MATH 463 SOLUTIONS TO EXAM 3 Spring '15

1

$$I = \int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{3+2\cos\theta}$$ since integrand is even in θ. (2pts)

Let $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta$ (2pts)

$$\cos\theta = \tfrac{1}{2}(z+z^{-1})$$

$$\Rightarrow I = \frac{1}{2}\oint_{|z|=1} \frac{dz}{iz}\,\frac{1}{3+z+\frac{1}{z}} = \frac{1}{2i}\oint_{|z|=1} \frac{1}{z^2+3z+1}\,dz$$ (2pts)

The integrand has poles of order 1 at z where: $z^2+3z+1=0 \Leftrightarrow z = \frac{-3\pm\sqrt{5}}{2}$ (2pts)

The pole $\frac{-3+\sqrt{5}}{2}$ lies inside the disk $|z|=1$, but $\frac{-3-\sqrt{5}}{2}$ does not. (2pts)

By Residue Theory:

$$I = \frac{1}{2i}\,2\pi i\,\operatorname{Res}\left(\frac{1}{z^2+3z+1};\ \frac{-3+\sqrt{5}}{2}\right) =$$ (2pts)

$$= \pi \lim_{z\to\frac{-3+\sqrt{5}}{2}} \left[\left(z-\frac{-3+\sqrt{5}}{2}\right)\frac{1}{z^2+3z+1}\right] = \pi \lim_{z\to\frac{-3+\sqrt{5}}{2}}\left(\frac{1}{2z+3}\right)$$ (2pts)

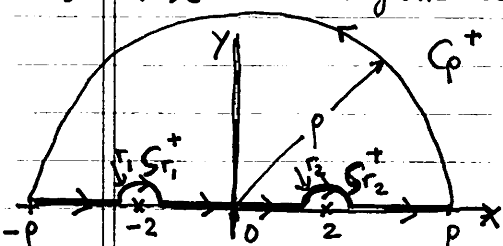
$$= \pi\,\frac{1}{-3+\sqrt{5}+3} = \frac{\pi}{\sqrt{5}}$$ (1pt)

Note: Here, we applied the formula $\operatorname{Res}(f;z_0) = \frac{g(z_0)}{h'(z_0)}$ if $f(z) = \frac{g(z)}{h(z)}$ and z_0 is a zero of 1st order of $h(z)$ while $g(z_0)\neq 0$; and $g(z)$ and $h(z)$ are analytic at z_0.

2

$$I = \lim_{\substack{\rho\to\infty\\ r_1,r_2\to 0^+}} \left\{ \int_{-\rho}^{-2-r_1} + \int_{-2+r_1}^{2-r_2} + \int_{2+r_2}^{\rho} \right\} \frac{xe^{i3x}}{x^2-4}\,dx = \lim_{\substack{\rho\to\infty\\ r_1,r_2\to 0^+}} \int_{\gamma_{\rho,r_1,r_2}} \frac{ze^{i3z}}{z^2-4}\,dz$$

- $x \Rightarrow z$: The integrand becomes $f(z) = \frac{ze^{i3z}}{z^2-4}$ w/ poles of order 1 at $z=\pm 2$ (2pts)



- Close path: $\Gamma_{\rho,r_1,r_2} = \gamma_{\rho,r_1,r_2} + S_{r_1}^+ + S_{r_2}^+ + C_\rho^+$ (3pts)

where $\gamma_{\rho,r_1,r_2} = (-\rho,-2-r_1) + (-2+r_1, 2-r_2) + (2+r_2,\rho)$

- Residue Thm: $\oint_{\Gamma_{\rho,r_1,r_2}} f(z)\,dz = 0$ (2pts)

where $\oint_{\Gamma_{\rho,r_1,r_2}} f(z)\,dz = \left(\int_{\gamma_{\rho,r_1,r_2}} + \int_{S_{r_1}^+} + \int_{S_{r_2}^+} + \int_{C_\rho^+}\right) f(z)\,dz.$; $f(z) = \frac{ze^{i3z}}{z^2-4}$ (1pt)

Note:

- $\lim_{\substack{\rho\to\infty\\ r_1,r_2\to 0^+}} \int_{\gamma_{\rho,r_1,r_2}} f(z)\,dz = I$; • $\lim_{\rho\to\infty}\int_{C_\rho^+} \frac{ze^{i3z}}{z^2-4}\,dz = 0$ by Jordan's Lemma. (2pts)
- $\lim_{r_1\to 0^+}\int_{S_{r_1}^+} \frac{ze^{i3z}}{z^2-4}\,dz = -i\pi\operatorname{Res}(f;-2) = -i\pi\frac{(-2)e^{-6i}}{-4}$ (2pts)
- $\lim_{r_2\to 0^+}\int_{S_{r_2}^+} \frac{ze^{i3z}}{z^2-4}\,dz = -i\pi\operatorname{Res}(f;2) = -i\pi\frac{2e^{6i}}{4}$ (2pts)

