6 Weak convergence of measures

6.1 Vitali theorem

We shall now prove Vitali’s Theorem 3.3.11 and Theorem 3.3.12. As we noted in the remark after the statement of Theorem 3.3.12, Vitali’s results give non-trivial necessary and sufficient conditions in order that
\[
\lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0,
\]
where \((X, \mathcal{A}, \mu)\) is a measure space and \(\{f_n : n = 1, \ldots\} \subseteq L^1_\mu(X)\). The following result was mentioned after the statement of Theorem 3.3.12 and is used to deduce LDC from Vitali’s theorem.

**Theorem 6.1.1. Lebesgue characterization of convergence in measure**

Let \((X, \mathcal{A}, \mu)\) be a measure space and let \(\{f_n : n = 1, \ldots\}\) be a sequence of \(\mu\)-measurable functions which converges \(\mu\text{-a.e.}\) to a \(\mu\)-measurable function \(f\).

If there is \(g \in L^1_\mu(X)\) such that
\[
\forall n = 1, \ldots, \quad |f_n| \leq g \quad \mu\text{-a.e.},
\]
then \(f_n \to f\) in measure.

**Proof.** Without loss of generality assume that
\[-\forall n = 1, \ldots \quad \text{and} \quad \forall x \in X, \quad |f_n(x)|, |f(x)| \leq g(x).\]

Then, for each \(\varepsilon > 0\), we define
\[
A_{n, \varepsilon} = \bigcup_{j=n}^{\infty} \{x : |f_j(x) - f(x)| \geq \varepsilon\} \subseteq \{x : |g(x)| \geq \varepsilon/2\};
\]
so that by the integrability of \(g\), \(\mu(A_{n, \varepsilon}) < \infty\) for \(n = 1, \ldots\). Since \(f_n \to f\) \(\mu\text{-a.e.}\) we have
\[
\mu \left( \bigcap_{n=1}^{\infty} A_{n, \varepsilon} \right) = 0.
\]
Thus, for each $\varepsilon > 0$,
\[
\lim_{n \to \infty} \mu(\{ x : |f_n(x) - f(x)| \geq \varepsilon \}) \leq \lim_{n \to \infty} \mu(A_{n,\varepsilon}) = 0,
\]
where the equality follows from Theorem 2.4.3b, from (6.2), and because $\mu(A_{1,\varepsilon}) < \infty$ and each $A_{n,\varepsilon} \subseteq A_{n-1,\varepsilon}$.

\[\square\]

**Theorem 6.1.2. Vitali uniform absolute continuity theorem (Theorem 3.3.11)**

Let $(X, A, \mu)$ be a finite measure space and let $\{f_n : n = 1, \ldots \} \subseteq L^1(X)$. (6.1) is valid for some $f \in L^1(X)$ if and only if

i. $\{f_n : n = 1, \ldots \}$ converges in measure to a $\mu$-measurable function $g$, and

ii. $\{f_n : n = 1, \ldots \}$ is uniformly absolutely continuous.

**Proof.** ($\iff$) Without loss of generality take each $f_n$ to be real-valued.

a. Given condition ii we first prove that $\{f_n : n = 1, \ldots \}$ is uniformly absolutely continuous. Let $\varepsilon > 0$. Choose $\delta > 0$ such that

\[\forall A \in \mathcal{A}, \text{ for which } \mu(A) < \delta, \text{ and } \forall n, \quad \left| \int_A f_n \, d\mu \right| < \frac{\varepsilon}{2}; \quad (6.3)\]

For each such $A$ we define the disjoint sets

\[A^+_n = \{ x \in A : f_n(x) \geq 0 \} \quad \text{and} \quad A^-_n = \{ x \in A : f_n(x) < 0 \}.\]

$A^+_n$ and $A^-_n$ are measurable since $f_n$ is measurable; and

\[\forall n = 1, \ldots, \mu(A^+_n) < \delta \quad \text{and} \quad \mu(A^-_n) < \delta,\]

since $\mu(A) < \delta$. By definition,

\[\int_{A^+_n} |f_n| \, d\mu = \left| \int_{A^+_n} f_n \, d\mu \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \int_{A^-_n} |f_n| \, d\mu = \left| \int_{A^-_n} f_n \, d\mu \right| < \frac{\varepsilon}{2}.\]

Consequently, for each $n$,

\[\int_A |f_n| \, d\mu = \int_{A^+_n \cup A^-_n} |f_n| \, d\mu = \int_{A^+_n} |f_n| \, d\mu + \int_{A^-_n} |f_n| \, d\mu < \varepsilon;\]

and so $\{ |f_n| : n = 1, \ldots \}$ is uniformly absolutely continuous.

b. Using the uniform absolute continuity of $\{|f_n|\}$ we shall now prove that $\{f_n : n = 1, \ldots \}$ is $L^1(X)$-Cauchy.

For each $\sigma > 0$, define

\[A_{m,n}(\sigma) = \{ x : |f_m(x) - f_n(x)| \geq \sigma \}\]

and
\[ B_{m,n}(\sigma) = \{ x : |f_m(x) - f_n(x)| < \sigma \} . \]

Then
\[
\int_X |f_m - f_n| \, d\mu = \int_{A_{m,n}(\sigma)} |f_m - f_n| \, d\mu + \int_{B_{m,n}(\sigma)} |f_m - f_n| \, d\mu
\leq \int_{A_{m,n}(\sigma)} |f_m - f_n| \, d\mu + \sigma \mu(B_{m,n}(\sigma)) \tag{6.4}
\leq \int_{A_{m,n}(\sigma)} |f_m| \, d\mu + \int_{A_{m,n}(\sigma)} |f_n| \, d\mu + \sigma \mu(X).
\]

For \( \varepsilon > 0 \) there is \( \sigma_0 > 0 \) such that
\[
\forall \sigma \leq \sigma_0, \quad \sigma \mu(X) < \frac{\varepsilon}{3} \tag{6.5}
\]
(here we use the fact that \( \mu(X) < \infty \)). From the first part of the proof there is \( \delta > 0 \) for which
\[
\int_A |f_n| \, d\mu < \frac{\varepsilon}{3} \tag{6.6}
\]
if \( A \in \mathcal{A} \) satisfies \( \mu(A) < \delta \). From \( i \) we can choose \( N \) where
\[
\forall m, n > N, \quad \mu(A_{m,n}(\sigma_0)) < \delta. \tag{6.7}
\]
Thus given \( \varepsilon > 0 \) we have picked \( \sigma_0 \) and \( \delta \) as above, i.e., (6.5) and (6.6), and then used \( i \) to find \( N \) as in (6.7). We apply (6.5) and (6.6) to (6.4) to obtain
\[
\forall m, n > N, \quad \int_X |f_m - f_n| \, d\mu < \varepsilon.
\]

c. Since \( L^1_1(X) \) is complete, we obtain (6.1). Moreover, \( g \) in part \( i \) equals \( f \) in (6.1) \( \mu \)-a.e. by the following argument. By Theorem 3.3.13a there exists a subsequence \( \{f_{n_k}\} \) which converges pointwise to \( g \) \( \mu \)-a.e. On the other hand, \( \{f_{n_k}\} \) converges in \( L^1_1(X) \) to \( f \), so we can choose a subsequence of \( \{f_{n_k}\} \) which converges pointwise to \( f \) \( \mu \)-a.e. (see Theorem 3.3.13).

(\( \Rightarrow \)) Fix \( \varepsilon > 0 \). Let
\[ A_n = \{ x : |f_n(x) - f(x)| \geq \varepsilon \} . \]

Then
\[
\int_X |f_n - f| \, d\mu \geq \int_{A_n} |f_n - f| \, d\mu \geq \varepsilon \mu(A_n)
\]
so that \( \mu(A_n) \rightarrow 0 \) from (6.1), and we have proved part \( i \).

Also, by (6.1), for any \( \varepsilon > 0 \) we can choose \( N \) for which
\[
\forall m, n \geq N, \quad \int_X |f_m - f_n| \, d\mu < \frac{\varepsilon}{3}.
\]
Since each $f_n \in L^1_{\mu}(X)$ we can find $\delta > 0$ such that if $\mu(A) < \delta$ for $A \in \mathcal{A}$ then
\[
\forall \ n = 1, \ldots, N, \quad \int_A |f_n| \, d\mu < \frac{\varepsilon}{3}.
\]
Consequently for $n > N$ and $\mu(A) < \delta, A \in \mathcal{A},$
\[
\left| \int_A f_n \, d\mu \right| \leq \int_A |f_n - f_N| \, d\mu + \int_A |f_N| \, d\mu < \varepsilon.
\]
Hence, for each $n$, if $\mu(A) < \delta$ then
\[
\left| \int_A f_n \, d\mu \right| < \varepsilon;
\]
and we have proved part ii.

\[\square\]

Remark. One is tempted to prove directly that $f \in L^1_{\mu}(X)$ using Fatou’s lemma, the uniform absolute continuity of $\{ |f_n| : n = 1, \ldots \}$, and the fact (from i) that $f_{n_k} \to f \mu$-a.e. for some subsequence of $\{ f_n : n = 1, \ldots \}$. By this method,
\[
\int_A |f| \, d\mu \leq \liminf_{n \to \infty} \int_A |f_{n_k}| \, d\mu \leq \sup_{n=1, \ldots} \int_A |f_n| \, d\mu < \varepsilon
\]
for $\mu(A) < \delta$. Thus $f \in L^1_{\mu}(A)$, so that since $\mu(X)$ is finite we write $X = \bigcup_{j=1}^{n} A_j$, $\mu(A_j) < \delta$, and we have $f \in L^1_{\mu}(X)$. This procedure works except when there are not enough sets of small enough measure to guarantee that $X = \bigcup_{j=1}^{n} A_j$.

The notions of uniform absolute continuity and Vitali equicontinuity are obviously closely related. The following result quantifies this assertion.

**Proposition 6.1.3.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let $M_b(X)$ be the space of complex measures associated with the measurable space $(X, \mathcal{A})$, and let $\{ \nu_n : n = 1, \ldots \} \subseteq M_b(X)$.

a. Assume that for each $n$, $\nu_n$ is a measure and $\nu_n \ll \mu$. If $\{ \nu_n : n = 1, \ldots \}$ is Vitali equicontinuous then it is uniformly absolutely continuous.

b. Assume $\mu(X) < \infty$. If $\{ \nu_n : n = 1, \ldots \}$ is uniformly absolutely continuous then it is Vitali equicontinuous.

**Proof.** a. We use the argument of Theorem 5.2.6. Assume $\{ \nu_n : n = 1, \ldots \}$ is not uniformly absolutely continuous. Then choose $\varepsilon > 0$, $\{ B_k : k = 1, \ldots \} \subseteq \mathcal{A}$, and integers $\{ n_k : k = 1, \ldots \}$ such that
\[
\mu(B_k) < \frac{1}{2^k+1} \quad \text{and} \quad \nu_{n_k}(B_k) \geq \varepsilon.
\]
Set $E_k = \bigcup_{j=k}^{\infty} B_j$ so that $E_{k+1} \subseteq E_k$. Also

$$\mu(E_k) \leq \sum_{j=k}^{\infty} \frac{1}{2^{j+1}} = \frac{1}{2^k}.$$ 

We write $A_k = E_k \setminus E_0$ where $E_0 = \bigcap_{k=1}^{\infty} E_k$. Thus, $A_{k+1} \subseteq A_k$ and $\bigcap_{k=1}^{\infty} A_k = \emptyset$.

By Vitali equicontinuity, there exists $N \in \mathbb{N}$ such that for all $k > N$

$$|\nu_{n_k}(A_k)| < \varepsilon.$$  \hspace{1cm} (6.9)

Since $\nu_{n_k} \ll \mu$ and $\mu(E_k) = 2^{-k}$, we have

$$\nu_{n_k}(E_0) = 0.$$ \hspace{1cm} (6.10)

(6.9) and (6.10) imply that

$$|\nu_{n_k}(E_k)| < \varepsilon.$$  

However, from (6.8), $\nu_{n_k}(E_k) \geq \nu_{n_k}(B_k) \geq \varepsilon$. This is the desired contradiction.

b. Fix $\varepsilon > 0$ and take $\{A_n : n = 1, \ldots\} \subseteq \mathcal{A}$ decreasing to $\emptyset$. By hypothesis there is $\delta > 0$ so that if $A \in \mathcal{A}$, $\mu(A) < \delta$, then $|\nu_{m}(A)| < \varepsilon$ for each $m$. Since $\mu(X) < \infty$, we have $\mu(A_n) \to 0$, and so there is $N$ for which $\mu(A_n) < \delta$ if $n > N$. Consequently $|\nu_{m}(A_n)| < \varepsilon$ for all $n > N$ and for all $m$.  

\[\Box\]

In light of Proposition 6.1.3 and the first part of the proof of the sufficient conditions in Theorem 6.1.2 we have:

**Proposition 6.1.4.** Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $\{f_n : n = 1, \ldots\} \subseteq L^1_{\mu}(X)$. Define $\nu_n$ by

$$\forall \, A \in \mathcal{A}, \quad \nu_n(A) = \int_A |f_n| \, d\mu.$$  

Then $\{\nu_n : n = 1, \ldots\}$ is Vitali equicontinuous if and only if $\{f_n : n = 1, \ldots\}$ is uniformly absolutely continuous.

The “only if” part of Proposition 6.1.4 does not require that $\mu(X) < \infty$.

**Theorem 6.1.5.** Vitali equicontinuity theorem (Theorem 3.3.12)

Let $(X, \mathcal{A}, \mu)$ be a measure space and choose a sequence $\{f_n : n = 1, \ldots\} \subseteq L^1_{\mu}(X)$. (6.1) is valid for some $f \in L^1_{\mu}(X)$ if and only if

i. $\{f_n : n = 1, \ldots\}$ converges in measure (to a $\mu$-measurable function $g$),

and

ii. $\{\nu_n : \forall \, A \in \mathcal{A}, \nu_n(A) = \int_A |f_n| \, d\mu\}$ is Vitali equicontinuous.
Proof. (\(\Rightarrow\)) If \(\mu(X) < \infty\) the result follows by Theorem 6.1.2 and Proposition 6.1.4. Note that \(\bigcup_n \{x : f_n(x) \neq 0\}\) is \(\sigma\)-finite (Problem 5.28) so that, without loss of generality, we take \(X\) to be \(\sigma\)-finite.

Let \(\{B_m : m = 1, \ldots\} \subseteq \mathcal{A}\) be an increasing sequence of sets that forms a cover of \(X\) and assume that each \(\mu(B_m) < \infty\). If we set \(A_m = B_m^c\) then \(A_{m+1} \subseteq A_m\) and \(\bigcap A_k = \emptyset\).

