BRIEF SOLUTIONS TO HOMEWORK 4

Spring 2013

(12) The kernel reads $K(x) = -\frac{1}{2}\,\mathrm{Ei}(|x|)$. Its Fourier transform is

$$\hat{K}(\zeta) = \frac{1}{2}\int_{-\infty}^{\infty} dx \left(\int_{|x|}^{\infty} dt\, \frac{e^{-t}}{t} \right) e^{-i\zeta x} \underset{t=|x|\rho}{=\!=} \frac{1}{2}\int_{-\infty}^{\infty} dx \left(\int_{1}^{\infty} d\rho\, \frac{e^{-|x|\rho}}{\rho} \right) e^{-i\zeta x}$$

$$= \frac{1}{2}\int_{1}^{\infty} \frac{d\rho}{\rho} \int_{-\infty}^{\infty} dx\; e^{-|x|\rho - i\zeta x} = \frac{1}{2}\int_{1}^{\infty} \frac{d\rho}{\rho} \left[\int_{-\infty}^{0} dx\; e^{-(i\zeta-\rho)x} + \int_{0}^{\infty} dx\; e^{-(i\zeta+\rho)x} \right] = \frac{1}{2}\int_{1}^{\infty} \frac{2}{\rho^2+\zeta^2}\, d\rho$$

$$= \frac{1}{\zeta}\tan^{-1}\zeta \qquad \text{where} \qquad \tan^{-1}\zeta = \frac{1}{2i}\ln\left(\frac{1+i\zeta}{1-i\zeta}\right), \text{ with singularities at } \zeta = \pm i.$$

Notice that $\hat{K}(\zeta)$ is analytic in the strip $\{\zeta \in \mathbb{C} : \underbrace{-1}_{-a} < \mathrm{Im}\,\zeta < \underbrace{1}_{b}\}$.

Thus, following the class notes, we assert that $\boxed{a = 1}$.

The next step is to determine the index, ν, of the Wiener-Hopf method.

Since $K(x)$ is even in x, we have $\nu = 0$ on real axis ($\mathrm{Im}\,\zeta = 0$).

Next, we move the path downward, from the real axis of the ζ-plane to close to the line $\mathrm{Im}\,\zeta = -1+\varepsilon$ $(0<\varepsilon \ll 1)$. Thus, we need to know how many zeroes or poles of $1-\lambda\hat{K}(\zeta)$ lie in $\{\zeta \in \mathbb{C} : -1 < \mathrm{Im}\,\zeta < 0\}$.

However, $1-\lambda\hat{K}(\zeta) = 1 - \frac{\lambda}{2i\zeta}\ln\left(\frac{1+i\zeta}{1-i\zeta}\right)$ has no pole [In fact, $\zeta=0$ is a regular pt.]

Thus, we only need to consider possible zeroes of $1-\lambda\hat{K}(\zeta)$ in $\{-1<\mathrm{Im}\,\zeta<0\}$.
Since $1-\lambda\hat{K}(\zeta)$ is even, its zeroes must occur in pairs symmetric w/ respect to 0.
Let us study two "extreme" cases first:

(i) $\lambda = \tilde{\varepsilon},\ 0<\tilde{\varepsilon}\ll 1$. Let us look for roots of $\hat{K}(\zeta) = \frac{1}{\tilde{\varepsilon}}$, or

$\tan^{-1}\zeta = \frac{\zeta}{\tilde{\varepsilon}}$. As $\tilde{\varepsilon}\downarrow 0$, if ζ is finite the LHS must approach ∞; thus, ζ must approach $\zeta = -i$. Hence, let us write

$\zeta = -i+\delta$, $|\delta| \ll 1$. By substitution in $\hat{K}(\zeta) = \frac{1}{\tilde{\varepsilon}}$ we find

$\delta \sim 2i\, e^{-2/\tilde{\varepsilon}}$. Thus, for λ close to 0, $1-\lambda\hat{K}(\zeta)$ has only <u>one zero</u>, pure imaginary, in $\{-1 < \mathrm{Im}\,\zeta < 0\}$.

(ii) <u>$\lambda = 1-\tilde{\varepsilon}$, $0 \le \tilde{\varepsilon} \ll 1$</u>: For $\lambda = 1$, we have $1-\lambda\hat{K}(\zeta) = 0 \Leftrightarrow \tan^{-1}\zeta = \zeta$.

