⑥ $u(x) = \lambda \int_0^1 dy \; K(x,y)\, u(y)$; $K(x,y) = \begin{cases} 3, & 0 \le y < x \le 1, \\ 2, & 0 \le x < y \le 1. \end{cases}$

ⓐ $u(x) = \lambda \int_0^x 3\, u(y)\, dy + \lambda \int_x^1 2\, u(y)\, dy \Rightarrow \frac{du}{dx} = \lambda u(x) \Rightarrow u(x) = C e^{\lambda x},$ $C \neq 0.$

From IE we get: $u(1) = \frac{3}{2} u(0)$. Thus, $e^{\lambda} = \frac{3}{2} = e^{\ln(3/2) + i2n\pi}$, $n = 0, \pm 1, \pm 2, \ldots$

Thus, $\lambda = \lambda_n = \ln\left(\frac{3}{2}\right) + i2n\pi$; $u_n(x) = C_n\, e^{\lambda_n x}$ (C_n: arbitrary)

ⓑ Clearly, $K(x,y) \neq K(y,x) = K^T(x,y) = \begin{cases} 3, & 0 \le x < y \le 1, \\ 2, & 0 \le y < x \le 1. \end{cases}$

Repeating the above procedure, we find: $v(x) = v_m(x) = F_m\, e^{-\lambda_m x}$,

where $\lambda = \lambda_m = \ln(3/2) + i2m\pi$ (m: integer $= 0, \pm 1, \pm 2, \ldots$), as in ⓐ

ⓒ $\int_0^1 dx \; u_n(x)\, v_m(x) = C_n F_m \int_0^1 e^{i2n\pi x}\, e^{-i2m\pi x}\, dx = 0$ if $n \neq m$.

⑦ Take $K(x,y) = \sum_n \frac{u_n(x)\, u_n(y)}{\lambda_n}$ with $\int_a^b dx \; u_n(x)\, u_\ell(x) = \delta_{n\ell}$; $a \le x, y \le b$.
u_n: eigenfunctions of K. ; u_n are real. (K: symm.)

Let us assume that $\check{\lambda} = \lambda_m$ for some $n = m$, and λ_m has multiplicity 1.
(is non-degenerate).

Then,
$$\check{\lambda} \int_a^b dy \; K(x,y)\, u(y) = \sum_n c_n \left(\frac{\lambda_m}{\lambda_n}\right) u_n(x), \quad \text{where } c_n := \int_a^b dy \; u_n(y)\, u(y),$$

by interchanging summation and integration.

The integral equation reads : $u(x) = f(x) + \sum_n c_n \left(\frac{\lambda_m}{\lambda_n}\right) u_n(x)$.

Multiply both sides of the last relation by $u_\ell(x)$ and integrate in $[a,b]$:

$$c_\ell = \int_a^b dx \; u_\ell(x)\, f(x) + c_\ell \frac{\lambda_m}{\lambda_\ell}, \quad \text{using} \quad \int_a^b dx \; u_n u_\ell = \delta_{n\ell}.$$

$$\therefore \quad c_\ell \left(1 - \frac{\lambda_m}{\lambda_\ell}\right) = \int_a^b dx \; u_\ell(x)\, f(x) \qquad \text{(QED)}$$

Notice that if there is a $u(x)$ satisfying the IE, the last relation must hold.
In particular, if $\ell = m$ then $f \perp u_\ell$. This argument can be extended to degenerate $\check{\lambda}$.

8 (a) Let $\varphi(x) := \psi(x)\sqrt{V(x)}$, assuming $V(x)>0$. The IE for $\varphi(x)$ reads

$$\varphi(x) = e^{ikx}\sqrt{V(x)} + \lambda\int_{-\infty}^{\infty} dy\ K(x,y)\,\varphi(y)\ ; \quad \underbrace{K(x,y)}_{\text{symm.}} = \frac{e^{ik|x-y|}}{2i}\sqrt{V(x)V(y)},\quad \lambda = 1/k.$$

For $k\to\infty$ $(\lambda\to 0^+)$, $\varphi(x)$ can be approximated via a rational function in λ.

