MATH 648M BRIEF SOLUTIONS TO HOMEWORK 1 Spring 13

1 (a) Notice that $(1+x+y+xy)^\nu = (1+x)^\nu (1+y)^\nu$, $0 \le x,y \le 1$.

Then, set $\alpha = \int_0^1 (1+y)^\nu u(y)\,dy$ to find $u(x) = 1+\alpha(1+x)^\nu$.

Substitute in the definition of α : $\alpha = \dfrac{2\nu+1}{2(\nu+1)-2^{2\nu+1}} \; \dfrac{2^{\nu+1}-1}{\nu+1}$ (after algebra)

(b) Set $\alpha = \int_0^1 dy\, u(y)^2 \Rightarrow u(x) = 1+\lambda\alpha$. Solve equation for α, given λ:

$\alpha = (1+\lambda\alpha)^2 \Rightarrow \alpha = \dfrac{1-2\lambda \pm \sqrt{1-4\lambda}}{2\lambda^2}$. Suppose we require real solutions.

Then $\lambda = \frac{1}{4}$ is a bifurcation point. [For $\lambda \le \frac{1}{4}$ both solutions are real; for $\lambda > 1/4$, these are complex]

As $\lambda \to 0$, one of the solutions blows up. $\lambda = 0$ is a singular pt, not bifurcation pt.

Homogeneous equation: $u = \lambda\alpha$ where $\alpha = \lambda^2\alpha^2 \Rightarrow \alpha(1-\lambda^2\alpha) = 0$ with $\lambda\alpha \neq 0$.
Thus, $u(x) = \frac{1}{\lambda}$, $\lambda \neq 0$.

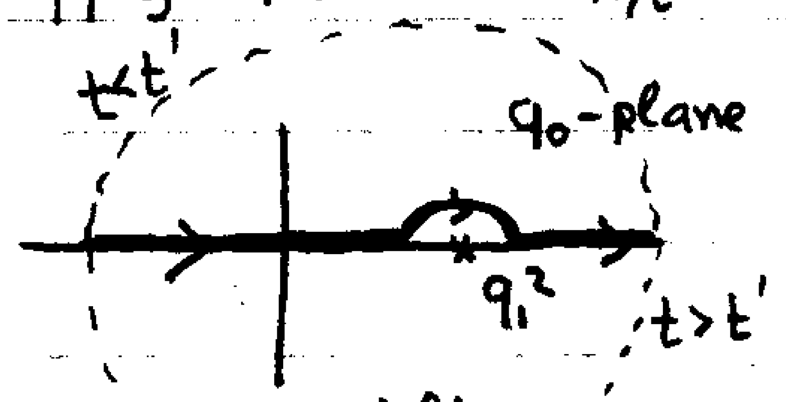
2 (a) Write $\Psi(x,t) = \Psi_0(x,t) + \Psi_p(x,t)$ where $\Psi(x,t) = 0$ for $t<0$ and :

- Ψ_0 : $i\Psi_{0,t} + \Psi_{0,xx} = 0$, $t>0$; $\Psi_0(x,t=0) = a(x)$.
- Ψ_p : $\Psi_p(x,t) = \int_0^\infty dt' \int_{-\infty}^\infty dx'\; G(x,t;x',t')\; V(x',t')\,\Psi(x',t')$ (by superposition)

where $iG_t + G_{xx} = \delta(x-x')\,\delta(t-t')$, $-\infty < x,t < \infty$ $(x', t' > 0$: fixed$)$

with $G(x,t=0;x',t') = 0$.

Apply FT in x,t : Compute $G(x,t;x',t') = \int_{-\infty}^\infty \frac{dq_1}{2\pi} \int_{-\infty}^\infty \frac{dq_0}{2\pi}\, \dfrac{e^{iq_1(x-x') - iq_0(t-t')}}{q_0 - q_1^2}$



Take path above pole at $q_0 = q_1^2$: $G = 0$ for $t<t'$.

After some algebra: $G(x,t;x',t') = \begin{cases} 0, & t<t' \\ -\dfrac{e^{i\pi/4}}{2\sqrt{\pi}}\, \dfrac{1}{\sqrt{t-t'}}\, e^{i\frac{(x-x')^2}{4(t-t')}}, & t>t' \end{cases}$

(by closing contour in lower q_0-plane)

2(b) To formulate the integral equation, we must determine $\Psi_0(x,t)$.

Write $\Psi_0(x,t) = \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iqx}\, \hat{\Psi}_0(q,t)$ where $i\hat{\Psi}_{0,t} - q^2 \hat{\Psi}_{0,xx} = 0$.

Thus, $\Psi_0(x,t) = \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iqx - iq^2 t}\, G(q)$, where $G(q) = \int_{-\infty}^{\infty} dx\, e^{-iqx} a(x)$ by initial data.

[Note : Here, we assume that the FT of $a(x)$ exists.]