We shall prove that \(\{f_n : n = 1, \ldots\}\) is \(L^1_\mu(X)\)-Cauchy. Take \(\varepsilon > 0\). From \(ii\) there is \(k\) such that

\[\forall m = 1, \ldots, \int_{A_k} |f_m| \, d\mu < \frac{\varepsilon}{4};\]

and so, for each \(m\) and \(n\),

\[\int_{A_k} |f_m - f_n| \, d\mu < \frac{\varepsilon}{2}. \tag{6.11}\]

Using Theorem 6.1.2 and the restriction measure space \((B_k, A, \mu)\), we see that there is \(N > 0\) such that

\[\forall m, n > N, \quad \int_{B_k} |f_m - f_n| \, d\mu < \frac{\varepsilon}{2}. \tag{6.12}\]

(6.11) and (6.12) yield the result.

(\(\Rightarrow\)) The proof of \(i\) from (6.1) is the same as in Theorem 6.1.2. To prove \(ii\) pick any \(\varepsilon > 0\) and take \(N\) such that

\[\forall n \geq N, \quad \int_X |f_n - f| \, d\mu < \frac{\varepsilon}{2}. \tag{6.13}\]

Let \(\{A_n : n = 1, \ldots\} \subseteq \mathcal{A}\) decrease to \(\emptyset\). Since \(f_n, f \in L^1_\mu(X), n = 1, \ldots, N\), and since \(1_{A_m} \to 0 \mu\text{-a.e.}\), we invoke LDC to assert that there is \(M > 0\) for which

\[\int_{A_m} |f| \, d\mu < \frac{\varepsilon}{2} \quad \text{and} \quad \int_{A_m} |f_n - f| \, d\mu < \frac{\varepsilon}{2} \tag{6.14}\]

if \(m > M\) and \(n = 1, \ldots, N\). Consequently, for each \(m > M\) and for all \(n\),

\[\int_{A_m} |f_n| \, d\mu \leq \int_{A_m} |f_n - f| \, d\mu + \int_{A_m} |f| \, d\mu < \varepsilon,\]

where the last inequality follows from (6.13) or the second part of (6.14) depending on whether \(n > N\) or \(n \leq N\).

\(\square\)

In Sections 6.2 and 6.3 we shall see that Vitali’s results are intimately related to subsequent important work by Hans Hahn, Otto Nikodym, Stanislaw Saks, Jean Dieudonné, and Alexandre Grothendieck in the area of weak convergence of measures.
A point of information and a corresponding word of caution are in order. We use “weak convergence” in the topological sense defined in Definition 10.9.2, also see Definition 6.3.1. Modern probabilists also have a notion of weak convergence of measures on complete separable metric spaces \( X \). They say that \( \mu_n \to \mu \) weakly, if for every bounded continuous function \( f \) on \( X \), \( \mu_n(f) \to \mu(f) \). In this setting our weak sequential convergence generates a stronger topology than “probabilistic weak sequential convergence”.

6.2 Nikodym and Hahn–Saks theorems

In this section the work of HAHN dates from 1922 and that of SAKS from 1933 [259], [258]. NIKODYM announced his results in 1931 and published them in 1933. Our presentation is due to JAMES BROOKS [45].

We use the following version of SCHUR’s lemma [46], cf., Theorem 10.7.2.

**Theorem 6.2.1. Schur lemma**

Let \( \{c_{i,j} : i, j = 1, \ldots \} \subseteq \mathbb{C} \) have the following properties:

i. \[ \forall i = 1, \ldots, \infty \quad \sum_{j=1}^{\infty} |c_{i,j}| < \infty, \]

ii. \[ \forall S \subseteq \mathbb{N}, \quad \exists \lim_{i \to \infty} \sum_{j \in S} c_{i,j}. \]

Then there is \( \{c_j : j = 1, \ldots \} \subseteq \mathbb{C} \) such that \( \sum |c_j| < \infty \) and

\[ \lim_{i \to \infty} \sum_{j=1}^{\infty} |c_{i,j} - c_j| = 0. \quad (6.15) \]

**Proof.** a. We first prove that if we are given \( \{z_1, \ldots, z_n\} \subseteq \mathbb{C} \), then there is \( S \subseteq \{j : 1 \leq j \leq n\} \) such that

\[ \sum_{j=1}^{n} |z_j| \leq 4 \sqrt{2} \left| \sum_{j \in S} z_j \right|. \quad (6.16) \]

To begin, divide the complex plane into its four “diagonal” quadrants \( Q_1, Q_2, Q_3, Q_4 \), see Figure ??.

Assume that

\[ \sum_{z_j \in Q_1} |z_j| \geq \frac{1}{4} \sum_{j=1}^{n} |z_j|, \]

and let \( S = \{j : z_j \in Q_1\} \). Noting that \( Q_1 = \{z = x + iy : |y| \leq x, x \geq 0\} \), we have
\[
\frac{|z|}{\sqrt{2}} \leq \text{Re}(z)
\]
since, for \(w = a + ib \in Q_1\),
\[
\sqrt{2} \text{Re}(w) = a\sqrt{2} = |a|\sqrt{2} = |a + ia| \geq |a + ib| = |w|.
\]
Thus,
\[
\left| \sum_{j \in S} z_j \right| \geq \sum_{j \in S} \text{Re}(z_j) \geq \frac{1}{\sqrt{2}} \sum_{j \in S} |z_j| \geq \frac{1}{4\sqrt{2}} \sum_{j=1}^{n} |z_j|,
\]
which is (6.16). (6.16) is a standard inequality and we refer as well to [252], page 119, and our remarks prior to Theorem 5.1.12.

b. We now prove (6.15) using the additional hypothesis that the limit in \(ii\) is 0 for any \(S\). Thus, in this case, we are taking each \(c_j = 0\) in (6.15). We assume (6.15) is false and shall obtain a contradiction. Since (6.15) is false, there is an \(\varepsilon > 0\) and a subsequence \(\{i_k : k = 1, \ldots\}\) so that
\[
\forall \, k = 1, \ldots, \, \sum_{j=1}^{\infty} |c_{i_k,j}| > \varepsilon > 0.
\]
Observe that there are strictly increasing sequences \(\{p_k : k = 1, \ldots\} \subseteq \mathbb{N}\) and \(\{n_k : k = 1, \ldots\} \subseteq \{i_k : k = 1, \ldots\}\) for which
\[
\forall \, k = 1, \ldots, \, \sum_{j=1}^{p_k} |c_{n_k,j}| < \frac{\varepsilon}{r} \tag{6.17}
\]
and
\[
\forall \, k = 1, \ldots, \, \sum_{j=p_{k+1}+1}^{\infty} |c_{n_k,j}| < \frac{\varepsilon}{r} \tag{6.18}
\]
where \(r > 2 + 8\sqrt{2}\). To see this start by taking any \(p_1\). Then, using (6.16), observe that for each \(n\) there is \(Y_n \subseteq \{0, \ldots, p_1\}\) such that
\[
\sum_{j=1}^{n} |c_{n,j}| \leq 4\sqrt{2} \left| \sum_{j \in Y_n} c_{n,j} \right|.
\]
Since there are only finitely many possible subsets \(Y_n\) we obtain (6.17) for some \(n_1\) from \(ii\). Then using \(i\) choose \(p_2\) for which (6.18) is true. Continue this process. From (6.17) and (6.18) we compute
\[
\sum_{j=p_k+1}^{p_{k+1}} |c_{n_k,j}| = \sum_{j=1}^{\infty} |c_{n_k,j}| - \sum_{j=1}^{p_k} |c_{n_k,j}| - \sum_{j=p_{k+1}+1}^{\infty} |c_{n_k,j}|
\]
\[
> \varepsilon - \sum_{j=1}^{p_k} |c_{n_k,j}| - \sum_{j=p_{k+1}+1}^{\infty} |c_{n_k,j}| > \varepsilon \left( 1 - \frac{2}{r} \right). \tag{6.19}
\]
We now use (6.16) in the following way. For each $k$ choose $S_k \subseteq \{j : p_k + 1 \leq j \leq p_{k+1}\}$ so that

$$4\sqrt{2} \left| \sum_{j \in S_k} c_{n_k,j} \right| \geq \sum_{j=p_k+1}^{p_{k+1}} |c_{n_k,j}|. \quad (6.20)$$

Letting $S = \bigcup S_k$ we combine (6.19) and (6.20), and obtain for each $k$ that

$$\left| \sum_{j \in S} c_{n_k,j} \right| \geq \left| \sum_{j \in S_k} c_{n_k,j} \right| - \sum_{j=1}^{p_k} |c_{n_k,j}| - \sum_{j=p_{k+1}+1}^{\infty} |c_{n_k,j}|$$

$$\geq \varepsilon \left( \frac{1}{4\sqrt{2}} - \frac{1}{2r\sqrt{2}} - \frac{r}{2} \right) > 0.$$

This is the desired contradiction (of ii).

c. We now reduce the general case to the setting of part b. Let $\{m_i : i = 1, \ldots\} \subseteq \mathbb{N}$ increase to infinity and define

$$a_{i,j} = c_{m_{i+1},j} - c_{m_i,j}.$$

$\{a_{i,j} : i, j = 1, \ldots\}$ satisfies the hypotheses of part b so that we can conclude

$$\lim_{i \to \infty} \sum_{j=1}^{\infty} |c_{m_{i+1},j} - c_{m_i,j}| = 0. \quad (6.21)$$

Since $\{m_i : i = 1, \ldots\}$ is arbitrary, (6.21) tells us that $\{c_i : c_i = \{c_{i,1}, c_{i,2}, \ldots\}\}$ is a Cauchy sequence in $\ell^1(\mathbb{N})$. The proof is finished by the completeness of $\ell^1(\mathbb{N})$.

$\square$

We cannot replace hypothesis ii in Theorem 6.2.1 with the condition that

$$\forall S \subseteq \mathbb{N}, \quad \exists \lim_{i \to \infty} \left| \sum_{j \in S} c_{i,j} \right|.$$

In fact, let $c_{m,n} = (1/2^m) e^{in\pi/2}$.

Theorem 10.7.2, which states the equivalence of weak and norm sequential convergence in $\ell^1(\mathbb{N})$, is an immediate consequence of Theorem 6.2.1. In fact, if $\{c_i : i = 1, \ldots\} \subseteq \ell^1(\mathbb{N})$, with $c_i = \{c_{i,1}, \ldots\}$, converges weakly to 0 then

$$\forall S \subseteq \mathbb{Z}, \quad \mathbbm{1}_S \in \ell^\infty(\mathbb{N})(\mathbb{Z}) \quad \text{and} \quad \lim_{i \to \infty} \int \mathbbm{1}_S c_i = 0.$$

Thus, the hypotheses of Theorem 6.2.1 are satisfied.
Theorem 6.2.2. Nikodym theorem

Let \((X, \mathcal{A})\) be a measurable space and let \(\{\mu_n : n = 1, \ldots\} \subseteq M_b(X)\). Assume
\[
\forall A \in \mathcal{A}, \quad \lim_{n \to \infty} \mu_n(A) = \mu(A)
\]
exists. Then \(\mu \in M_b(X)\) and \(\{\mu_n : n = 1, \ldots\}\) is Vitali equicontinuous.

Proof. a. Clearly \(\mu\) is finitely additive on \(\mathcal{A}\). Let \(\{A_j : j = 1, \ldots\} \subseteq \mathcal{A}\) decrease to \(\emptyset\). We shall prove
\[
\lim_{j \to \infty} \mu(A_j) = 0. \tag{6.22}
\]

\(a.i\). Set \(E_k = A_k \setminus A_{k+1}\) so that \(\{E_k : k = 1, \ldots\}\) is a disjoint family and
\[
A_k = (A_k \setminus A_{k+1}) \cup (A_{k+1} \setminus A_{k+2}) \cup \ldots = \bigcup_{j=k}^{\infty} E_j.
\]

Our immediate task is to verify (6.23) below.

Define
\[
c_{i,j} = \mu_i(E_j).
\]

We shall check that \(\{c_{i,j} : i, j = 1, \ldots\}\) satisfies the hypotheses of Theorem 6.2.1. For each \(i = 1, \ldots\),
\[
\sum_{j=1}^{\infty} |c_{i,j}| = \sum_{j=1}^{\infty} |\mu_i(E_j)| \leq \sum_{j=1}^{\infty} |\mu_i|(E_j) = |\mu_i| \left( \bigcup_{j=1}^{\infty} E_j \right) = |\mu_i|(A_1) < \infty,
\]
and condition \(i\) of Theorem 6.2.1 is satisfied. For condition \(ii\), let \(S \subseteq \mathbb{N}\) and note that
\[
\lim_{i \to \infty} \sum_{j \in S} c_{i,j} = \lim_{i \to \infty} \mu_i \left( \bigcup_{j \in S} E_j \right) = \mu \left( \bigcup_{j \in S} E_j \right) \in \mathbb{C}.
\]

By assumption, we have \(\lim_{i \to \infty} \mu_i(E_j) = \mu(E_j)\). Combining this hypothesis with the existence of \(\{c_j\}\) from Theorem 6.2.1, we see that \(c_j = \mu(E_j)\) for all \(j = 1, \ldots\). Consequently, from Theorem 6.2.1, we have
\[
\lim_{i \to \infty} \sum_{j=1}^{\infty} |\mu_i(E_j) - \mu(E_j)| = 0. \tag{6.23}
\]

\(a.ii\). We now show that
\[
\lim_{n \to \infty} \sum_{j=n}^{\infty} \mu(E_j) = 0. \tag{6.24}
\]
To this end we calculate
\[
\sum_{j=n}^{\infty} |\mu(E_j)| \leq \sum_{j=n}^{\infty} |\mu_i(E_j) - \mu(E_j)| + \sum_{j=n}^{\infty} |\mu_i(E_j)| \\
\leq \sum_{j=1}^{\infty} |\mu_i(E_j) - \mu(E_j)| + \sum_{j=n}^{\infty} |\mu_i(E_j)|.
\]
Thus,
\[
\lim_{n \to \infty} \sum_{j=n}^{\infty} |\mu(E_j)| \leq \sum_{j=1}^{\infty} |\mu_i(E_j) - \mu(E_j)| + \lim_{n \to \infty} |\mu_i| \left( \bigcup_{j=n}^{\infty} E_j \right),
\]
so that by our hypothesis on \( \{A_n : n = 1, \ldots\} \) and the fact that \( \mu_i \) is bounded,
\[
\lim_{n \to \infty} \sum_{j=n}^{\infty} |\mu(E_j)| \leq \sum_{j=1}^{\infty} |\mu_i(E_j) - \mu(E_j)|.
\]
Therefore, (6.24) follows from (6.23).

