1 (a) Let $S_n(z) = 1 + z + z^2 + \dots + z^n$

$\Rightarrow \quad z S_n(z) = z + z^2 + \dots + z^{n+1} = S_n(z) - 1 + z^{n+1}$ (2pts)

Thus, $z S_n = S_n - 1 + z^{n+1} \Leftrightarrow S_n(z) = \dfrac{1 - z^{n+1}}{1 - z}$. (2pts)

(b) Let $F_n(\theta) = 1 + \cos\theta + \cos(2\theta) + \dots + \cos(n\theta)$

Clearly, $F_n(\theta) = \operatorname{Re} S_n(e^{i\theta})$. Thus, it suffices to compute $S_n(e^{i\theta})$ (2pts)

$$S_n(e^{i\theta}) = \frac{1 - (e^{i\theta})^{n+1}}{1 - e^{i\theta}} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \quad \text{[from part (a)]} \quad \text{(1pt)}$$

$$= \frac{e^{i(n+1)\theta/2}}{e^{i\theta/2}} \; \frac{e^{i(n+1)\theta/2} - e^{-i(n+1)\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} \quad \text{(1pt)}$$

$$= e^{in\theta/2} \; \frac{\cancel{2i}\sin[(n+1)\theta/2]}{\cancel{2i}\sin(\theta/2)} \quad \text{(1pt)}$$

$$\Rightarrow \quad F_n(\theta) = \operatorname{Re} S_n(e^{i\theta}) = \cos\left(\frac{n\theta}{2}\right) \frac{\sin[(n+1)\theta/2]}{\sin(\theta/2)} \quad \text{(1pt)}$$

2 (a) A function $u(x,y)$ is harmonic in $\mathcal{D}$ if it has continuous 2nd partial derivatives in $\mathcal{D}$ and $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ in $\mathcal{D}$. (1pt)

(b) Consider analytic $f(z) = u(x,y) + i v(x,y)$ in $\mathcal{D}$

By Cauchy-Riemann eqs:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{in } \mathcal{D}.$$

Take $\frac{\partial}{\partial x}(\cdot)$ of 1st eqn, $\frac{\partial}{\partial y}(\cdot)$ of 2nd eqn and add them: (1pt)

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0 \quad \text{because } v_{yx} = v_{xy} \text{ by continuity of 2nd deriv's.}$$

Take $\frac{\partial}{\partial y}(\cdot)$ of 1st eqn, $\frac{\partial}{\partial x}(\cdot)$ of 2nd eqn and subtract them: (1pt)

$$0 = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(\partial u/\partial y\right) = v_{yy} + v_{xx}.$$ Thus u, v are harmonic.

(c) $\dfrac{\partial v}{\partial x} = 6xy + 2x \,, \quad \dfrac{\partial^2 v}{\partial x^2} = 6y + 2$ (1pt)

$\dfrac{\partial v}{\partial y} = 3x^2 - 3y^2 - 2y \,, \quad \dfrac{\partial^2 v}{\partial y^2} = -6y - 2.$ (1pt)

$\Rightarrow \quad \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 6y + 2 - (6y+2) = 0.$ (1pt)

(d) By C-R eqns: (1) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2 - 2y$ (1pt)

arbitrary real function of y.

$\Rightarrow \quad u(x,y) = \underbrace{x^3 - 3y^2x - 2yx + C_1(y)}_{\partial/\partial y}$ (1pt)

(2) $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \iff -6yx - 2x + C_1'(y) = \overbrace{-6xy - 2x}^{-\partial v/\partial x}$ (1pt)

$\iff \quad C_1'(y) = 0 \quad \iff \quad C_1(y) = K =$ real const. (1pt)

Thus, $\quad u(x,y) = x^3 - 3y^2x - 2yx + K$

$\boxed{3}$ Let $w = e^{iz}$: $\quad \sin z = \dfrac{w - w^{-1}}{2i} \,, \quad \cos z = \dfrac{w + w^{-1}}{2}.$ (2pts)

