5. Determine whether the improper integral $\int_{-5}^{2} \frac{2x+3}{(x-2)(x+5)} dx$ diverges or converges. If it converges, determine its value; if it diverges, give the reason.

(10 pts) Find the antiderivative. We compute it using partial fractions but note that a substitution with u being the denominator works as well.

$$\frac{2x+3}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$$

$$\begin{aligned} 2x+3 &= A(x+5)+B(x-2) \\ 2 &= A+B \\ 3 &= 5A-2B \end{aligned}$$

Solving the last system of equation yields $A = B = 1$. So we have that

$$\begin{aligned} \int \frac{2x+3}{(x-2)(x+5)}\, dx &= \int \left(\frac{1}{x-2} + \frac{1}{x+5}\right) dx \\ &= \ln|x-2| + \ln|x+5| + C \end{aligned}$$

Note that the absolute value signs are essential in this problem because without them you end up trying to take the natural log of negative numbers which we cannot do.

(10 pts) The next step is to evaluate the improper integral. Note that the integrand is continuous over $(-5,2)$ but is unbounded near both ends of the interval. To proceed then we must pick some $-5 < c < 2$ and try evaluating the integrals from -5 to c and from c to 2.

WARNING: THIS IS AN IMPORTANT STEP IN THE PROBLEM AND NOT SIMPLY A FORMALITY !

We treat these types of integrals in this way because it avoids things like $\infty - \infty$ which are meaningless (and definitely not 0) and give us no information.

We pick $c = 0$ for convenience and try evaluating

$$\int_{-5}^{0} \frac{2x+3}{(x-2)(x+5)}\, dx \text{ and } \int_{0}^{2} \frac{2x+3}{(x-2)(x+5)}\, dx\ .$$

If either of these diverge then we say that the entire integral diverges. If both converge our integral converges and the sum of their values is the value of the entire integral.
Starting with the first, we have

$$\begin{aligned} \lim_{d\to -5^+} \int_{d}^{0} \frac{2x+3}{(x-2)(x+5)}\, dx &= \lim_{d\to -5^+} \int_{d}^{0} \left(\frac{1}{x-2} + \frac{1}{x+5}\right) dx \\ &= \lim_{d\to -5^+} \left(\ln|-2| + \ln|5| - \ln|d-2| - \ln|d+5|\right) \\ &= \infty \end{aligned}$$

since $\lim_{d \to -5^+} \ln|d-2| = \ln 7$ and $\lim_{d \to -5^+} \ln|d+5| = -\infty$.

Therefore the integral diverges.

Another common mistake was to try the following:

$$\int_{-5}^{2} \frac{2x+3}{(x-2)(x+5)}\,dx \;=\; \int_{-5}^{2} \left(\frac{1}{x-2} + \frac{1}{x+5}\right) dx \;=\; \int_{-5}^{2} \frac{1}{x-2}\,dx \;+\; \int_{-5}^{2} \frac{1}{x+5}\,dx$$

However this too results in the dreaded $\infty - \infty$ and is not allowed. In fact, the linearity that we have for definite integrals no longer holds for improper integrals (see for example problem 61 from section 8.7 in your text).