\( a.iii \) Our next step is to prove
\[
\lim_{j \to \infty} \left( \mu_i(A_j) - \sum_{k=j}^{\infty} \mu(E_k) \right) = 0, \quad \text{uniformly in } i. \tag{6.25}
\]
(6.25) is certainly true for each \( i \) (not uniformly) by (6.24) and the hypothesis on \( \{A_j : j = 1, \ldots\} \). Given \( \varepsilon > 0 \), use (6.23) to choose \( I > 0 \) such that
\[
\forall i > I, \quad \sum_{j=1}^{\infty} |\mu_i(E_j) - \mu(E_j)| < \varepsilon.
\]
Next, take \( J > 0 \) with the property that
\[
\forall i = 1, \ldots, I, \quad \sum_{j=J}^{\infty} |\mu_i(E_j) - \mu(E_j)| < \varepsilon.
\]
Hence,
\[
\forall j > J \text{ and } \forall i = 1, \ldots, \quad \left| \mu_i(A_j) - \sum_{k=j}^{\infty} \mu(E_k) \right| = \left| \sum_{k=j}^{\infty} (\mu_i(E_k) - \mu(E_k)) \right| \\
\leq \sum_{k=J}^{\infty} |\mu_i(E_k) - \mu(E_k)| < \varepsilon,
\]
and this is (6.25).
Finally, to obtain (6.22), we first use the Moore–Smith theorem (Theorem 10.4.1), in conjunction with \textit{a.iii}, and the fact that
\[
\lim_{i \to \infty} \left( \mu_i(A_j) - \sum_{k=j}^{\infty} \mu_k(E_k) \right) = \mu(A_j) - \sum_{k=j}^{\infty} \mu(E_k),
\]
to compute
\[
\lim_{j \to \infty} \left( \mu(A_j) - \sum_{k=j}^{\infty} \mu(E_k) \right) = \lim_{i \to \infty} \lim_{j \to \infty} \left( \mu_i(A_j) - \sum_{k=j}^{\infty} \mu_k(E_k) \right). \tag{6.26}
\]
The right-hand side of (6.26) is 0 as we observed after (6.25). Consequently we can apply (6.24) (again) to the left-hand side of (6.26), and (6.22) follows.

\textbf{Remark.} Consider the measurable space \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\), and define
\[
\forall \ n = 1, \ldots \quad \text{and} \quad \forall \ A \in \mathcal{B}(\mathbb{R}), \quad \mu_n(A) = m(A \cap [n, \infty)).
\]
Then each \(\mu_n\) is a measure. Since \(\mu_n \geq \mu_{n+1}\) on \(\mathcal{B}(\mathbb{R})\), we define \(\mu\) as
\[
\forall \ A \in \mathcal{B}(\mathbb{R}), \quad \lim_{n \to \infty} \mu_n(A) = \mu(A), \tag{6.27}
\]
and note that each \(\mu(A)\) exists with possibly infinite values. \(\mu\) is not a measure. In fact,
\[
\mu \left( \bigcup_{n=1}^{\infty} [n, n+1) \right) = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \mu([n, n+1)) = 0.
\]
It is interesting to note that if \(\{\mu_n : n = 1, \ldots\}\) is a sequence of measures on a measurable space \((X, \mathcal{A})\) and \(\mu_n \leq \mu_{n+1}\) then the limit \(\mu\) in (6.27) is a measure.

We use Theorem 6.2.2 to prove the following result.

\textbf{Theorem 6.2.3. Hahn–Saks theorem}
\begin{enumerate}
\item Let \((X, \mathcal{A})\) be a measurable space, let \(\nu\) be a measure, and let \(\{\mu_n : n = 1, \ldots\} \subseteq \text{M}_b(X)\). Assume
\begin{enumerate}
\item \(\forall \ A \in \mathcal{A}, \lim_{n \to \infty} \mu_n(A) = \mu(A)\) exists,
\item \(\forall \ n = 1, \ldots, \mu_n \ll \nu\).
\end{enumerate}
Then \(\{\mu_n : n = 1, \ldots\}\) is uniformly absolutely continuous.
Proof. Assume that \( \{\mu_n : n = 1, \ldots\} \) is not uniformly absolutely continuous. Then there is \( \varepsilon > 0 \), a subsequence \( \{n_m : m = 1, \ldots\} \subseteq \mathbb{N} \), and \( \{A_m : m = 1, \ldots\} \subseteq \mathcal{A} \) such that, for each \( m = 1, \ldots \),

\[
\nu(A_m) < \frac{1}{2^m}
\]

and

\[
|\mu_{n_m}(A_m)| \geq \varepsilon. \tag{6.29}
\]

For simplification of notation write \( n_m \equiv m \) so that (6.29) is replaced by

\[
\forall \ m = 1, \ldots, \ |\mu_m(A_m)| \geq \varepsilon. \tag{6.30}
\]

Using this we shall obtain a contradiction.

a. For each \( k \geq 0 \) we first prove that there is a strictly increasing subsequence \( \{n^k_i : i = 1, \ldots\} \subseteq \mathbb{N} \) such that \( \{n^k_i : i = 1, \ldots\} \) is a subsequence of \( \{n^k_i : i = 1, \ldots\} \),

\[
n^k_1 > n^k_2, \quad n^k_2 > n^k_3, \quad n^k_3 > n^k_4, \quad \ldots
\]

and

\[
\sum_{i=1}^{\infty} |\mu_{n^k_i}|(A_{n^k_{i+1}}) < \frac{\varepsilon}{2}. \tag{6.31}
\]

where the elements of the sum decrease to 0.

In order to do this first observe that (6.28) and the hypothesis \( \mu_1 \ll \nu \) yield

\[
\lim_{m \to \infty} |\mu_1|(A_m) = 0.
\]

Consequently, there is a strictly increasing sequence \( \{n^1_i : i = 1, \ldots\} \), for which \( |\mu_1|(A_{n^1_i}) \) is decreasing, \( n^1_i \to \infty \), and

\[
|\mu_1|(A_{n^1_1}) < \frac{\varepsilon}{2+1}.
\]

Now since \( |\mu_{n^1_1}| \ll \nu \) we use (6.28) again to obtain a strictly increasing sequence \( \{n^2_j : i = 1, \ldots\} \subseteq \{n^1_i : i = 1, \ldots\} \) such that

\[
\forall \ j = 1, \ldots, \ n^1_{j+1} < n^2_j, \quad \text{and} \quad \forall \ i = 1, \ldots, \ |\mu_{n^1_i}|(A_{n^2_j}) < \frac{\varepsilon}{2^2+1},
\]

and \( |\mu_{n^2_j}|(A_{n^2_j}) \) decreases as \( n^2_j \to \infty \). Next we examine \( |\mu_{n^2_j}| \) and construct \( \{n^3_i : i = 1, \ldots\} \) in the same way, and continue the process.

b. Define \( \nu_i = \mu_{n^i_j} \) and \( B_j = A_{n^i_j} \). Observe that when we chose \( \{n^k_{j+1} : j = 1, \ldots\} \) from \( \{n^k_j : j = 1, \ldots\} \) we had

\[
n^k_1 > n^k_2 > n^k_3.
\]
Thus \( n^{k+1} \geq n^k + 1 \) and so \( n^k \geq k \). Therefore, by 6.28,
\[
\nu(B_i) < \frac{1}{2^i}. \tag{6.32}
\]
Also, we compute
\[
\sum_{j=m+1}^{\infty} |\nu_m|(B_j) = \sum_{j=1}^{\infty} |\nu_m|(B_{m+j}) \leq \sum_{j=1}^{\infty} |\nu_m|(A_{n^{m+1}}) < \frac{\epsilon}{2}, \tag{6.33}
\]
where the last two inequalities follow from \( a \). The latter inequality is a consequence of (6.31). For the former, note that for fixed \( m \), \( \{\nu_m|(A_{n^{m+1}}) : j = 1, \ldots \} \) is decreasing and
\[
n^{m+1}_j < \ldots < n^{m+j}_2 < n^{m+j}_1.
\]
\( c.i. \) Clearly \( C_n = \bigcup_{j=n}^{\infty} B_j \) and \( D = \bigcap_{j=1}^{\infty} \bigcup_{j=n}^{\infty} B_j = \bigcap_{j=n}^{\infty} C_n \).
\( c.i. \) Clearly \( C_n \subseteq C_{n-1} \) so that \( \{C_n : n = 1, \ldots \} \) decreases to \( D \). Consequently, \( E_n = C_n \setminus D \) decreases to \( \emptyset \). Using (6.32) we compute \( \nu(D) = 0 \) since
\[
\nu(D) \leq \nu(C_{n+1}) \leq \sum_{j=n+1}^{\infty} \nu(B_j) < \sum_{j=n+1}^{\infty} \frac{1}{2^j} = \frac{1}{2^n}. \]
Thus, by hypothesis \( ii \),
\[
\forall i = 1, \ldots, \nu_i(D) = 0.
\]
We therefore conclude that
\[
\forall i = 1, \ldots \text{ and } \forall n = 1, \ldots, \nu(E_n) = \nu(C_n). \tag{6.34}
\]
\( c.ii. \) From hypothesis \( i \) and Theorem 6.2.2 we have
\[
\lim_{n \to \infty} \nu_m(E_n) = 0, \text{ uniformly in } m. \tag{6.35}
\]
Applying (6.34) to (6.35),
\[
\lim_{n \to \infty} \nu_m(C_n) = 0, \text{ uniformly in } m.
\]
Hence choose \( N > 0 \) such that
\[
\forall n \geq N \text{ and } \forall m = 1, \ldots, |\nu_m(C_n)| < \frac{\epsilon}{2}. \tag{6.36}
\]
\( c.iii. \) Note that
\[
C_n = B_n \cup \left( B_n^c \cap \left( \bigcup_{j=n+1}^{\infty} B_j \right) \right).
\]
where $B_n$ and $B_n^\sim \cap \left( \bigcup_{j=n+1}^\infty B_j \right)$ are disjoint.

d. For $n \geq N$ and $m = n$ we use (6.30) and $c.iii$ to compute

$$
\varepsilon \leq |\nu_n(B_n)| \leq |\nu_n(C_n)| + |\nu_n\left( B_n^\sim \cap \left( \bigcup_{j=n+1}^\infty B_j \right) \right)|;
$$

so that from (6.36)

$$
\varepsilon < \frac{\varepsilon}{2} + |\nu_n| \left( \bigcup_{j=n+1}^\infty B_j \right) \leq \frac{\varepsilon}{2} + \sum_{j=n+1}^\infty |\nu_n|(B_j). \tag{6.37}
$$

We obtain a contradiction ($\varepsilon < \varepsilon$) using (6.33) in (6.37).

\[ \square \]

Remark. The Hahn-Saks theorem (Theorem 6.2.3) was originally proved by Saks independent of Theorem 6.2.2. He then deduced Theorem 6.2.2 in the following way. Given $\{\mu_n : n = 1, \ldots\} \subseteq M_b(X)$ and the hypothesis of Theorem 6.2.2. Define

$$
\nu(A) = \sum_{n=1}^\infty \frac{1}{2^n \|\mu_n\|_1} |\mu_n|(A). \tag{6.38}
$$

$\nu$ is a bounded measure and $\mu_n \ll \nu$ for each $n$. Consequently, we can apply Theorem 6.2.3 to obtain the uniform absolute continuity of $\{\mu_n : n = 1, \ldots\}$ with respect to $\nu$. Thus, using Proposition 6.1.3b and the uniform absolute continuity, we conclude that $\mu \in M_b(X)$ by a double limit argument. Also, $\mu \ll \nu$ (this is the $\mu$ defined in the hypothesis of Theorem 6.2.2).

### 6.3 Weak sequential convergence

**Definition 6.3.1. Weak sequential convergence in $L^p_b(X)$**

- Let $(X, A, \mu)$ be a measure space and let $1/p + 1/q = 1$. Consider a sequence $\{f_n : n = 1, \ldots\} \subseteq L^p_b(X)$ and $f \in L^p_b(X)$, $1 \leq p < \infty$. We say that $\{f_n : n = 1, \ldots\}$ converges weakly to $f$, and we write $f_n \rightharpoonup f$ weakly, if

$$
\forall g \in L^q_b(X), \quad \lim_{n \to \infty} \int_X f_n g \, d\mu = \int_X fg \, d\mu.
$$

- Let $(X, A, \mu)$ be a $\sigma$-finite measure space. A sequence $\{f_n : n = 1, \ldots\} \subseteq L^\infty_b(X)$ converges weakly to $f \in L^\infty_b(X)$ if for each set function $\nu$, finitely additive on $A$ and vanishing on $\{A \in A : \mu(A) = 0\}$,

$$
\lim_{n \to \infty} \int_X f_n \, d\nu = \int_X f \, d\nu.
$$
As we pointed out in Example 3.6.5, weak convergence does not necessarily imply norm convergence, cf., Theorem 10.7.2. In fact, weak convergence does not imply pointwise a.e. convergence, uniform convergence, or convergence in measure.

The weak sequential convergence defined above is actually sequential convergence for a certain topology, called the weak topology. We discuss the weak topology generally in Section 10.9, but for now we consider some special results. In particular, we have the following characterization of weak sequential convergence in $L^1_\mu(X)$.

**Theorem 6.3.2. Characterization of weak sequential convergence**

Let $(X, \mathcal{A}, \mu)$ be a measure space and consider a sequence $\{f_n : n = 1, \ldots\} \subseteq L^1_\mu(X)$ and $f \in L^1_\mu(X)$. Then, $f_n \rightharpoonup f$ weakly if and only if

i. $\forall A \in \mathcal{A}, \int_A (f_n - f) \, d\mu \to 0$,

ii. $\sup_{n \in \mathbb{N}} \|f_n\|_1 = K < \infty$.

**Proof.** ($\implies$) Part i is immediate by definition of weak convergence. To prove ii we first define $F_n : L^\infty_\mu(X) \to \mathbb{C}$ by

$$
\forall g \in L^\infty_\mu(X), \quad F_n(g) = \int_X (f_n - f) g \, d\mu.
$$

Each $F_n$ is continuous by LDC. Since $F_n(g) \to 0$ for each $g \in L^\infty_\mu(X)$ by hypothesis, we know that

$$
\forall g \in L^\infty_\mu(X), \quad \exists N_g \text{ such that } \forall n \geq N_g, \quad |F_n(g)| \leq 1;
$$

and so

$$
\forall g \in L^\infty_\mu(X), \quad \exists M_g = \max\{1, |F_1(g)|, \ldots, |F_{N_g - 1}(g)|\}
$$

such that

$$
\forall n = 1, \ldots, \quad |F_n(g)| \leq M_g.
$$

Therefore, we can apply the Uniform Boundedness Principle (Theorem 10.7.1 or Theorem 10.8.5) and conclude that

$$
\exists M \text{ such that } \forall n = 1, \ldots \text{ and } g \in L^\infty_\mu(X), \quad |F_n(g)| \leq M \|g\|_\infty.
$$

Clearly,

$$
\|f_n\|_1 = \int_X f_n g_n \, d\mu,
$$

for some $g_n \in L^\infty_\mu(X)$, where, without loss of generality, we take $|g_n| = 1$ on $X$. Therefore ii follows.

