

$$\boxed{\text{I}} \quad \vec{F}(x, y, z) = yz^2\hat{i} + xz^2\hat{j} + [2xyz + 2\cos(2z)]\hat{k}$$

$$(a) \quad \text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2 & 2xyz + 2\cos(2z) \end{vmatrix} \quad \underline{\underline{[1pt]}}$$

$$= \left\{ \frac{\partial}{\partial y} [2xyz + 2\cos(2z)] - \frac{\partial}{\partial z} (xz^2) \right\} \hat{i} - \left[ \frac{\partial}{\partial x} (2xyz + 2\cos(2z)) - \frac{\partial}{\partial z} (yz^2) \right] \hat{j} \\ + \left[ \frac{\partial}{\partial x} (xz^2) - \frac{\partial}{\partial y} (yz^2) \right] \hat{k} \\ = (2xz - 2xz) \hat{i} - (2yz - 2yz) \hat{j} + (z^2 - z^2) \hat{k} = \vec{0} \quad \underline{\underline{[1pt]}}$$

It is indeed possible to have  $\vec{F} = \nabla f$ .

$$\text{We must have: } yz^2 = \frac{\partial f}{\partial x} \quad \textcircled{1} \quad xz^2 = \frac{\partial f}{\partial y} \quad \textcircled{2} \quad 2xyz + 2\cos(2z) = \frac{\partial f}{\partial z} \quad \textcircled{3}$$

$$\underline{\underline{[1pt]}}$$

$$\textcircled{1}: \quad yz^2 = \frac{\partial f}{\partial x} \Rightarrow f(x, y, z) = yz^2x + C_1(y, z) \quad \underline{\underline{[1pt]}}$$

$$\textcircled{2}: \quad xz^2 = \frac{\partial f}{\partial y} = z^2x + \frac{\partial C_1(y, z)}{\partial y} \Rightarrow \frac{\partial C_1}{\partial y} = 0 \Rightarrow C_1(y, z) = C(z) \quad \underline{\underline{[1pt]}}$$

$$\textcircled{3}: \quad 2xyz + 2\cos(2z) = \frac{\partial f}{\partial z} = 2yzx + \frac{dC}{dz} \Rightarrow \frac{dC}{dz} = 2\cos(2z)$$

$$\Rightarrow C(z) = \sin(2z) + K \quad \underline{\underline{[1pt]}}$$

$$\text{Hence: } f(x, y, z) = yz^2x + \sin(2z) + K$$

(b) By the Fundamental Theorem of Line Integrals:

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(t=1)) - f(\vec{r}(t=-1)) \quad \underline{\underline{[1pt]}}$$

$$\vec{r}(t=-1) = -\hat{i} + \hat{j} - \frac{\pi}{4}\hat{k} : f(-1, 1, -\frac{\pi}{4}) = -\left(\frac{\pi}{4}\right)^2 - 1 + K \quad \} \quad \underline{\underline{[1pt]}}$$

$$\vec{r}(t=1) = -\hat{i} + \hat{j} + \frac{\pi}{4}\hat{k} : f(-1, 1, \frac{\pi}{4}) = -\left(\frac{\pi}{4}\right)^2 + 1 + K \quad \} \quad \underline{\underline{[1pt]}}$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \left\{ -\left(\frac{\pi}{4}\right)^2 + 1 + K \right\} - \left\{ -\left(\frac{\pi}{4}\right)^2 - 1 + K \right\} = 2 \quad \underline{\underline{[1pt]}}$$

[2]  $I = \int_C M dx + N dy$  where  $M = y^4$ ,  $N = x^3$ .

