BRIEF SOLUTION TO HOMEWORK 5

Spring 2013

(16) Suppose the particle is at state $|q_i\rangle$ at $t=T_i$ and state $|q_f\rangle$ at $t=T_f$.

Divide $[T_i, T_f]$ into N subintervals $[t_j, t_{j+1}]$, $j=1,\dots,N$ $(N \gg 1)$; $t_1 = T_i$, $t_{N+1} = T_f$, with $t_{j+1} - t_j = \varepsilon > 0$. By unitary evolution, the particle will be at state

$$\lim_{\varepsilon \downarrow 0} \exp\left[-\frac{i}{\hbar}\varepsilon H(t_N)\right] \exp\left[-\frac{i}{\hbar}\varepsilon H(t_{N-1})\right] \cdots \exp\left[-\frac{i}{\hbar}\varepsilon H(t_1)\right] |q_i\rangle \qquad (1)$$

at time $t = T_f$. Here, we have $H(t) = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(x,t)$, $x \in \mathbb{R}$.

The amplitude associated with the probability that the particle will end up with state $|q_f\rangle$ is the inner product

$$\mathcal{A} = \lim_{\varepsilon \downarrow 0} \langle q_f | \exp\left[-\frac{i}{\hbar}\varepsilon H(t_N)\right] \exp\left[-\frac{i}{\hbar}\varepsilon H(t_{N-1})\right] \cdots \exp\left[-\frac{i}{\hbar}\varepsilon H(t_1)\right] | q_i \rangle. \qquad (2)$$

It suffices to show that the amplitude (2) is the known path integral.

Assuming that the eigenstates of the position operator, $\hat{q}$, form a basis ("complete set" of states), we have

$$\langle q_f | \hat{A}\hat{B} | q_i \rangle = \int_{-\infty}^{\infty} dq \, \langle q_f | \hat{A} | q \rangle \langle q | \hat{B} | q_i \rangle.$$

Thus,

$$\mathcal{A} = \iint_{-\infty}^{+\infty} \cdots \int dq_{N-1} \cdots dq_1 \, \langle q_f | \exp\left[-\frac{i}{\hbar}\varepsilon H(t_N)\right] | q_{N-1} \rangle \langle q_{N-1} | \exp\left[-\frac{i}{\hbar}\varepsilon H(t_{N-1})\right] | q_{N-2} \rangle$$
$$\cdots \langle q_1 | \exp\left[-\frac{i}{\hbar}\varepsilon H(t_1)\right] | q_i \rangle \qquad (3)$$

By using eigenstates of the momentum operator, we have

$$\langle q_n | e^{-\frac{i}{\hbar}\varepsilon H(t_n)} | q_{n-1} \rangle = \int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \langle q_n | e^{-\frac{i}{\hbar}\varepsilon H(t_n)} | p_n \rangle \underbrace{\langle p_n | q_{n-1} \rangle}_{e^{-\frac{i}{\hbar} p_n q_{n-1}}}$$

(since $\langle q, p \rangle = e^{\frac{i}{\hbar} q p}$)

where $e^{-\frac{i}{\hbar}\varepsilon H} \simeq 1 - \frac{i}{\hbar}\varepsilon H$ (which suffices to leading order)

Thus, for $H = \frac{p^2}{2m} + V(x,t)$,

$$\langle q_n | e^{-\frac{i}{\hbar}\varepsilon H(t_n)} | p_n \rangle \simeq \left[1 - \frac{i}{\hbar}\varepsilon \frac{p_n^2}{2m} - \frac{i}{\hbar}\varepsilon V(q_n, t_n) \right] e^{\frac{i}{\hbar} p_n q_n}$$

$$\simeq \exp\left[-\frac{i}{\hbar}\varepsilon \frac{p_n^2}{2m} - \frac{i}{\hbar}\varepsilon V(q_n, t_n) \right] e^{i/\hbar\, p_n q_n}.$$

Hence,

$$\langle q_n | e^{-\frac{i}{\hbar}\varepsilon H(t_n)} | q_{n-1} \rangle \underset{\varepsilon \downarrow 0}{\simeq} \int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \exp\left[i/\hbar\, p_n (q_n - q_{n-1}) - \frac{i}{\hbar}\varepsilon \frac{p_n^2}{2m} - \frac{i}{\hbar}\varepsilon V(q_n, t_n) \right].$$