($\Longleftarrow$) Take any $\varepsilon > 0$. We prove that for any fixed $g \in L^\infty_\mu(X)$,

$$
\lim_{n \to \infty} \left| \int_X (f_n - f) g \, d\mu \right| \leq \varepsilon. \quad (6.39)
$$
Choose a simple function $h$ such that $\|g - h\|_\infty \leq \varepsilon / (2 \max(K, \|f\|_1))$, see Theorem 2.5.5 and Definition 2.5.9. Then

$$\left| \int_X (f_n - f) g \, d\mu \right| \leq \|g - h\|_\infty \|f_n - f\|_1 + \left| \int_X (f_n - f) h \, d\mu \right|,$$

and this gives (6.39).

The incredible thing is that we can drop condition $ii$ in Theorem 6.3.2, even in the setting of quite general measure spaces. This is the content of Theorem 6.3.7.

**Proposition 6.3.3.** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\{f_n : n = 1, \ldots\} \subseteq L^1(X)$ have the property that

$$\forall A \in \mathcal{A}, \quad \lim_{n \to \infty} \int_A f_n \, d\mu$$

exists and is finite. Then

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall n = 1, \ldots, \quad \mu(A) < \delta \implies \int_A |f_n| \, d\mu < \varepsilon.$$

**Proof.** We define the following family of measures $\mu_n$:

$$\forall A \in \mathcal{A}, \quad \mu_n(A) = \int_A f_n \, d\mu.$$

Thus, according to Theorem 5.3.2,

$$\forall n = 1, \ldots, \quad \mu_n \in M_{ac}(X, \mu).$$

Moreover, by our hypothesis, we have

$$\forall A \in \mathcal{A}, \quad \exists \lim_{n \to \infty} \mu_n(A).$$

These two statements are the assumptions of Theorem 6.2.3. Therefore we conclude that $\{\mu_n : n = 1, \ldots\}$ is uniformly absolutely continuous on $X$, i.e.,

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ such that } \forall A \in \mathcal{A},$$

$$\mu(A) < \delta \implies \forall n = 1, \ldots, \quad \left| \int_A f_n \, d\mu \right| < \varepsilon.$$

In other words, the sequence $\{f_n : n = 1, \ldots\}$ is uniformly absolutely continuous. Therefore, to finish the proof, we only need to observe that in the proof of the "if" part of Theorem 6.1.2 we have, in particular, established that $\{f_n : n = 1, \ldots\}$ is uniformly absolutely continuous if and only if $\{|f_n| : n = 1, \ldots\}$ is uniformly absolutely continuous.

$\square$
Remark. It is an interesting observation that in the proof of Proposition 6.3.3 we may assume without loss of generality that

$$\forall A \in \mathcal{A}, \lim_{n \to \infty} \int_A f_n \, d\mu = 0. \tag{6.40}$$

To see this we proceed as follows.

First we shall prove that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) and \( k \in \mathbb{N} \) such that

$$\forall m, n \geq k \quad \text{and} \quad \forall A \in \mathcal{A}, \text{for which } \mu(A) < \delta, \quad \int_A |f_m - f_n| \, d\mu < \frac{\varepsilon}{2}. \tag{6.41}$$

In fact, if (6.41) were not true then there exist \( \varepsilon > 0 \) and sequences \( \{m_i : m_i \geq i, i = 1, \ldots\} \), \( \{n_i : n_i \geq i, i = 1, \ldots\} \), and \( \{A_i : \mu(A_i) < 1/i, i = 1, \ldots\} \), such that

$$\forall i = 1, \ldots, \int_{A_i} |f_{m_i} - f_{n_i}| \, d\mu \geq \frac{\varepsilon}{2}. \tag{6.42}$$

On the other hand, the sequence \( \{f_{m_i} - f_{n_i} : i = 1, \ldots\} \subseteq L^1(\mu) \), satisfies the assumption of our proposition with \( \lim_{i \to \infty} \int_A (f_{m_i} - f_{n_i}) \, d\mu = 0 \) for all sets \( A \in \mathcal{A} \). Thus, if the conclusion of Proposition 6.3.3 was true for 0 limits, it would yield a contradiction with (6.42). Hence, (6.41) is established.

Second, we note that for any \( \varepsilon > 0 \) there exists \( \delta' > 0 \) such that

$$\forall A \in \mathcal{A}, \text{for which } \mu(A) < \delta', \quad \text{and} \quad \forall j \leq k, \quad \int_A |f_j| \, d\mu < \frac{\varepsilon}{2}. \tag{6.43}$$

This is a consequence of Proposition 3.3.9.

(6.40) follows from (6.41) and (6.43) because

$$\forall \varepsilon > 0, \exists \delta, \delta' > 0 \text{ such that } \forall n > k,
\mu(A) < \min(\delta, \delta') \implies \int_A |f_n| \, d\mu \leq \int_A |f_n - f_k| \, d\mu + \int_A |f_k| \, d\mu < \varepsilon.$$

Using Proposition 6.3.3 we may strengthen Theorem 6.3.2.

Theorem 6.3.4. Sufficient conditions for weak sequential convergence in \( L^1(\mu) \)

Let \( X \) be a locally compact Hausdorff space, let \( (X, \mathcal{A}, \mu) \) be a measure space, and let \( \mu \) be regular. Consider the sequence \( \{f_n : n = 1, \ldots\} \subseteq L^1(\mu) \).

i. \( \forall A \in \mathcal{A}, \lim_{n \to \infty} \int_A f_n \, d\mu \) exists and is finite,

ii. \( \sup_{n \in \mathbb{N}} \|f_n\|_1 = K < \infty \),
then \( f_n \rightharpoonup f \) weakly for some \( f \in L^1(\mu) \).

Proof. Fix \( g \in L^1(\mu) \). We first show that \( \{\int_X f_n g \, d\mu : n = 1, \ldots\} \) is a Cauchy sequence. Let \( \varepsilon > 0 \). There exists a simple function \( h \), such that \( \|g - h\|_\infty < \varepsilon/(3K) \). Thus,
6.3 Weak sequential convergence 281

\[ \forall n = 1, \ldots, \left| \int_X f_n g \, d\mu - \int_X f_n h \, d\mu \right| \leq \|g - h\|_\infty \|f_n\|_1 < \frac{\varepsilon}{3} \quad (6.44) \]

and

\[ \lim_{n \to \infty} \int_X f_n h \, d\mu \]

exists and is finite. By hypothesis, choose \( M \) so that for all \( i, j \geq M \),

\[ \left| \int_X f_i h \, d\mu - \int_X f_j h \, d\mu \right| < \frac{\varepsilon}{3} \quad (6.45) \]

Then, by (6.44) and (6.45) we have

\[ \forall i, j \geq M, \quad \left| \int_X f_i g \, d\mu - \int_X f_j g \, d\mu \right| < \varepsilon, \]

and so \( \{ \int_X f_n g \, d\mu \} \) is a Cauchy sequence. We may thus conclude that

\[ \forall g \in L_\mu^\infty(X), \quad \lim_{n \to \infty} \int_X f_n g \, d\mu \]

exists and is finite. We write

\[ \lambda(g) = \lim_{n \to \infty} \int_X f_n g \, d\mu, \]

for \( g \in L_\mu^\infty(X) \). Because of assumption \( ii \) we conclude that \( \lambda \) is a bounded linear functional on \( L_\mu^\infty(X) \). In particular, \( \lambda_n(A) = \int_A f_n \, d\mu \) defines \( \lambda_n \in M_b(X) \) and hence \( \lambda \in M_b(X) \) by the proof of Theorem 6.2.2. Since \( X \) is a locally compact Hausdorff space, this assertion is also a consequence of the Riesz Representation theorem proved in Chapter 7, since \( \lambda \) is a bounded linear functional on \( C_0(X) \).

Proposition 6.3.3, together with Theorem 5.2.7 and the observation following it, yields that \( \lambda \ll \mu \). Thus, according to the Radon–Nikodym theorem (Theorem 5.3.1), there exists \( f \in L_\mu^1(X) \) such that

\[ \forall A \in \mathcal{A}, \quad \lambda(A) = \int_A f \, d\mu. \]

Hence, \( \lambda(h) = \int_A h f \, d\mu \) for all simple functions \( h \). Since these functions are dense in \( L_\mu^\infty(X) \) and since \( \lambda \) is bounded on \( L_\mu^\infty(X) \), it follows that

\[ \forall g \in L_\mu^\infty(X), \quad \lim_{n \to \infty} \int_X f_n g \, d\mu = \int_X f g \, d\mu. \]

Indeed, fix \( g \in L_\mu^\infty(X) \) and \( \varepsilon > 0 \). Choose a simple function \( h \) such that \( \|g - h\|_\infty < \varepsilon/(2 \max(K, \|f\|_1)) \). Then,
\[
\lim_{n \to \infty} \left| \int_X f_n g \, d\mu - \int_X f g \, d\mu \right| \leq \lim_{n \to \infty} \left| \int_X f_n g \, d\mu - \int_X f_n h \, d\mu \right| + \lim_{n \to \infty} \left| \int_X f_n h \, d\mu - \int_X f h \, d\mu \right| + \left| \int_X f h \, d\mu - \int_X f g \, d\mu \right| < \varepsilon.
\]

Therefore, \( f_n \) converges weakly to \( f \).

The following result is due to Dieudonné [80] in his work on Köthe spaces (a term Dieudonné introduced), extending work of Gottfried Köthe and his coauthors from the setting of bilinear forms, cf., [87] and Theorem 6.3.8.

**Proposition 6.3.5.** Let \( X \) be a locally compact Hausdorff space, let \( (X, \mathcal{A}, \mu) \) be a measure space, and let \( \mu \) be regular. Assume the sequence \( \{f_n : n = 1, \ldots\} \subseteq L^1(X) \) has the property that

\[
\forall A \in \mathcal{A}, \quad \lim_{n \to \infty} \int_A f_n \, d\mu
\]

exists and is finite. Then

\[
\forall \varepsilon > 0, \exists K \subseteq X, \text{compact, such that } \forall n = 1, \ldots, \int_{K^c} |f_n| \, d\mu < \varepsilon.
\]

**Proof.** As in the Remark following Proposition 6.3.3 we may assume without loss of generality that the limits \( \lim_{n \to \infty} \int_A f_n \, d\mu \) are 0.

i. We start with the observation that, for every \( f \in L^1(X) \), there exists a countable collection \( \{A_n : n = 1, \ldots\} \subseteq \mathcal{A} \) of measurable sets with each \( \mu(A_n) < \infty \), such that

\[
\forall x \in \left( \bigcup_{n=1}^{\infty} A_n \right)^c, \quad f(x) = 0.
\]

Thus, there exist a measurable set \( B \in \mathcal{A} \) such that \( B = \bigcup_{n=1}^{\infty} B_n \), where \( \mu(B_n) < \infty \), for all \( n = 1, \ldots, \) and where

\[
\forall x \in B^c \text{ and } \forall n = 1, \ldots, \quad f_n(x) = 0.
\]

Since the measure \( \mu \) is regular, we can write \( B = N \cup (\bigcup K_j) \), where each \( K_j, j = 1, \ldots, \) is compact, \( K_j \subseteq K_{j+1} \), and \( \mu(N) = 0 \).

Our goal is to show that

\[
\forall \varepsilon > 0, \exists j \in \mathbb{N}, \text{ such that } \forall n = 1, \ldots, \int_{K_j^c} |f_n| \, d\mu < \varepsilon.
\]

ii. Let \( H \subseteq L^1(X) \) be the set consisting of characteristic functions of measurable subsets of \( B \). For each \( j = 1, \ldots, \), set
\( \rho_j(g_1, g_2) = \int_{K_j} |g_1 - g_2| \, d\mu \)

for any two elements \( g_1, g_2 \in H \). We note that if \( \rho_j(g_1, g_2) = 0 \) for every \( j = 1, \ldots \), then \( g_1 - g_2 = 0 \) \( \mu \)-a.e. in \( B \) and therefore in \( X \). Thus, we may define a metrizable uniform structure on \( H \), see Definition 10.1.8.

Let us begin by showing that this metrizable uniform space \( H \) is complete. First, if \( \{g_n : n = 1, \ldots \} \) is a Cauchy sequence in \( H \), then it is a Cauchy sequence for each metric space \((H, \rho_j), j = 1, \ldots \). This, in turn, implies that, for each \( j = 1, \ldots \), \( \{g_n \mathbb{1}_{K_j} : n = 1, \ldots \} \) is a Cauchy sequence in \( L^1_\mu(X) \), and so it has a limit \( g^j \in L^1_\mu(X) \).

Second, from the sequence \( \{g_n \mathbb{1}_{K_1} : n = 1, \ldots \} \), we can choose a subsequence \( \{g_{m_n} \mathbb{1}_{K_1} : n = 1, \ldots \} \) which converges pointwise \( \mu \)-a.e. to \( g^1 \). In particular, we observe that \( g^1 \) equals 0 or 1 \( \mu \)-a.e. Next, the sequence \( \{g_{m_n} \mathbb{1}_{K_2} : n = 1, \ldots \} \) converges in norm to \( g^2 \), and so we can choose a subsequence that converges pointwise \( \mu \)-a.e. to \( g^2 \), and \( g^2 \) is also a characteristic function of a measurable set in \( B \).