$\sin z = -i\lambda \cos z \iff \dfrac{w - w^{-1}}{\cancel{2}i} = -i\lambda \dfrac{w + w^{-1}}{\cancel{2}}$

$\iff \quad w - \frac{1}{w} = \lambda\left(w + \frac{1}{w}\right) \overset{w \neq 0}{\iff} w^2 - 1 = \lambda(w^2 + 1)$ (1pt)

$\iff \quad w^2(1 - \lambda) = 1 + \lambda$ or $\quad w^2 = \dfrac{1+\lambda}{1-\lambda}.$ (1pt)

Thus, we arrive at the equation: $\quad e^{2iz} = \dfrac{1+\lambda}{1-\lambda} > 1 \quad \left(\mathrm{Arg}\left(\dfrac{1+\lambda}{1-\lambda}\right) = 0\right)$ (2pts)

Thus,

$2iz = \log\left(\dfrac{1+\lambda}{1-\lambda}\right) = \mathrm{Log}\left(\dfrac{1+\lambda}{1-\lambda}\right) + i2k\pi \,; \quad k = 0, \pm 1, \pm 2, \ldots$ (3pts)

$\iff \quad z = \dfrac{1}{2i} \underbrace{\mathrm{Log}\left(\dfrac{1+\lambda}{1-\lambda}\right)}_{>0} + k\pi$ (1pt)

$\boxed{4}$ (a) $\displaystyle f(z) = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z+3}$. (2pts)

$$A = \lim_{z\to 0} [z f(z)] = \lim_{z\to 0} \frac{2+i}{(z+1)(z+3)} = \frac{2+i}{3} \qquad (1pt)$$

$$B = \lim_{z\to -1} [(z+1) f(z)] = \lim_{z\to -1} \frac{2+i}{z(z+3)} = \frac{2+i}{-1\cdot 2} = -\frac{1}{2}(2+i) \qquad (1pt)$$

$$C = \lim_{z\to -3} [(z+3) f(z)] = \lim_{z\to -3} \frac{2+i}{z(z+1)} = \frac{2+i}{-3(-2)} = \frac{1}{6}(2+i) \qquad (1pt)$$

(b) $$f(z) = \frac{2i}{[(z+i)(z-i)]^2} = \frac{2i}{(z+i)^2(z-i)^2} = \frac{A}{z+i} + \frac{B}{(z+i)^2} + \frac{C}{z-i} + \frac{D}{(z-i)^2}. \quad 1$$
(1pt)

$$B = \lim_{z\to -i} \left[\cancel{(z+i)^2} \frac{2i}{\cancel{(z+i)^2}(z-i)^2} \right] = \frac{2i}{(-i-i)^2} = \frac{2i}{-4} = -i/2 \qquad (1pt)$$

$$D = \lim_{z\to i} [(z-i)^2 f(z)] = \lim_{z\to i} \frac{2i}{(z+i)^2} = \frac{2i}{(i+i)^2} = -i/2 \qquad (1pt)$$

$$A = \lim_{z\to -i} \frac{d}{dz} \left[\cancel{(z+i)^2} \frac{2i}{\cancel{(z+i)^2}(z-i)^2} \right] = \lim_{z\to -i} [-2\cdot 2i\ (z-i)^{-3}] \qquad (1pt)$$

$$= -4i\ (-i-i)^{-3} = \frac{-4i}{(-2i)^3} = \frac{-4i}{-8(-i)} = -1/2$$

$$B = \lim_{z\to i} \frac{d}{dz} \left[\cancel{(z-i)^2} \frac{2i}{(z+i)^2\cancel{(z-i)^2}} \right] = \lim_{z\to i} [2i(-2)(z+i)^{-3}] \qquad (1pt)$$

$$= 2i(-2)(2i)^{-3} = \frac{-4i}{-8i} = 1/2.$$