( \mu(X) = 1 \), cf., Corollary 7.4.2.