To carry out the integration over p_n, we note that

$$-\frac{i}{\hbar}\varepsilon \frac{p_n^2}{2m} + \frac{i}{\hbar} p_n (q_n - q_{n-1}) = -\frac{i}{\hbar} \varepsilon/_{2m} \left(p_n - m \frac{q_n - q_{n-1}}{\varepsilon} \right)^2 + i \frac{m}{2\hbar} \frac{(q_n - q_{n-1})^2}{\varepsilon}.$$

Thus, we have

$$\int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \exp\left[i/\hbar\, p_n (q_n - q_{n-1}) - i/\hbar\, \varepsilon \frac{p_n^2}{2m} \right] = \exp\left[i \frac{m}{2\hbar} \frac{(q_n - q_{n-1})^2}{\varepsilon} \right] \frac{1}{2\pi\hbar} e^{-i\pi/4} \sqrt{\frac{2m\hbar\pi}{\varepsilon}},$$

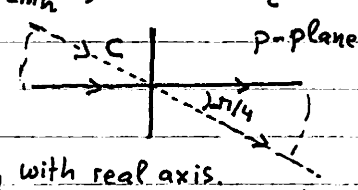
which yields

$$\langle q_n | e^{-i/\hbar\, \varepsilon H(t_n)} | q_{n-1} \rangle \underset{\varepsilon \downarrow 0}{\simeq} \sqrt{\frac{m}{2\pi\hbar i \varepsilon}} \exp\left\{ \frac{i}{\hbar}\varepsilon \left[\frac{m}{2} \left(\frac{q_n - q_{n-1}}{\varepsilon} \right)^2 - V(q_n, t_n) \right] \right\}.$$

The substitution of this formula into (3) leads to the known path integral.

NOTE : In the above, we had to compute the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-i\alpha(p-\beta)^2} dp = \int_{-\infty}^{\infty} e^{-i\alpha p^2} dp \; ; \; \alpha = \frac{\varepsilon}{2m\hbar}, \; \beta = m \frac{q_n - q_{n-1}}{\varepsilon}.$$



To carry out this integral, rotate the path by $-\pi/4$ and perform the integration along the line C that forms angle $-\pi/4$ with real axis.
[The integral along real axis equals the integral along C by Cauchy integral theorem.)

(17) (a) The PDE can be written as

$$(\Box + m^2)\Psi(x) = -ie\left(\Psi\partial_\mu A^\mu + 2A^\mu\partial_\mu\Psi\right) + e^2 A_\mu A^\mu \Psi \quad ; \quad \Box := \frac{\partial^2}{\partial t^2} - \Delta$$

To simplify the right-hand side, we can choose to work with a convenient gauge. For example, in the Lorentz gauge, we have $\partial_\mu A^\mu = 0$.

By fixing the Lorentz gauge, (1) becomes

$$(\Box + m^2)\Psi(x) = \underbrace{-2ie\, A^\mu\partial_\mu\Psi + e^2 A_\mu A^\mu \Psi}_{\rho(x)}.$$

To convert this PDE into an integral equation, we define the Green function $G(x;\, x') = G(x-x')$

$$(\Box + m^2)\, G(x-x') = \overset{(3)}{\delta}(\vec{x}-\vec{x}')\,\delta(t-t') \quad , \quad \vec{x}', \vec{x} \in \mathbb{R}^3 ;\ t, t' \in \mathbb{R}.$$

By applying Fourier transform in $x-x'$, we get

$$G(x-x') = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-x')}\,\hat{G}(p) \quad , \text{ where } \quad \hat{G}(p) = \frac{1}{m^2-p^2} \ ;$$

$p^2 = p_\mu p^\mu = p_0^2 - \vec{p}^{\,2}$, $p\cdot x = p_\mu x^\mu = p_0 t - \vec{p}\cdot\vec{x}$.

For Feynman boundary conditions, the contour of integration in the p_0-plane must be <u>above</u> the pole at $p_0 = \sqrt{m^2+\vec{p}^{\,2}} =: E(\vec{p})$ and <u>below</u> the pole at $p_0 = -E(\vec{p})$.

Equivalently, we can write

$$G(x-x') = \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-x')}\, \frac{1}{m^2-p^2-i\varepsilon} \ ,$$

and keep integration paths on real axes.