$$D(\lambda) \simeq 1 + D'(0)\,\lambda = 1 + \frac{1}{k}D'(0), \qquad D'(0) = -\frac{1}{2i}\int_{-\infty}^{\infty} dx\ V(x).$$

$$N(x,y,\lambda) \simeq K(x,y) - \lambda N_1(x,y) = K(x,y) - \lambda\int_{-\infty}^{\infty} dx_1\ K\begin{pmatrix} x & x_1 \\ y & x_1\end{pmatrix}$$ ← from classnotes

$$= \frac{e^{ik|x-y|}}{2i}\sqrt{V(x)V(y)} + \frac{1}{4k}\Big[e^{ik|x-y|}\sqrt{V(x)V(y)}\int_{-\infty}^{\infty}dx_1\,V(x_1)$$
$$- \sqrt{V(x)V(y)}\int_{-\infty}^{\infty}dx_1\ e^{ik(|x-x_1|+|x_1-y|)}\,V(x_1)\Big]$$

The solution $\varphi(x) = \varphi(x;\lambda)$ is approximated by

$$\varphi(x;\lambda) \simeq e^{ikx}\sqrt{V(x)} + \frac{1}{k}\int_{-\infty}^{\infty}dy\ \underbrace{H(x,y,1/k)}_{\text{resolvent}}\,e^{iky}\sqrt{V(y)}\ ; \quad H(x,y,1/k) = \frac{N(x,y,1/k)}{D(1/k)}$$

or

$$\psi(x;k) \simeq e^{ikx} + \int_{-\infty}^{\infty}dy\ \frac{e^{ik(|x-y|+y)}}{2ik}\ \frac{1 - \frac{1}{2ik}\int_{-\infty}^{\infty}dx_1\left[1 - e^{ik(|x-x_1|+|x_1-y|-|x-y|)}\right]V(x_1)}{1 - \frac{1}{2ik}\int_{-\infty}^{\infty}dx_1\,V(x_1)}\,V(y).$$

(b) The kernel $J_0(2ka\sin\frac{\varphi-\varphi'}{2})$ is translation invariant and periodic on $(0,2\pi)$.

Thus, the eigenfunctions of the homogeneous equation are $\boxed{j_n(\varphi) = e^{in\varphi}}$, $n = 0,\pm1,\pm2\ldots$ (from classnotes).

The corresponding eigenvalues are $\lambda_n = -\alpha_n = \frac{1}{k_n}$ where k_n are Fourier coefficients of the kernel.

Notice, from the given formula for $J_n(x)$, that: $J_0(2ka\sin\frac{\varphi}{2}) = \int_0^{2\pi}\frac{d\vartheta}{2\pi}\,e^{ika\sin(\varphi-\vartheta)}\,e^{ika\sin\vartheta}$,

i.e., $J_0(2ka\sin\frac{\varphi}{2})$ is the convolution of $e^{ika\sin\varphi}$ with itself. From the integral

$$J_n(x) = \int_0^{2\pi}\frac{d\varphi'}{2\pi}\,e^{ix\sin\varphi'}\,e^{-in\varphi'},$$

we assert that $e^{ix\sin\varphi}$ has Fourier coefficients $J_n(x)$, $n\in\mathbb{Z}$.

It follows that $J_0(2ka\sin\frac{\varphi}{2})$, which is the autoconvolution of $e^{ika\sin\varphi}$, has Fourier coefficients $J_n(ka)^2$.

Thus, the eigenvalues of the homogeneous IE are $\boxed{\lambda_n = -\alpha_n = \frac{1}{J_n(ka)^2}}$.

To find $j(\varphi)$ in the <u>non-homogeneous</u> IE, it suffices to expand

$$f(\varphi) = e^{ika\sin\varphi} = \sum_{n=-\infty}^{\infty} f_n e^{in\varphi}.$$

From the previous discussion, $f_n = J_n(ka)$.

We expand $\boxed{j(\varphi) = \sum_{n=-\infty}^{\infty} j_n e^{in\varphi}.}$

By substitution in the IE, we find

$$\boxed{j_n = \frac{J_n(ka)}{1+\alpha J_n(ka)^2}}, \quad n \in \mathbb{Z}$$

This gives the unique $L^2(0,2\pi)$ solution provided $\alpha \neq \alpha_n = -\frac{1}{J_n(ka)^2}$.

[In fact, in physical applications $\alpha > 0$].