By Green's theorem we have

$$\begin{aligned} I &= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA & [2 \text{ pts}] \\ &= \iint_R (3x^2 - 4y^3) dA & [2 \text{ pts}] \end{aligned}$$

where  $R$  is the interior of the square, defined by  
 $-1 \leq x \leq 1$  and  $-1 \leq y \leq 1$

[Grader]: Students do not need to sketch  $R$ . ]

Hence  $I = \int_{-1}^1 \int_{-1}^1 (3x^2 - 4y^3) dy dx$  [2 pts]

$$\begin{aligned} &= \int_{-1}^1 (3x^2 y - y^4) \Big|_{y=-1}^1 dx \\ &= \int_{-1}^1 6x^2 dx & [1 pt] \\ &= 6 \left( \frac{x^3}{3} \right) \Big|_{-1}^1 = 4 & [1 pt] \end{aligned}$$

[3] By Stokes' Theorem :

$$I = \int_C \vec{F} \cdot d\vec{r} = \iint_{\Sigma} \text{curl } \vec{F} \cdot \hat{n} \, dS \quad [1pt]$$

where the unit normal vector  $\hat{n}$  is directed upward. [1pt]

Parametrize  $\Sigma$  by using  $(x, y)$ :

$$\vec{r}(x, y) = x \hat{i} + y \hat{j} + (1-x-y) \hat{k} \quad [1pt]$$

where  $(x, y)$  lies in region  $R$  that is the interior of a triangle, in  $xy$ -plane, with vertices  $(0, 0), (1, 0), (0, 1)$ . [1pt]

[Grader]: Students do not need to sketch  $R$ . ]

$$\begin{aligned} \vec{r}_x &= \hat{i} - \hat{k} \\ \vec{r}_y &= \hat{j} - \hat{k} \end{aligned} \quad : \quad \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \hat{i} + \hat{j} + \hat{k} \quad [1pt]$$

$$\text{Unit normal vector to } \Sigma : \hat{n} = \pm \frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|} = \pm \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$$

For  $\hat{n}$  being directed upward: Choose  $(+)$  sign. [1pt]

[Grader]: Students may use any other reasonable method to express  $\vec{r}_x \times \vec{r}_y$  and find right sign for  $\hat{n}$ ]

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 2z & -x \end{vmatrix} = -2\hat{i} + \hat{j} - 3\hat{k} \quad [1pt]$$

$$I = (+) \iint_R (-2\hat{i} + \hat{j} - 3\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) \, dA \quad [1pt]$$

$$= + \iint_R (-2 + 1 - 3) \, dA = -4 \iint_R \, dA \quad [1pt]$$

$$= -4 \left( \text{Area of } R \right) = -4 \cdot \frac{1}{2} = -2 \quad [1pt]$$

[Grader]: Students may use any other reasonable method to compute the double integral over region  $R$ . ]

4 By the Divergence Theorem:

$$\iint_{\Sigma} \vec{F} \cdot \hat{n} dS = \iiint_D \operatorname{div} \vec{F} dV \quad [3 \text{ pts}]$$

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (3y) + \frac{\partial}{\partial z} (5z) = 10 \quad [3 \text{ pts}]$$

$$\begin{aligned} \iint_{\Sigma} \vec{F} \cdot \hat{n} dS &= 10 \iiint_D dV \\ &= 10 (\text{Volume of } D) \quad [2 \text{ pts}] \\ &= 10 \cdot \underbrace{1}_{\text{height}} \cdot \underbrace{\pi 1^2}_{\text{area of base}} = 10\pi \quad [2 \text{ pts}] \end{aligned}$$

Grader: Students may use any other reasonable method to compute the triple integral, e.g., as an iterated integral.

For example, by use of cylindrical coordinates:

$$\begin{aligned} \iint_{\Sigma} \vec{F} \cdot \hat{n} dS &= 10 \int_0^{2\pi} \int_0^1 \int_1^2 r dz dr d\theta \\ &= 10 \int_0^{2\pi} \int_0^1 r dr d\theta \quad \text{If any student follows this} \\ &= 10 \int_0^{2\pi} \left( \frac{r^2}{2} \right) \Big|_0^1 d\theta \quad \text{procedure, distribute points consistently.} \\ &= S \cdot 2\pi = 10\pi \quad ] \end{aligned}$$