MATH 648M: Advanced Analytic Methods with Applications
Department of Mathematics, UMCP **Spring 2013**
Homework 3 **Posted**: Thursday, 03/07/13
Due: Thursday, 03/28/13

If you take the course for 3 credits, do all problems. If you take the course for 1 credit, do Problems 10 and 11. Computers are not allowed.

9. [10pts] Consider the equation $u(x) = \lambda \int_{-\infty}^{\infty} dy\, e^{-ixy}\, u(y)$, $-\infty < x < \infty$, where λ is complex. Consider $u \in L^2(-\infty, +\infty)$. Solutions to this equation are "Fourier transforms of themselves."
(a) Show that there are only 4 eigenvalues λ of the kernel e^{-ixy}. What are they?
(b) Show by an explicit calculation that the functions $u_n(x) = e^{-x^2/2}\, H_n(x)$, where $H_n(x) = (-1)^n\, e^{x^2}\, \frac{d^n}{dx^n}\, e^{-x^2}$ are Hermite polynomials, are eigenfunctions of the above equation. What are the corresponding eigenvalues? Conclude that the eigenvalues of part (a) are infinitely degenerate.
(c) Using the result in (b) and the fact that u_n are known to form a basis, show that any L^2 solution is of the form $u(x) = f(x) + \mathcal{C}\, \hat{f}(x)$, where $f(x)$ is an (arbitrary) odd or even, L^2 function with Fourier transform $\hat{f}(k)$, and $\mathcal{C}$ is a suitable constant. Evaluate $\mathcal{C}$ and relate its value(s) to the eigenvalues found in part (a).
(d) From (c), construct a solution to the original integral equation by taking $f(x) = e^{-ax^2/2}$ (Gaussian, $a > 0$).

10. [10pts] For a viscous fluid past a semi-infinite thin plate, the steady-state flux $u(x)$ on the plate satisfies a Wiener-Hopf integral equation of the 1st kind, viz.,
$$\int_0^{\infty} dy\, K_0(x-y)\, u(y) = 2\pi, \quad x \geq 0,$$
where $K_0(x)$ is the modified Hankel function of zeroth order, given by the integral formula
$$K_0(x) = \int_{-\infty}^{\infty} dt\, e^{-itx}\, (1+t^2)^{-1/2}.$$
Determine $u(x)$ exactly by applying the Wiener-Hopf method.

11. [10pts] A field $\phi(x, y)$ satisfies the PDE $\partial_{xx}\phi + \partial_{yy}\phi - p^2\phi = 0$ everywhere in the (x, y)-plane ($\mathbb{R}^2$) except on the semi-infinite cut (negative x-axis) $\mathcal{L} = \{(x, y) \in \mathbb{R}^2 |\, x < 0,\ y = 0\}$. Consider $p > 0$. In addition, $\phi(x, y)$ obeys the following boundary conditions. First, $\phi(x, 0) = e^x$ if $x < 0$ (on both sides of the cut, $y = 0^{\pm}$). Second,
$$\phi(x, y) \to 0 \qquad \text{as } \sqrt{x^2 + y^2} \to \infty.$$
We look for an $\phi(x, y)$ that is continuous everywhere; in addition, $\partial_y\phi(x, y)$ is everywhere continuous except across the cut $\mathcal{L}$. Determine $\phi(x, y)$ exactly for all points (x, y) by using the Wiener-Hopf method. **Note:** Your final answer should be given in terms of a (single) Fourier integral in x, whose integrand involves a known function.