Finally: $0 = I - i\pi e^{-6i}/2 - i\pi e^{6i}/2 \Longleftrightarrow \boxed{I = i\pi\cos 6}$ (1pt)

3 (a) $\lambda=0, \mu=1$: $f(z)=\frac{1}{1-z}$

$z=1$ is a pole of order 1. (1pt)

$f(z)= 1+z+\dots+z^n+\dots = \sum_{n=0}^{\infty} z^n, \quad |z|<1.$ (Taylor series) (1pt)

(b) $\lambda=1, \mu=0$: $f(z)=\cos(1/z^2)$

$z=0$ is an essential singularity (1pt)

$f(z) = 1 - \frac{1}{2!}\left(\frac{1}{z^2}\right)^2 + \frac{1}{4!}\left(\frac{1}{z^2}\right)^4 + \dots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{z^2}\right)^{2n}, \quad |z|>0$ (2pts)

(Laurent series)

using $\cos w = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!} w^{2n}$ which converges for $|w|<\infty$; $w = 1/z^2$

(c) $\lambda=1$ and $\mu=1$: $f(z) = \frac{\cos(1/z^2)}{1-z}$;

Method I: $I = \frac{1}{2\pi i}\int_C \frac{\cos(1/z^2)}{1-z} = \text{Res}(f;0)$ (1pt)

We need to find the residue of $f(z)$ at the essential singularity $z=0$

To do this, we need to multiply the Taylor series of $\frac{1}{1-z}$ with the Laurent series for $\cos(1/z^2)$; and single out the $1/z$ term!

$$f(z) = \left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{z^2}\right)^{2n}\right]\cdot\left(\sum_{m=0}^{\infty} z^m\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{1}{z^{2n-m}} \quad \text{(1pt)}$$

The a_{-1}/z term comes from setting: $2n-m=1 \Leftrightarrow \underline{m=2n-1}, \underline{n\geq 1}$ (because $m\geq 0$)

Thus: $$\frac{a_{-1}}{z} = \sum_{\substack{n=1\\(n\geq 1)}}^{\infty}\frac{(-1)^n}{(2n)!}\left.\frac{1}{z^{2n-m}}\right|_{2n-m=1 \Leftrightarrow m=2n-1}$$ [setting $m=2n-1$]

$$= \frac{1}{z}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}$$ (1pt)

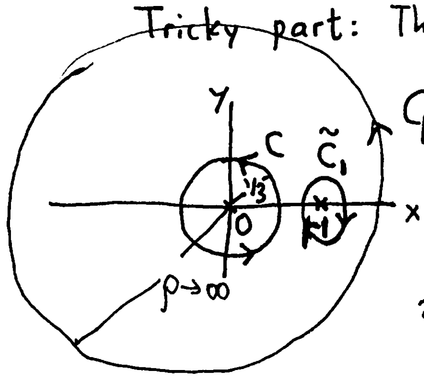
$$\Rightarrow a_{-1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}1^{2n} = \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}1^{2n}}_{\text{expansion of } \cos(w) \text{ for } w=1} - \left[\frac{(-1)^n}{(2n)!}1^{2n}\right]\Big|_{n=0}$$

$$= \cos 1 - 1$$ (2pts)

Thus: $I = \cos 1 - 1$

Method II: Deform the contour C to a large circle w/ radius $\rho \to \infty$.

~~Tricky part~~: This large circle has a nonzero contribution as $\rho \to \infty$.



By deforming C to C_ρ we have to take into account the simple pole at $z=1$

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi i}\oint_{\tilde{C}_1} f(z)\,dz + \frac{1}{2\pi i}\oint_{C_\rho} f(z)\,dz \qquad (1)$$

where $$\frac{1}{2\pi i}\oint_{\tilde{C}_1} f(z)\,dz = -\operatorname{Res}(f;1) = -\lim_{z\to 1}\left[(z-1)\,\frac{\cos(1/z^2)}{1-z}\right]$$

$$= +\cos 1.$$

Consider the integral over C_ρ as $\rho \to \infty$.

Parametrize C_ρ: $z = \rho e^{it}$, $0 \le t \le 2\pi$

$$\frac{1}{2\pi i}\oint_{C_\rho} f(z)\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\cos\left(1/(\rho e^{it})^2\right)}{1-\rho e^{it}}\,\underbrace{i\rho e^{it}\,dt}_{dz}$$

If ρ is large ($\rho\to\infty$): $$\frac{1}{1-\rho e^{it}} \approx \frac{1}{-\rho e^{it}}$$

$$\lim_{\rho\to\infty}\frac{1}{2\pi i}\oint_{C_\rho} f(z)\,dz = \lim_{\rho\to\infty}\frac{1}{2\pi \cancel{i}}\int_0^{2\pi}\frac{\cos 0}{-\cancel{\rho}\cancel{e^{it}}}\,\cancel{i}\cancel{\rho}\cancel{e^{it}}\,dt$$

$$= -\frac{1}{2\pi}\cdot 2\pi = -1.$$

Thus: (1) $\Rightarrow$ $I = \cos 1 - 1$ (as $\rho \to \infty$)