MATH 648M: Advanced Analytic Methods with Applications
Department of Mathematics, UMCP **Spring 2008**
Homework 5 **Posted**: Saturday, 04/12/08
Due: Tuesday, 04/24/08

For usual credit, solve Problems 14–16. The last problem (17) is a BONUS problem: extra credit will be given to students who work on it; but no points will be taken away from those who do not touch it.

14. (20 pts) [Homogenization in one space dimension.] Consider the problem

$$\partial_x(D\partial_x u^\epsilon) = \partial_t u^\epsilon + f(x, x/\epsilon), \quad 0 < x < 1,$$

where $u^\epsilon = 0$ at $x = 0, 1$ and $u^\epsilon = 0$ at $t = 0$; assume $u^\epsilon = u^\epsilon(x,t)$ and $D = D(x, x/\epsilon)$.

(a) (10 pts) Find the homogenized problem for the steady state ($\partial_t u^\epsilon \equiv 0$). **Hint:** Make the suitable assumptions for f.

(b) (10 pts) Find the homogenized problem for the time-dependent solution. **Hint**: Define slow anf fast space and *time* variables.

15. (20 pts) [Homogenization in 2 space dimensions with periodicity.] Consider the two-dimensional problem with periodic microstructure:

$$\nabla \cdot (D\nabla u^\epsilon) = f(\mathbf{x}) \qquad \mathbf{x} \in \Omega \subset \mathbf{R}^2,$$

where $u^\epsilon = g(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$. The diffusivity $D = D(\mathbf{x}, \mathbf{x}/\epsilon)$ is positive and periodic in $\mathbf{y} = \mathbf{x}/\epsilon$ with period $\mathbf{y}_p$, as assumed in class.

Find explicitly the homogenized diffusivity $\mathbf{D}^{eff}$ (2×2 matrix) if $D(\mathbf{x}, \mathbf{y}) = D_0(x_1, x_2)\, e^{\alpha(y_1)}\, e^{\beta(y_2)}$ where D_0, α and β are smooth functions; $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Explain the origin of anisotropy in $\mathbf{D}^{eff}$. **Hint:** You do NOT need to repeat the procedure described in class for part (a). Simply apply the results of that procedure to the D given here.

16. (20 pts) Let η_1, η_2 be Random Variables (RV's) representing the positions of two interacting particles in a gas. Their joint probability density function (pdf) is

$$f(x_1, x_2) = Z^{-1} \exp(-x_1^2 - x_2^2 - g x_1^2 x_2^2),$$

where Z is a normalization constant and g is a positive coupling constant ($g > 0$).

Calculate exactly the conditional expectation value $h(x; g) := E[\eta_1 \eta_2^2 | \eta_1 = x]$. Find an expansion for $h(x; g)$ in powers of g. What is the condition for the validity of this expansion?

17. [20 pts: **BONUS PROBLEM.**] Let η be a Random Variable (RV), and $g(x)$ be a nonnegative, non-decreasing function of real x. Show that for any a, $P(\eta \geq a) \leq E[g(\eta)]/g(a)$. Use this result to show that $P(|\eta - E[\eta]| \geq \sigma k) \leq k^{-2}$ where σ is the standard deviation of η and k is any positive integer. (Thus, conclude: it is unlikely that η deviates from its expected value more than a few standard deviations.) **Hint:** Start with the definition of $E[g(\eta)]$ assuming that η has pdf $f(x)$, and use a lower bound for the integral involved.