Finally, continuing this diagonal process, we find a subsequence of \( \{g_n : n = 1, \ldots \} \subseteq H \) which converges pointwise \( \mu \)-a.e. to \( g^j \) on each \( K_j, j = 1, \ldots \). If we let

\[
g(x) = \begin{cases} g^j(x), & x \in K_j, \\ 0, & x \in \left( \bigcup_{j=1}^\infty K_j \right)^c, \end{cases}
\]

then \( g \in H \) and

\[
\forall j = 1, \ldots, \lim_{n \to \infty} \rho_j(g, g_n) = 0.
\]

This proves that \( g \) is the limit of \( \{g_n : n = 1, \ldots \} \) on \( B \) in the topology of the uniform structure, and we have proved \( H \) is complete.

iii. Let \( f \in L^1_\mu(X) \) and suppose \( f = 0 \) on \( B^- \). The linear mapping

\[
T_f : H \to \mathbb{C} \\
g \mapsto \int_X g f \, d\mu
\]

is continuous. In fact, for each \( \varepsilon > 0 \) there exists \( j \in \mathbb{N} \) such that

\[
\int_{K_j^c} |f| \, d\mu < \varepsilon; \tag{6.46}
\]

and there exists \( n \in \mathbb{N} \) such that if \( F_n = \{x : |f(x)| \leq n\} \subseteq X \) then

\[
\int_{F_n^c} |f| \, d\mu < \varepsilon. \tag{6.47}
\]

If \( g_1, g_2 \in H \) one has, by (6.46), that
Weak convergence of measures

\[ \left| \int_{K_J} (g_1 - g_2) f \, d\mu \right| < 2\varepsilon, \]

and, by (6.47), that

\[ \left| \int_{F_n} (g_1 - g_2) f \, d\mu \right| < 2\varepsilon. \]

Therefore, we can conclude that

\[ \left| \int_B (g_1 - g_2) f \, d\mu \right| \leq \left| \int_{K_J \cap F_n} (g_1 - g_2) f \, d\mu \right| + \left| \int_{K_J \cap F_n^c} (g_1 - g_2) f \, d\mu \right| + \left| \int_{K_J^c} (g_1 - g_2) f \, d\mu \right| \]

\[ \leq n \int_{K_J} \left| g_1 - g_2 \right| d\mu + \int_{K_J} (g_1 - g_2) f \, d\mu \]

\[ < n \rho_p(g_1, g_2) + 2\varepsilon + 2\varepsilon = n \rho_p(g_1, g_2) + 4\varepsilon; \]

and so the continuity of \( T_f \) is established thanks to the property of convergence in the uniform structure.

iv. We shall apply Baire’s theorem (Theorem 10.6.1 and Theorem 10.6.2) to the complete metric space \( H \) to establish the following fact. For every \( \varepsilon > 0 \), there exist \( m, n_0 \in \mathbb{N} \), a measurable set \( A_0 \subseteq B \), and \( \delta > 0 \) such that, for each measurable set \( A \in A \) satisfying

\[ \int_{K_m} |\mathbb{1}_A - \mathbb{1}_{A_0}| \, d\mu < \delta, \]  

we have

\[ \forall n \geq n_0, \quad \left| \int_A f_n \, d\mu \right| < \varepsilon. \]  

Indeed, let

\[ G_n = \left\{ g \in H : \forall m \geq n, \left| \int_X g f_m \, d\mu \right| \leq \varepsilon \right\} \subseteq H, \]

for \( n = 1, \ldots \). Then, \( H = \bigcup_{n=1}^{\infty} G_n \), where \( G_n \subseteq G_{n+1} \) for each \( n = 1, \ldots \), since we are assuming \( \lim_{n \to \infty} \int_A f_n \, d\mu = 0 \). Also, each \( G_n \) is closed in \( H \) by the continuity of the \( T_{f_n} \)’s and the fact that the intersection of closed sets is closed. We use these properties to apply Baire’s theorem. Consequently, there exists \( n_0 \in \mathbb{N} \) such that \( G_{n_0} \) contains a non-empty open set \( U \subseteq G_{n_0} \). Thus, \( G_{n_0} \) contains an open ball of the form defined in (6.48). Therefore, by the definition of \( G_{n_0} \), the elements of this ball satisfy (6.49).

v. If we now take \( A = A_0 \cap K_m \), we have
\[ \int_{K_m} |1_{A_0 \cap K_m} - 1_{A_0^c}| \, d\mu = 0. \]

Therefore, (6.48) holds for \( A = A_0 \cap K_m \), and, hence,

\[ \forall n \geq n_0, \quad \left| \int_{A_0 \cap K_m} f_n \, d\mu \right| < \varepsilon. \]

Let \( M \subseteq K_m^\sim \) be any measurable set. If \( B = M \cup (A_0 \cap K_m) \), one has

\[ \int_{K_m} |1_B - 1_{A_0^c}| \, d\mu = 0 \]

which again implies that

\[ \forall n \geq n_0, \quad \left| \int_B f_n \, d\mu \right| < \varepsilon, \]

and so

\[ \forall n \geq n_0, \quad \left| \int_M f_n \, d\mu \right| < 2\varepsilon. \quad (6.50) \]

Without loss of generality, assume each \( f_n \) is real-valued. Let \( M^+_{m,n} = \{ x \in K_m^\sim : f_n(x) \geq 0 \} \) and \( M^-_{m,n} = \{ x \in K_m^\sim : f_n(x) < 0 \} \). Since \( K_m^\sim = M^+_{m,n} \cup M^-_{m,n} \), we conclude from (6.50) that

\[ \forall n \geq n_0, \quad \int_{K_m^\sim} |f_n| \, d\mu < 4\varepsilon. \]

On the other hand, for each \( n < n_0 \), there exists \( m_n \) such that

\[ \int_{K_m^\sim} |f_n| \, d\mu < \varepsilon, \]

by the integrability of \( f_n \) and the facts that \( \{ K_j \} \) is increasing and \( B = N \cup (\bigcup K_j) \). Thus, for \( j = \max(m, m_1, \ldots, m_{n_0}-1) \),

\[ \forall \varepsilon > 0, \exists j \in \mathbb{N}, \text{ such that } \forall n = 1, \ldots, \int_{K_j^\sim} |f_n| \, d\mu < \varepsilon. \]

\[ \square \]

Using the theorems in Section 6.1 and 6.2 it is now not difficult to prove Theorem 6.3.7, taken from [80], page 89. To this end we need the following result.

**Proposition 6.3.6.** Let \( X \) be a locally compact Hausdorff space, let \((X, \mathcal{A}, \mu)\) be a measure space, and let \( \mu \) be continuous and regular. Then, for all \( 0 \leq t < \mu(X) \), there exists \( A \in \mathcal{A} \) such that

\[ \mu(A) = t. \]
Proof. First, by the regularity of \( \mu \), we may assume without loss of generality that \( X \) is compact.

Let \( 0 \leq t < \mu(X) \). We need to construct a decreasing sequence of measurable sets \( A_n \subseteq X \), \( n = 1, \ldots \), with the property that

\[
\forall n = 1, \ldots, \quad t \leq \mu(A_n) < t + \frac{\mu(X)}{2^n}.
\]

We show that this is possible using induction. Assume that \( A_1, \ldots, A_n \) have been constructed and consider a finite open covering of \( X \) by sets \( M_j \), \( j = 1, \ldots, N \), where each \( \mu(M_j) \leq \mu(X)/2^{n+1} \). This is accomplished by the hypotheses that \( X \) is compact and \( \mu \) is continuous. By dividing these open sets appropriately, we may assume that \( M_1, \ldots, M_N \) is a finite disjoint covering (no longer an open covering) of \( X \) and that each \( \mu(M_j) \leq \mu(X)/2^{n+1} \).

Define \( B_{n,j} = A_n \cap M_j \), \( j = 1, \ldots, N \). We can choose a subfamily \( B_{n,j_k} \), \( k = 1, \ldots, K \), for which

\[
t \leq \sum_{k=1}^{K} \mu(B_{n,j_k}) \leq t + \frac{\mu(X)}{2^{n+1}}.
\]

Take \( A_{n+1} = \bigcup_{k=1}^{K} B_{n,j_k} \) and set \( A = \bigcap_{n=1}^{\infty} A_n \).

\[\square\]

Theorem 6.3.7. Characterization of weak sequential convergence of functions

Let \( X \) be a locally compact Hausdorff space, let \( (X, \mathcal{A}, \mu) \) be a measure space, and let \( \mu \) be regular. A sequence \( \{f_n : n = 1, \ldots \} \subseteq L^1_\mu(X) \) converges weakly to some \( f \in L^1_\mu(X) \) if and only if

\[
\forall A \in \mathcal{A}, \quad \lim_{n \to \infty} \int_A f_n \, d\mu
\]

exists and is finite, cf., Theorem 6.3.2.

Proof. \((\Rightarrow)\) For \( A \in \mathcal{A} \) let \( g = \mathbb{1}_A \in L^\infty_\mu(X) \). Thus

\[
\lim_{n \to \infty} \int_A f_n \, d\mu = \lim_{n \to \infty} \int_X f_n g \, d\mu = \int_X f g \, d\mu = \int_A f \, d\mu.
\]

\((\Leftarrow)\) According to Theorem 6.3.4, it is enough to show that

\[
\sup_{n \in \mathbb{N}} \|f_n\|_1 < \infty.
\]

In view of Proposition 6.3.5, there exists a compact set \( K \subseteq X \) such that

\[
\forall n = 1, \ldots, \quad \int_K |f_n| \, d\mu < 1. \tag{6.51}
\]
Since \( \mu \) is regular, in particular \( \mu(K) < \infty \). Thus, there exists at most a countable collection of points \( D = \{ a_j : j = 1, \ldots \} \subseteq K \) with the property that \( \mu(\{a_j\}) > 0 \) for all \( j = 1, \ldots \). Moreover,

\[
\forall \, \delta > 0, \exists D_\delta \subseteq D, \text{ a finite set, for which } \mu(D \setminus D_\delta) \leq \delta.
\]  

(6.52)

On the other hand, for any \( \delta > 0 \) we can partition the set \( K \setminus D \) into a finite collection of measurable subsets \( K_i, i = 1, \ldots, m \), such that

\[
\forall \, i = 1, \ldots, m, \quad \mu(K_i) \leq \delta.
\]  

(6.53)

This is a consequence of Proposition 6.3.6 in the following way. Choose \( K_1 \in \mathcal{A} \) such that \( K_1 \subseteq K \setminus D \) and \( \mu(K_1) = \delta \). If \( \mu((K \setminus D) \setminus K_1) \leq \delta \) then let \( K_2 = (K \setminus D) \setminus K_1 \) and we are done. If \( \mu((K \setminus D) \setminus K_1) > \delta \) then choose a \( K_2 \) analogous to our choice of \( K_1 \). The procedure is complete in a finite number of steps since \( \mu(K \setminus D) < \infty \).

From Proposition 6.3.3 and for \( \varepsilon = 1 \), there is \( \delta_1 > 0 \) such that

\[
\forall \, n = 1, \ldots, \quad \mu(A) \leq \delta_1 \implies \int_A |f_n| \, d\mu \leq 1.
\]  

(6.54)

Let \( \delta = \delta_1 \) in (6.52) and (6.53). Thus, taking \( A = K_i \) and \( A = D \setminus D_{\delta_1} \) in (6.54) we have

\[
\forall \, i = 1, \ldots, m \text{ and } \forall \, n = 1, \ldots, \quad \int_{K_i} |f_n| \, d\mu, \int_{D \setminus D_{\delta_1}} |f_n| \, d\mu \leq 1.
\]  

(6.55)

Finally, we observe, by hypothesis, that \( \lim_{n \to \infty} f_n(a_j) \) exists and is finite for each \( a_j \in D_{\delta_1} \). Since there are only finitely many such points,

\[
\exists \, N > 0 \text{ such that } \forall \, n = 1, \ldots, \quad \int_{D_{\delta_1}} |f_n| \, d\mu \leq N.
\]  

(6.56)

Combining (6.51), (6.55), and (6.56) yield that

\[
\exists \, M < \infty \text{ such that } \forall \, n = 1, \ldots, \quad \int_X |f_n| \, d\mu \leq M.
\]

\( \square \)

**Remark.** Since \( 21_A = 1_X + (1_A - 1_{A^-}) \) it is clear that the assumption that the limits,

\[
\forall \, A \in \mathcal{A}, \quad \lim_{n \to \infty} \int_A f_n \, d\mu,
\]

exist and are finite is equivalent to the assumption that the limits,

\[
\forall \, A \in \mathcal{A}, \quad \lim_{n \to \infty} \left( \int_A f_n \, d\mu - \int_{A^-} f_n \, d\mu \right)
\]
exist and are finite. This observation is important in view of the result of John Rainwater (nom de plume) [232], that a sequence \( \{x_n : n = 1, \ldots\} \subseteq B \) of elements of a Banach space converges weakly to an element \( x \in B \) if and only if \( \lim_{n \to \infty} f(x_n) = f(x) \), for each extremal point \( f \) on the unit sphere in the dual Banach space \( B' \). Extremal points of a convex set are the points that are not interior points of any line segment contained in this set.

The following theorem is a corollary of Theorem 6.3.4.

**Theorem 6.3.8. Dunford–Pettis theorem**

Let \( X \) be a locally compact Hausdorff space, let \((X, \mathcal{A}, \mu)\) be a measure space, and let \( \mu \) be regular. A sequence \( \{f_n : n = 1, \ldots\} \subseteq L^1_\mu(X) \) converges weakly to some \( f \in L^1_\mu(X) \) if and only if

\[
\forall \varepsilon > 0, \exists \delta > 0, \text{ such that } \mu(A) < \delta \implies \forall n = 1, \ldots, \int_A |f_n| \, d\mu < \varepsilon,
\]

\[
\forall \varepsilon > 0, \exists K \subseteq X, \text{ compact, such that } \forall n = 1, \ldots, \int_{K^c} |f_n| \, d\mu < \varepsilon,
\]

\[
\forall U \subseteq X, \text{ open, } \lim_{n \to \infty} \int_U f_n \, d\mu \text{ exists and is finite,}
\]

\[
\sup_{n \in \mathbb{N}} \|f_n\|_1 = K < \infty.
\]

**Proof.** (\( \implies \)) This direction is a consequence of Theorem 6.3.7, Proposition 6.3.3, Proposition 6.3.5, and Theorem 6.3.2.

(\( \Leftarrow \)) We shall invoke Theorem 6.3.4 to prove weak sequential convergence. Let \( A \in \mathcal{A} \). It is sufficient to prove that \( \lim_{n \to \infty} \int_A f_n \, d\mu \) exists and is finite. Let \( \varepsilon > 0 \). By hypothesis there is \( K \), compact, such that

\[
\int_{K^c} |f_n| \, d\mu < \varepsilon, \quad \text{uniformly in } n \in \mathbb{N}.
\]  

(6.57)

By regularity, for each \( \delta > 0 \) there is open \( U_\delta \subseteq K \) such that \( A \cap K \subseteq U_\delta \) and \( \mu(U_\delta \setminus (A \cap K)) < \delta \). Choose \( \delta \) from the first of the necessary conditions to obtain

\[
\forall n = 1, \ldots, \int_{U_\delta \setminus (A \cap K)} |f_n| \, d\mu < \varepsilon.
\]  

(6.58)

By hypothesis, \( \lim_{n \to \infty} \int_{U_\delta} f_n \, d\mu \) exists and is finite. This fact, combined with (6.57), (6.76), and a straightforward interchanging of limits argument, yields the desired result.
In fact, the fundamental result of Dunford and Pettis [87], pages 376–378, is their characterization of weakly compact subsets of $L^1_\mu(X)$ for a regular measure space $(X, \mathcal{A}, \mu)$. A version of their theorem, due to Dieudonné [80], pages 93–94, is the following: $H \subseteq L^1_\mu(X)$ is relatively weakly compact if and only if

i. $H \subseteq L^1_\mu(X)$ is bounded;

ii. $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$\forall A \in \mathcal{A}, \mu(A) < \delta \implies \forall f \in H, \int_A |f| \, d\mu < \varepsilon;$$

iii. $\forall \varepsilon > 0, \exists Y \subseteq X$ compact, such that

$$\forall f \in H, \int_Y |f| \, d\mu < \varepsilon.$$

### 6.4 Dieudonné-Grothendieck theorem

The Dieudonné–Grothendieck theorem, Theorem 6.4.2, characterizes the weak convergence of a sequence $\{\mu_n : n = 1, \ldots\} \subseteq M_b(X)$, where $X$ is a locally compact space. We require locally compact Hausdorff spaces because of their property that if $K_1$ and $K_2$ are disjoint compact sets, then there are disjoint open sets $V_1$ and $V_2$ such that $K_1 \subseteq V_1$ and $K_2 \subseteq V_2$.

