MATH 463 Spring '15

SOLUTIONS TO QUIZ 2

NAME:

MATH463, Sec. 0101: In–class Quiz # 2
Tuesday, March 30, 2015

Solve the following 2 problems. Justify your answers. Cross out what is not meant to be part of your final answer. Total number of points: 20.

I. (10 pts) Show how you can calculate the integral

$$I = \int_\Gamma e^z \sin(5z)\ \mathrm{d}z\ ,$$

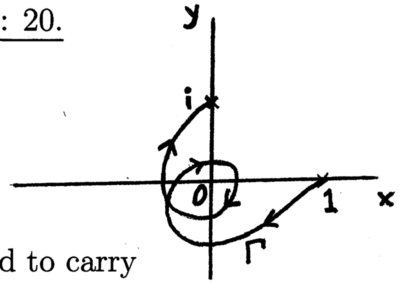


FIG. 1

where the contour Γ is shown in Figure 1. **Note:** You do not need to carry out all the messy algebra; but be sure to show ALL the steps for deriving a definitive answer.

Write $\sin(5z) = \frac{1}{2i}\left(e^{i5z} - e^{-i5z}\right)$ (3pts)

$$I = \int_\Gamma e^z \frac{1}{2i}\left(e^{i5z} - e^{-i5z}\right) dz = \frac{1}{2i}\int_\Gamma \left[e^{(1+5i)z} - e^{(1-5i)z}\right] dz \quad \text{(5pts)}$$

$$= \frac{1}{2i}\left\{ \frac{1}{1+5i}\, e^{(1+5i)z}\Big|_{z=1}^{i} - \frac{1}{1-5i}\, e^{(1-5i)z}\Big|_{z=1}^{i} \right\} \quad \text{(2pts)}$$

$$= \frac{1}{2i}\left\{ \frac{1}{1+5i}\left[e^{(1+5i)i} - e^{1+5i}\right] - \frac{1}{1-5i}\left[e^{(1-5i)i} - e^{1-5i}\right] \right\}$$

In other words, we compute I by finding the anti-derivative of $e^z \sin(5z)$.

CONTINUED ON p. 2

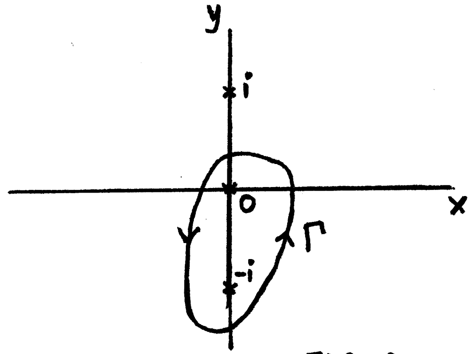


FIG. 2

II.(10pts) Compute the integral

$$I = \int_\Gamma \frac{1}{z(z^2+1)}\, dz \ ,$$

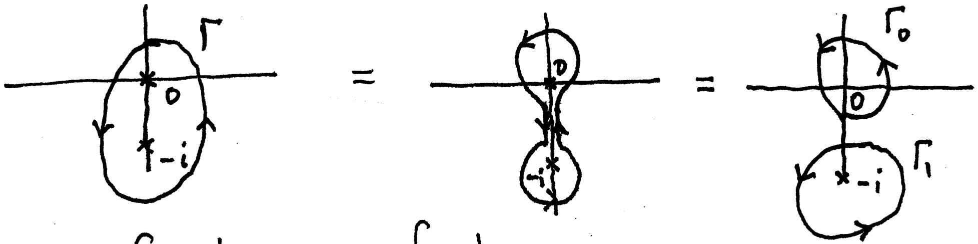
where the contour Γ is shown in Figure 2. **Note:** You should give your answer in the form $I = a + ib$.

The integrand has singularities at $z = 0, \pm i$. (1pt)

The contour Γ encloses two singularities of the integrand:

$z = 0, -i$. (2pts)

We compute the contribution from each singularity



$$\int_\Gamma \frac{1}{z(z^2+1)}\, dz = \int_{\Gamma_0} \frac{dz}{z(z^2+1)} + \int_{\Gamma_1} \frac{dz}{z(z^2+1)}$$ (2pts)

where Γ_0: encircles $z=0$ only and Γ_1 : encircles $z=-i$ only.

$$\int_{\Gamma_0} \frac{dz}{z(z^2+1)} = \int \frac{(1/z^2+1)}{z}\, dz \ \underset{\text{Integral formula}}{\overset{\text{Cauchy}}{=\!=}} \ 2\pi i \left(\frac{1}{z^2+1}\right)\Big|_{z=0} = 2\pi i$$ (2pts)

$$\int_{\Gamma_1} \frac{dz}{\underbrace{z(z^2+1)}_{(z+i)(z-i)}} = \int \frac{\left(1/z(z-i)\right)}{z+i}\, dz \ \underset{\text{IF}}{\overset{\text{Cauchy}}{=\!=}} \ 2\pi i \left(\frac{1}{z(z-i)}\right)\Big|_{z=-i} = 2\pi i \frac{1}{-i(-2i)}$$

$$= -i\pi$$ (2pts)

Thus : $I = 2\pi i + (-i\pi) = i\pi$ (1pt)