MATH462, Sec. 0101: Partial Differential Equations
Department of Mathematics, UMCP **Fall 2018**
Exam 1 **Handed out**: Tuesday, 10/16/18

Answer all questions. Make sure that you explain all your steps and justify your answers. Cross out what is not meant to be part of your answer.

1. [30pts] Consider the PDE $au_x + bu_y = 0$ obeyed by the unknown function $u(x, y)$; a and b are nonzero real constants.

   (a)[15pts] Find the general solution $u(x, y)$ for the above PDE. **Note:** You may apply any reasonable method. Explain what you do clearly.

   (b) Suppose that the function $u(x, y)$ satisfies the above PDE and, in addition, it also obeys a certain condition. Consider separately the following cases for this condition.

   (b.i)[8pts] $u(x, 0) = \phi(x)$, where ϕ is a given function. Find $u(x, y)$. Is it unique?

   (b.ii)[7pts] Now suppose that u satisfies the condition $u = 1$ along the whole line described by $bx = ay$ in the (x, y)-plane. Is the resulting solution $u(x, y)$ unique in this case? Explain.

2. [35pts] Suppose that $u(x, t)$ satisfies the following (Cauchy) problem for the wave equation:

$$u_{tt} = c^2 u_{xx} \ , \ -\infty < x < \infty \ , \ t > 0 \ ; \quad u(x, 0) = g(x) \ , \quad u_t(x, 0) = h(x) \ ,$$

   where $g(x)$ and $h(x)$ are given functions.

   (a)[15pts] Derive an expression for $u(x, t)$ in terms of the functions g and h. **Note:** In this part, you are asked to determine from scratch the d'Alembert solution for the wave equation.

   (b)[10pts] In particular, consider the case with $g(x) = x$ and $h(x) = x + 1$. By using your result from part 2(a), determine $u(x, t)$. Simplify your final result as much as possible. **Note:** In this part, you are asked to write down an explicit formula for $u(x, t)$ via the d'Alembert formula.

   (c)[10pts] Now consider the Cauchy problem displayed in the beginning of Problem 2. For what $g(x)$ and $h(x)$ entering the initial conditions would it be possible to have an exact solution given by the formula $u(x, t) = g(x) + t\, h(x)$ for all x and all $t > 0$? **Hint:** Assume that the last formula for u holds, and enforce the wave equation. What do you obtain for g and h?

3. [35pts] In this problem, you are asked to solve an inhomogeneous-Dirichlet diffusion problem for $w(x, t)$ on the *half-line*:

$$w_t - kw_{xx} = f(x, t) \ , \ \text{for } x > 0 \ , \ t > 0 \qquad (k > 0) \ ,$$
$$w(0, t) = h(t) \quad \text{for } t > 0 \ , \quad w(x, 0) = \phi(x) \quad \text{for } x > 0 \ ; \ f(x, t),\ \phi(x),\ h(t) \text{ are given functions.}$$

   (a)[10pts] Define a suitable function $V(x, t)$ in terms of $w(x, t)$ so that $V(0, t) = 0$. What PDE and initial condition are satisfied by this $V(x, t)$ for $x > 0$? Explain. **Note:** For full credit, choose the simplest possible definition for V.

   (b)[15pts] By the reflection method, construct a diffusion problem for $V(x, t)$ extended to the whole real axis, $-\infty < x < \infty$. **Note:** For full credit, explain how the reflection method works!

   (c)[10pts] Derive a formula for $V(x, t)$ and then for $w(x, t)$, $x > 0$, by use of 3(b). You may use, without proof, the formula solving the diffusion equation with a source on the whole real axis.