This equation has only the (double) root $\zeta = 0$ that lies in the first branch of $\tan^{-1}\zeta$. For $\lambda = 1-\tilde{\varepsilon}$, we apply perturbation theory (assuming there is no bifurcation, due to the analyticity of $\tan^{-1}\zeta$ in the neighborhood of $\zeta = 0$).

So, we look for root $\zeta = \delta$, $0 < |\delta| \ll 1$. By substitution in $1-\lambda\hat{K}(\zeta) = 0$

we get: $\frac{1}{1-\tilde{\varepsilon}} = \frac{1}{\delta}\tan^{-1}\delta \underset{\delta\to 0}{\sim} 1 - \frac{\delta^2}{3} \Rightarrow \frac{\delta^2}{3} \sim -\frac{\tilde{\varepsilon}}{1-\tilde{\varepsilon}} \Leftrightarrow \delta \sim \pm i\sqrt{3\tilde{\varepsilon}}$.

Thus, $1-\lambda\hat{K}(\zeta)$ has the zero $\delta \sim -i\sqrt{3(1-\lambda)}$ in $\{\zeta \in \mathbb{C} : -1 < \mathrm{Im}\,\zeta < 0\}$.

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For each of these two extreme cases for λ, the index ν is

$\nu = 1 > 0$ along $\mathrm{Im}\,\zeta = -1+\varepsilon$ [where $\varepsilon < 2e^{-2/\tilde{\varepsilon}}$ in (i)]

Then, the original IE has non-trivial solutions of the form

$u(x) = c\, \varphi_0(x)$ (c : arbitrary constant, $c \in \mathbb{R}$ real)

where $\varphi_0(x)$ is found by the procedure described in class (for $\nu > 0$).

Heuristically: For $\lambda = O(1)$ or $1-\lambda = O(1)$, we can further expand as in (i) and (ii).
The correction terms so obtained give: a zero on the imaginary axis in $\{-1 < \mathrm{Im}\,\zeta < 0\}$.
The result $\nu = 1 > 0$ remains intact. [Rigorously, one has to construct an analytic function whose values depend on the number of zeroes and use analytic continuation]

(13) Let $S[x] = \int_{T_i}^{T_f} dt \; L(x,\dot{x},t)$; $x = x(t) = x_c(t) + \varepsilon\,\varphi(t)$, $\varphi \in C^\infty(T_i,T_f)$, $\varphi(T_i) = 0 = \varphi(T_f)$.

$$S[x_c+\varepsilon\varphi] - S[x_c] = \int_{T_i}^{T_f} dt \left[\varepsilon \frac{\partial L}{\partial x}\varphi + \varepsilon\frac{\partial L}{\partial \dot{x}}\dot{\varphi} + \frac{1}{2}\varepsilon^2\frac{\partial^2 L}{\partial x^2}\varphi^2 + \frac{1}{2}\varepsilon^2\frac{\partial^2 L}{\partial \dot{x}^2}\dot{\varphi}^2 + \varepsilon^2\frac{\partial^2 L}{\partial x\partial\dot{x}}\varphi\dot{\varphi} + O(\varepsilon^3)\right]\Bigg|_{x=x_c}$$

By setting the first variation equal to zero: $\dfrac{\partial L}{\partial x} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}}$ at $x = x_c$.

Then,

$$S[x_c+\varepsilon\varphi] - S[x_c] = \frac{\varepsilon^2}{2}\int_{T_i}^{T_f} dt \left[\frac{\partial^2 L}{\partial x^2}\varphi^2 + \frac{\partial^2 L}{\partial \dot{x}^2}\dot{\varphi}^2 + 2\frac{\partial^2 L}{\partial x\partial\dot{x}}\varphi\dot{\varphi}\right]\Bigg|_{x=x_c} + O(\varepsilon^3) .$$

In particular, the second variation is $\lim_{\varepsilon\to 0}\varepsilon^{-2}\{S[x_c+\varepsilon\varphi] - S[x_c]\} =: \dfrac{\delta^2 S}{\delta x^2}\Big|_{x=x_c}$.