Then, the poles on the p_0-plane are located at $p_0 = \pm\sqrt{E(\vec{p})^2 - i\varepsilon}$

$$\simeq \begin{cases} E(\vec{p}) - i\tilde{\varepsilon} \\ -E(\vec{p}) + i\tilde{\varepsilon} \end{cases} \text{ as } \varepsilon \downarrow 0, \text{ where } \tilde{\varepsilon} = \frac{\varepsilon}{2E(\vec{p})}.$$

So, the position of each pole relative to the integration path (real axis) is in agreement with Feynman boundary conditions.

The integral equation for $\Psi(x)$ reads

$$\Psi(x) = \Psi_0(x) - 2ie \int_{\mathbb{R}^4} G(x-x') \; A^\mu(x') \; [\partial_\mu \Psi(x') + \tfrac{i}{2} e A_\mu(x')\Psi(x')] \, d^4x'$$

where $\Psi_0(x)$ solves $(\Box + m^2)\Psi_0(x) = 0$, e.g. $\Psi_0(x) = e^{-iE(\vec{k})t} e^{i\vec{k}\cdot\vec{x}}$ (plane wave of momentum $\vec{k}$ and positive energy $E(\vec{k})$).

(b) The Dirac equation coupled with the EM field reads

$$(i\gamma_\mu \partial^\mu - e\gamma_\mu A^\mu - m)\,\Psi(x) = 0. \qquad (1)$$

To show that $\Psi^\dagger(x)\Psi(x)$ is conserved, it suffices to show that there exists some $\vec{J}(x)$ such that $\frac{\partial}{\partial t}[\Psi^\dagger(x)\Psi(x)] + \vec{\nabla}\cdot\vec{J} = 0$.

From (1) we have:

$$(i\partial^\mu + eA^\mu)\,\bar{\Psi}\gamma_\mu + m\bar{\Psi} = 0, \qquad (1')$$

where $\bar{\Psi} = \Psi^\dagger\gamma^0$. Multiplying (1) by $\bar{\Psi}$ from left yields

$$\left.\begin{aligned} &\bar{\Psi}(i\gamma_\mu\partial^\mu - eA^\mu\gamma_\mu)\Psi - m\bar{\Psi}\Psi = 0. \\ &i(\partial^\mu\bar{\Psi})\gamma_\mu\Psi + eA^\mu\bar{\Psi}\gamma_\mu\Psi + m\bar{\Psi}\Psi = 0 \end{aligned}\right\} \overset{\text{Add}}{(+)} \Rightarrow i\bar{\Psi}\gamma_\mu\partial^\mu\Psi + i(\partial^\mu\bar{\Psi})\gamma_\mu\Psi = 0$$

Similarly: by (1')

or, $$\partial^\mu(\bar{\Psi}\gamma_\mu\Psi) = 0 \iff \frac{\partial}{\partial t}(\Psi^\dagger\Psi) + \vec{\nabla}\cdot(\overbrace{\Psi^\dagger\vec{\gamma}\,\Psi}^{\vec{J}}) = 0.$$

(c) • $\not{p}\not{k} + \not{k}\not{p} = p_\mu \gamma^\mu k_\nu \gamma^\nu + k_\nu \gamma^\nu p_\mu \gamma^\mu = p_\mu k_\nu (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$

$= p_\mu k_\nu \, 2 g^{\mu\nu} = 2 p \cdot k = 2 p_\mu k^\mu$

• $\mathrm{Tr}(\not{k}\not{p}) = \mathrm{Tr}(k_\mu \gamma^\mu p_\nu \gamma^\nu) = k_\mu p_\nu \mathrm{Tr}(\gamma^\mu \gamma^\nu)$

$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \underbrace{\mathbb{1}}_{\text{4x4 unit matrix}} \Rightarrow \mathrm{Tr}(\gamma^\mu \gamma^\nu) + \mathrm{Tr}(\gamma^\nu \gamma^\mu) = 2 g^{\mu\nu} \mathrm{Tr}(\mathbb{1}) = 2 g^{\mu\nu} 4$

$\Rightarrow \mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$.

Thus, $\mathrm{Tr}(\not{k}\not{p}) = 4 k_\mu p_\nu g^{\mu\nu} = 4 k_\mu p^\mu = 4 k \cdot p$