By our convention (e.g., Section 5.1 and Section 7.2), $M_b(X)$ consists of the bounded regular Borel measures on $X$. An analogous characterization of weak sequential convergence was established in Section 6.3 for sequences $\{f_n : n = 1, \ldots\} \subseteq L^1_\mu(X)$. By definition of the weak topology in Section 10.9 and the fact that $L^1_\mu(X)' \subseteq L^\infty_\mu(X)$, the proof of this latter characterization might seem relatively concrete, compared with the complexity of $M_b(X)'$.

However, employing the technique that Saks used to prove Nikodym’s theorem (see Section 6.2), we can transfer weak convergence of $\{\mu_n\}$ to that of a sequence of functions. As such we rewrite Theorem 6.3.7 in the following way.

**Theorem 6.4.1. Characterization of weak sequential convergence of measures**

*Let $X$ be a locally compact Hausdorff space and let $(X, \mathcal{A})$ be a measurable space. A sequence $\{\mu_n : n = 1, \ldots\} \subseteq M_b(X)$ converges weakly to some $\mu \in M_b(X)$ if and only if*

$$\forall A \in \mathcal{A}, \lim_{n \to \infty} \mu_n(A)$$

*exists and is finite."

**Proof.** We begin as suggested in the Remark after the proof of Theorem 6.2.3, i.e., for a given sequence of measures $\{\mu_n : n = 1, \ldots\} \subseteq M_b(X)$, we define
\[ \forall A \in \mathcal{A}, \quad \nu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n||\mu_n||_1} |\mu_n|(A), \]

and we note that \( \mu_n \ll \nu \) for all \( n \). Thus we may use \( R^N \) to find a sequence of functions \( h_n \in L^1_b(X) \), \( n = 1, \ldots \), such that \( \mu_n = h_n \nu \). This last equality can be used to define a linear injective isometric isomorphism \( i : L^1_b(X) \to M_b(X) \), cf., Theorem 5.3.2.

We now assert that the weak topology on \( L^1_b(X) \) is equivalent to the topology on \( L^1_b(X) \) induced by the weak topology on \( M_b(X) \) by means of the mapping \( i \). The space \( i(L^1_b(X)) \), in both induced weak topologies, from \( M_b(X) \) and \( L^1_b(X) \), has topological bases of closed convex sets. Theorem 10.9.4 implies that these topologies on \( i(L^1_b(X)) \) are identical. This gives our assertion by the injectivity of \( i \).

Consequently, \( \mu_n \to \mu \) weakly in \( M_b(X) \) if and only if \( \{h_n\} \) converges weakly in \( L^1_b(X) \). Our result now follows by a direct application of Theorem 6.3.7.

\[ \]

The transference technique mentioned before Theorem 6.4.1 does not imply that Theorem 6.4.2 is not a significant strengthening of Theorem 6.4.1 - on the contrary. Theorem 6.4.2 is due to Dieudonné [79], pages 35–36, and Grothendieck [124], pages 146–150, cf., [80].

**Theorem 6.4.2. Dieudonné–Grothendieck theorem**

*Let \( X \) be a locally compact Hausdorff space and let \( \{\mu_n : n = 1, \ldots\} \subseteq M_b(X) \) be a bounded sequence. Then, \( \{\mu_n : n = 1, \ldots\} \) converges weakly to some \( \mu \in M_b(X) \) if and only if

\[ \forall U \subseteq X, \quad \text{open}, \quad \lim_{n \to \infty} \mu_n(U) \]

exists and is finite.*

**Proof.** It is clear we only need to show the sufficiency of the condition for weak convergence. We shall assume without loss of generality that the measures \( \mu_n \) are real-valued.

\( i. \) Let \( \{A_n : n = 1, \ldots\} \subseteq \mathcal{A} \) be a disjoint sequence of open subsets of \( X \) and let \( J \subseteq \mathbb{N} \) be a set of positive integers. Observe that, by our assumption,

\[ \sum_{j \in J} \mu_n(A_j) = \mu_n \left( \bigcup_{j \in J} A_j \right) \]

converges to a finite limit as \( n \to \infty \). We also note that for each \( n \), \( \{\mu_n(A_k) : k = 1, \ldots\} \in l^1(\mathbb{N}) \), and that

\[ \int J \mu_n(A_j) \, dc(j) = \sum_{j \in J} \mu_n(A_j). \]
Thus, in particular, we may use Theorem 6.3.7 for \((\mathbb{N}, \mathcal{P}(\mathbb{N}), c)\) to deduce that for each \(n\), \(\{\mu_n : n = 1, \ldots\}\) is weakly convergent in \(\ell^1(\mathbb{N})\).

Similarly, using Proposition 6.3.5, we conclude that
\[
\forall \varepsilon > 0, \exists k \in \mathbb{N}, \text{ such that } \forall n = 1, \ldots, \sum_{j > k} |\mu_n(A_j)| \leq \varepsilon;
\]
and so, for all \(j > k\), we have
\[
\forall n = 1, \ldots, |\mu_n(A_j)| \leq \varepsilon.
\]
Thus, for each disjoint sequence \(\{A_j : j = 1, \ldots\}\) of open sets in \(X\),
\[
\lim_{j \to \infty} \mu_n(A_j) = 0, \text{ uniformly for } n \in \mathbb{N}. \quad (6.59)
\]

ii. We shall now prove that the condition \((6.59)\) implies that
\[
\forall \varepsilon > 0, \forall K \subseteq X, \text{ compact, } \exists U \text{ open, } K \subseteq U, \text{ such that, } \\
\forall n = 1, \ldots, |\mu_n|(U \setminus K) < \varepsilon. \quad (6.60)
\]
Indeed, assume that \((6.60)\) is not true. Then
\[
\exists \varepsilon > 0 \text{ and } \exists K \subseteq X, \text{ compact, such that } \exists n \text{ for which } |\mu_n|(U \setminus K) \geq \varepsilon. \quad (6.61)
\]
For this \(\varepsilon > 0\) and compact set \(K \subseteq X\), we shall prove that there are a decreasing sequence \(\{A_m : m = 1, \ldots\}\) of open sets containing \(K\), a subsequence \(\{\mu_{n_m} : m = 1, \ldots\}\) of \(\mu_n\) whose closures are compact (i.e., relatively compact sets), such that
\[
\forall m = 1, \ldots, \overline{B_{m+1}} \subseteq A_{m+1} \setminus A_m, \text{ and } |\mu_{n_m}(B_{m+1})| > \varepsilon/2. \quad (6.62)
\]
To this end, assume the construction for \((6.62)\) is done up to step \(m\). By our assumption \((6.61)\), there exists \(\mu_{n_{m+1}}\) such that \(|\mu_{n_{m+1}}|(A_m \setminus K) > \varepsilon/2\). Thus, there is a compact set \(F \subseteq A_m \setminus K\) such that \(|\mu_{n_{m+1}}|(F) > \varepsilon/2\) by the regularity of \(\mu_{n_{m+1}}\). This, in turn, implies that there exists an open, relatively compact set \(V\) such that \(F \subseteq V \subseteq \overline{V} \subseteq A_m \setminus K\) and \(|\mu_{n_{m+1}}|(V) > \varepsilon/2\), since \(|\mu_{n_{m+1}}|\) is a positive measure. Because \(\mu_{n_{m+1}}\) is real-valued, there exists \(A \in A, A \subseteq V, \text{ such that } |\mu_{n_{m+1}}(A)| > \varepsilon/2\), see Problem 5.22. This inequality and the regularity of \(\mu_{n_{m+1}}\) imply that there exists an open relatively compact set \(B_{m+1}\) such that
\[
A \subseteq B_{m+1} \subseteq V \text{ and } |\mu_{n_{m+1}}(B_{m+1})| > \varepsilon/2. \quad (6.63)
\]
To verify \((6.63)\), first note that \(\mu_{n_{m+1}} = \mu_{n_{m+1}}^+ - \mu_{n_{m+1}}^-\), where \(\mu_{n_{m+1}}^+, \mu_{n_{m+1}}^-\) are regular positive measures, see Problem 5.21. Let \(B_{m+1}\) be open relatively compact sets for which \(A \subseteq B_{m+1}\) and for which
Then set $B_{m+1} = B^+ \cap B^-$ to obtain (6.63). Finally, we take $A_{m+1}$ to be any open neighborhood of $K$, contained in $A_m$ and disjoint with $B_{m+1}$. (6.62) is obtained, see Figure ??.

It is now elementary to see that this construction for (6.62) contradicts (6.59).

iii. Analogous to part ii, condition (6.59) also implies that
\[
\forall \varepsilon > 0, \exists K \subseteq X, \text{compact, such that } \forall n = 1, \ldots, |\mu_n|(K^c) < \varepsilon. \tag{6.64}
\]

To verify (6.64) assume it is not true. Then
\[
\exists \varepsilon > 0 \text{ such that } \forall K \subseteq X, \text{compact, } \exists n \text{ such that } |\mu_n|(K^c) > \varepsilon. \tag{6.65}
\]

For any such $K_1$ in (6.65) choose $n_1$ for which $|\mu_{n_1}|(K_1^c) \geq \varepsilon$. In part ii, we showed that if $A \in \mathcal{A}$ and $|\mu|(A) > \varepsilon$, then there is an open set $V \supseteq A$ such that $|\mu(V)| > \varepsilon$. Symmetrically, in this part iii, we use the regularity of $\mu_{n_1}$ to assert the existence of a compact set $K_2 \subseteq K_1^c$ such that $|\mu_{n_1}(K_2)| > \varepsilon/2$. Then, as in part ii, we proceed by induction to obtain a contradiction of (6.59).

iv. As with the argument to prove Theorem 6.4.1, there exists a measure $\nu$ such that, for all $n \in \mathbb{N}$, $\mu_n \ll \nu$; and the question of weak convergence of $\{\mu_n\}$ in $M_b(X)$ can be reduced to the question of weak convergence of a sequence $\{h_n : n = 1, \ldots\}$ of functions in $L_1(X)$, where each $\mu_n = h_n\nu$.

According to Theorem 6.3.8, to prove that $\{\mu_n : n = 1, \ldots\}$ is weakly convergent it is enough to show that (6.60) and (6.64) imply
\[
\forall \varepsilon > 0, \exists \delta > 0, \text{ such that } \mu(A) \leq \delta \Longrightarrow \forall n = 1, \ldots, \int_A |h_n| \, d\nu \leq \varepsilon \tag{6.66}
\]
and
\[
\forall \varepsilon > 0, \exists K \subseteq X, \text{compact, such that } \forall n = 1, \ldots, \int_{K^c} |h_n| \, d\nu \leq \varepsilon. \tag{6.67}
\]

Since $\nu$ is non-negative, according to Theorem 5.3.4b, (6.64) is equivalent to (6.67).

To prove (6.66), we first observe that, because of the regularity of $\nu$, it is sufficient to prove it for open sets.

To this end, we assume that (6.66) does not hold for open sets, and we shall obtain a contradiction to (6.60), as written in (6.70). Thus, there exist $\varepsilon > 0$ and a sequence of open sets $A_n$ such that $\mu(A_n) < \varepsilon$ and
\[
\int_{A_n} |h_n| \, d\nu > \varepsilon.
\]
Let $B_n = \bigcup_{m=n}^{\infty} A_m$, so that

$$\int_{B_n} |h_n| \, d\nu > \varepsilon. \quad (6.68)$$

Assumption (6.64) allows us to assume without loss of generality that $X$ is compact. Thus, (6.60) implies that for each open set $B$ and for each $\varepsilon > 0$ there exists a compact set $F \subseteq B$ such that

$$\forall n = 1, \ldots, \int_{B \setminus F} |h_n| \, d\nu < \varepsilon.$$

In fact, let $B = K^-$ and $F = U^-$ in (6.60). Therefore we can find a sequence of compact sets $F_n \subseteq B_n$ with the property that

$$\forall m = 1, \ldots, \int_{B_n \setminus F_n} |h_m| \, d\nu < \frac{\varepsilon}{2^{n+1}}.$$

Let $L_n = \bigcap_{m=1}^{n} F_m$. (6.68) implies that

$$\int_{L_n} |h_n| \, d\nu = \int_{B_n} |h_n| \, d\nu - \int_{B_n \setminus L_n} |h_n| \, d\nu > \varepsilon - \int_{B_n \setminus L_n} |h_n| \, d\nu.$$

Moreover, because $B_n \setminus L_n = \bigcup_{m=1}^{n} (B_n \setminus F_m)$, we have

$$\int_{B_n \setminus L_n} |h_n| \, d\nu \leq \sum_{m=1}^{n} \int_{B_n \setminus F_m} |h_n| \, d\nu \leq \sum_{m=1}^{n} \frac{\varepsilon}{2^{m+1}} < \frac{\varepsilon}{2}.$$

The last two inequalities imply that

$$\int_{L_n} |h_n| \, d\nu > \frac{\varepsilon}{2}. \quad (6.69)$$

Let $F = \bigcap_{n=1}^{\infty} F_n$. (6.60) implies that there exists an open set $U \supseteq F$ such that

$$\forall n = 1, \ldots, \int_{U \setminus F} |h_n| \, d\nu < \frac{\varepsilon}{2}. \quad (6.670)$$

Recall that the sets $B_n$ form a decreasing sequence with $\nu$-measure approaching 0. Since $F_n \subseteq B_n$, we conclude that $\nu(F) = 0$, and so

$$\forall n = 1, \ldots, \int_{U} |h_n| \, d\nu < \frac{\varepsilon}{2}. \quad (6.70)$$

On the other hand, because the $L_n$s are compact and $\{L_n\}$ is decreasing, and since $F = \bigcap_{n=1}^{\infty} L_n$, if $U$ contains $F$, then there exists $n_0$ such that $L_{n_0} \subseteq U$. This implies that

$$\int_{U} |h_{n_0}| \, d\nu \geq \int_{L_{n_0}} |h_{n_0}| \, d\nu > \frac{\varepsilon}{2},$$

which contradicts (6.70).