Now let us specify : $L = \dfrac{\dot{x}^2}{2} - \dfrac{1}{2}kx^2$ (harmonic oscillator)

The Euler-Lagrange equation yields $\ddot{x}_c + kx_c = 0$.

$$\frac{\delta^2 S}{\delta x^2}\Big|_{x=x_c} = \frac{1}{2}\int_{T_i}^{T_f} dt \left[\left(\frac{d\varphi}{dt}\right)^2 - k\varphi^2\right] = \frac{1}{2}\int_{T_i}^{T_f} dt \; \varphi\left(-\frac{d^2}{dt^2} - k\right)\varphi , \quad \varphi(T_i)=0=\varphi(T_f)$$

by integration by parts.

To check the sign of $\dfrac{\delta^2 S}{\delta x^2}$ we expand φ in $\{\varphi_n\}_{n=1}^{\infty}$, the basis set that solves $\left(\dfrac{d^2}{dt^2} + k\right)\varphi_n = \lambda_n\varphi_n$, $\varphi_n(T_i) = \varphi_n(T_f) = 0$. [Eigenvalue problem]

The $\{\varphi_n\}_{n=1}^{\infty}$ consists of eigenfunctions $\varphi_n(t) = A_n \sin\left[\sqrt{k-\lambda_n}\,(t-T_i)\right]$

where $\sqrt{k-\lambda_n}\,(T_f - T_i) = n\pi$; $n = 1, 2, \ldots$ (λ_n: eigenvalues) .

$$\frac{\delta^2 S}{\delta x^2}\Big|_{x=x_c} = -\int_{T_i}^{T_f} dt \underbrace{\left(\sum_{n'} a_{n'}\varphi_{n'}(t)\right)}_{\varphi(t)}\left(\sum_n a_n\lambda_n\varphi_n(t)\right) = -\sum_{n',n}\lambda_n a_{n'}a_n\int_{T_i}^{T_f} dt\; \varphi_n(t)\varphi_{n'}(t)$$

$= -\sum_{n=1}^{\infty} a_n^2\lambda_n$ by orthogonality of φ_n's.

Thus, $\dfrac{\delta^2 S}{\delta x^2}\Big|_{x=x_c} > 0$ if and only if $\lambda_1 < 0 \Leftrightarrow k < \left(\dfrac{\pi}{T_f - T_i}\right)^2$.

(14) ⓐ $\hat{x}\, e^{\frac{i}{\hbar}\hat{p}a} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{\hbar}a\right)^n \hat{x}\hat{p}^n$ where $\hat{x}\hat{p}^n = \hat{p}^n\hat{x} + i\hbar n\,\hat{p}^{n-1}$ (by induction).

Thus, $\hat{x}\, e^{\frac{i}{\hbar}\hat{p}a} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{\hbar}a\right)^n \hat{p}^n\hat{x} + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}a\right)^n i\hbar n\,\hat{p}^{n-1}$

$= e^{\frac{i}{\hbar}\hat{p}a}\,\hat{x} - a\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(\frac{i}{\hbar}a\right)^{n-1}\hat{p}^{n-1} = e^{\frac{i}{\hbar}\hat{p}a}\,(\hat{x}-a)$

ⓑ Heisenberg equation of motion: $\frac{d}{dt}\hat{O}_H(t) = \frac{i}{\hbar}[\hat{H},\hat{O}_H(t)] = \frac{i}{\hbar}e^{\frac{i}{\hbar}\hat{H}t}[\hat{H},\hat{O}]e^{-\frac{i}{\hbar}\hat{H}t}$.

So, it suffices to compute commutators in "Schrödinger picture".
In the following, we drop the "^" over operators, for convenience; set $\underline{\hbar=1}$.

ⓑ i)

$[H,p] = \left[\frac{1}{2m}p^2 + V(x), p\right] = [V(x),p] = i\,\partial_x V$

$[H,x] = \left[\frac{1}{2m}p^2 + V(x), x\right] = \frac{1}{2m}[p^2,x] = \frac{1}{2m}(-2ip) = -i\frac{p}{m}$

Equations of motion: $\boxed{\frac{d}{dt}x_H(t) = \frac{p_H}{m}}$, $\boxed{\frac{d}{dt}p_H(t) = -\partial_x V(x_H)}$

ii) • $[H,x] = \left[\frac{1}{2}p\frac{1}{m(x)}p, x\right] = \frac{1}{2}p\left[\frac{1}{m(x)}p, x\right] + \frac{1}{2}[p,x]\frac{1}{m(x)}p$

