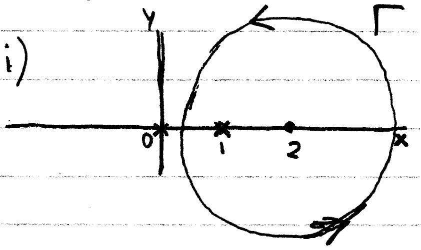
1 (a) Possible singular points are: $z=0,1$.

• $z=0$ is a pole of order 1: Notice that $f(z)=\dfrac{g(z)}{z}$ where $g(z)$ is analytic at $z=0$ and $g(0)\neq 0$. [1pt]

• $z=1$ is a pole of order 2: $f(z)=\dfrac{g(z)}{(z-1)^2}$ where $g(1)\neq 0$ and $g(z)$ is analytic at $z=1$. [1pt]

(b) i)



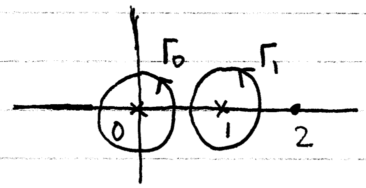
The closed contour Γ encloses only the double pole at $z=1$. By the Generalized Cauchy Integral Theorem:

$$I=\int_\Gamma \frac{(z^2+z-1)/z}{(z-1)^2}\,dz$$

$$=\frac{2\pi i}{1!}\lim_{z\to 1}\frac{d}{dz}\left(\frac{z^2+z-1}{z}\right)= \quad \text{[2pts]}$$

$$=2\pi i\lim_{z\to 1}\frac{(2z+1)z-(z^2+z-1)}{z^2}=2\pi i\lim_{z\to 1}\frac{z^2+1}{z^2}=4\pi i \quad \text{[2pts]}$$

ii) The close contour encloses both singularities at $z=0,1$



$$I=\left(\int_{\Gamma_0}+\int_{\Gamma_1}\right)\frac{z^2+z-1}{z(z-1)^2}\,dz \quad \text{[1pt]}$$

$$=\int_{\Gamma_0}\frac{z^2+z-1}{z(z-1)^2}\,dz+\int_{\Gamma_1}\frac{z^2+z-1}{z(z-1)^2}\,dz=\int_{\Gamma_0}\frac{(z^2+z-1)/(z-1)^2}{z}\,dz+\int_{\Gamma_1}\frac{(z^2+z-1)/z}{(z-1)^2}\,dz \quad \text{[1pt]}$$

$$=\underbrace{2\pi i\left(\frac{z^2+z-1}{(z-1)^2}\right)\Bigg|_{z=0}}_{\text{simple pole (order 1)}}+\underbrace{\frac{2\pi i}{1!}\lim_{z\to 1}\frac{d}{dz}\left(\frac{z^2+z-1}{z}\right)}_{\text{from part b.i) - pole of order 2}} \quad \text{[2pts]}$$

$$=-2\pi i+4\pi i=2\pi i$$

2 (a)

$$H(0) = \frac{1}{2\pi i}\int_C \frac{e^{\zeta} + \zeta^{-1}}{\zeta}\,d\zeta = \frac{1}{2\pi i}\int_C \frac{e^{\zeta}}{\zeta}\,d\zeta + \frac{1}{2\pi i}\int_C \frac{1}{\zeta^2}\,d\zeta \qquad [1pt.]$$

$$= \underbrace{e^0}_{1.5pts} + \underbrace{0}_{1.5pts} = 1$$

by Cauchy Integral Formula (and its Generalized version) [3pts]

(b) Suppose z lies <u>outside z</u>.

$$H(z) = \frac{1}{2\pi i}\int_C \frac{e^{\zeta} + \zeta^{-1}}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_C \frac{e^{\zeta}}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_C \frac{1}{\zeta(\zeta - z)}\,d\zeta \qquad [1pt]$$

The first integral is 0 because z is outside C and, thus, the integrand is analytic inside C. **(Cauchy Integral Theorem)** [2pts]