[Note: If you replace the formula for $G(q)$ in the integral over q and carry out the q-integration, you may find $\Psi_0(x,t) = \frac{e^{-i\pi/4}}{2\pi}\sqrt{\frac{\pi}{t}} \int_{-\infty}^{\infty} dx'\, a(x')\, e^{i\frac{(x-x')^2}{4t}}$.]

The desired integral equation reads

$$\Psi(x,t) = \Psi_0(x,t) - \frac{e^{i\pi/4}}{2\sqrt{\pi}} \int_0^t dt' \int_{-\infty}^{\infty} dx'\, (t-t')^{-1/2}\, e^{i\frac{(x-x')^2}{4(t-t')}}\, V(x',t')\, \Psi(x',t').$$

Suppose V is bounded. This equation can be solved approximately if $\sup|V|$ is sufficiently small : Apply iteration :

To zeroth order : $\Psi(x,t) \simeq \Psi_0(x,t)$

Next order : $\Psi(x,t) \simeq \Psi_0(x,t) - \frac{e^{i\pi/4}}{2\sqrt{\pi}} \int_0^t dt' \int_{-\infty}^{\infty} dx'\, (t-t')^{-1/2}\, e^{i\frac{(x-x')^2}{4(t-t')}}\, V(x',t')\, \Psi_0(x',t')$

[Born approximation]

3 A Hint was given in class :

Split $(0,a)$ into intervals $(0,\varepsilon_1), (\varepsilon_1,\varepsilon_2)\ldots, (\varepsilon_{n-1},\varepsilon_n), \ldots, (\varepsilon_{N-1}, a)$ such that

$|\lambda|\, \|K\|_{L^2((\varepsilon_{n-1},\varepsilon_n)\times(\varepsilon_{n-1},\varepsilon_n))} < \mu < 1/2$ (by suitable choice of N). ; $\varepsilon_0 = 0$, $\varepsilon_N = a$.

Then construct u by iteration in each of these intervals, applying the Banach Fixed Pt Thm.

Examples :

$(0,\varepsilon_1)$: The iteration series for u converges since $|\lambda|\,\|K\|<1$; also, we can show that $\|u\|_{L^2} \le \|f\|_{L^2} + |\lambda|\, \|K\|_{L^2} \|u\|_{L^2}$

$\Rightarrow \|u\|_{L^2(0,\varepsilon_1)} \le \frac{\|f\|}{1-|\lambda|\,\|K\|} \le 2\|f\|_{L^2(0,a)}$ by $|\lambda|\,\|K\| < 1/2$.

$(\varepsilon_1,\varepsilon_2)$: Write iteration scheme as: $u_k(x) = \underbrace{f(x) + \lambda\int_0^{\varepsilon_1} K(u_{k-1}-u)\,dy + \lambda\int_0^{\varepsilon_1} K u\, dy}_{=: f_{(1)}(x)} + \lambda\int_{\varepsilon_1}^{x} K\, u_{k-1}\, dy$, k: large, $x\in(\varepsilon_1,\varepsilon_2)$

where u is the solution in $(0,\varepsilon_1)$.

It can be easily shown that $\|f_{(1)}\|_{L^2} < \infty$ (by $\|u\|_{L^2(0,\varepsilon_1)} < 2\|f\|$ found above). In fact, as $k \to \infty$,

$$f_{(1)}(x) \simeq f(x) + \lambda \int_0^{\varepsilon_1} K(x,y)\, u(y)\, dy \,, \quad \text{since} \quad \lambda \int_0^{\varepsilon_1} K(u_{k-1} - u)\, dy \text{ is arbitrarily small (in } L^2\text{-sense)}$$

Thus, the iteration scheme converges in $(\varepsilon_1, \varepsilon_2)$. Also, $\|u\|_{L^2(\varepsilon_1,\varepsilon_2)} \leq 2\|f\|_{(1)} \leq C\|f\|_{L^2}$.

Repeat this argument for each interval $(\varepsilon_{n-1}, \varepsilon_n)$, $n = 2, \dots, N$.

4 Apply the Laplace Transform (LT) in x: By $K(x) = \ln x$,

$$\mathring{f}(s) = \mathring{K}(s)\, \mathring{u}(s) \Rightarrow \mathring{u}(s) = \frac{\mathring{f}(s)}{\mathring{K}(s)} \quad \text{where}$$

$$\mathring{K}(s) = \int_0^\infty \ln x \; e^{-sx}\, dx \overset{sx=t}{=} \frac{1}{s}\int_0^\infty \ln\left(\frac{t}{s}\right) e^{-t}\, dt = -\frac{\gamma + \ln s}{s}.$$

$$\Rightarrow \quad u(x) = \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \; e^{sx} \frac{s}{\gamma + \ln s}\, \mathring{f}(s)$$

where c is a positive number such that the path of integration lies to the right of all singularities of $\frac{s}{\gamma+\ln s}\mathring{f}(s)$.