$= \frac{1}{2}p\frac{1}{m(x)}(-i) + \frac{1}{2}(-i)\frac{1}{m(x)}p = -\frac{i}{2}\left[p\frac{1}{m(x)} + \frac{1}{m(x)}p\right]$

where $\left(p\frac{1}{m(x)} + \frac{1}{m(x)}p\right)\phi = -i\frac{\partial}{\partial x}\frac{1}{m(x)}\phi - i\frac{1}{m(x)}\frac{\partial}{\partial x}\phi = i\frac{m'(x)}{m(x)^2}\phi - 2i\frac{1}{m(x)}\frac{\partial\phi}{\partial x}$

$= i\frac{m'(x)}{m(x)^2}\phi + \frac{1}{m(x)}p$

Thus: $\boxed{\frac{dx_H}{dt} = \frac{i}{2}\frac{m'(x_H)}{m(x_H)^2} + \frac{1}{m(x_H)}p_H}$

• $[H,p] = \left[\frac{1}{2}p\frac{1}{m(x)}p, p\right] = \frac{1}{2}p\left[\frac{1}{m(x)}, p\right]p$ where

$\left[\frac{1}{m(x)}, p\right]\phi = -i\frac{m'(x)}{m(x)^2}\phi$

Thus: $\boxed{\frac{dp_H}{dt} = \frac{1}{2}p_H\frac{m'(x_H)}{m(x_H)^2}p_H}$

iii) We write $H = \frac{1}{2m}[\vec{p}^2 - 2e\vec{A}(\vec{x},t)\cdot\vec{p}] + \mathcal{U}(\vec{x},t)$; $\mathcal{U}(\vec{x},t) := \frac{1}{2m}\left[-ie\nabla\cdot A + e^2A(\vec{x},t)^2\right] + eA_0(\vec{x},t)$

where we used the commutator $[A_j(\vec{x},t), p_j] = i\,\partial_{x_j}A_j$ $(j=1,2,3)$

Let us remove the "→"'s for convenience; so, $p\in\mathbb{R}^3$, $x\in\mathbb{R}^3$, $A\in\mathbb{R}^3$.

$$[H, x_j] = \frac{1}{2m}[p^2, x_j] - \frac{e}{m}\sum_{\ell=1}^{3} A_\ell(x,t)\,[p_\ell, x_j] = \frac{1}{2m}[p_j^2, x_j] - \frac{e}{m}A_j(x,t)[p_j, x_j]$$

$$= \frac{1}{2m}(-2i\,p_j) - \frac{e}{m}A_j(x,t)(-i) = \frac{1}{i}\,\frac{1}{m}\,[p_j - e A_j(x,t)] \quad \text{using } [p_\ell, x_j] = -i\delta_{\ell j}$$

Thus:

$$\boxed{\frac{d}{dt}x_H(t) = \frac{1}{m}\left[p_H(t) - e A(x_H, t)\right]}\,.$$

$$[H, p_j] = -\frac{e}{m}\sum_{\ell=1}^{3}[A_\ell(x,t), p_j]\,p_\ell + [\mathcal{U}(x,t), p_j]$$

where $[A_\ell(x,t), p_j]\,\phi = -i\,A_\ell(x,t)\dfrac{\partial\varphi}{\partial x_j} + i\dfrac{\partial A_\ell}{\partial x_j}\varphi + i\,A_\ell(x,t)\dfrac{\partial\varphi}{\partial x_j} = i\,\partial_{x_j}A_\ell\,\varphi$

Thus, $[H, p_j] = -i\dfrac{e}{m}\displaystyle\sum_{\ell=1}^{3}(\partial_{x_j}A_\ell)p_\ell + i\,\partial_{x_j}\mathcal{U}(x,t) = -i\dfrac{e}{m}\,\partial_{x_j}A\cdot p + i\,\partial_{x_j}\mathcal{U}$

Hence:

$$\boxed{\frac{d}{dt}x_{H,j}(t) = \frac{e}{m}\,\partial_{x_j}A(x_H,t)\cdot p_H(t) - \partial_{x_j}\mathcal{U}(x_H,t)} \quad (j=1,2,3)$$

More compactly:

$$\frac{d}{dt}x_H(t) = \frac{e}{m}\,\nabla_x A(x_H,t)\cdot p_H(t) - \nabla_x\mathcal{U}(x_H,t)$$

where $\mathcal{U}(x,t) = \dfrac{1}{2m}\left[-ie\,\nabla_x\cdot A(x,t) + e^2 A(x,t)^2\right] + e\,A_0(x,t)$