MATH 648M

BRIEF SOLUTIONS TO HOMEWORK 3

Spring 2013

9(a) Write the integral equation as

$$u(x) = \lambda\, \mathcal{F}u(x) \quad \text{where} \quad \mathcal{F}u := \int_{-\infty}^{\infty} dy \; e^{-ixy}\, u(y)$$

$$\left.\begin{array}{l} \mathcal{F}^4 u = (2\pi)^2 u \\ \mathcal{F}u = 1/\lambda\, u \end{array}\right\} \Rightarrow \quad \lambda^4 = \frac{1}{(2\pi)^2} \Rightarrow \lambda = \frac{\pm 1}{\sqrt{2\pi}},\ \pm \frac{i}{\sqrt{2\pi}}$$

(b) Consider $u_n(x) = e^{-x^2/2} H_n(x)$; $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

$$\Rightarrow \hat{u}_n(k) = \int_{-\infty}^{\infty} dx \; e^{-ikx}\; e^{-x^2/2} (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = (-1)^n \int_{-\infty}^{\infty} dx \; e^{-ikx} \left(\frac{d}{dx} - x\right)^n e^{-x^2/2}$$

After integration by parts, we obtain

$$\hat{u}_n(k) = (-1)^n \int_{-\infty}^{\infty} dx \; e^{-x^2/2} \left(-\frac{d}{dx} - x\right)^n e^{-ikx} = (-1)^n i^n \int_{-\infty}^{\infty} dx \; e^{-x^2/2} \left(k - \frac{d}{dk}\right)^n e^{-ikx}$$

$$= (-i)^n \left(k - \frac{d}{dk}\right)^n \int_{-\infty}^{\infty} dx \; e^{-x^2/2}\; e^{-ikx} = \sqrt{2\pi}\, (-i)^n\, u_n(k)$$

The given IE gives

$$u_n(x) = \lambda\, \hat{u}_n(x) = \lambda \sqrt{2\pi}\, (-i)^n u_n(x) \Rightarrow \lambda = \lambda_n = \frac{i^n}{\sqrt{2\pi}},$$

in agreement with 9(a) ; check $n = 0, 1, 2, 3$.

(c) We will answer this question for eigenvalue $\lambda = \lambda_0 = 1/\sqrt{2\pi}$. The proof is analogous for the other eigenvalues $(n = 1, 2, 3)$.

- Let $u(x)$ correspond to λ_0. Then we can write

$$u(x) = \sum_{\substack{n=0 \\ n=4m}}^{\infty} c_n u_n(x) = \sum_{\substack{n=0 \\ n:\text{even}}}^{\infty} \frac{1}{2} c_n \left[1 + (-i)^n\right] u_n(x), \text{ where } \sum_{n:\text{even}} |c_n|^2 < \infty$$

(c_n's have been extended to $n=2m$).

$$u(x) = \sum_{n:\text{even}} \frac{c_n}{2} u_n(x) + \sum_{n:\text{even}} \frac{c_n}{2\sqrt{2\pi}} \hat{u}_n(x)$$

$$= \sum_{n:\text{even}} \frac{c_n}{2} u_n(x) + \frac{1}{\sqrt{2\pi}} FT\left\{ \sum_{n:\text{even}} \frac{c_n}{2} u_n(x) \right\} = f(x) + \frac{1}{\sqrt{2\pi}} \hat{f}(x) ;\ f \in L^2(0,\infty)$$

$$C = 1/\sqrt{2\pi}$$

Similarly: $C = \frac{i}{\sqrt{2\pi}}$ for $\lambda = \lambda_1 = \frac{i}{\sqrt{2\pi}}$; $C = \frac{-1}{\sqrt{2\pi}}$ for $\lambda = \lambda_2 = \frac{-1}{\sqrt{2\pi}}$;

$C = {-i}/\sqrt{2\pi}$ for $\lambda = \lambda_3 = {-i}/\sqrt{2\pi}$.

(d) Let $f(x) = e^{-ax^2/2}$ (f: even)

$$\Rightarrow \hat{f}(k) = \int_{-\infty}^{\infty} dx\ e^{-ikx}\ e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}}\ e^{-k^2/2a}$$

Then, we write

$$u^{(\pm)}(x) = f(x) \pm \frac{1}{\sqrt{2\pi}} \hat{f}(x) = e^{-ax^2/2} \pm \frac{1}{\sqrt{a}}\ e^{-x^2/2a}$$

which satisfies the IE with eigenvalues $\lambda = \frac{\pm 1}{\sqrt{2\pi}}$ $(n=0,2)$.