$\square$
The following example should caution us from reading too much into Theorem 6.4.2.

**Example 6.4.3. Convergence for open intervals but not all open sets**

Let \( f_n : [0, 1] \to \mathbb{R}^+ \), \( n = 1, \ldots \), have the form

\[
 f_n(x) = \sum_{k=0}^{2^n-1} \Delta_k(x),
\]

where \( \Delta_k \) is an “isosceles triangle function” having its base of length \( 1/2^{2n+1} \) centered in \( [k/2^n, (k+1)/2^n] \), \( \Delta_j \) is congruent to \( \Delta_k \), and \( \int_0^1 f_n = 1 \). It is easy to see that for all \((a, b) \subseteq [0, 1] \),

\[
 \lim_{n \to \infty} \int_a^b f_n \, dx = \int_a^b \, dx.
\]

In light of Theorem 6.3.7 we would like to conclude that the sequence \( \{ f_n : n = 1, \ldots \} \) converges weakly to 1 (cf., Theorem 6.4.2 noting that, after all, open sets in \([0, 1] \) are just countable unions of open intervals). Such is not the case. Let \( A = \{ x : \forall n, f_n(x) = 0 \} \). Since \( m(\{ x : f_n(x) > 0 \}) = 1/2^{n+1} \), then \( m(\{ x : \exists n, \text{ for which } f_n(x) > 0 \}) \leq 1/2 \); and thus \( m(A) \geq 1/2 \). Consequently,

\[
 \lim_{n \to \infty} \int_A f_n \, dx \neq \int_A \, dx \geq \frac{1}{2}
\]

because \( \int_A f_n = 0 \).

An interesting problem with applications to harmonic analysis (e.g., [21], Chapters 2 and 7, and [47]) is to find other families of sets, besides the open sets, for which Theorem 6.4.2 is true. The best results for compact Hausdorff spaces are due to Wells [313], Pfanzagl (Manuscripta Math. 4 (1971) 91–98) has proved the analogue of Theorem 6.4.2 for the case of regular complex measures on any Hausdorff space, cf., Problem 2.19.

Because of Theorem 6.4.2 and the Riesz Representation Theorem (Chapter 7), it is interesting to investigate the relation between weak and weak * convergence in \( M_b(X) \). Grothendieck proved: if \( X \) is compact and the closure of every open set is open then any weak * convergent sequence in \( M_b(X) \) is weak convergent, cf., Theorem 10.9.6. This has been generalized by Seevers and Shaefer [263].

**Example 6.4.4. \( B([0, 1]) \)**

Consider the measurable space \( ([0, 1], B([0, 1])) \). Choose a sequence \( \{ \mu_n : n = 1, \ldots \} \subseteq M_b([0, 1]) \) such that \( \mu_n \to \mu \) in the weak * topology, card supp \( \mu_n < \infty \), and supp \( \mu_n \subseteq \mathbb{Q} \). For example we could take \( \mu_n = \sum_{k=1}^{n} (1/n) \delta_{k/n} \). Let \( A \subseteq [0, 1] \) be a closed set of irrationals with positive Lebesgue measure, e.g.,
Problem 1.4a. If $f = 1_A$ then $\int f \, d\mu_n = 0$ and $\int f = m(A)$. Obviously, $f \in (M_b([0,1]))'$. Consequently, weak * convergence does not entail weak convergence in $M_b([0,1])$. Also, we have proved that, generally, the following statement is false: let $X$ be a compact space, let $A \subseteq X$ be closed, and assume that for $\mu_n, \mu \in M_b(X)$, $\mu_n \rightharpoonup^* \mu$ in the weak * topology; then

$$\forall f \in C(X), \quad \int_A f \, d\mu_n \to \int_A f \, d\mu.$$ 

6.5 Norm and weak sequential convergence

In terms of weak convergence some of the results of Sections 6.1 and 6.2 can be rephrased as follows.

**Theorem 6.5.1. Norm and weak sequential convergence**

Let $(X, \mathcal{A}, \mu)$ be a measure space and assume that the sequence $\{f_n : n = 1, \ldots\} \subseteq L^1(X)$ converges weakly to $f \in L^1(X)$. Then, (6.1) is valid if and only if $f_n \rightharpoonup^* f$ in measure on each $A \in \mathcal{A}$ satisfying $\mu(A) < \infty$, cf., Example 3.3.15 and Example 3.6.5.

**Proof.** $(\Longrightarrow)$ This is immediate from Theorem 6.1.5.

$(\Longleftarrow)$ The characterization of weak sequential convergence in Theorem 6.3.2 yields that the assumptions of Theorem 6.2.2 are satisfied. Theorem 6.2.2 implies, in turn, that condition ii in Theorem 6.1.5 holds. The result follows now from Theorem 6.1.5 by noting that in its “if” part one only needs that $\{f_n : n = 1, \ldots\}$ converges in measure on sets of finite measure.

**Proposition 6.5.2.** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\{f_n, f : n = 1, \ldots\} \subseteq L^1(X)$. Assume $f_n \rightharpoonup^* f$ $\mu$-a.e. and $\|f_n\|_1 \to \|f\|_1$. Then,

$$\forall A \in \mathcal{A}, \quad \lim_{n \to \infty} \int_A |f_n| \, d\mu = \int_A |f| \, d\mu.$$ 

**Proof.** Take $A \in \mathcal{A}$. From Fatou’s lemma,

$$\limsup_{n \to \infty} \int_A |f_n| \, d\mu \geq \int_A |f| \, d\mu \geq \int_X |f| \, d\mu - \liminf_{n \to \infty} \int_{A^c} |f_n| \, d\mu.$$  

(6.71)

Since $\|f_n\|_1 \to \|f\|_1$ we have

$$\int_X |f| \, d\mu \geq \liminf_{n \to \infty} \int_{A^c} |f_n| \, d\mu + \lim_{n \to \infty} \int_A |f_n| \, d\mu.$$ 

(6.72)

Combining (6.71) and (6.72) gives the result.  

\(\square\)
Theorem 6.5.3b is due to Radon (1913) and F. Riesz (1928).

**Theorem 6.5.3. Radon–Riesz theorem**

Let \((X, \mathcal{A}, \mu)\) be a measure space and let \(\{f_n, f : n = 1, \ldots \} \subseteq L^p_\mu(X), 1 \leq p < \infty\). Assume \(\|f_n\|_p \to \|f\|_p\).

a. If \(1 \leq p < \infty\) and \(f_n \to f\) \(\mu\text{-a.e.}\), then

\[
\lim_{n \to \infty} \|f_n - f\|_p = 0.
\]

b. If \(1 < p < \infty\) and \(f_n \to f\) weakly, then

\[
\lim_{n \to \infty} \|f_n - f\|_p = 0.
\]

**Proof.**
a. For any \(a, b \geq 0\), \((a + b)^p \leq 2^p (a^p + b^p)\). Let \(|f_n| = a\) and \(|f| = b\). Then the non-negative functions

\[
g_n = 2^p (|f_n|^p + |f|^p) - |f_n - f|^p, \quad n = 1, \ldots ,
\]

converge \(\mu\text{-a.e.} \) to \(2^{p+1}|f|^p\). Using Fatou’s lemma we have

\[
2^{p+1} \int_X |f|^p \, d\mu \leq \liminf_{n \to \infty} \int_X g_n \, d\mu
\]

\[
= 2^{p+1} \int_X |f|^p \, d\mu - \lim_{n \to \infty} \int_X |f_n - f|^p \, d\mu,
\]

where the equality follows since \(\|f_n\|_p \to \|f\|_p\). Therefore,

\[
\lim_{n \to \infty} \int_X |f_n - f|^p \, d\mu = 0.
\]

b. There are different proofs of part b, e.g., in [141], page 233, or [238], pages 78–80. We shall follow the latter proof.

i. Assume that \(p \geq 2\). We shall consider the following function:

\[
h(y) = \frac{|1 + y|^p - 1 - py}{|y|^p} = \frac{g(y)}{|y|^p}.
\]

Since \(p \geq 2\), \(g''\) exists on \(\mathbb{R}\) and

\[
\forall y \in \mathbb{R}, \quad g''(y) = p(p - 1)|1 + y|^{p-2}.
\]

Thus, \(g'' > 0\) on \(\mathbb{R} \setminus \{-1\}\). (If \(p = 2\) then \(g'' > 0\) on \(\mathbb{R}\), and if \(p > 2\) then \(g''(-1) = 0\).) Using this fact and applying the mean value theorem two times, we see that \(g > 0\) on \(\mathbb{R} \setminus \{0\}\). Further, \(g(0) = 0\). Therefore, \(h > 0\) on \(\mathbb{R} \setminus \{0\}\). If \(p = 2\) then \(\lim_{|y| \to 0} h(y) = 1\) and if \(p > 2\) then \(\lim_{|y| \to 0} h(y) = \infty\). Clearly, \(\lim_{|y| \to \infty} h(y) = 1\). Combining these facts we see that

\[
\exists C > 0 \text{ such that } \forall y \in \mathbb{R}, \quad |1 + y|^p \geq 1 + py + C|y|^p. \quad (6.73)
\]
In particular, (6.73) is valid when $C$ is replaces by $\min(C, 1)$.

We now replace $y$ in (6.73) by

$$\frac{f_n(x) - f(x)}{f(x)}$$
on $P = \{x \in X : f(x) \neq 0\}$. Multiplying both sides of (6.73), with this substitution, by $|f(x)|^p$, we obtain

$$|f_n(x)|^p \geq |f(x)|^p + p|f(x)|^{p-2}\overline{f(x)}(f_n(x) - f(x)) + C|f_n(x) - f(x)|^p \quad (6.74)$$
on $P$. However, (6.74) is also true for $f(x) = 0$ in the case $0 < C \leq 1$. For any such $C$, we integrate (6.74) over $X$ to obtain

$$\int_X |f_n|^p \, d\mu \geq \int_X |f|^p \, d\mu + p \int_X |f|^{p-2}\overline{f}(f_n - f) \, d\mu + C \int_X |f_n - f|^p \, d\mu.$$ 

Since $|f|^{p-2}f \in L^q_p(X)$, where $1/p + 1/q = 1$, it follows that

$$\lim_{n \to \infty} \int_X |f|^{p-2}\overline{f}(f_n - f) \, d\mu = 0,$$

by the weak convergence of $\{f_n : n = 1, \ldots\}$ to $f$. Combining this observation with the assumption that $\|f_n\|_p \to \|f\|_p$, we obtain

$$0 \geq \limsup_{n \to \infty} \int_X |f_n - f|^p \, d\mu.$$

Consequently, part $b$ of the theorem is proved for $p \geq 2$.

$i$. Let $1 < p < 2$. Consider the function

$$F(y) = \begin{cases} 
\frac{|1+y|^{p-1} - py}{|y|^{p-2}}, & |y| \geq 1, \\
\frac{|1+y|^{p-1} - py}{|y|^2}, & |y| < 1.
\end{cases} \quad (6.75)$$

By arguments similar to those of part $i$,

$$\exists C \in (0, 1] \text{ such that } \forall y \in \mathbb{R}, \quad F(y) \geq C. \quad (6.76)$$

Let

$$X_n = \{x \in X : |f_n(x) - f(x)| \geq |f(x)|\}.$$ 

Setting $y = (f_n(x) - f(x))/f(x)$ on $P$ and multiplying by $|f|^p$, (6.76) becomes

$$|f_n(x)|^p \geq |f(x)|^p + p|f(x)|^{p-2}\frac{|f(x)|}{f(x)}(f_n(x) - f(x)) + C|f_n(x) - f(x)|^p \quad (6.77)$$

for $x \in P \cap X_n$. As in (6.73) we can take $0, C \leq 1$. For $x \in P$, write $g(x) = |f(x)|/f(x)$ and extend it to a $\mu$-measurable function $g$ on $X$ for which $|g| \leq 1$. Then, (6.77) is also valid for $x \in X_n \setminus P$ in the case $0 < C \leq 1$. 


For \( x \in P \cap X_n^- \), (6.76) becomes

\[
|f_n(x)|^p \geq |f(x)|^p + p|f(x)|^{p-1}g(x)(f_n(x) - f(x)) + C |f_n(x) - f(x)|^2 / |f(x)|^2 |f(x)|^p.
\]

(6.78)

Since \( P^- \cap X_n^- = \emptyset \), then (6.78) is valid for all \( x \in X_n^- \).

We now integrate (6.77) and (6.78) over \( X_n^- \) and \( X_n^- \), respectively, to obtain

\[
\int_{X^-} |f_n|^p \, d\mu \geq \int_X |f|^p \, d\mu + p \int_{X^-} |f|^{p-1}g(f_n - f) \, d\mu + C \left( \int_{X^-} |f_n - f|^p \, d\mu + \int_{X_n^-} |f_n - f| \, |f| \, d\mu \right).
\]

Combining this observation with our assumptions yields

\[
\lim_{n \to \infty} \int_{X_n^-} |f_n - f|^p \, d\mu = 0 = \lim_{n \to \infty} \int_{X_n^-} |f_n - f|^2 \, |f|^{p-2} \, d\mu.
\]

(6.79)

Moreover, we have that

\[
\int_{X_n^-} |f_n - f|^p \, d\mu \leq \int_{X_n^-} |f_n - f||f|^{p-1} \, d\mu
\]

\[
\leq \left( \int_{X_n^-} |f_n - f|^2 |f|^{p-2} \, d\mu \right)^{1/2} \left( \int_{X_n^-} |f|^p \, d\mu \right)^{1/2}
\]

\[
\leq \left( \int_{X_n^-} |f_n - f|^2 |f|^{p-2} \, d\mu \right)^{1/2} \left( \int_X |f|^p \, d\mu \right)^{1/2}.
\]

(6.80)

(6.79) together with (6.80) yield the conclusion of part b for \( 1 < p < 2 \).

\[\square\]

**Corollary 6.5.4.** Let \((X, \mathcal{A}, \mu)\) be a measure space, and take \(\{f_n : n = 1, \ldots\} \subseteq L^p_\mu(X)\), \(1 \leq p < \infty\), such that \(f_n \to f \mu\text{-a.e.}\). Then, \(\|f_n\|_p \to \|f\|_p\) if and only if \(\|f_n - f\|_p \to 0\).

**Theorem 6.5.5.** Pointwise and weak sequential convergence

Let \((X, \mathcal{A}, \mu)\) be a measure space, \(1 < p < \infty\), and let \(\{f_n : n = 1, \ldots\} \subseteq L^p_\mu(X)\). Assume \(f_n \to f \mu\text{-a.e.}, f\) is \(\mu\)-measurable, and \(\sup_n \|f_n\|_p = K < \infty\).

Then \(f \in L^p_\mu(X)\) and \(f_n \to f\) weakly in \(L^p_\mu(X)\), cf., Theorem 10.9.11.

