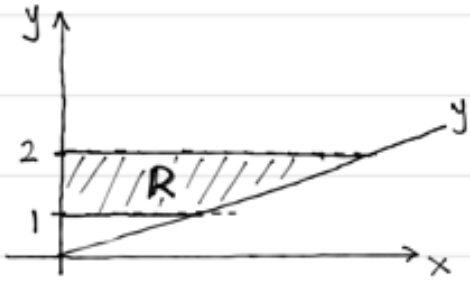
1.



[Grader: Students are not required to sketch R.]

Write M as an iterated integral.

$$M = \iint_R y\,dA = \int_1^2 \int_0^{2y} y\,dx\,dy \qquad \text{[5pts]}$$

$$= \int_1^2 y\,(x)\Big|_{x=0}^{2y} dy = \int_1^2 2y^2\,dy \qquad \text{[3 pts]}$$

$$= 2\left[\frac{y^3}{3}\right]\Big|_1^2 = 2\,\tfrac{7}{3} = \tfrac{14}{3} \qquad \text{[2pts]}$$

2. $V = \iint_R (4+x+2y)\,dA$ where R is the region bounded by the circle $x^2+y^2=1$ in the xy-plane. [3pts]

[Grader: Students are not required to sketch R.]

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, where $0 \le r \le 1$, $0 \le \theta < 2\pi$ [1pt]

$$V = \int_0^{2\pi} \int_0^1 (4 + r\cos\theta + 2r\sin\theta)\,r\,dr\,d\theta \qquad \text{[2pts]}$$

$$= \int_0^{2\pi} \left[4\frac{r^2}{2} + (\cos\theta + 2\sin\theta)\frac{r^3}{3}\right]\Big|_{r=0}^{1} d\theta = \int_0^{2\pi} \left[2 + \frac{1}{3}(\cos\theta + 2\sin\theta)\right] d\theta \qquad \text{[2pts]}$$

$$= \left[2\theta + \frac{1}{3}(\sin\theta - 2\cos\theta)\right]\Big|_0^{2\pi} = 2\cdot 2\pi = 4\pi \qquad \text{[2pts]}$$

3. (a) The mass density of the object is $\delta(x,y,z) = \sqrt{x^2+y^2+z^2}$ [1pt]

The total mass of the object is

$$m = \iiint_D \delta(x,y,z)\, dV = \iiint_D \sqrt{x^2+y^2+z^2}\, dV$$ [1pt]

(b) We use spherical coordinates (ρ,φ,θ), where

$x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$; $0 \le \theta < 2\pi$, $0 \le \varphi \le \varphi_0$, $1 \le \rho \le 2$ [1pt]

[Grader: The students may use any other reasonable coordinate system. They are not required to sketch solid region D.]

The angle φ_0 must be determined by the equation for the cone.

$$3z^2 = x^2+y^2 \Rightarrow 3\rho^2\cos^2\varphi_0 = \rho^2\sin^2\varphi_0$$

$$\Rightarrow \tan^2\varphi_0 = 3 \Rightarrow \tan\varphi_0 = \sqrt{3} \Rightarrow \varphi_0 = \pi/3$$ [2pts]

(since $0 < \varphi_0 < \pi/2$)

$$m = \int_0^{2\pi}\int_0^{\pi/3}\int_1^2 \rho\ \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$ [2pts]

$$= \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^4}{4}\right]\Big|_1^2 \sin\varphi\, d\varphi\, d\theta = \frac{15}{4}\int_0^{2\pi}\int_0^{\pi/3}\sin\varphi\, d\varphi\, d\theta$$ [1pt]

$$= \frac{15}{4}\int_0^{2\pi}[-\cos\varphi]\Big|_0^{\pi/3} d\theta = \frac{15}{4}\cdot(1-\cos\pi/3)\,2\pi$$ [1pt]

$$= \frac{15}{4}\left(1-\tfrac{1}{2}\right)2\pi = \frac{15\pi}{4}$$ [1pt]

4 (a) The desired transformation is

$$u = x/4 \ , \quad v = y/3 \ , \qquad \text{[2pts]}$$

by which

$$x = 4u \quad \text{and} \quad y = 3v \qquad \text{[1pt]}$$

(b) The double integral over the region S of uv-plane reads

$$I = \iint_S (1+u^2+v^2)^{5/2} \underbrace{|J|}_{\text{Jacobian}} \, dA \qquad \text{[1pt]}$$

where S is the region bounded by the circle $u^2+v^2 = 1$.

[Grader: Students are not required to sketch region S.]

Jacobian: $$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 0 & 3 \end{vmatrix} = 12 . \qquad \text{[2pts]}$$

Thus,

$$I = 12 \iint_S (1+u^2+v^2)^{5/2} \, dA$$

To compute I, we use polar coordinates: $u = r\cos\theta$, $v = r\sin\theta$ where $0 \le r \le 1$, $0 \le \theta < 2\pi$ [1pt]

$$I = 12 \int_0^{2\pi} \int_0^1 (1+r^2)^{5/2} \, r \, dr \, d\theta \qquad \text{[2pts]}$$

$$= 12 \cdot 2\pi \left[\frac{1}{7} (1+r^2)^{7/2} \right] \Big|_0^1 = 24\pi \, \frac{1}{7} \left(2^{7/2} - 1\right) \qquad \text{[1pt]}$$