

1) $z(x,y) = y^{-1} e^{-x/y}, \quad y \neq 0.$

$$\frac{\partial z}{\partial x} = -\frac{1}{y^2} e^{-x/y} \quad , \quad [2 \text{ pts}]$$

$$\frac{\partial z}{\partial y} = -\frac{1}{y^2} e^{-x/y} + y^{-1} \cdot \frac{x}{y^2} e^{-x/y} = (-y^{-2} + xy^{-3}) e^{-x/y} \quad , \quad [2 \text{ pts}]$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{y^2} \left(-\frac{1}{y}\right) e^{-x/y} = \frac{1}{y^3} e^{-x/y} \quad [2 \text{ pts}]$$

$$x \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} = \frac{x}{y^3} e^{-x/y} - \frac{1}{y^2} e^{-x/y} = (xy^{-3} - y^{-2}) e^{-x/y} \quad . \quad [3 \text{ pts}]$$

This is equal to $\frac{\partial z}{\partial y}$. Thus, $z(x,y)$ obeys $[1 \text{ pt}]$

$$\frac{\partial z}{\partial y} = x \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} .$$

2(a) $f(x,y) = \sin(xy)$

f increases most rapidly at $(1,\pi)$ in the direction of $\text{grad } f(1,\pi)$ $[1 \text{ pt}]$

$$\text{grad } f(x,y) = \hat{i} f_x + \hat{j} f_y = y \cos(xy) \hat{i} + x \cos(xy) \hat{j} \quad [1 \text{ pt}]$$

$$\Rightarrow \text{grad } f(1,\pi) = \pi \cos(\pi) \hat{i} + \cos(\pi) \hat{j} = -\pi \hat{i} - \hat{j} \quad [1 \text{ pt}]$$

Maximal directional derivative at $(1,\pi)$ is $\|\text{grad } f(1,\pi)\|$: $[1 \text{ pt}]$

$$\|\text{grad } f(1,\pi)\| = \sqrt{\pi^2 + 1} \quad [1 \text{ pt}]$$

(b) $F(x,y,z) = \sin(xy) + \cos(xz) = 0$

The tangent plane has equation

$$f_x(x_0, y_0, z_0)(x-x_0) + f_y(x_0, y_0, z_0)(y-y_0) + f_z(x_0, y_0, z_0)(z-z_0) = 0. \quad [1 \text{ pt}]$$

We compute:

$$\begin{aligned} f_x(x,y,z) &= y \cos(xy) - z \sin(xz) \\ f_y(x,y,z) &= x \cos(xy), \quad f_z = -x \sin(xz) \end{aligned} \quad [1 \text{ pt}]$$

$[1 \text{ pt}]$

Equation of tangent plane:

$$f_x\left(\frac{\pi}{4}, 4, 2\right)(x-\frac{\pi}{4}) + f_y\left(\frac{\pi}{4}, 4, 2\right)(y-4) + f_z\left(\frac{\pi}{4}, 4, 2\right)(z-2) = 0$$

$$\Leftrightarrow [4 \cos(4\pi/4) - 2 \sin(2\pi/4)](x-\pi/4) + \frac{\pi}{4} \cos(\pi/4)(y-4) - \frac{\pi}{4} \sin(\pi/4)(z-2) = 0$$

$$\Leftrightarrow -6(x-\pi/4) + \pi/4(y-4) + \pi/4(z-2) = 0 \quad [2 \text{ pts}]$$

3 (a) To find critical points, we compute:

$$\begin{aligned} f_x &= -4x + 3y - 4 \\ f_y &= 3x + 2y + 3 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad [1pt]$$

Critical pt (x,y) : Set $f_x = f_y = 0$. [1pt]

System: $\begin{cases} -4x + 3y - 4 = 0 \\ 3x + 2y + 3 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -1 \\ y = 0 \end{cases}$. Only cp is $(-1, 0)$. [2pts]

We apply Second Partial Test:

Take $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2$ [1pt]

where $f_{xx} = -4$, $f_{yy} = 2$, $f_{xy} = 3$

$\Rightarrow D(-1,0) = (-4)2 - 3^2 = -17 < 0$. Thus, $(-1,0)$ is a saddle point. [1pt]

(b) Region R is a closed, bounded region. By the Min-Max

Theorem (given in class), f must have both a maximum and a

minimum value on R since f is continuous.

[2pts]

By part (a), the only cp of f in interior (inside) R is a saddle point.

Hence, f must have both a minimum and a maximum value at

the boundary of R . [2pts]

4 Let $u = y-x+a$, $v = y-z+b$, $s = z-x+c$; $w = f(u,v,s)$. [1pt]

By the Chain Rule, we have: [1pt]

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial x} = -f_u - f_s, \quad [2pts]$$

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial y} = f_u + f_v, \quad [2pts]$$

$$\frac{\partial w}{\partial z} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial z} = -f_v + f_s. \quad [2pts]$$

Thus:

$$A = (-f_u - f_s) + (f_u + f_v) + (-f_v + f_s) = 0. \quad [2pts]$$