<u>Note</u>: For $a=1$, $u^{(\pm)}(x) = \begin{cases} e^{-x^2/2} + e^{-x^2/2} = 2\,e^{-x^2/2}, & \lambda = 1/\sqrt{2\pi} \\ e^{-x^2/2} - e^{-x^2/2} = 0, & \lambda = -1/\sqrt{2\pi}. \end{cases}$

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(10) Write

$$\int_{-\infty}^{\infty} dy\ K_0(x-y)\, u(y) = g(x) + h(x)\ , \qquad -\infty<x<\infty \qquad (1)$$

where $g(x) = \begin{cases} 0\ , & x \geq 0 \\ \int_0^{\infty} dy\ K_0(x-y)\, u(y), & x<0 \end{cases}$, $h(x) = \begin{cases} 2\pi, & x \geq 0, \\ 0, & x<0. \end{cases}$

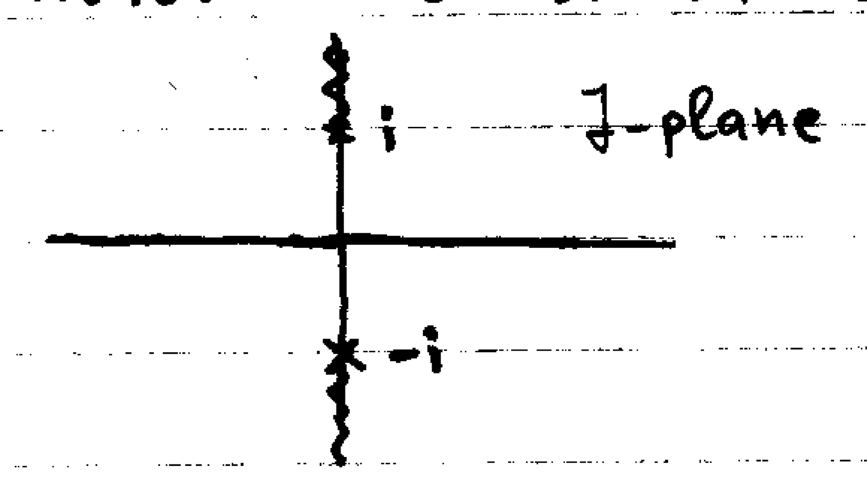
We have $\hat{h}(\zeta) = \int_0^{\infty} e^{-i\zeta x}\, 2\pi\ dx = \frac{2\pi}{i\zeta} = \lim_{\varepsilon \downarrow 0} \frac{2\pi}{i(\zeta - i\varepsilon)}$, moving the pole to $\{\mathrm{Im}\,\zeta > 0\}$.

FT of (1) : for <u>$0<\varepsilon\ll 1$</u>

$$\frac{2\pi}{\sqrt{1+\zeta^2}}\ \underset{-}{\hat{u}}(\zeta) = \underset{+}{\hat{g}}(\zeta) + \underbrace{\frac{2\pi}{i(\zeta - i\varepsilon)}}_{-} \qquad \text{assuming } (\zeta \in \mathbb{R}) \qquad (2)$$

Factorize by inspection: $\frac{2\pi}{\sqrt{1+\zeta^2}} = 2\pi\ \underbrace{\frac{1}{\sqrt{\zeta+i}}}_{+}\ \underbrace{\frac{1}{\sqrt{\zeta-i}}}_{-}$

[Note: The branch cuts for $(1+\zeta^2)^{-1/2}$ must be chosen as shown below:



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Equation (2) entails: $\frac{2\pi}{\sqrt{\zeta - i}}\ \underset{-}{\hat{u}}(\zeta) = \underset{+}{\sqrt{\zeta+i}}\ \underset{+}{\hat{g}}(\zeta) + \underset{-}{\frac{2\pi}{i(\zeta - i\varepsilon)}}\ \underset{+}{\sqrt{\zeta+i}}$

By further decomposing the mixed $(-+)$ term, we wind up with

$$\underset{-}{\frac{2\pi}{\sqrt{\zeta-i}}}\;\underset{-}{\hat{u}}(\zeta) - \underset{-}{\frac{2\pi}{i(\zeta-i\varepsilon)}}\;\underset{-}{\sqrt{i\varepsilon+i}} = \underset{+}{\sqrt{\zeta+i}}\;\hat{g}_{+}(\zeta) + \underbrace{\frac{2\pi}{i(\zeta-i\varepsilon)}\left(\sqrt{\zeta+i}-\sqrt{i\varepsilon+i}\right)}_{+}$$

$=: E(\zeta)$: entire function

We assert that $E(\zeta)\to 0$ as $\zeta\to\infty$ in the upper and lower half plane.

By Liouville's theorem: $E(\zeta)\equiv 0$ (everywhere)

Thus,

$$\hat{u}_{-}(\zeta) = \sqrt{i(1+\varepsilon)}\;\frac{\sqrt{\zeta-i}}{i(\zeta-i\varepsilon)} \qquad (\varepsilon>0)$$

Invert to get $u(x)$ and let $\varepsilon\downarrow 0$:

$$u(x) = e^{i\pi/4}\int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\,e^{i\zeta x}\,\frac{\sqrt{\zeta-i}}{i\zeta}, \quad x>0 \;;\quad u(x)\equiv 0,\; x<0.$$

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(11) Let

$$\varphi(x,y) = \int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\,e^{i\zeta x}\,\hat{\varphi}(\zeta,y)$$