**Proof.** From Fatou’s lemma, for the case \(|f_n|^p\) and \(|f|^p\), we have

\[
\|f\|_p \leq K^p,
\]

and so \(f \in L^p_\mu(X)\). Take \(\varepsilon > 0\) and \(g \in L^p_\mu(X)\). We shall find \(N\) such that
\[ \forall n \geq N, \quad \left| \int_X (f - f_n)g \, d\mu \right| < \varepsilon. \] (6.81)

Since \(|g|^q \in L^1_\mu(X)\) there is \(\delta > 0\) such that

\[ \forall A \in \mathcal{A} \text{ for which } \mu(A) < \delta, \quad \left( \int_A |g|^q \, d\mu \right)^{1/q} < \frac{\varepsilon}{6K}, \] (6.82)

see Proposition 3.3.9. Also, from the integrability of \(|g|^q\) there is \(B \in \mathcal{A}\) such that \(\mu(B) < \infty\) and

\[ \left( \int_{B^c} |g|^q \, d\mu \right)^{1/q} < \frac{\varepsilon}{6K}. \] (6.83)

Because of Egorov’s theorem (Theorem 2.5.7) there is \(E \subseteq B\), such that \(E \in \mathcal{A}\) and \(\mu(E) > 0\), for which

\[ \mu(B \cap E^c) < \delta \quad \text{and} \quad f_n \rightharpoonup f \text{ uniformly on } E. \]

From the uniform convergence we can choose \(N \in \mathbb{N}\) so that

\[ \forall n \geq N \text{ and } \forall x \in E, \quad |f(x) - f_n(x)| < \frac{\varepsilon}{3\|g\|_q(\mu(E))^{1/p}}. \]

Consequently,

\[ \forall n \geq N, \quad \left( \int_E |f - f_n|^p \, d\mu \right)^{1/p} \|g\|_q < \frac{\varepsilon}{3}. \] (6.84)

Thus, taking \(A = B \cap E^c\), and using Hölder’s and Minkowski’s inequalities, we obtain

\[ \int_X |f_n - f| |g| \, d\mu = \int_A + \int_{B^c} + \int_E |f_n - f| |g| \, d\mu < \varepsilon, \]

which, in turn, gives (6.81). In fact, \(\int_A |f_n - f| |g| \, d\mu < \varepsilon/3\) by Hölder, Minkowski, and (6.82); \(\int_{B^c} |f_n - f| |g| \, d\mu < \varepsilon/3\) by Hölder, Minkowski, and (6.83); and \(\int_E |f_n - f| \, d\mu < \varepsilon/3\) by Hölder and (6.84).

\[ \square \]

In Theorem 6.3.2 we note that weak convergence in \(L^1_\mu(X)\) implies norm boundedness. This phenomenon is rather general because of the Uniform Boundedness Principle (Theorem 10.8.5 and Corollary 10.8.6). Thus, we can restate Theorem 6.5.5 as follows.

**Corollary 6.5.6.** Let \((X, \mathcal{A}, \mu)\) be a measure space, \(1 < p < \infty\), and let \(\{f_n : n = 1, \ldots\} \subseteq L^p_\mu(X)\). Assume \(f_n \rightharpoonup f \mu\text{-a.e. and assume } f \text{ is } \mu\text{-measurable. Then } f \in L^p_\mu(X), \text{ and } f_n \rightharpoonup f \text{ weakly in } L^p_\mu(X) \text{ if and only if } \sup_{n \in \mathbb{N}} \|f_n\|_p < \infty.\)
Example 6.5.7. Failure of Theorem 6.5.5 for $p = 1$

Theorem 6.5.5 is false when $p = 1$. Let $f_n : [0, 1] \to \mathbb{R}^+$ be 0 on $[1/n, 1]$, $n$ at $x = 0$, and linear on $[0, 1/n]$. Thus $f_n \to 0$ $m$-a.e., and $\|f_n\|_1 = 1/2$, whereas $f \not\to 0$ weakly.

Because of Theorem 10.9.4 we know that if $f_n \to f$ weakly in $L^p(X)$ then certain linear combinations of the $f_n$’s converge to $f$ in the $L^p(X)$ topology. The Banach–Saks theorem [17], [238], pages 78–81, which we now state is much finer in that instead of rather arbitrary linear combinations we can use arithmetic means.

**Theorem 6.5.8. Banach–Saks theorem**

Let $(X, \mathcal{A}, \mu)$ be a measure space, $1 \leq p < \infty$, and let $\{f_n, f : n = 1, \ldots \} \subseteq L^p(X)$. Assume $f_n \to f$ weakly. Then there is a subsequence $\{f_{n_k} : k = 1, \ldots \}$ whose arithmetic means $(1/m) \sum_{k=1}^{m} f_{n_k}$ converge in the $L^p(X)$ topology to $f$.

Banach and Saks only proved Theorem 6.5.8 for the $1 < p < \infty$ cases. The result was proved for $L^1(X)$ in 1965 by Szenk. Schreier (1930) showed that $C([0, 1])$ does not have the property of the Banach–Saks theorem.

Because of Problem 3.21d we see that if $\|f_n - f\|_1 \to 0$ then $f_{n_k} \to f$ $\mu$-a.e., for some subsequence $\{f_{n_k} : k = 1, \ldots \}$. We then ask whether $f_n \to f$ weakly yields the same result. Generally the answer is negative as we have seen in Theorem 3.3.14, Example 3.3.15, Example 3.6.5, Problem 3.28, and Problem 3.29. Note that if we take $\{f_n : n = 1, \ldots \}$ as in Theorem 3.3.14 then $f_n \to \alpha$ weakly, where $\alpha = \int_0^1 f$, cf., [323], pages 87–88.

We close this section by noting that $L^1(X)$ and $M_0(X)$ are weakly sequentially complete. This is proved using Schur’s lemma (Theorem 6.2.1) and the results of Sections 6.1 and 6.2, see, e.g., [80]. On the other hand these spaces are never weakly complete as uniform spaces. In fact, no infinite dimensional normed space is weakly complete.

### 6.6 Potpourri and titillation

1. Giuseppe Vitali (August 26, 1875–February 29, 1932), the oldest of five children, was born in Ravenna. After graduating from the “liceo” in Ravenna, he studied mathematics at the University of Bologna in 1895. He then received a scholarship to the Scuola Normale Superiore, cf., Section 3.8.1. Dini and Bianchi were there at the time. He graduated in 1899 and in his thesis he extended a theorem of Mittag–Leffler to Riemann surfaces. His next work was devoted to Abelian integrals. In 1901 he was Dini’s assistant, and then he began his teaching career in a “liceo”. After two brief appointments in Sassari and Voghera he taught at the Liceo Colombo in Genova from
1904 to 1922. This of course, does not match Dedekind’s record of undergraduate teaching. Also, Weierstrass’ “defenders” would point out that his (Weierstrass) service in teaching penmanship and gymnastics, besides mathematics, should count for something extra. Finally, in 1923, Vitali received a position at the University of Modena (a weak counterexample to one of the fundamental theorems of life that “you can keep a good person down”). In 1924 he went to the University of Padova where, at the end of 1926, he was struck with hemiplegia, a paralysis resulting from injury to the motor center of the brain. His intellectual powers were unaffected. He left Padova in 1930 for the University of Bologna. For more, see Vitali’s obituary [299].

2. The Arzela–Ascoli theorem (Theorem 10.4.3) provides criteria to characterize compact subsets of the Banach space $C([0,1])$: the Dunford–Pettis theorem (Theorem 6.3.8) gives criteria to characterize weakly compact subsets of the Banach space $L_1(X)$; and the Alaoglu theorem (Theorem 10.9.5) asserts that the closed unit ball of the dual of a normed vector space is weak* compact.

It is natural to ask if there are workable criteria to characterize compact subsets of $L_p^m(R^d)$, $p \geq 1$, are characterized in the following result, whose proof uses the Arzela–Ascoli theorem.

**Theorem 6.6.1. Kolmogorov compactness theorem**

Let $p \geq 1$. A subset $K \subseteq L_p^m(R^d)$ is relatively compact in the $L^p$-norm topology if and only if the following conditions hold:

i. $\sup_{f \in K} \|f\|_p < \infty$;

ii. $\lim_{T \to \infty} \int_{|x| > T} |f(x)|^p \, dx = 0$ uniformly for $f \in K$;

iii. $\forall \varepsilon > 0, \exists r = r(\varepsilon) > 0$ such that $\forall y \in B(0,r)$ and $\forall f \in K,$

$$\int_{R^d} |f(x + y) - f(x)|^p \, dx < \varepsilon.$$ 

$B(0,r) \subseteq R^d$ is the closed Euclidean ball centered at $0 \in R^d$ and with radius $r$.

The conditions i, ii, iii use norm-boundedness of $K$, uniform “small tails” at infinity, and uniform translation continuity, respectively. Kolmogorov [173], [174] (1931) proved the result for $L^p_m([a,b])$, $p > 1$, and with the $f(x+y)$ in condition iii replaced by

$$f_y(x) = \frac{1}{2y} \int_{x-y}^{x+y} f, \quad y > 0.$$ (6.85)

In fact, he proved that, for each $y > 0$, the set $\{f_y : f \in K\}$ is equicontinuous. His proof required $p > 1$. A. N. Tulaikov [300] (1933) dealt with the case
302 6 Weak convergence of measures

$p = 1$. Marcel Riesz [243] (1933) introduced condition iii. Accessible proofs of the Kolmogorov compactness theorem can be found in [219], pages 212–216, [196], pages 41–44, [163], pages 292–299, and [322], pages 274–277. The statement for $\mathbb{R}^d$ can be generalized to any locally compact group using the same proof.

With either iii as written or using the mean (6.85), we are reminded of the FTC.

An ingenious application of Theorem 6.6.1 to Heisenberg uncertainty principle for classes of functions is due to Harold Shapiro [273] (1991). For example, he used Theorem 6.6.1 to prove that if $\{f_n\} \subseteq L^2_0(\mathbb{R})$ is an orthonormal sequence, then it is not possible that all four of the sequences $(t_n)$, $(\gamma_n)$, $(\sigma^2 f_n)$, and $(\sigma^2 \hat{f}_n)$ are bounded, where the expected values $t_n$ and $\gamma_n$ are

$$t_n = \int_{\mathbb{R}} t |f_n(t)|^2 \, dt \quad \text{and} \quad \gamma_n = \int_{\mathbb{R}} \gamma |\hat{f}_n(\gamma)|^2 \, d\gamma,$$

where the variances $\sigma^2 f_n$ and $\sigma^2 \hat{f}_n$ are

$$\sigma^2 f_n = \int_{\mathbb{R}} (t - t_n)^2 |f_n(t)|^2 \, dt \quad \text{and} \quad \sigma^2 \hat{f}_n = \int_{\mathbb{R}} (\gamma - \gamma_n)^2 |\hat{f}_n(\gamma)|^2 \, d\gamma,$$

and where the Fourier transform $\hat{f}_n$ defined on $\hat{\mathbb{R}} = \mathbb{R}$ is defined in Appendix 11, see [28], Chapter 7, [137], Chapter 5, and [109] for perspective on the Heisenberg uncertainty principle.

3. The proof of the Dieudonné–Grothendieck theorem (Theorem 6.4.2) depends on the structure of the weakly compact sets in $M_b(X)$, cf., the Dunford–Pettis theorem (Theorem 6.3.8). Besides those already referenced, we mention work, relevant to the point of view of this chapter, of Bartle, Dunford, and J. Schwartz (1955), Darst (1967), Topsoe (1970), Adamski, Gänslers, and Kaiser (1976), Brooks and Chacon (1980), Brooks (1980), Pan- Chapagesan (1998), Brooks, Saitô, and Wright (2003). We emphasize that this general topic is also of interest in probability theory.

4. In “The Scottish Book”, Hugo Steinhaus asked the following question, see [209] Problem 126: Does there exist a family $\mathcal{F}$ of measurable functions on a measure space $X$, such that

i) $\forall f \in \mathcal{F}, |f| = 1$,

ii) for each sequence $\{f_n : n = 1, \ldots\} \subseteq \mathcal{F}$, the sequence

$$\frac{1}{m} \sum_{k=1}^{m} f_k(x)$$

is divergent for a.e. $x \in X$?

Related to this problem, János Komlós in [177] showed that every bounded sequence of integrable random variables $\{\xi_n : n = 1, \ldots\}$ has a
subsequence \( \{ \xi_{n_k} : k = 1, \ldots \} \) such that there exists an integrable random variable \( \eta \), for which
\[
\frac{\xi_{n_1} + \cdots + \xi_{n_m}}{m} \to \eta \text{ a.e.},
\]
converges to \( \eta \) with probability 1. In the language of analysis, we can restate a stronger version of this result as follows.

**Theorem 6.6.2. Komlós theorem**

Let \((X, \mathcal{A}, \mu)\) be a finite measure space and let \( \{f_n : n = 1, \ldots \} \subseteq L^1_\mu(X) \) be a bounded sequence of integrable functions \( f_n : X \to \mathbb{R} \). Then there exists a subsequence \( \{f_{n_k} : k = 1, \ldots \} \) and a function \( f \in L^1_\mu(X) \) such that
\[
\frac{1}{m} \sum_{l=1}^{m} f_{n_{k_l}} \to f \text{ } \mu\text{-a.e.,}
\]
for each subsequence \( \{f_{n_{k_l}} : l = 1, \ldots \} \).

The convergence of arithmetic means in (6.86) is called Césaro summability or Césaro convergence. The theorem, both hypotheses and conclusion, should be compared with the Banach–Saks theorem (Theorem 6.5.8).

The theorem of Komlós has many generalizations and applications, see, e.g., [78] or [256] for characterizations of weak compactness by means of Komlós’ theorem. In other developments, a recent paper of Heinrich von Weizsäcker [304] studies the question of necessity of the integrability assumption in Theorem 6.6.2 for sequences of nonnegative functions which are allowed to assume the value \(+\infty\). In this case, the following converse to Komlós theorem holds.

**Theorem 6.6.3. Converse Komlós theorem**

Let \((X, \mathcal{A}, \mu)\) be a finite measure space and let \( \{f_n : n = 1, \ldots \} \) be a sequence of measurable functions \( f_n : X \to \mathbb{R}^+ \cup \{+\infty\} \). If there exists a function \( f \in L^\infty_\mu(X) \) such that
\[
\frac{1}{m} \sum_{l=1}^{m} f_n \to f \text{ } \mu\text{-a.e.,}
\]
then there exists a subsequence \( \{f_{n_l} : l = 1, \ldots \} \) and an equivalent measure \( \nu \) such that \( \{f_{n_l} : l = 1, \ldots \} \) is bounded in \( L^1_\nu(X) \).