For the second integral: The closed contour Γ encloses only the pole at $z=0$. By the Cauchy Integral Formula

$$\frac{1}{2\pi i}\int_C \frac{1}{\zeta(\zeta - z)}\,d\zeta = \frac{1}{2\pi i}\int_C \frac{1/(\zeta - z)}{\zeta}\,d\zeta = \frac{1\cdot 2\pi i}{2\pi i}\left(\frac{1}{\zeta - z}\right)\Bigg|_{\zeta=0} = -\frac{1}{z}. \qquad [3pts]$$

Thus:

$$\lim_{z\to i} H(z) = \lim_{z\to i}\left[0 - \frac{1}{z}\right] = -\frac{1}{i} = i \qquad [1pt]$$

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3 (a) The function $f(z)$ has a singularity at $z=3$.
Since $|3 - z_0| = 3$, the radius R of convergence of the Taylor series at $z_0 = 0$ should be $R=3$: $|z| < 3$ [3pts]

$$f(z) = \frac{z-1}{3-z} = \frac{1}{3}\,\frac{z-1}{1 - z/3} = \frac{1}{3}(z-1)\sum_{j=0}^{\infty}\lambda^j = \frac{1}{3}(z-1)\sum_{j=0}^{\infty}\frac{z^j}{3^j} \qquad (\lambda = z/3) \qquad [3pts]$$

$$= \frac{1}{3}\left(\sum_{j=0}^{\infty}\frac{z^{j+1}}{3^j} - \sum_{j=0}^{\infty}\frac{z^j}{3^j}\right) = \frac{1}{3}\left(\sum_{j=1}^{\infty}\frac{z^j}{3^{j-1}} - 1 - \sum_{j=1}^{\infty}\frac{z^j}{3^j}\right) \qquad [1pt]$$

$$= \frac{1}{3}\left(3\sum_{j=1}^{\infty}\frac{z^j}{3^j} - 1 - \sum_{j=1}^{\infty}\frac{z^j}{3^j}\right) = -\frac{1}{3} + \frac{2}{3}\sum_{j=1}^{\infty}\frac{z^j}{3^j} \quad \text{if } \left|\frac{z}{3}\right| < 1. \qquad [1pt]$$

(b)

$$e^{f(z)} = 1 + f(z) + \frac{1}{2!}f(z)^2 + \dots + \frac{1}{n!}f(z)^n + \dots \qquad [1pt]$$

$$= 1 + \left(\frac{z-1}{3-z}\right) + \frac{1}{2!}\left(\frac{z-1}{3-z}\right)^2 + \dots + \frac{1}{n!}\left(\frac{z-1}{3-z}\right)^n + \dots$$

This series has infinite number of negative powers of $z-3$.
Thus, $z=3$ is an <u>essential singularity</u> for $f(z)$. [1pt]

$\boxed{4}$ Decompose in partial fractions:

$$f(z) = \frac{1}{z(z-i)} = \frac{A}{z} + \frac{B}{z-i} \qquad \text{[3pts]}$$

$$A = \lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{1}{z-i} = i \qquad \text{[1pt]}$$

$$B = \lim_{z\to i} (z-i) f(z) = \lim_{z\to i} \frac{1}{z} = -i \qquad \text{[1pt]}$$

Thus, $f(z) = \frac{i}{z} - \frac{i}{z-i}$

For $|z| > 1 \iff \left|\frac{1}{z}\right| < 1$ we try to expand in geometric series.

The expansion parameter should be λ with $|\lambda| = \left|\frac{1}{z}\right|$. $(|\lambda| < 1)$

$$\frac{i}{z-i} = \frac{i}{z} \frac{1}{1-\underbrace{i/z}_{\lambda}} = \frac{i}{z} \sum_{j=0}^{\infty} \left(\frac{i}{z}\right)^j \quad \text{if } \left|\frac{i}{z}\right| < 1 \iff |z| > 1 \qquad \text{[3pts]}$$

$$\iff \frac{i}{z-i} = i \sum_{j=0}^{\infty} \frac{i^j}{z^{j+1}}$$

$$f(z) = \frac{i}{z} - i\left(\frac{1}{z} + \sum_{j=1}^{\infty} \frac{i^j}{z^{j+1}}\right) = -i \sum_{j=1}^{\infty} \frac{i^j}{z^{j+1}} \qquad \text{[2pts]}$$