MATH462, Sec. 0101: Partial Differential Equations
Department of Mathematics, UMCP **Fall 2018**
Exam 2 **Handed out**: Tuesday, 11/13/18

Answer all questions. Make sure that you explain all your steps and justify your answers. Cross out what is not meant to be part of your answer.

1. [35pts] You are asked to study the following problem by separation of variables:

$$u_t = ku_{xx} \ , \ \text{for } 0 < x < l \ , \ t > 0 \qquad (k > 0) \ ;$$
$$u(0,t) = 0 \ , \ u_x(l,t) = 0 \ (\text{for } t > 0) \ ;$$
$$u(x,0) = \phi(x) \ (\text{for } 0 < x < l) \ .$$

(a)[5pts] By trying $u(x,t) = X(x)T(t)$, derive an ODE and boundary conditions for $X(x)$, and derive an ODE for $T(t)$. **Note:** Notice the different types of boundary conditions at $x = 0$, l.

(b)[10pts] Determine the suitable eigenfunctions and eigenvalues for $X(x)$ from part 1(a).

(c)[5pts] Solve the ODE for $T(t)$ from part 1(a).

(d)[15pts] By using the results of parts 1(a)-(c) above, derive an expression for the solution $u(x,t)$, which must satisfy $u(x,0) = \phi(x)$ for $0 < x < l$ with any given function $\phi(x)$.
Note: In part 1(d), you might want to use the identities $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ and $(\sin A)^2 = [1 - \cos(2A)]/2$.

2. [30pts] Consider a string that is fixed at the end $x = 0$ and is free at the end $x = l$ except that a load is attached to the right end, at $x = l$. The displacement $u(x,t)$ satisfies the problem

$$u_{tt} = c^2 u_{xx} \quad \text{for } 0 < x < l, \ t > 0 \ ; \ u(0,t) = 0 \ , \ u_{tt}(l,t) = -ku_x(l,t) \ ,$$

where k is a positive constant ($k > 0$).

(a)[10pts] By applying separation of variables, formulate the related eigenvalue problem.

(b)[20pts] Find the equation for the positive eigenvalues λ and obtain the corresponding eigenfunctions of the problem formulated in part 2(a). Are there any negative eigenvalues λ? *Explain in detail.*

3. [35pts] Consider the $u(x,t)$ that obeys the diffusion equation $u_t = u_{xx}$, for $0 < x < \pi$, $t > 0$.

(a)[15pts] Find all separated solutions of this equation that satisfy $u_x(0,t) = 0$ and $u_x(\pi,t) = 0$.

(b)[20pts] Find the unique $u(x,t)$ that satisfies $u_t = u_{xx}$ and $u_x(0,t) = 0 = u_x(\pi,t)$ and the initial condition $u(x,0) = \sin^4 x$ for $0 < x < \pi$. *Simplify your answer as much as possible.*
Note: In part 3(b) you should use the result of 3(a). By use of trigonometric identities you may wish to write $(\sin^2 x)^2$ as a finite sum involving convenient trigonometric functions (i.e., convenient cosines or sines). For example, you may use $(\cos x)^2 = [1+\cos(2x)]/2$ and $(\sin x)^2 = [1 - \cos(2x)]/2$.