By the given PDE:

$$\hat{\varphi}(\zeta,y) = \begin{cases} \hat{F}_1(\zeta)\,e^{-\sqrt{\zeta^2+p^2}\,y}, & y>0 \\ \hat{F}_2(\zeta)\,e^{\sqrt{\zeta^2+p^2}\,y}, & y<0, \end{cases}$$

where $\hat{F}_{1,2}$ are to be determined

$$\varphi(x,\underbrace{y=0^+}_{\text{from } y>0}) = \int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\,e^{i\zeta x}\,\hat{F}_1(\zeta) \Rightarrow \hat{F}_1(\zeta) = \underbrace{\int_{-\infty}^{0}dx\,e^{-i\zeta x}\,e^{x}}_{=\frac{1}{1-i\zeta}} + \underbrace{\int_{0}^{\infty}dx\,e^{-i\zeta x}\,\varphi(x,0^+)}_{=:\hat{F}_{-}(\zeta)}$$

$$\varphi(x,\underbrace{y=0^-}_{\text{from } y<0}) = \int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\,e^{i\zeta x}\,\hat{F}_2(\zeta) \Rightarrow \hat{F}_2(\zeta) = \int_{-\infty}^{0}dx\,e^{-i\zeta x}\,e^{x} + \int_{0}^{\infty}dx\,e^{-i\zeta x}\,\varphi(x,0^-)$$

Since $\varphi(x,y)$ is continuous $\Rightarrow$ $\hat{F}_1(\zeta) = \hat{F}_2(\zeta) = \frac{1}{1-i\zeta} + \hat{F}_{-}(\zeta)$ (1)

$(\zeta\in\mathbb{R})$

We now turn attention to $\varphi_y(x,y)$. From the above results, we have:

$$\varphi_y(x,0^+) = -\int_{-\infty}^{\infty} \frac{d\zeta}{2\pi}\, e^{i\zeta x}\, \hat{F}_1(\zeta)\, \sqrt{\zeta^2+p^2} \quad ; \quad \varphi_y(x,0^-) = \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi}\, e^{i\zeta x}\, \hat{F}_1(\zeta)\, \sqrt{\zeta^2+p^2}.$$

Thus, $\quad \varphi_y(x,0^+) = -\,\varphi_y(x,0^-) \quad$ <u>all</u> x.

In particular, since $\varphi_y(x,y)$ must be continuous across $\{y=0,\ x\geq 0\}$:

$$\varphi_y(x,0^\pm) = 0 \qquad \text{for } x \geq 0.$$

$$\Rightarrow \quad -\hat{F}_1(\zeta)\,\sqrt{\zeta^2+p^2} = \int_{-\infty}^{0} dx\ \varphi_y(x,0^+)\, e^{-i\zeta x} =: \hat{\Psi}_+(\zeta) \qquad (\zeta\in\mathbb{R}) \qquad ②$$

By eliminating $\hat{F}_1(\zeta)$ from Eqs. (1) and (2), we arrive at:

$$\hat{F}_1(\zeta) = -\frac{\hat{\Psi}_+(\zeta)}{\sqrt{\zeta^2+p^2}} = \underbrace{\frac{1}{1-i\zeta}}_{+} + \hat{F}_-(\zeta) \qquad (\zeta\in\mathbb{R}) \qquad ③$$

By factorizing $\sqrt{\zeta^2+p^2} = \sqrt{(\zeta+ip)(\zeta-ip)}$ we rewrite (3) as

$$-\underset{+}{\frac{\hat{\Psi}_+(\zeta)}{\sqrt{\zeta+ip}}} = \frac{i\sqrt{\zeta-ip}}{\zeta+i} + \underset{-}{\sqrt{\zeta-ip}\ \hat{F}_-(\zeta)} \qquad (\zeta\in\mathbb{R})$$

or

$$\underset{+}{-\frac{\hat{\Psi}_+(\zeta)}{\sqrt{\zeta+ip}}} \underset{+}{-\frac{i}{\zeta+i}\sqrt{-i-ip}} = \underset{-}{\frac{i}{\zeta+i}\left(\sqrt{\zeta-ip}-\sqrt{-i-ip}\right)} + \underset{-}{\sqrt{\zeta-ip}\ \hat{F}_-(\zeta)}$$

$$= E(\zeta): \text{entire}$$

Observe that $E(\zeta)\to 0$ as $\zeta\to\infty$.

Thus, $\quad \hat{\Psi}_+(\zeta) = -\dfrac{i}{\zeta+i}\sqrt{-i-ip}\ \sqrt{\zeta+ip}$

$$\overset{②}{\Rightarrow} \quad \hat{F}_1(\zeta) = \frac{i\sqrt{-i-ip}}{(\zeta+i)\sqrt{\zeta-ip}} = \frac{e^{i\pi/4}\sqrt{1+p}}{(\zeta+i)\sqrt{\zeta-ip}}$$

Finally, $\quad \varphi(x,y) = \displaystyle\int_{-\infty}^{\infty} \frac{d\zeta}{2\pi}\, e^{i\zeta x}\, e^{-\sqrt{\zeta^2+p^2}\,|y|}\, \frac{e^{i\pi/4}\sqrt{1+p}}{(\zeta+i)(\zeta-ip)^{1/2}}$