Homework #1 Due: Thursday 09/08/22; IN CLASS

1. Compute the following integrals:

$$\int_0^1 \log(x)dx$$

where log*(x)* denotes the natural logarithm of *x* also known as ln*(x)*; and

$$\int_1^\infty \frac{1}{x^a}dx$$

for a fixed parameter $a>1$. What happens for $a=1$?

2. Consider the following function:

$$f : R \to R \ , \ f(x) = \begin{cases} x\log(x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   a. Is this function continuous on R (that is of class C^0)?
   b. Is this function differentiable on R ?
   c. Is this function of class C^1 (that is continuous, differentiable, and has continuous first derivative)?
   d. What is the largest integer k so that f is of class C^k ?

Note: A function is said of class C^k if it is k times differentiable, and the k^{th} derivative is continuous. Thus a function is of class C^0 if it is continuous; a function is of class C^1 if it is continuous, differentiable, and has continuous first derivative; a function is of class C^2 if it is continuous, differentiable, its first derivative is continuous, differentiable, and its second derivative is also continuous. In general a function f is of class C^k if its derivative f' is of class C^{k-1}.

3. Recall Euler's formula $e^{i\theta} = \cos(\theta) + isin(\theta)$ and the geometric series sum formula

$$1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z}.$$

Solve the following equations where the unknown is θ:
   a. $1 + e^{i\theta} = 0$, $-\infty < \theta < \infty$
   b. $1 + e^{i\theta} + e^{2i\theta} = 0$, $-\infty < \theta < \infty$
   c. $1 + e^{i\theta} + e^{2i\theta} + e^{3i\theta} = 0$, $-\infty < \theta < \infty$
   d. $1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = 0$ where n is a fixed positive integer and $-\infty < \theta < \infty$

4. Compute the following integrals:
   a. $\int_{-\pi}^{\pi} e^{-3it}dt$
   b. $\int_0^1 e^{2\pi ix}dx$
   c. $\int_0^1 e^{-2\pi i\omega}e^{4\pi i\omega}d\omega$
   d. $\int_0^1 e^{-2\pi int}dt$, where n is an integer.

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5. Which of the following sets is an orthonormal basis in $\boldsymbol{R}^3$? Which set is a basis? Why?
   a. $e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}, e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$
   b. $f_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, f_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}, f_3 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}$
   c. $g_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}, g_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, g_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}, g_4 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$
   d. $h_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, h_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, h_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$

6. Consider the following orthonormal basis (ONB) in $\boldsymbol{C}^4$ (verify it is indeed an ONB):

$$e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, e_3 = \frac{1}{\sqrt{5}}\begin{pmatrix}0\\0\\2\\1\end{pmatrix}, e_4 = \frac{1}{\sqrt{5}}\begin{pmatrix}0\\0\\-1\\2\end{pmatrix}$$

   a. Decompose the following vector with respect to this basis (that is, find coefficients c_1,c_2,c_3,c_4 so that $x=c_1e_1+c_2e_2+c_3e_3+c_4e_4$):

$$x = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

   b. Find the projection of x onto the linear span of e_1 and e_2.
   c. Find the vector y that lays in the span of e_3 and e_4 and is closest in Euclidian norm to vector x (that is, y that minimizes $||x-z||$ for all z in the linear span of e_3 and e_4)

7. Which of the following sequences are in $\boldsymbol{l}^2(\boldsymbol{Z})$?
   a. $x = (x_n)_{n\in Z}\ , x_n = \frac{1}{1+n^2}$
   b. $y = (y_n)_{n\in Z}\ , y_n = e^n$
   c. $z = (z_n)_{n\in Z}\ , z_n = \begin{cases} e^n, if\ |n| < 100 \\ 0\ , otherwise \end{cases}$.

Note: $l^2(\boldsymbol{Z})$ is defined as the set of square-summable sequences indexed by $\boldsymbol{Z}$. These are sequences $\{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$ so that $\sum_{n=-\infty}^{\infty}|x_n|^2 < \infty$.

8. Solve the following initial-value problem:

$$\frac{df}{dx} = -2xf(x)\,, \qquad f(0) = 3.$$